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Semesterproject

Examination of measure valued solutions for the magnetohydrodynamic equations

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Chapter 1

Introduction

1.1 The ideal Magnetohydrodynamic equations

The magnetohydrodynamic (MHD) equations describe the behaviour of electrically conducting fluids. For the *ideal MHD equations* we have to couple the Euler equations of fluid dynamics and the Maxwell equations of electrodynamics. A semi-conservative form of the ideal MHD equations is given by

Continuity equation:
$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0$$
 (1.1)
 $\frac{\partial}{\partial t} = 0$

$$Equation of motion: \qquad \frac{\partial}{\partial t}(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + (p + \frac{1}{2}(\mathbf{B} \cdot \mathbf{B}))I - \mathbf{B} \otimes \mathbf{B}) = -\mathbf{B}\operatorname{div}(\mathbf{B})$$

$$Induction \ equation: \qquad \frac{\partial \mathbf{B}}{\partial t} + \operatorname{div}(\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) = -\mathbf{u}\operatorname{div}(\mathbf{B}) \qquad (1.3)$$

$$Energy \ equation: \qquad \frac{\partial E}{\partial t} + \operatorname{div}\left((E + p + \frac{1}{2}\mathbf{B}^{2})\mathbf{u} - (\mathbf{u} \cdot \mathbf{B})\mathbf{B}\right) = -(\mathbf{u} \cdot \mathbf{B})\operatorname{div}(\mathbf{B}\emptyset1.4)$$

where $\rho = \rho(\mathbf{x}, t)$ denotes the mass density, **j** the current density, **B** the magnetic field, p the pressure, γ the ratio of specific heats and **u** is the vector of velocities. The RHS (right hand side) of (1.1) - (1.4) is called the *Godunov-Powell source term*. Since div(**B**) = 0 one could neglect it to receive a conservative form of the MHD equations. However, we keep it to make the equations Galilean invariant and to gain certain properties, like stability, for numerical schemes.

We will now give a derivation of this equations. First we will look at the conservation of mass (1.1). Therefore let $W \subset \mathbb{R}^3$ be a compact set with piecewise smooth boundary ∂W and outer normal $\mathbf{n}(\mathbf{x})$. The mass in W is given by $\int_W \rho(\mathbf{x}, t) d\mathbf{x}$. The rate of flow over a given point \mathbf{x} , the flux $f(\mathbf{x}, t)$, is given by velocity times the mass density

$$f(\mathbf{x},t) = \rho \mathbf{u}.\tag{1.5}$$

Therefore the flow over the boundary ∂W is given by the surface integral $\int_{\partial W} f(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) \, dS\mathbf{x}$. If we assume now that no mass is created or destroyed in W, then the quantity can only change due to flow through the boundary ∂W . Hence we get that

$$\frac{d}{dt} \int_{W} \rho(\mathbf{x}, t) \, d\mathbf{x} = -\int_{\partial W} \mathbf{f}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) \, dS(\mathbf{x}). \tag{1.6}$$

If we assume that ${\bf u}$ and ρ are differentiable functions, then by the divergence theorem it follows that

$$\int_{W} \frac{\partial}{\partial t} \rho(\mathbf{x}, t) \, d\mathbf{x} = -\int_{\partial W} \mathbf{f}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) \, dS(\mathbf{x}) = \int_{W} \operatorname{div}(\mathbf{f}(\mathbf{x}, t)) \, d\mathbf{x}.$$
(1.7)

Since W is arbitrary, this implies

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}) = 0, \qquad (1.8)$$

We have thus proven the *Continuity* equation (1.1).

Similar momentum is also conserved. Momentum $\rho \mathbf{u}$ is also advected by velocity, so the flux due to advection is the tensor product $\rho \mathbf{u} \otimes \mathbf{u}$. In addition to advection, momentum is affected by the pressure, given by the stress tensor pI with $I \in \mathbb{R}^{3\times 3}$ denoting the identity matrix. We get the flux

$$\mathbf{f}(\mathbf{x},t) = \rho \mathbf{u} \otimes \mathbf{u} + pI. \tag{1.9}$$

Furthermore we have to take forces that are acting on the medium, the body forces, into account. This is the *Lorentz force* $F = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$ acting on a particle carrying an electric charge q moving with velocity \mathbf{u} under the magnetic field \mathbf{B} and electric field \mathbf{E} . We will not include the electric force $q\mathbf{E}$, since it is negligible as we shall shortly show. Therefore the magnetic body force acting on the volume W is

$$\sum_{k: p_k \text{ in } W} q_k(\mathbf{u}_k \times \mathbf{B}) = \left(\sum_{k: p_k \text{ in } W} q_k \mathbf{u}_k\right) \times \mathbf{B} = \mathbf{j} \times \mathbf{B}$$
(1.10)

where p_k denotes a particle with electric charge q_k moving with speed \mathbf{u}_k and \mathbf{j} the current density. We get that the change of momentum equals the change over the boundary plus the body forces acting on the medium inside the Volume W

$$\frac{d}{dt} \int_{W} \rho \mathbf{u} \, d\mathbf{x} = -\int_{\partial W} (\rho \mathbf{u} \otimes \mathbf{u} + pI) \cdot \mathbf{n}(\mathbf{x}) \, dS(\mathbf{x}) + \mathbf{j} \times \mathbf{B}$$
(1.11)

Assuming differentiability as before, we get with the divergence theorem and since W was arbitrary

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + pI) = \mathbf{j} \times \mathbf{B}.$$
(1.12)

By using standard vector identities and the continuity equation (1.1) we get an equivalent form

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + J_{\mathbf{u}}\mathbf{u}\right) = -\nabla p + \mathbf{j} \times \mathbf{B}.$$
(1.13)

where $J_{\mathbf{u}}$ denotes the Jacobian matrix of \mathbf{u} Before we continue our derivation of the magneto-hydrodynamic equations we will state the classic Maxwell equation and derive them.

Faraday's law of induction
$$\operatorname{rot}(\mathbf{E}) = -\frac{\partial \mathbf{E}}{\partial t}$$
 (1.14)
 $\partial \mathbf{E}$

Ampere's law
$$\operatorname{rot}(\mathbf{B}) = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$
 (1.15)

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Gauss's law
$$\operatorname{div}(\mathbf{E}) = \frac{\rho_e}{\epsilon_0}$$
 (1.16)

Gauss's law for magnetism
$$\operatorname{div}(\mathbf{B}) = 0$$
 (1.17)

We will give a short derivation of these equations. Gauss's law for magnetism (1.17) follows from the fact that due to observations it's assumed that there are no magnetic monopoles in nature. Therefore if we take an arbitrary volume $W \in \mathbb{R}^3$ the flux through the boundary ∂W is zero. Using the divergence theorem we get

$$0 = \int_{\partial W} \mathbf{B} \, dS(\mathbf{x}) = \int_{W} \operatorname{div}(\mathbf{B}) \tag{1.18}$$

and since W was arbitrary this implies (1.17).

Next we will use Coulomb's law to show Gauss's law (1.16). Coulombs law is an experimental fact. One of its versions states that the force acting on a particle at rest with charge q and position \mathbf{r} is

$$\mathbf{F}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \rho_e(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d\mathbf{r}'$$
(1.19)

where ρ_e is the electric charge density and ϵ_0 is the permittivity of free space. By the definition of the electric field $\mathbf{E} = \frac{\mathbf{F}}{q}$ and by talking the divergence on both sides in the distributional sense we get (1.16)

$$\operatorname{div}(\mathbf{E})(\mathbf{r}) = \frac{1}{\epsilon_0} \int_{\mathbb{R}^3} \rho_e(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') d\mathbf{r}' = \frac{\rho_e(\mathbf{r})}{\epsilon_0}$$
(1.20)

where $\delta(\mathbf{r})$ denotes the delta distribution with pole at \mathbf{r} . We used that $\operatorname{div}(\frac{\mathbf{r}}{|\mathbf{r}|^3}) = 4\pi\delta(\mathbf{r})$.

For Faraday's induction law we will look at two inertial frames. Let our laboratory frame O be at rest with coordinates \mathbf{x} , electric field \mathbf{E} and a time independent magnetic field \mathbf{B} . Now we will move a conductor loop, encasing the area S, with constant speed

 \mathbf{u}_0 across this magnetic field. We denote the reference frame O' co-moving with the conductor loop. Then its coordinates are denoted by $\mathbf{x}' = \mathbf{x} - \mathbf{u}_0 t$. If we look at a particle with charge e moving with speed \mathbf{u} in the laboratory frame O it receives the Lorentz force

$$\mathbf{F} = e(\mathbf{E} + \mathbf{u} \times \mathbf{B}). \tag{1.21}$$

In the moving frame O' the particle with charge e is moving with speed $\mathbf{u} - \mathbf{u}_0$ receives the force

$$\mathbf{F}' = e(\mathbf{E}' + \mathbf{u} \times \mathbf{B}' - \mathbf{u}_0 \times \mathbf{B}'). \tag{1.22}$$

Since the forces **F** and **F'** must be the same and $\mathbf{B}'(\mathbf{x}') = \mathbf{B}'(\mathbf{x} - \mathbf{u}_0 t) = \mathbf{B}(\mathbf{x})$ it holds that

$$\mathbf{E}'(\mathbf{x}') = \mathbf{E}(\mathbf{x}) + \mathbf{u}_0 \times \mathbf{B}'(\mathbf{x}') = \mathbf{E}(\mathbf{x}) + \mathbf{u}_0 \times \mathbf{B}(\mathbf{x}).$$
(1.23)

Hence the electromotive force acting on the conductor loop, that is located at ∂S , is the line integral

$$\int_{\partial S} \mathbf{E}' \cdot d\mathbf{x}' = \int_{S} \operatorname{rot}(\mathbf{E} + \mathbf{u}_0 \times \mathbf{B}) \cdot \mathbf{n} \, dS(\mathbf{x})$$
(1.24)

$$= \int_{S} (\mathbf{u}_0 \operatorname{div}(\mathbf{B}) - \mathbf{B} \operatorname{div}(\mathbf{u}_0) + J_{\mathbf{B}} \mathbf{u}_0 - J_{\mathbf{u}_0} \mathbf{B}) \cdot \mathbf{n} \, dS(\mathbf{x}) \quad (1.25)$$

$$= \int_{S} J_{\mathbf{B}} \mathbf{u}_0 \cdot \mathbf{n} \, dS(\mathbf{x}) \tag{1.26}$$

where we used Stokes theorem for the first equality. Further we used that for the time independent fields \mathbf{E}, \mathbf{B} we have the static field equations $\operatorname{div}(\mathbf{B}) = 0$ and $\operatorname{rot}(\mathbf{E}) = 0$. On the other hand \mathbf{B}' is not time independent, since $\mathbf{B}'(\mathbf{x}', t) = \mathbf{B}(\mathbf{x}) = \mathbf{B}(\mathbf{x}' - \mathbf{u}_0 t)$. Therefore we get in O' by the chain rule

$$\frac{\partial \mathbf{B}'}{\partial t} = -J_{\mathbf{B}}\mathbf{u}_0. \tag{1.28}$$

We get

$$\int_{S} \operatorname{rot} \mathbf{E}' \cdot \mathbf{n} \ dS(\mathbf{x}') = \int_{\partial S} \mathbf{E}' \cdot d\mathbf{x}' = -\int_{S} \frac{\partial \mathbf{B}'}{\partial t} \cdot \mathbf{n} \ dS(\mathbf{x}').$$
(1.29)

