Numerical Study of Small-Jump Regularization on Exotic Contracts in Lévy Markets

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Abstract

Standard numerical methods to calculate option prices in Lévy markets require the truncation and subsequent regularization of the small jumps component of the underlying Lévy process. These methods include Monte Carlo techniques as well as finite difference schemes. The regularization of the Lévy process is accomplished by adding a diffusion component based on the level of truncation. Because the error induced by truncating a Lévy process converges weakly to a Gaussian process, this regularization is justified for European options pricing. However, in exotic contracts, the option prices do not behave as would a diffusion-driven option. In this thesis, we consider American and Barrier contracts. Using a finite element method and tempered stable Lévy processes, we show the pricing error of small-jump regularization in American contracts near the free exercise boundary and Barrier contracts near the barrier. We demonstrate and conclude that it is not appropriate to replace the small jumps of the Lévy process with a diffusion component in exotic contracts.

Contents

1. Introduction

In 1973, Fischer Black and Myron Scholes published their seminal paper on options pricing, *The Pricing of Options and Corporate Liabilities* [7]. They proposed that price processes could be modeled by geometric Brownian motion, and derived a closed-form equation to price options. Since 1973, researchers have debated the quality of modeling price processes with a geometric Brownian motion. For example, stock returns exhibit skew and kurtosis that is not captured by Brownian motion dynamics, whose increments follow a normal distribution. Furthermore, stock prices are known to contain jumps, therefore it is possible that a continuous process such as Brownian motion is not suitable. See Figure 1.1 for an example of discontinuous price paths in market data.