Since we can choose S and the conductor loop arbitrary we get for any inertial system Faraday's induction law (1.14)

$$\operatorname{rot}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.30}$$

For the derivation of Ampere's law we will start with a experimental fact, the Biot-Savart law. It is a magnetic equivalent to Coulomb's law for magnetic fields and that one is obtained by dividing both sides of (1.19) by q. The Biot-Savart law states that

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \mathbf{j}(\mathbf{r}') \times \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d\mathbf{r}'.$$
 (1.31)

Since $\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\nabla_{\mathbf{r}}(\frac{1}{|\mathbf{r}-\mathbf{r}'|})$ and with the product rule for curls

$$\operatorname{rot}\left(\mathbf{j}(\mathbf{r}')\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{rot}(\mathbf{j}(\mathbf{r}')) + \nabla_{\mathbf{r}}\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \times \mathbf{j}(\mathbf{r}') = (1.32)$$

$$= -\mathbf{j}(\mathbf{r}') \times \nabla_{\mathbf{r}} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right)$$
(1.33)

where we used that \mathbf{j} does not depend on \mathbf{r} . We get

$$\mathbf{B}(\mathbf{r}) = \operatorname{rot}\underbrace{\left(\frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}'\right)}_{=: \mathbf{A}(\mathbf{r})}.$$
(1.34)

This implies that

$$\operatorname{rot}(\mathbf{B}) = \nabla(\operatorname{div}(\mathbf{A})) - \Delta(\mathbf{A}).$$
(1.35)

Since

$$\nabla_{\mathbf{r}} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) = -\nabla_{\mathbf{r}'} \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right)$$
(1.36)

$$\Delta\left(\frac{1}{\mathbf{r}-\mathbf{r}'}\right) = -4\pi\delta(\mathbf{r}-\mathbf{r}') \tag{1.37}$$

it follows

$$\operatorname{rot}(\mathbf{B}) = -\nabla \left(\frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \mathbf{j}(\mathbf{r}') \cdot \nabla_{\mathbf{r}'} \frac{1}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \right) + \mu_0 \int_{\mathbb{R}^3} \mathbf{j}(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') d\mathbf{r}'$$
(1.38)

. We are proofing the classical Ampere's law for the magneto static case and therefore

$$\operatorname{div}(\mathbf{j}) = 0 \tag{1.39}$$

holds. By performing partial integration on the first term of the right hand side of (1.38) and using (1.39) we get the classical law of Ampere

$$\operatorname{rot}(\mathbf{B}) = \mu_0 \mathbf{j} \tag{1.40}$$

So far we have looked at results from Gauss, Faraday and Ampere. Maxwell's contribution to the equations (1.14) - (1.17) was that he realized that in the general dynamic

case Ampere's law violates the continuity equations of electrodynamics

$$\operatorname{div}(\mathbf{j}) = \operatorname{div}(\rho_e \mathbf{u}) = -\frac{\partial \rho_e}{\partial t}$$
(1.41)

which is similar to the continuity equation of hydrodynamics (1.8). If we plug $-\frac{\partial \rho_e}{\partial t}$ into (1.38) for div(**j**) instead of setting it zero as in the static case (1.39) we get

$$-\nabla\left(\frac{\mu_0}{4\pi}\int_{\mathbb{R}^3}\mathbf{j}(\mathbf{r}')\cdot\nabla_{\mathbf{r}'}\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\right) = \nabla\left(\frac{\mu_0}{4\pi}\int_{\mathbb{R}^3}\operatorname{div}(\mathbf{j}(\mathbf{r}'))\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\right) = (1.42)$$
$$= \nabla\left(\frac{\mu_0}{4\pi}\int_{\mathbb{R}^3}\operatorname{div}(\mathbf{j}(\mathbf{r}'))\frac{1}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\right) = (1.42)$$

$$= -\nabla \left(\frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{\partial \rho_e}{\partial t} \frac{1}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}'\right) = (1.43)$$

$$= -\frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{\partial \rho_e}{\partial t} \nabla_{\mathbf{r}} \frac{1}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' =$$
(1.44)

$$= \frac{\mu_0}{4\pi} \int_{\mathbb{R}^3} \frac{\partial \rho_e}{\partial t} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d\mathbf{r}' =$$
(1.45)

$$= \mu_0 \epsilon_0 \frac{\partial E}{\partial t}.$$
 (1.46)

This term is called the *displacement current*. With the second term of (1.38) we get Ampere's law for the dynamic case

$$\operatorname{rot}(\mathbf{B}) = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$
(1.47)

This equation completes the Maxwell equations (1.14) - (1.17).

After we have derived the Maxwell equations, we now have to incorporated them into the MHD-equations. First we will use Ampere's law (1.15) without the displacement current. An explanation why we can neglect the displacement current is given later in [.....LINK......]. In the following we will chose units such that the magnetic permeability $\mu_0 = 1$. Therefore we get $\mathbf{j} = \operatorname{rot}(\mathbf{B})$ and it follows that

$$\mathbf{j} \times \mathbf{B} = \operatorname{rot}(\mathbf{B}) \times \mathbf{B} = -\mathbf{B} \times \operatorname{rot}(\mathbf{B}) = -\frac{1}{2}\nabla(\mathbf{B} \cdot \mathbf{B}) + J_{\mathbf{B}}\mathbf{B}$$
 (1.48)

with $J_{\mathbf{B}}$ denoting the Jacobi matrix. Further, using the product rule, one can show that

$$-\frac{1}{2}\nabla(\mathbf{B}\cdot\mathbf{B}) + J_{\mathbf{B}}\mathbf{B} = -\frac{1}{2}\nabla(\mathbf{B}\cdot\mathbf{B}) + \operatorname{div}(\mathbf{B}\otimes\mathbf{B}) - \mathbf{B}\operatorname{div}(\mathbf{B}).$$
(1.49)

Replacing $\mathbf{j} \times \mathbf{B}$ in (1.12) according to (1.49) and using $\nabla(\mathbf{B} \cdot \mathbf{B}) = \operatorname{div}(\mathbf{B} \cdot \mathbf{B})I$ we get the MHD-approximated Equation of motion

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u} + (p + \frac{1}{2}(\mathbf{B} \cdot \mathbf{B}))I - \mathbf{B} \otimes \mathbf{B}) = -\mathbf{B}\operatorname{div}(\mathbf{B})$$
(1.50)

This is exactly equation (1.2).

Next we will look at the magnetic induction equation. We start with Faraday's induction law (1.14)

$$\operatorname{rot}(\mathbf{E}) = -\frac{\partial \mathbf{B}}{\partial t}.$$
(1.51)

Further we use a simple version of Ohm's law that states

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j} \tag{1.52}$$

where η is the resistivity. Substituting this equation into Faraday's induction law leads to

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{rot}(\mathbf{u} \times \mathbf{B}) - \operatorname{rot}(\eta \mathbf{j})$$
(1.53)

the so called resistive magnetic induction equation. We are looking at the ideal MHDequations and therefore assume that the resistivity is so small that we can neglect the term $rot(\eta \mathbf{j})$. By using standard vector identities we get

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{rot}(\mathbf{u} \times \mathbf{B}) = -\operatorname{div}(\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u})$$
(1.54)

Hence, by adding $0 = -\mathbf{u} \operatorname{div}(\mathbf{B})$ on the right hand side we get the magnetic induction equation (1.3)

$$\frac{\partial \mathbf{B}}{\partial t} + \operatorname{div}(\mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u}) = -\mathbf{u}\operatorname{div}(\mathbf{B})$$
(1.55)

Next we will look at the energy equation (1.4). We start with the hydrodynamic conservation law for energy

$$\frac{\partial E_h}{\partial t} + \operatorname{div}((E_h + p)\mathbf{u}) = 0 \tag{1.56}$$

where E_h denotes the hydrodynamic energy of an ideal gas. It is composed of the internal energy and the kinetic energy

$$E_h = \frac{p}{\gamma - 1} + \frac{1}{2}\rho \mathbf{u}^2.$$

Equation (1.56) is part of the Euler equations. In the magneto hydrodynamic case we also have to take the work resulting from the electromagnetic force (1.10) acting on the fluid particles into account. We get

$$\frac{\partial E_h}{\partial t} + \operatorname{div}((E_h + p)\mathbf{u}) = \mathbf{u} \cdot (\mathbf{j} \times \mathbf{B})$$
(1.57)

Using Ohm's law without resistivity we get

$$\mathbf{u} \cdot (\mathbf{j} \times \mathbf{B}) = -\mathbf{j} \cdot (\mathbf{u} \times \mathbf{B}) = \mathbf{j} \cdot \mathbf{E}$$
(1.58)

Further by multiplying Faraday's induction law (1.14) with **B** and adding Ampere's law (1.15) multiplied with $-\mathbf{E}$ without displacement current we get

$$\underbrace{\operatorname{rot}(\mathbf{E}) \cdot \mathbf{B} - \operatorname{rot}(\mathbf{B}) \cdot \mathbf{E}}_{= \operatorname{div}(\mathbf{E} \times \mathbf{B})} = -\frac{1}{2} \frac{\partial \mathbf{B}^2}{\partial t} - \mathbf{j} \cdot \mathbf{E}$$
(1.59)

Since $\operatorname{div}(\mathbf{E} \times \mathbf{B}) = \mathbf{B}^2 \mathbf{u} + (\mathbf{u} \cdot \mathbf{B}) \cdot \mathbf{B}$ we get with (1.58) and (1.59) that

$$\mathbf{u} \cdot (\mathbf{j} \times \mathbf{B}) = -\operatorname{div}(\mathbf{B}^2 \mathbf{u} + (\mathbf{u} \cdot \mathbf{B}) \cdot \mathbf{B}) - \frac{1}{2} \frac{\partial \mathbf{B}^2}{\partial t}$$

Plugging this into (1.57) and defining the total energy as the hydrodynamic energy plus the magnetic pressure $E = E_h + \frac{1}{2}\mathbf{B}^2$ we get the *energy equation*

$$\frac{\partial E}{\partial t} + \operatorname{div}\left((E + p + \frac{1}{2}\mathbf{B}^2)\mathbf{u} - (\mathbf{u} \cdot \mathbf{B})\mathbf{B}\right) = -(\mathbf{u} \cdot \mathbf{B})\operatorname{div}(\mathbf{B})$$
(1.60)

where we added $-(\mathbf{u} \cdot \mathbf{B}) \operatorname{div}(\mathbf{B}) = 0.$

In (1.10) we neglected the electric force term $\rho_e \mathbf{E}$. Additionally we used Ampere's law without the displacement current. We have to discuss why we can neglect those terms, if $u_a \ll c_0$, with the Alfvén speed $u_a = \frac{|B|}{\sqrt{\mu_0 \rho}}$. We start by solving equation (1.15) for **j** and substituting the result into the equation of motion (1.13). We get

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + J_{\mathbf{u}}\mathbf{u}\right) = -\nabla p + \frac{1}{\mu_0} \operatorname{rot}(\mathbf{B}) \times \mathbf{B} - \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}.$$
 (1.61)

The displacement current produces an extra $-\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}$ term. We will show that this term is small compared to $\rho \frac{\partial \mathbf{u}}{\partial t}$. For the ideal MHD equations it is assumed that one can ignore resistivity. Hence Ohm's law (1.52) is $E = -\mathbf{u} \times \mathbf{B}$ and therefore

$$-\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B} = \epsilon_0 \frac{\partial}{\partial t} ((\mathbf{u} \times \mathbf{B}) \times \mathbf{B}) \approx \epsilon_0 |B|^2 \frac{\partial \mathbf{u}}{\partial t} = \frac{|B|^2}{c_0^2 \mu_0} \frac{\partial \mathbf{u}}{\partial t}.$$
 (1.62)

Therefore the $\frac{\partial \mathbf{E}}{\partial t}$ term is by the factor $\frac{|B|^2}{c_0^2 \mu_0 \rho} = \frac{u_a^2}{c_0^2}$ smaller than the first term $\rho \frac{\partial \mathbf{u}}{\partial t}$ in (1.61). Next we show that the electric force term $\rho_e \mathbf{E}$ is small compared to $\rho J_{\mathbf{u}} \mathbf{u}$ term. By Gauss's law (1.15) and Ohm's law $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$ we have

$$\rho_e \mathbf{E} = \epsilon_0 \mathbf{E} \operatorname{div}(\mathbf{E}) = \frac{1}{c_0^2 \mu_0} (\mathbf{u} \times \mathbf{B}) \operatorname{div}(\mathbf{u} \times \mathbf{B}) \approx \frac{|B|^2}{c_0^2 \mu_0 \rho} \rho J_{\mathbf{u}} \mathbf{u}$$
(1.63)

In making the last approximation we assume that the length scale of the third term is comparable to the last term. According to [.....LINK to Plasma for Astroph.] its almost always the case that these length scales are at least of same order of magnitude. We have that $\rho_e \mathbf{E}$ is smaller than $\rho J_{\mathbf{u}} \mathbf{u}$ by the same factor $\frac{u_a^2}{c_0^2}$ and therefore we can drop $\rho_e \mathbf{E}$ and $-\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \times \mathbf{B}$, if $u_a \ll c_0$.

1.2 The entropy equation

(Warum .. kurze Erklärung)

We will now derive a evolution equation for the pressure. Therefore we start with the energy equation (1.4) and substitute for the total energy $E = \frac{1}{2}\rho \mathbf{u}^2 + \frac{p}{\gamma-1} + \frac{1}{2}\mathbf{B}^2$.

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho \mathbf{u}^2 + \frac{p}{\gamma - 1} + \frac{1}{2} \mathbf{B}^2 \right) + \operatorname{div} \left(\left(\frac{1}{2} \rho \mathbf{u}^2 + \frac{\gamma p}{\gamma - 1} + \mathbf{B}^2 \right) \mathbf{u} - (\mathbf{u} \cdot \mathbf{B}) \mathbf{B} \right) = -(\mathbf{u} \cdot \mathbf{B}) \operatorname{div}(\mathbf{B})$$
(1.64)

With the product rule we get that

$$\frac{\partial}{\partial t} \left(\frac{1}{2}\rho \mathbf{u}^2\right) + \operatorname{div}\left(\frac{1}{2}\rho \mathbf{u}^2 \mathbf{u}\right) = \frac{\mathbf{u}^2}{2} \frac{\partial \rho}{\partial t} + \rho \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{u} + \frac{\mathbf{u}^2}{2} \operatorname{div}(\rho \mathbf{u}) + \rho(J_{\mathbf{u}}\mathbf{u}) \cdot \mathbf{u}$$

$$= \rho \frac{\partial \mathbf{u}}{\partial t} \cdot \mathbf{u} + \rho(J_{\mathbf{u}}\mathbf{u}) \cdot \mathbf{u}$$
(1.65)

For the second equality we used the conservation of mass. Next we will use Ohm's law without resistivity $\mathbf{E} = -\mathbf{u} \times \mathbf{B}$ and standard vector identities to get that

$$\mathbf{B}^{2}\mathbf{u} - (\mathbf{u} \cdot \mathbf{B})\mathbf{B} = \mathbf{B} \times (\mathbf{u} \times \mathbf{B}) = \mathbf{E} \times \mathbf{B}.$$
 (1.66)

Hence

$$\operatorname{div}(\mathbf{B}^{2}\mathbf{u} - (\mathbf{u} \cdot \mathbf{B})\mathbf{B}) = \operatorname{div}(\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot \operatorname{rot}(\mathbf{E}) - \mathbf{E} \cdot \operatorname{rot}(\mathbf{B})$$
$$= -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{E} \cdot \mathbf{j}$$
$$= -\frac{1}{2} \frac{\partial \mathbf{B}^{2}}{\partial t} - (\mathbf{j} \times \mathbf{B}) \cdot \mathbf{u}$$
(1.67)

where we used Faraday's law of induction (1.14), Amperes's law (1.15) without the displacement current and Ohm's law. Applying the two equations (1.65) and (1.67) to the energy equation (1.64) we get

$$\underbrace{\rho\left(\frac{\partial \mathbf{u}}{\partial t} + J_{\mathbf{u}}\mathbf{u}\right) \cdot \mathbf{u} - (\mathbf{j} \times \mathbf{B}) \cdot \mathbf{u} + \nabla p \cdot \mathbf{u}}_{= 0} - \nabla p \cdot \mathbf{u} + \frac{1}{\gamma - 1} \frac{\partial p}{\partial t} + \operatorname{div}\left(\frac{\gamma p \mathbf{u}}{\gamma - 1}\right) = -(\mathbf{u} \cdot \mathbf{B})\operatorname{div}(\mathbf{B})$$

where we used the equation of motion in the form (1.13). By applying the product rule we get the *pressure equation*

$$\frac{\partial p}{\partial t} + \gamma \operatorname{div}(\mathbf{u})p + \nabla p \cdot \mathbf{u} = -(\gamma - 1)(\mathbf{u} \cdot \mathbf{B})\operatorname{div}(\mathbf{B}).$$
(1.68)

With this pressure equation it is possible to derive an entropy equality if we define the entropy S and the corresponding entropy flux \mathbf{Q} as

$$S = -\frac{\rho s}{\gamma - 1}$$
 and $\mathbf{Q} = -\frac{\rho s \mathbf{u}}{\gamma - 1}$ with $s = \log(p) - \gamma \log(\rho)$.

We want to prove the following entropy equality

$$\frac{\partial S}{\partial t} + \operatorname{div}(\mathbf{Q}) = \frac{\rho(\mathbf{u} \cdot \mathbf{B})}{p} \operatorname{div}(\mathbf{B})$$
(1.69)

By the product rule we get

$$\frac{\partial S}{\partial t} + \operatorname{div}(\mathbf{Q}) = -\frac{1}{\gamma - 1} \left(\rho \frac{\partial s}{\partial t} + \underbrace{\frac{\partial \rho}{\partial t} s + \operatorname{div}(\mathbf{u})\rho s + (\nabla \rho \cdot \mathbf{u})s}_{= (\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{u}))s = 0} + (\nabla s \cdot \mathbf{u})\rho \right)$$

where we used the conservation of mass (1.1). Furthermore by substituting $s = \log(p) - \gamma \log(\rho)$ and adding $0 = \frac{\gamma}{\gamma - 1}\rho \operatorname{div}(\mathbf{u}) - \frac{\gamma}{\gamma - 1}\rho \operatorname{div}(\mathbf{u})$ we get that the left side of (1.69) is equal to

$$-\frac{1}{\gamma-1}\left(\frac{\rho}{p}\left(\frac{\partial p}{\partial t}+\nabla p\cdot\mathbf{u}\right)+\gamma\rho\mathrm{div}(\mathbf{u})\underbrace{-\gamma\rho\mathrm{div}(\mathbf{u})-\gamma\left(\frac{\partial \rho}{\partial t}+\nabla\rho\cdot\mathbf{u}\right)}_{=-\gamma\left(\frac{\partial \rho}{\partial t}+\mathrm{div}(\rho\mathbf{u})\right)=0}\right)$$

where we used the conservation of mass another time. We are left with $-\frac{\rho}{(\gamma-1)p}$ times the pressure equality (1.68)

$$-\frac{\rho}{(\gamma-1)p}\left(\frac{\partial p}{\partial t}+\nabla p\cdot\mathbf{u}+\gamma\mathrm{div}(\mathbf{u})p\right)=\frac{\rho(\mathbf{u}\cdot\mathbf{B})}{p}\mathrm{div}(\mathbf{B}).$$

which we just derived from the energy equality. This proves the entropy equality (1.69).

1.3 Characteristic analysis in one space dimension

We can transform the MHD equations in conservative form into the following form

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = 0 \tag{1.70}$$

with the *primitive variables* $\mathbf{U} = [\rho, u_1, u_2, u_3, B_1, B_2, B_3, p]^{\top}$ and the partial derivative of \mathbf{U} with respect to t is denoted by \mathbf{U}_t . Because of the one-dimensional assumptions B_1 is a constant. By the chain rule (1.70) is equivalent to

$$\mathbf{U}_t + A\mathbf{U}_x = 0 \tag{1.71}$$

where $A = \mathbf{F}_{\mathbf{U}}$ is the Jacobian of \mathbf{F} . Here we don't force div $(\mathbf{B}) = 0$ since we want to keep the Powell source term for numerical properties of some schemes. Otherwise it would follow that $(B_1)_x = 0$ and we could reduce the matrix (1.72) to a 7 × 7 matrix. In our case

$$A = \begin{bmatrix} u_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_1 & 0 & 0 & 0 & \frac{B_2}{\rho} & \frac{B_3}{\rho} & \frac{1}{\rho} \\ 0 & 0 & u_1 & 0 & 0 & -\frac{B_1}{\rho} & 0 & 0 \\ 0 & 0 & 0 & u_1 & 0 & 0 & -\frac{B_1}{\rho} & 0 \\ 0 & 0 & 0 & 0 & u_1 & 0 & 0 \\ 0 & B_2 & -B_1 & 0 & 0 & u_1 & 0 \\ 0 & B_3 & 0 & -B_1 & 0 & 0 & u_1 & 0 \\ 0 & a^2\rho & 0 & 0 & 0 & 0 & 0 & u_1 \end{bmatrix}$$
(1.72)

where $a = \sqrt{\frac{\gamma p}{\rho}}$ is the sound speed. By looking at the kernel of det $(A - \lambda I)$ we get the eigenvalues of A

$$\lambda_{1,2} = u_1 \pm c_f, \qquad \lambda_{3,4} = u_1 \pm c_A, \qquad \lambda_{5,6} = u_1 \pm c_s, \qquad \lambda_{7,8} = u_1 \tag{1.73}$$

where c_f is the fast magneto-sonic wave speed and c_s denotes the slow magneto-sonic wave speed. The waves corresponding to these speeds are longitudinal waves with variations in density and pressure. The waves belonging to c_A , the Alfvén speed, are transverse waves with no variation in pressure and density. The wave that belongs to λ_7 is called the entropy wave and it is a contact discontinuity with no variation in pressure and velocity. Further we have an additional *divergence wave* with wave speed u_1 . The values of c_f, c_s and c_A are given by

$$c_{f}^{2} = \frac{1}{2} \left(a^{2} + \frac{\mathbf{B}^{2}}{\rho} + \sqrt{\left(a^{2} + \frac{\mathbf{B}^{2}}{\rho}\right)^{2} - 4a^{2}\frac{B_{1}^{2}}{\rho}} \right)$$
$$c_{A}^{2} = \frac{B_{1}^{2}}{\rho}$$
$$c_{s}^{2} = \frac{1}{2} \left(a^{2} + \frac{\mathbf{B}^{2}}{\rho} - \sqrt{\left(a^{2} + \frac{\mathbf{B}^{2}}{\rho}\right)^{2} - 4a^{2}\frac{B_{1}^{2}}{\rho}} \right)$$

It holds that

$$c_s \le c_A \le c_f.$$

Since the eigenvalues (1.73) are real the MHD equations are hyperbolic. In our case of eight eigenvalues $\lambda_7 = \lambda_8$ and therefore the MHD system of equations is not strictly hyperbolic. Furthermore there are several cases where some of the other wave speeds can be equal. For example if $B_1^2 = a^2$ and $B_2^2 + B_3^2 = 0$ then $c_s = c_A = c_f$. This case is called the "triple umbilic".

1.4 Numerical schemes

We can write the two dimensional semi-conservative form of the MHD equations (1.1) - (1.4) in the following form

$$\mathbf{W}_t + \mathbf{F}(\mathbf{W})_x + \mathbf{G}(\mathbf{W})_y = S^1(\mathbf{W}, \mathbf{W}_x) + S^2(\mathbf{W}, \mathbf{W}_y)$$
(1.74)

where the vector of *conserved variables* is given by

$$\mathbf{W} = [\rho, \rho u_1, \rho u_2, \rho u_3, B_1, B_2, B_3, E]^\top$$

The fluxes are

$$\mathbf{F}(\mathbf{W}) = \begin{pmatrix} \rho u_1 \\ \rho u_1^2 + \pi_1 - \frac{B_1^2}{2} \\ \rho u_1 u_2 - B_1 B_2 \\ \rho u_1 u_3 - B_1 B_3 \\ 0 \\ u_2 B_1 - u_1 B_2 \\ u_3 B_1 - u_1 B_3 \\ (E + \pi_1) u_1 - u_1 \frac{B_1^2}{2} - B_1 (u_2 B_2 + u_3 B_3) \end{pmatrix},$$

and

$$\mathbf{G}(\mathbf{W}) = \begin{pmatrix} \rho u_2 \\ \rho u_1 u_2 - B_1 B_2 \\ \rho u_2^2 + \pi_2 - \frac{B_2^2}{2} \\ \rho u_3 u_2 - B_3 B_2 \\ u_1 B_2 - u_2 B_1 \\ 0 \\ u_3 B_2 - u_2 B_3 \\ (E + \pi_2) u_2 - u_2 \frac{B_2^2}{2} - B_2 (u_1 B_1 + u_3 B_3) \end{pmatrix},$$

where we have defined

$$\pi_1 = p + \frac{B_2^2 + B_3^2}{2}$$
, and $\pi_2 = p + \frac{B_1^2 + B_3^2}{2}$

For the Godunov-Powell source term we get

$$\mathbf{S}^{1}(\mathbf{W}, \mathbf{W}_{x}) = \begin{pmatrix} 0 \\ -\left(\frac{B_{1}^{2}}{2}\right)_{x} \\ -B_{2}(B_{1})_{x} \\ -B_{3}(B_{1})_{x} \\ -u_{1}(B_{1})_{x} \\ -u_{2}(B_{1})_{x} \\ -u_{2}(B_{1})_{x} \\ -u_{3}(B_{1})_{x} \\ -u_{1}\left(\frac{B_{1}^{1}}{2}\right)_{x} - (u_{2}B_{2} + u_{3}B_{3})(B_{1})_{x} \end{pmatrix}$$
(1.75)

and

$$\mathbf{S}^{2}(\mathbf{W}, \mathbf{W}_{y}) = \begin{pmatrix} 0 \\ -B_{1}(B_{2})_{y} \\ -\left(\frac{B_{2}^{2}}{2}\right)_{y} \\ -B_{3}(B_{2})_{y} \\ -B_{3}(B_{2})_{y} \\ -u_{1}(B_{2})_{y} \\ -u_{2}(B_{2})_{y} \\ -u_{2}(B_{2})_{y} \\ -u_{2}\left(\frac{B_{2}^{1}}{2}\right)_{y} - (u_{1}B_{1} + u_{3}B_{3})(B_{2})_{y} \end{pmatrix}$$
(1.76)

We will approximate the solution of the problem on the domain $[X_L, X_R] \times [Y_L, Y_R]$ and therefore define a uniform grid with gridsizes Δx and Δy . We set

$$x_i = X_L + i\Delta x, \quad y_j = y_L + j\Delta y \quad \text{and} \quad I_{i,j} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}).$$
 (1.77)

We denote the cell average of \mathbf{W} over $I_{i,j}$ at time t^n with $\mathbf{W}_{i,j}^n$. A standard finite volume scheme is obtained by integrating the balance law (1.74) over the the cell $I_{i,j}$ and the time interval $[t^n, t^{n+1})$ with $t^{n+1} = t^n + \Delta t^n$, where Δt_n is determined by a suitable CFL condition. The resulting fully discrete form of the scheme is

$$\mathbf{W}_{i,j}^{n+1} = \mathbf{W}_{i,j}^{n} - \frac{\Delta t^{n}}{\Delta x} (\mathbf{F}_{i+\frac{1}{2},j}^{n} - \mathbf{F}_{i-\frac{1}{2},j}^{n}) - \frac{\Delta t^{n}}{\Delta y} (\mathbf{G}_{i,j+\frac{1}{2}}^{n} - \mathbf{G}_{i,j-\frac{1}{2}}^{n}) + \Delta t^{n} (\mathbf{S}_{i,j}^{1} + \mathbf{S}_{i,j}^{2}).$$

The flux in x-direction

$$\mathbf{F}_{i+\frac{1}{2},j}^n = \mathbf{F}(\mathbf{W}_{i,j}^n, \mathbf{W}_{i,j}^{n+1}).$$

and the source term $S_{i,j}^1$ are given by solutions to the Riemann problem

$$\mathbf{W}_t + \mathbf{f}(\mathbf{W})_x = \mathbf{S}^1(\mathbf{W}, \mathbf{W}_x), \qquad \mathbf{W}(x, 0) = \begin{cases} \mathbf{W}_L, & x < 0\\ \mathbf{W}_R, & x > 0 \end{cases}$$

and the flux and source term in y-direction are obtain by the corresponding Riemann problem. [..... Think about it again ... local conservation but not other problem ... maybe leave it away + local conservation]

1.4.1 A three-wave HLL solver

We approximate the Riemann problem by three waves with wave speeds s_L , s_R and s_M . The approximate solutions and fluxes are given by

$$\mathbf{W}^{\text{HLL3}} = \begin{cases} \mathbf{W}_L, & \text{if } \frac{x}{t} \leq s_L \\ \mathbf{W}_L^*, & \text{if } s_L < \frac{x}{t} \leq s_M \\ \mathbf{W}_R^*, & \text{if } s_M < \frac{x}{t} \leq s_R \\ \mathbf{W}_R, & \text{if } s_R < \frac{x}{t} \end{cases}$$
$$\mathbf{F}^{\text{HLL3}}(\mathbf{W}_L, \mathbf{W}_R) = \begin{cases} \mathbf{F}_L, & \text{if } \frac{x}{t} \leq s_L \\ \mathbf{F}_L^*, & \text{if } s_L < \frac{x}{t} \leq s_M \\ \mathbf{F}_R^*, & \text{if } s_M < \frac{x}{t} \leq s_R \\ \mathbf{F}_R, & \text{if } s_R < \frac{x}{t} \end{cases}$$

The outer wave speeds are chosen as in [..... LINK zu Gurski]. The left wave speed s_L for example is chosen as the minimum of the smallest eigenvalue of the matrix (1.72) evaluated once at \mathbf{W}_L and once at the Roe averaged state $\overline{\mathbf{W}}$.

$$s_L = \min(u_{1L} - c_{fL}, \overline{u_1} - \overline{c_f})$$
 $s_R = \min(u_{1R} + c_{fR}, \overline{u_1} + \overline{c_f})$

In this solver the middle wave models a material contact discontinuity with similar properties than the compound entropy/divergence wave. Hence, the velocity field and the tangential magnetic fields are assumed to be constant across the middle wave. Therefore

$$\mathbf{u}^* = \mathbf{u}_L^* = \mathbf{u}_R^*, \quad B_2^* = B_{2L}^* = B_{2R}^* \text{ and } B_3^* = B_{3L}^* = B_{3R}^*$$

We do not fix $\operatorname{div}(\mathbf{B}) = 0$ and let the normal magnetic field B_1 jump across the middle wave. Across the outer waves it is constant. Since the $/di(\mathbf{B})$ does only change across the middle wave the source term does only affect the Rankine-Hugoniot conditions for the middle wave. We get

$$s_L(\mathbf{W}_L^* - \mathbf{W}_L) = \mathbf{F}_L^* - \mathbf{F}_L, \qquad (1.78)$$

$$s_R(\mathbf{W}_R^* - \mathbf{W}_R) = \mathbf{F}_R^* - \mathbf{F}_R, \qquad (1.79)$$

$$s_M(\mathbf{W}_R^* - \mathbf{W}_L^*) = \mathbf{F}_R^* - \mathbf{F}_L^* + \mathbf{S}^{1,*}$$
(1.80)

with

$$\mathbf{S}^{1,*} = \begin{pmatrix} 0 \\ -\frac{(B_{1R})^2 - (B_{1R})^2}{2} \\ -B_2^*(B_{1R} - B_{1L}) \\ -B_3^*(B_{1R} - B_{1L}) \\ -\mathbf{u}^*(B_{1R} - B_{1L}) \\ -u_1^* \frac{(B_{1R})^2 - (B_{1R})^2}{2} - (u_2^* B_2^* + u_3^* B_3^*)(B_{1R} - B_{1L}) \end{pmatrix}$$
(1.81)

We will now explain, why we must have the jump condition (1.80). Since B_2, B_3 and **u** are constant across the middle wave, the source term (1.75) may be written in the form $S^1(\mathbf{W}, \mathbf{W}_x) = (\mathbf{T}(\mathbf{u}, \mathbf{B})B_1)_x$ with

$$\mathbf{T}(\mathbf{u}, \mathbf{B}) = \left(0, \frac{B_1}{2}, B_2, B_3, \mathbf{u}, \frac{u_1 B_1}{2} + u_2 B_2 + u_3 B_3\right)^{\mathsf{T}}$$

Therefore

$$\mathbf{W}_t + \mathbf{F}(\mathbf{W})_x + (\mathbf{T}(\mathbf{u}, \mathbf{B})B_1)_x = 0$$

and local conservation gives

$$s_M(\mathbf{W}_R^* - \mathbf{W}_L^*) = \mathbf{F}_R^* - \mathbf{F}_L^* + \mathbf{T}_R^* B_{1R}^* - \mathbf{T}_L^* B_{1L}^*.$$

Relation (1.80) then follows from $\mathbf{T}_R^* B_{1R}^* - \mathbf{T}_L^* B_{1L}^* = \mathbf{S}^{1,*}$. We define

$$\mathbf{S}_{i}^{1,n} = \mathbf{S}_{i-\frac{1}{2}}^{1,*} \mathbf{1}_{(s_{M,i-\frac{1}{2}} \ge 0)} + \mathbf{S}_{i+\frac{1}{2}}^{1,*} \mathbf{1}_{(s_{M,i+\frac{1}{2}} < 0)}$$
(1.82)

We set $u_1^* = s_M$, since the middle wave of the exact problem has speed u_1 . Therefore we get with (1.78) and (1.79)

$$\rho_{\theta}^* = \rho_{\theta} \frac{u_{1\theta} - s_{\theta}}{s_M - s_{\theta}}, \qquad \theta \in \{L, R\}.$$
(1.83)

Combining all three conservation equation results in

$$\mathbf{F}_{R} - \mathbf{F}_{L} = s_{R}\mathbf{W}_{R} - s_{L}\mathbf{W}_{L} + (s_{M} - s_{R})\mathbf{W}_{R}^{*} + (s_{L} - s_{M})\mathbf{W}_{L}^{*} + S^{1,*}.$$

We can solve the second component of this equation for s_M . Therefore we use equation (1.83) and that $s_M = u_{1L}^* = u_{1R}^* = u_1^*$ to get

$$s_M = u_1^* = \frac{\pi_{1R} - \pi_{1L} + \rho_R u_{1R}(u_{1R} - s_R) - \rho_L u_{1L}(u_{1L} - s_L)}{\rho_R(u_{1R} - s_R) - \rho_L(u_{1L} - s_L)}.$$

Similarly, by using local conservation across the outer waves (1.78) and (1.79) we get

$$\pi_{1\theta}^* = \pi_{1\theta} + \rho_{\theta}(u_{1\theta} - s_{\theta})(u_{1\theta} - s_M)$$

Further we get that

$$u_{\sigma}^* = \frac{\zeta c_{\sigma} - \beta d_{\sigma}}{\alpha \zeta + \beta^2}, \qquad B_{\sigma}^* = \frac{-\alpha d_{\sigma} - \beta c_{\sigma}}{\alpha \zeta + \beta^2}$$

with

$$c_{\sigma} = \rho_R u_{\sigma R} (u_{1R} - s_R) - \rho_L u_{\sigma L} (u_{1L} - s_L) - (B_{1R} B_{\sigma R} - B_{1L} B_{\sigma L}),$$

$$d_{\sigma} = B_{\sigma R} (s_R - u_{1R}) - B_{\sigma L} (s_L - u_{1L}) - (B_{1L} u_{\sigma L} - B_{1R} u_{\sigma R}),$$

$$\alpha = \rho_R (u_{1R} - s_R) - \rho_L (u_{1L} - s_L), \quad \zeta = s_R - s_L, \quad \beta = B_{1R} - B_{1L}$$

with $\sigma \in \{2,3\}$. It can be ensured that the denominator $\alpha \zeta + \beta^2 \neq 0$ by modifying the outer wave speeds slightly

$$s_R \ge u_{1R} + \frac{1}{2} \max\{(u_{1L} - u_{1R}), 0\} + \tilde{c}_{fR},$$

$$s_L \le u_{1L} - \frac{1}{2} \max\{(u_{1L} - u_{1R}), 0\} - \tilde{c}_{fL},$$

where

$$\tilde{c}_{f\theta}^2 = \frac{\gamma p_{\theta}}{\rho_{\theta}} + \frac{B_{1\theta}^2}{\rho_{\theta}} (1+\epsilon) + \frac{B_{2\theta}^2 + B_{3\theta}^2}{\rho_{\theta}} + \sqrt{\left(\frac{\gamma p_{\theta} + \mathbf{B}_{\theta}^2}{\rho_{\theta}}\right)^2 - 4\frac{\gamma p_{\theta} B_{1\theta}^2}{\rho_{\theta}^2}}, \quad \theta \in \{L, R\}$$

for some small $\epsilon > 0$. Finally we can compute the intermediate energy

$$E_{\theta}^{*} = \frac{E_{\theta}(u_{1\theta} - s_{\theta}) + \pi_{1\theta}u_{1\theta} - \pi_{1\theta}^{*}s_{M} + \frac{B_{1\theta}^{2}}{2}(u_{1\theta} - s_{M}) + B_{1\theta}(B_{2\theta}u_{2\theta} + B_{3\theta}u_{1\theta} - B_{2\theta}u_{2\theta} - B_{3\theta}u_{1\theta})}{s_{M} - s_{\theta}}$$

Hence, all the intermediate states are determined explicitly. The intermediate fluxes are now obtained by local conservation (1.78) and (1.79)

$$\mathbf{F}_L^* = \mathbf{F}_L + s_L(\mathbf{W}_L^* - \mathbf{W}_L), \qquad \mathbf{F}_R^* = \mathbf{F}_R + s_R(\mathbf{W}_R^* - \mathbf{W}_R)$$

Therefore we get the fluxes for the three wave solver

$$\mathbf{F}_{i+\frac{1}{2},j}^{H_3} = \begin{cases} \mathbf{F}_{i,j}, & \text{if } s_{L,i+\frac{1}{2},j} > 0 \\ \mathbf{F}_{i,j}^*, & \text{if } s_{L,i+\frac{1}{2},j} \le 0 \text{ and } s_{M,i+\frac{1}{2},j} \ge 0 \\ \mathbf{F}_{i+1,j}^*, & \text{if } s_{M,i+\frac{1}{2},j} < 0 \text{ and } s_{R,i+\frac{1}{2},j} \ge 0 \\ \mathbf{F}_{i+1,j}, & \text{if } s_{R,i+\frac{1}{2},j} < 0 \end{cases}$$

As we will see in later numerical experiments the three wave solver does not model Alfvén waves precisely. Therefore one can introduce a five wave HLL solver. The derivation, that is similar to the one of the three wave HLL solver, of such a solver, can be found in [.....Link to Fuchs].

1.4.2 Higher order reconstruction

Instead of using constant functions $\mathbf{W}_{i,j}$ on each cell, one can use linear functions to obtain second order accuracy in space. For a second order approximation in time we can use the strong-stability preserving Runge-Kutta scheme.

$$\begin{split} W^*_{i,j} &= W^n_{i,j} + \Delta t^n \mathbb{L}^n_{i,j}, \\ W^{**}_{i,j} &= W^{*}_{i,j} + \Delta t^n \mathbb{L}^*_{i,j}, \\ W^{n+1}_{i,j} &= \frac{1}{2} (W^n_{i,j} + W^{**}_{i,j}). \end{split}$$

with

$$\mathbb{L}_{i,j}^{n} = \mathbf{W}_{i,j}^{n} - \frac{1}{\Delta x} (\mathbf{F}_{i+\frac{1}{2},j}^{n} - \mathbf{F}_{i-\frac{1}{2},j}^{n}) - \frac{1}{\Delta y} (\mathbf{G}_{i,j+\frac{1}{2}}^{n} - \mathbf{G}_{i,j-\frac{1}{2}}^{n}) + \tilde{\mathbf{S}}_{i,j}^{1} + \tilde{\mathbf{S}}_{i,j}^{2}.$$
(1.84)

We will now define the numerical fluxes \mathbf{F}, \mathbf{G} and later the source terms $\tilde{\mathbf{S}}^1$ and $\tilde{\mathbf{S}}^2$.

ENO reconstruction

The ENO (Essentially Non-Oscillatory) reconstruction is second order accurate for smooth solutions. We reconstruct in the primitive variables

$$\mathbf{U}_{i,j} = [\rho_{i,j}, \mathbf{u}_{i,j}, \mathbf{B}_{i,j}, p_{i,j}],$$

that can be obtained by transforming the conservative variables. The ENO-differences in each direction are given as

$$\overline{D}^{x}\mathbf{U}_{i,j} = \begin{cases} \mathbf{U}_{i+1,j} - \mathbf{U}_{i,j}, & \text{if } \Gamma_{i,j}^{x} \leq 1 \\ \mathbf{U}_{i,j} - \mathbf{U}_{i-1,j}, & \text{otherwise} \end{cases} \quad \overline{D}^{y}\mathbf{U}_{i,j} = \begin{cases} \mathbf{U}_{i,j+1} - \mathbf{U}_{i,j}, & \text{if } \Gamma_{i,j}^{y} \leq 1 \\ \mathbf{U}_{i,j} - \mathbf{U}_{i,j-1}, & \text{otherwise} \end{cases}$$

$$(1.85)$$

where

$$\Gamma_{i,j}^{x} = \frac{|\psi(\mathbf{U}_{i+1,j}) - \psi(\mathbf{U}_{i,j})|}{|\psi(\mathbf{U}_{i,j}) - \psi(\mathbf{U}_{i-1,j})|}, \quad \Gamma_{i,j}^{y} = \frac{|\psi(\mathbf{U}_{i,j+1}) - \psi(\mathbf{U}_{i,j})|}{|\psi(\mathbf{U}_{i,j}) - \psi(\mathbf{U}_{i,j-1})|}$$

and ψ is some function called the global smoothness indicator. We use $\psi(\mathbf{U}) = E$. The reconstructed linear function in the cell $I_{i,j}$ is

$$\overline{\mathbf{U}}_{i,j}(x,y) = \mathbf{U}_{i,j} + \frac{\overline{D}^x \mathbf{U}_{i,j}}{\Delta x} (x - x_i) + \frac{\overline{D}^y \mathbf{U}_{i,j}}{\Delta y} (y - y_j)$$
(1.86)

The reconstructed conservative variables can be obtained by transforming the reconstructed primitive variables.

WENO reconstruction

Alternatively to the ENO-reconstruction, one can look at the following one. Consider the cell differences

$$\widetilde{D}^{x}\mathbf{U}_{i,j} = \left(\omega_{i,j}^{x}(\mathbf{U}_{i+1,j} - \mathbf{U}_{i,j}) + (1 - \omega_{i,j}^{x})(\mathbf{U}_{i,j} - \mathbf{U}_{i-1,j})\right)$$

$$\widetilde{D}^{y}\mathbf{U}_{i,j} = \left(\omega_{i,j}^{y}(\mathbf{U}_{i,j+1} - \mathbf{U}_{i,j}) + (1 - \omega_{i,j}^{y})(\mathbf{U}_{i,j} - \mathbf{U}_{i,j-1})\right)$$
(1.87)

with the weights

$$\begin{split} \omega_{i,j}^x &= \frac{a_{i,j}^0}{a_{i,j}^0 + a_{i,j}^1}, \quad a_{i,j}^0 = \frac{1}{3(\epsilon + \beta_{i,j}^{x,0})}, \quad a_{i,j}^1 = \frac{2}{3(\epsilon + \beta_{i,j}^{x,1})}\\ \omega_{i,j}^y &= \frac{b_{i,j}^0}{b_{i,j}^0 + b_{i,j}^1}, \quad b_{i,j}^0 = \frac{1}{3(\epsilon + \beta_{i,j}^{y,0})}, \quad b_{i,j}^1 = \frac{2}{3(\epsilon + \beta_{i,j}^{y,1})} \end{split}$$

where $\epsilon > 0$ is small and

$$\beta_{i,j}^{x,0} = (\psi(\mathbf{U}_{i+1,j}) - \psi(\mathbf{U}_{i,j}))^2, \qquad \beta_{i,j}^{x,1} = (\psi(\mathbf{U}_{i,j}) - \psi(\mathbf{U}_{i-1,j}))^2, \beta_{i,j}^{y,0} = (\psi(\mathbf{U}_{i,j+1}) - \psi(\mathbf{U}_{i,j}))^2, \qquad \beta_{i,j}^{y,1} = (\psi(\mathbf{U}_{i,j}) - \psi(\mathbf{U}_{i,j-1}))^2.$$

The indicator function is as well $\psi(\mathbf{V}) = E$. The approximated solution on each cell $I_{i,j}$ is

$$\widetilde{\mathbf{U}}_{i,j}(x,y) = \mathbf{U}_{i,j} + \frac{\widetilde{D}^x \mathbf{U}_{i,j}}{\Delta x}(x-x_i) + \frac{\widetilde{D}^y \mathbf{U}_{i,j}}{\Delta y}(y-y_j)$$

The WENO reconstruction is third-order accurate for smooth solutions.

The ENO and WENO reconstruction suffer from a common problem. The reconstructed pressure and density may not be positive. For obtaining physically meaningful results it is essential that these quantities are positive. Therefore we have to modify the differences (1.85) and (1.87). We won't go further into detail and refer to [.... 51 in fuchs = waagan // fuchs....].

We will now define the second order numerical fluxes (1.84).

$$\mathbf{F}_{i+\frac{1}{2},j} = \mathbf{F}(\mathbf{W}^E_{i,j},\mathbf{W}^W_{i+1,j}), \qquad \mathbf{G}_{i,j+\frac{1}{2}} = \mathbf{G}(\mathbf{W}^N_{i,j},\mathbf{W}^S_{i,j+1})$$

with

$$\begin{split} \mathbf{W}_{i,j}^E &= \widehat{\mathbf{W}}_{i,j}(x_{i+\frac{1}{2}}, y_j), \qquad \mathbf{W}_{i,j}^W &= \widehat{\mathbf{W}}_{i,j}(x_{i-\frac{1}{2}}, y_j), \\ \mathbf{W}_{i,j}^N &= \widehat{\mathbf{W}}_{i,j}(x_i, y_{j+\frac{1}{2}}), \qquad \mathbf{W}_{i,j}^S &= \widehat{\mathbf{W}}_{i,j}(x_i, y_{j-\frac{1}{2}}). \end{split}$$

Here $\widehat{\mathbf{W}}$ denotes the positivity preserving modifications of the ENO or WENO reconstructed conserved variables $\overline{\mathbf{W}}$ or $\widetilde{\mathbf{W}}$.

The second order source term can be calculated as in (1.82)

$$\mathbf{S}_{i,j}^1 = \mathbf{S}_{i-\frac{1}{2},j}^{1,*} \mathbf{1}_{(s_{M,i-\frac{1}{2},j} \ge 0)} + \mathbf{S}_{i+\frac{1}{2},j}^{1,*} \mathbf{1}_{(s_{M,i+\frac{1}{2},j} < 0)}$$

with $\mathbf{S}_{i-\frac{1}{2},j}^{1,*}$ as in (1.81) but with $\mathbf{W}_{i,j}$, $\mathbf{W}_{i+1,j}$ replaced by $\mathbf{W}_{i,j}^{E}$, $\mathbf{W}_{i+1,j}^{W}$. Since for smooth solutions the discretized source $S_{i,j}^{n}$ vanishes with $(B_{1}^{W})_{i+1,j} - (B_{1}^{E})_{i,j}$, we have to add an extra term to obtain second order consistency. In [....LINK to Fuchs...] it was suggested to modify the source term in the following way

$$\mathbf{S}_{i,j}^{1,\text{mod}} = \mathbf{S}_{i,j}^{1} + \begin{pmatrix} 0\\ \mathbf{B}_{i,j}\\ \mathbf{u}_{i,j}\\ \mathbf{u}_{i,j} \cdot \mathbf{B}_{i,j} \end{pmatrix} \frac{1}{\Delta x} \widehat{D}^{x} B_{i,j}^{1}$$

where \widehat{D}^x is the positivity preserving modification of the differences (1.87).

1.5 Numerical experiments

1.5.1 Brio-Wu shock tube

This is one of the classical one-dimensional test problems for the ideal MHD. The initial data is given by

$$[\rho, \mathbf{u}, \mathbf{B}, p] = \begin{cases} [1, 0, 0, 0, 0.75, 1, 0, 1] & \text{if } x < 0.5 \\ [0.125, 0, 0, 0, 0.75, -1, 0, 0.1] & \text{else} \end{cases}$$

The computational domain is $(x,t) \in [0,1] \times [0,0.2]$. Further we have Neumann boundary conditions. From left to right the solution (see Figure 1.1) involves five waves: a fast rarefaction wave, a slow compound wave, a contact wave, a slow shock and a fast rarefaction wave that already left the computational domain at t = 0.2. The compound wave at $x \approx 0.45$ consists of a slow compressive shock and a rarefaction. This behaviour is a consequence of the fact that the MHD equations are not strictly hyperbolic.



FIGURE 1.1: Density ρ of the Brio-Wu shock tube problem computed by different schemes on a grid with 256 points at time t = 0.2.



(a) L^1 -differences $||\rho_N - \rho_{N/2}||_{L^1}$ of the Brio-Wu shock tube problem at t = 0.2 plotted against the degrees of freedom $N = 2^i$ for $i = 6, \ldots, 11$

(b) L^1 -error in comparison to a reference solution $||\rho_{\rm ref} - \rho_N||_{L^1}$ plotted against the degrees of freedom $N = 2^i$ for $i = 5, \ldots, 11$. The reference solution was computed with the HLL5-WENO scheme on 2^{13} grid points.

FIGURE 1.2: Convergence study of different solvers for the Brio-Wu shock tube.

1.5.2 Powell-magnetic advection

This one dimensional problem is used to see how a non-constant normal magnetic field is treated. Schemes that follow the convergence constrain $\operatorname{div}(\mathbf{B}) = 0 \Rightarrow B_1 = \operatorname{const.}$ typically fail. The initial data is given by

$$[\rho, \mathbf{u}, \mathbf{B}, p] = [1, 1, 0, 0, 1 - \sin(2\pi x), 0.5, 0, 0.5]$$

With this data the equation for the magnetic field component B_1 reduces to the linear advection equation.



FIGURE 1.3: B_1 at time t = 1 computed with different schemes on a grid with 64 grid points



FIGURE 1.4: L^1 -difference to the exact solution $||(B_1)_{\text{ex}} - (B_1)_N||_{L^1}$ plotted against the degrees of freedom $N = 2^i$ for i = 5, ..., 12 computed for the Powell-magnetic advection problem.

1.5.3 Orszag-Tang vortex

In this standard two dimensional test problem. The initial data leads to supersonic turbulences. It tests how the code handles the formation of shocks and shock-shock interactions. Further the problem is invariant under 180 degree rotation, providing a symmetry test for the code. The initial data is given by

$$[\rho, \mathbf{u}, \mathbf{B}, p] = [\gamma^2, -\sin(\pi y), \sin(\pi x), 0, -\sin(\pi y), \sin(2\pi x), 0, \gamma]$$

where $\gamma = \frac{5}{3}$ is the heat capacity ratio. The computational domain is $(x, t) \in [0, 2]^2 \times [0, 1]$. We use periodic boundary conditions. In Figure (1.9a) we can see the solution of the problem at t = 1. Since the solution contains shocks we expect at most half of the order the schemes would have for problems with continuous solutions. Therefore the first order schemes, such as the HLL3 and HLL5 solver obtain order $\frac{1}{2}$ and the second order schemes obtain order 1. Figure (1.9b) verifies this convergence speeds.





(a) Pressure p of the Orszag-Tang vortex at t = 1, computed with the HLL5 solver with second order WENO reconstruction on 1024^2 grid points.

(b) L^{1} -error $||p_N - p_{ref}||_{L^1([0,2]^2)}$ plotted against the grid size $N = 2^i$ for $i = 5, \ldots, 9$. The reference solution was computed with the HLL5-WENO scheme on 1024^2 gridpoints.

FIGURE 1.5: Pressure p of the Orszag-Tang vortex at t = 1

1.5.4 Kelvin-Helmholz instability

Let us first look at the hydrodynamic case of the problem. Therefore we set the magnetic field equal zero. The MHD equations reduce to the Euler equations of gas dynamics. We will look at a perturbed shear flow and observe how the different layers mix. The



FIGURE 1.6: Pressure p of the Orszag-Tang vortex at t = 1 computed with different schemes on 256^2 grid points.

shear flow initial data on the domain $(x, y) \in [0, 1]^2$ is given by

$$\mathbf{U}_{0}(\mathbf{x}) = \begin{cases} \mathbf{U}_{\text{mid}} & \text{if } I_{1} < x_{2} < I_{2} \\ \mathbf{U}_{\text{out}} & \text{if } x_{2} \le I_{1} \text{ or } I_{2} \le x_{2} \end{cases}$$
(1.88)

where $I_1(x_1) = 0.25$ and $I_2(x_1) = 0.75$ are the two interface profiles. We have periodic boundary conditions on each boundary. Since we chose two interfaces, there is no jump across the boundary. In the hydrodynamic case the states $\mathbf{U}_{\theta} = [\rho_{\theta}, \mathbf{u}_{\theta}, \mathbf{B}_{\theta}, p_{\theta}], \quad \theta \in$ {mid,out} are given by

 $\mathbf{U}_{mid} = [1, 0.5, 0, 0, 0, 0, 0, 2.5], \quad \text{and} \quad \mathbf{U}_{out} = [2, -0.5, 0, 0, 0, 0, 0, 2.5].$

We can perturb this problem in two ways. First we will look at a perturbation of the two interfaces and later we will perturb the initial value of u_2 . For the interface perturbation

we set

$$I_j^* = I_j^*(x_1, \omega) = I_j + \epsilon Y_j(x_1, \omega), \qquad j = 1, 2$$

where ϵ is the amplitude and

$$Y_j(x_1, \omega) = \sum_{n=1}^m a_j^n(\omega) \cos(2n\pi(b_j^n(\omega) + x_1)), \qquad j = 1, 2$$

The corresponding problem is then

$$\mathbf{U}_{0}(\mathbf{x},\omega) = \begin{cases} \mathbf{U}_{\text{mid}} & \text{if } I_{1}^{*}(x_{1},\omega) < x_{2} < I_{2}^{*}(x_{1},\omega) \\ \mathbf{U}_{\text{out}} & \text{if } x_{2} \le I_{1}^{*}(x_{1},\omega) \text{ or } I_{2}^{*}(x_{1},\omega) \le x_{2} \end{cases}$$
(1.89)

In Figure (1.7) we see the approximated density of the initial value problem (1.89) at t = 1 plotted for grid sizes between 128^2 and 2048^2 . The second order accurate HLL5-WENO scheme was used. The figure suggests that the solution does not converge as the mesh is refined. We can observe more small scale structures as the mesh is refined. To further verify the lack of sample convergence we looked in Figure (1.8) at the L^1 -differences

$$||\rho_N - \rho_{N/2}||_{L^1([0,1])}, \text{ for } N = \{2^i : i = 6, \dots, 11\}$$

of these approximations.

The second perturbation uses slip interfaces (1.88). The velocity in x_2 -direction is perturbed the following way

$$u_2 = A\sin(2\pi x_1)\sin(2\pi x_2) \tag{1.90}$$

where A is the Amplitude of the perturbation. In Figure we look at the L1-differences $||\rho_N - \rho_{N/2}||_{L^1}$ of approximations computed by the HLL5-WENO scheme. We can see here as well that these differences don't converge.





(e) 2048^2

FIGURE 1.7: Approximate density for the initial data (1.89) for one sample (fixed ω) and with $\epsilon = 0.02$, computed with the second order HLL5-WENO scheme at t = 1 for different mesh resolutions.



FIGURE 1.8: L^1 -differences $||\rho_N - \rho_{N/2}||_{L^1}$ with initial data (1.89) computed for a single sample with the HLL5-WENO scheme plotted against the grid sizes $N = 2^i$ for $i = 6, \ldots, 11$





(a) Approximate density for the perturbation (1.90) of the initial data (1.88) for one sample (fixed ω) and with A = 0.01, computed with the second order HLL5-WENO scheme at t = 1 on a 1024^2 grid.

(b) L^1 -differences $||\rho_N - \rho_{N/2}||_{L^1}$ with initial data (1.89) computed for a single sample with the HLL5-WENO scheme plotted against the grid sizes $N = 2^i$ for $i = 6, \ldots, 10$

FIGURE 1.9: Pressure p of the Orszag-Tang vortex at t = 1

Chapter 2

Entropy measure valued solutions

In order to get a more general notion for solutions we will use the concept of Young measures. First we will give an introduction to Young measure. In Section [......] we will look at the measure valued Cauchy problem, a weak, more general formulation of the Cauchy problem. The approximation of solutions to this measure valued Cauchy problem are discussed in Section [.....].

2.1 Young measures

Let $\mathcal{M}(\mathbb{R}^N)$ be the Banach space of all finite Radon measures on \mathbb{R}^N , which are inner regular Borel measures μ with finite variation $\|\mu\|_{\mathcal{M}(\mathbb{R}^N)} = |\mu|(\mathbb{R}^N) < \infty$. Further let $C_0(\mathbb{R}^N)$ be the Banach space of all continuous real-valued functions on \mathbb{R}^N which vanish at infinity, equipped with the supremum norm. Then by a well known form of the Riesz representation theorem [...link to real and abstract analysis p.364...] $\mathcal{M}(\mathbb{R}^N)$ can be identified with $C_0(\mathbb{R}^N)'$ the dual space of $C_0(\mathbb{R}^N)$ by the isometric isomorphism

$$\Phi: \begin{cases} \mathcal{M}(\mathbb{R}^N) & \to & C_0(\mathbb{R}^N)^{\prime} \\ \mu & \mapsto & \langle \mu, \cdot \rangle \end{cases}$$

with the dual pairing $\langle \mu, g \rangle = \int_{\mathbb{R}^N} g(\xi) d\mu(\xi)$.

The duality between those two spaces induces a weak-* topology on $\mathcal{M}(\mathbb{R}^N)$. A sequence $\mu^n \in \mathcal{M}(\mathbb{R}^N)$ converges with respect to this topology to $\mu \in \mathcal{M}(\mathbb{R}^N)$ if $\langle \mu^n, g \rangle \to \langle \mu, g \rangle$ for all $C_0(\mathbb{R}^N)$. This is also known as narrow convergence.

The set of all probability measures on \mathbb{R}^N is the subset

$$\mathcal{P}(\mathbb{R}^N) := \left\{ \mu \in \mathcal{M}(\mathbb{R}^N) : \mu \ge 0, \ \mu(\mathbb{R}^N) = 1 \right\}.$$

A Young measure on $D \subset \mathbb{R}^k$ is a weak-* measurable function

$$\nu: \begin{cases} D & \to & \mathcal{P}(\mathbb{R}^N) \\ z & \mapsto & \nu_z := \nu(z) \end{cases}$$

where a function is said to be weak-* measurable if the mapping $z \to \langle \nu_z, g \rangle$ is Borel measurable for every $g \in C_0(\mathbb{R}^N)$. The set of all Young measures from D into \mathbb{R}^N is denoted by $\mathbf{Y}(D, \mathbb{R}^N)$. A sequence of Young measures $\nu^n \in \mathbf{Y}(D, \mathbb{R}^N)$ converges narrowly to $\nu \in \mathbf{Y}(D, \mathbb{R}^N)$ if

$$\int_D \varphi(z) \langle \nu_z^n, g \rangle dz \to \int_D \varphi(z) \langle \nu_z, g \rangle dz \quad \forall \ \varphi \in L^1(D), \ \forall \ g \in C_0(D).$$

This weak-* convergence with respect to the weak-* topology $\sigma(L^{\infty}(D), L^{1}(D))$ is denoted by $\langle \nu^{n}, g \rangle \rightharpoonup [] * \langle \nu, g \rangle$.

Next we will state a version of the fundamental theorem of Young measures.

Theorem 2.1. Let $\nu^n \in \mathbf{Y}(D, \mathbb{R}^N)$ for $n \in \mathbb{N}$ be a sequence of Young measures. Then there exist a subsequence ν^m and a nonnegativ measure-valued function $\nu : D \to \mathcal{M}_+(\mathbb{R}^N)$ such that

(i)
$$\int_D \varphi(z) \langle \nu_z^m, g \rangle dz \to \int_D \varphi(z) \langle \nu_z, g \rangle dz \quad \forall \ \varphi \in L^1(D), \ \forall \ g \in C_0(D).$$

and further satisfies

- (ii) $\|\nu_z\|_{\mathcal{M}(\mathbb{R}^N)} \leq 1$ for a.e. $z \in D$;
- (iii) If $K \subset \mathbb{R}^N$ is closed and $supp(\nu_z^n) \subset K$ for a.e. $z \in D$ and n large, then $supp(\nu_z) \subset K$ for a.e. $z \in D$.

If one can additionally find for every bounded, measurable $E \in D$ a nonnengativ $\kappa \in C(\mathbb{R}^N)$ with $\lim_{|\zeta|\to\infty} \kappa(\zeta) = \infty$ such that

$$\sup_{n} \int_{E} \langle \nu_{z}^{n}, \kappa \rangle dz < \infty$$
(2.1)

then

(iv)
$$\|\nu_z\|_{\mathcal{M}(\mathbb{R}^N)} \leq 1$$
 for a.e. $z \in D$;

and hence $\nu \in \mathbf{Y}(D, \mathbb{R}^N)$.

Proof. Let $L^{\infty}_{\omega}(D; \mathcal{M}(\mathbb{R}^N))$ be the space of equivalent classes of weak-* measurable functions $\mu: D \to \mathcal{M}(\mathbb{R}^N)$, equipped with the norm

$$\|\mu\|_{\infty,\mathcal{M}} := \operatorname{ess\,sup}_{z\in D} \|\mu_z\|_{\mathcal{M}}.$$

It can be shown [...ref to Ball /4...] that $L^{\infty}_{\omega}(D; \mathcal{M}(\mathbb{R}^N))$ is isometrically isomorph to the dual of $L^1(D; C_0(\mathbb{R}^N))$ and therefore a Banach space. Since $\nu^n \in \mathbf{Y}(D, \mathbb{R}^N)$ we have $\|\nu^n\|_{\infty,\mathcal{M}} = 1$ for all $n \in \mathbb{N}$ and therefore the sequence ν^n is bounded in $L^{\infty}_{\omega}(D; \mathcal{M}(\mathbb{R}^N))$. Hence there exists a $\nu \in L^{\infty}_{\omega}(D; \mathcal{M}(\mathbb{R}^N))$ and a weak-* convergent subsequence ν_m of ν_n . This weak-* convergence with respect to the weak-* topology on $L^{\infty}_{\omega}(D; \mathcal{M}(\mathbb{R}^N))$ is

$$\lim_{m \to \infty} \int_D \langle \nu_z^m, \Psi(z, \cdot) \rangle \, dz = \int_D \langle \nu_z, \Psi(z, \cdot) \rangle \, dz$$

for all $\Psi \in L^1(D; C_0(\mathbb{R}^N))$. In particular, letting $\Psi(x, \xi) = \varphi(x)g(\xi)$ for $\varphi(x) \in L^1(D)$ and $g \in C_0(\mathbb{R}^N)$ proofs (i). Next, we claim that $\nu_z \ge 0$ for a.e. $z \in D$. If not, then there would exist a non-negative function $\Psi \in L^1(D; C_0(\mathbb{R}^N))$ such that $\int_D \langle \nu_z, \Psi(z, \cdot) \rangle dz < 0$. But then, since $\nu_z^m \ge 0$

$$0 > \int_D \langle \nu_z, \Psi(z, \cdot) \rangle \, dz = \lim_{m \to \infty} \int_D \langle \nu_z^m, \Psi(z, \cdot) \rangle \, dz \ge 0$$

is a contradiction. (ii) follows from the weak-* semi continuity of the norm $\|\cdot\|_{-\infty}$, \mathcal{M} . To show (iii) let $g \in C_0(\mathbb{R}^N)$ with $g|_K = 0$. Therefore $\langle \nu_z^m, g \rangle = 0$ for almost every $z \in D$ and m large enough. Hence

$$\int_D \varphi(z) \langle \nu_z, g \rangle \, dz = \lim_{m \to \infty} \int_D \varphi(z) \langle \nu_z^m, g \rangle \, dz = 0$$

for all $\varphi \in L^1(D)$, and therefore $\langle \nu_z, g \rangle = 0$ for a.e. $z \in D$. This is (iii). Assume now that (2.1) holds. Fix a set $E \subset D$ of finite, nonzero Lebesgue measure $\lambda(E)$ and denote the average integral over E by $f_E = \frac{1}{\lambda(E)} \int_E$. For every R > 0 we define

$$\theta_{R}(\xi) = \begin{cases} 1 & \kappa(\xi) \le R \\ 1 + R - \kappa(\xi) & R < \kappa(\xi) \le R + 1 \\ 0 & R + 1 < \kappa(\xi) \end{cases}$$

Since $\theta_R(\xi) \in C_0(\mathbb{R}^N)$, we have with (ii)

$$\int_E \langle \nu_z^m, \theta_R \rangle \, dz = \lim_{m \to \infty} \int_E \langle \nu_z, \theta_R \rangle \, dz \le \int_E \| \nu_z \|_{\mathcal{M}} \, dz \le 1.$$

Further $1 - \theta_R(\xi) \leq \frac{\kappa(\xi)}{R}$ and therefore

$$\int_{E} 1 - \langle \nu_{z}^{m}, \theta_{R} \rangle \, dz = \lim_{m \to \infty} \int_{E} \langle \nu_{z}^{m}, 1 - \theta_{R} \rangle \, dz \leq \frac{1}{R} \int_{E} \langle \nu_{z}^{m}, \kappa \rangle \, dz$$

From the above inequality it follows that

$$1 \leq \lim_{R \to \infty} \lim_{m} \int_{E} \langle \nu_{z}^{m}, \theta_{R} \rangle \, dz + \lim_{R \to \infty} \sup_{m} \frac{1}{R} \int_{E} \langle \nu_{z}^{m}, \kappa \rangle \, dz$$
$$= \lim_{R \to \infty} \int_{E} \langle \nu_{z}, \theta_{R} \rangle \, dz$$
$$\leq \int_{E} \|\nu_{z}\|_{\mathcal{M}} \, dz \leq 1$$

where we used (2.1) for the equality. Therefore $\int_E \|\nu_z\|_{\mathcal{M}} dz = 1$ and since $E \subset D$ was arbitrary we proofed (iv).

Let (Ω, \mathcal{F}, P) be a probability space, $D \in \mathbb{R}^d$ a Borel set and $u : \Omega \times D \to \mathbb{R}^N$ a random field, then we can define its law ν by the induced measure $\nu_z := P \circ u(\cdot, z)^{-1}$ and therefore

$$\langle \nu_z, g \rangle = \int_{\mathbb{R}^N} g(\xi) d\nu_z(\xi) = \int_{\mathbb{R}^N} g(\xi) d(P \circ u(\cdot, z)^{-1})(\xi) = \int_{\Omega} g(u(\omega, z)) \, dP(\omega) \quad (2.2)$$

for $g \in C_0(\mathbb{R}^N)$. This defines a Young measure:

Theorem 2.2. If $u : \Omega \times D \to \mathbb{R}^N$ is jointly measurable then $\nu_z := P \circ u(\cdot, z)^{-1}$ defines a Young measure from $D \to \mathbb{R}^N$.

Proof. Since u is jointly measurable it follows for fixed z that the set $\{\omega : u(\omega, z) \in U\} = u(z, \cdot)^{-1}(U)$ is P-measurable.

We need to show that the definition of ν is independent of the choice of the mapping in the equivalent class of jointly measurable functions $u : \Omega \times D \to \mathbb{R}^N$. Let \tilde{u} and \hat{u} be two functions such that $\tilde{u} = \hat{u}$ for $P \times \lambda$ -a.e. (ω, z) , then by applying Tonelli's theorem we get

$$0 = \int_{\Omega \times D} \mathbb{1}_{\{\tilde{u} \neq \hat{u}\}}(\omega, z) d(P \otimes \lambda)(\omega, z) = \int_{D} \int_{\Omega} \mathbb{1}_{\{\tilde{u} \neq \hat{u}\}}(\omega, z) dP(\omega) d\lambda(z) = \int_{D} P(\{\omega : \tilde{u}(\omega, z) \neq \hat{u}(\omega, z)\}) d\lambda(z)$$

Therefore $P(\{\omega : \tilde{u}(\omega, z) \neq \hat{u}(\omega, z)\}) = 0$ for a.e. $z \in D$. Hence for every Borel set $U \in \mathbb{R}^N$

$$P(\{\omega: \tilde{u}(\omega, z) \in U\}) = P(\{\omega: \hat{u}(\omega, z) \in U\})$$

for a.e. zinD. Finally ν is weak-* measurable since

$$\langle \nu_z, g \rangle = \int_{\Omega} g(u(\omega, z)) \ dP(\omega)$$

which is measurable in z for every $g \in C_0(\mathbb{R}^N)$.

Next we construct a measurable random field from a given Young measure.

Theorem 2.3. For every Young measure $\nu \in \mathbf{Y}(D, \mathbb{R}^N)$ there exists a probability space (Ω, \mathcal{F}, P) and a function $u : \Omega \times D \to \mathbb{R}^N$ such that u is measurable on the product σ -algebra on $\Omega \times D$ and such that u has law ν .

In particula we can choose the probability space as $([0,1), \mathcal{B}, \lambda)$ with \mathcal{B} denoting the Borel σ -algebra on [0,1) and λ the Lebesgue-measure.

Proof. We will proof the theorem on the probability space $([0,1), \mathcal{B}, \lambda)$ and N = 1. For $n \in \mathbb{N}$ and $j \in \mathbb{Z}$ we define

$$F_n^j = \begin{cases} (-\infty, -2^n) & \text{if } j = -2^{2n} \\ [-2^n(j-1), -2^n j) & \text{if } j = -2^{2n} + 1, \dots, 2^{2n} \\ [2^n, \infty) & \text{if } j = 2^{2n} + 1 \\ \emptyset & \text{else.} \end{cases}$$

a partition of \mathbb{R} into two outer intervalls and 2^{2n} inner intervalls of Lebesgue measure 2^{-n} Let $p_n^j(z) = \sum_{l \leq j} \nu_z(F_n^l)$. Since $C_0(\mathbb{R})$ is dense in $L^1(\mathbb{R})$ and $\nu \in \mathbf{Y}(D,\mathbb{R})$ we have that $p_n^j(z) : \mathbb{R} \to [0,1)$ is measureable for all n, j. Further $p_n^j(z) = 0$ for $j < -2^{2n}$ and $p_n^j(z) = 1$ for $j > -2^{2n} + 1$. For n arbitrary but fixed we can define a function

$$u_n(\omega, z) := \xi_n^j$$
 with j such that $p_n^{j-1}(z) \le \omega < p_n^j(z)$

where $\xi_n^j \in F_n^j$ can be chosen arbitrarily. This function is well-defined for all $(\omega, z) \in \Omega \times D$ and for fixed z piecewise constant in ω .

Next we will proof that u_n is measureable on the product σ -algebra between \mathcal{F} and the Borel σ -algebra on D. Since u_n only takes finitly many values ξ_n^j , it suffices to show that $u_n^{-1}(\xi_n^j)$ is measureable for each ξ_n^j . Indeed,

$$\begin{split} u_n^{-1}(\xi_n^j) &= \left\{ (\omega, z) \in \Omega \times D : p_n^{j-1}(z) \le \omega < p_n^j(z) \right\} \\ &= (\Omega \times D) \cap \left\{ (\omega, z) \in \Omega \times \mathbb{R} : p_n^{j-1}(z) \le \omega \right\} \cap \left\{ (\omega, z) \in \Omega \times \mathbb{R} : \omega < p_n^j(z) \right\}, \end{split}$$

the intersection between the epigraph of p_n^{j-1} and the hypograph of p_n^j which are measureable, since all p_j^n are. Since the partition $F_{j \ j \in \mathbb{Z}}^m$ is a refinement of $F_{j \ j \in \mathbb{Z}}^n$ whenever

m > n it follows $|u_n(\omega, z) - u_m(\omega, z)| < \lambda(F_j^n) = 2^{-n}$ with m, n chosen, depending on (ω, z) , large enough. Hence u_n converges pointwise to some function $u : \Omega \times D \to \mathbb{R}$ and therefore u is measurable by the measurability of each u_n .

Finally by Lebesgue's theorem of dominated convergence and the definition of the Lebesgue integral we have for every $g \in C_0(\mathbb{R})$ and $z \in D$ that

$$\int_{\Omega} g(u(\omega, z)) dP(\omega) = \lim_{n} \int_{\Omega} g(u_n(\omega, z)) dP(\omega) = \lim_{n} \sum_{j} \nu_z(F_n^j) g(\xi_n^j) = \int_{\mathbb{R}} g(\xi) d\nu_z$$

2.2 The measure valued Cauchy problem

Since standard numerical schemes may not converge [...Link to MVS..] and new structures are found at smaller and smaller scale it is reasonable to look at a different (weaker) notion of solutions. Entropy solutions, whenever they exist should be included in this class of solutions.

Instead of looking for an integrable function which solves the Cauchy problem

$$\frac{\partial}{\partial t}u + \operatorname{div}(f(u)) = s(u)$$

$$u(x, 0) = u_0$$
(2.3)

we generalize the problem and require the solution to be a Young measure. With the notation of the previous section we introduce the following generalized problem

$$\frac{\partial}{\partial t} \langle \nu, \mathrm{id} \rangle + \mathrm{div}(\langle \nu, f \rangle) = \langle \nu, s \rangle$$

$$\nu_{(x,0)} = \sigma_x$$
(2.4)

with given initial data $\sigma_x \in \mathbf{Y}(\mathbb{R}^d, \mathbb{R}^N)$. A Young measure $\nu \in \mathbf{Y}(\mathbb{R}^d \times \mathbb{R}_+, \mathbb{R}^N)$ is a *measure valued (MV) solution* of (2.3) if (2.3) holds in the sense of distributions. That is

$$\int_{\mathbb{R}_{+}} \int_{\mathbb{R}^{N}} \frac{\partial}{\partial t} \varphi(x, t) \langle \nu_{(x,t)}, \mathrm{id} \rangle + \nabla_{x} \varphi(x, t) \cdot \langle \nu_{(x,t)}, f \rangle \, dx dt + \int_{\mathbb{R}^{N}} \langle \sigma_{x}, \mathrm{id} \rangle \varphi(x, 0) \, dx \quad (2.5)$$
$$= \int_{\mathbb{R}_{+}} \int_{\mathbb{R}^{N}} \langle s, \mathrm{id} \rangle \varphi(x, t) \, dx dt \quad (2.6)$$

for all test functions $\varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}_+)$. If $\nu \in \mathbf{Y}(\mathbb{R}^d \times \mathbb{R}_+, \mathbb{R}^N)$ additionally satisfies for every entropy pair (η, q) and all non-negative $0 \leq \varphi \in C_c^1(\mathbb{R}^d \times \mathbb{R}_+)$ the entropy inequality

$$\int_{\mathbb{R}_{+}} \int_{\mathbb{R}^{N}} \frac{\partial}{\partial t} \varphi(x, t) \langle \nu_{(x,t)}, \eta \rangle + \nabla_{x} \varphi(x, t) \cdot \langle \nu_{(x,t)}, q \rangle \, dx dt + \int_{\mathbb{R}^{N}} \langle \sigma_{x}, \eta \rangle \varphi(x, 0) \, dx \quad (2.7)$$

$$\geq \int_{\mathbb{R}_{+}} \int_{\mathbb{R}^{N}} \langle \nu_{(x,t)}, \eta' s \rangle \varphi(x,t) \, dxdt \tag{2.8}$$

where η' denotes the derivative with respect to the conserved variables of the entropy function η , then ν is called an *entropy measure valued (EMV) solution*.

Let $\delta_{u(x,t)}$ denote the Delta distribution with a singularity at u(x,t) so that $\langle \delta_{u(x,t)}, f \rangle = f(u(x,t))$. Then every entropy solution u of (2.3) give rise to an EMV solution ν of (2.3) by defining $\nu_{(x,t)} := \delta_{u(x,t)}$. Therefore the set of EMV solutions with initial data $\sigma = \delta_{u_0}$ is at least as large as the set of entropy solutions of (2.3) with initial data u_0 .

2.3 Construction of approximate EMV solutions for conservation laws

In this section we will only look at conservation laws in one space dimension. The generalization to more dimensions is straightforward and can be found in [.....Ulrik-diss......]. The source term s in is set to zero and therefore in the definition of the EMV solution (2.6), (2.8) the right hand sides are zero.

For the construction of approximate EMV solutions we will begin with a suitable numerical scheme. Lets look at a finite volume/difference scheme with locally Lipschitz continuous numerical flux function $F_{i+1/2}^{\Delta x}$, where Δx is the mesh size. Further let the numerical flux be consistent with the given flux function f. Let $F_{i+1/2}^{\Delta x}$ depend on the (2p+1)-points around $u_i^{\Delta x}$, where $u^{\Delta x}(t) := \{u_i^{\Delta x}(t)\}_{i=1}^N$ denotes the numerical approximation generated by the semi discrete (continuous in time) scheme. Let the discretized initial data be denoted by $u_0^{\Delta x}$ and write for the evolution operator $\mathcal{S}^{\Delta x}$ such that $u^{\Delta x} = \mathcal{S}^{\Delta x}(u_0^{\Delta x})$.

The following Algorithm explains how we obtain a approximate EMV solution.

Step 1: Let $\sigma \in \mathbf{Y}(\mathbb{R}, \mathbb{R}^N)$ be the initial Young measure in (2.3). According to Theorem 2.3 there exists a probability space (Ω, \mathcal{F}, P) and a random field $u_0 : \Omega \to L^{\infty}()$ such that σ is it's law.

Step 2: We evolve the dicretized initial random field $u_0^{\Delta x}$ by applying the described evolution operator to each $\omega \in \Omega$ such that $u^{\Delta x}(\omega) = S^{\Delta x}(u_0^{\Delta x}(\omega))$ to obtain a approximation to the solution $u(\omega)$, with corresponding random field $u_0(\omega)$.

Step 3: Define the approximate measure valued solution $\nu^{\Delta x} \in \mathbf{Y}(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}^N)$ as the law of $u^{\Delta x}$ with Theorem 2.2. Next we will show that if the numerical scheme satisfies

a set of criteria the approximate EMV solutions obtained by the above procedure will converge narrowly to a EMV solution of (2.3).

Bibliography

- [1] S. Mishra. Numerical methodes for conservation laws and related equations. 2013.
- [2] U. S. Fjordholm. Structure preserving finite volume methods for the shallow water equations. Master's thesis, University of Oslo, 2009.
- [3] E. Tadmor. The numerical viscosity of entropy stable schemes for systems of conservation laws. I. Math. Comp., 49:91–103, 1987.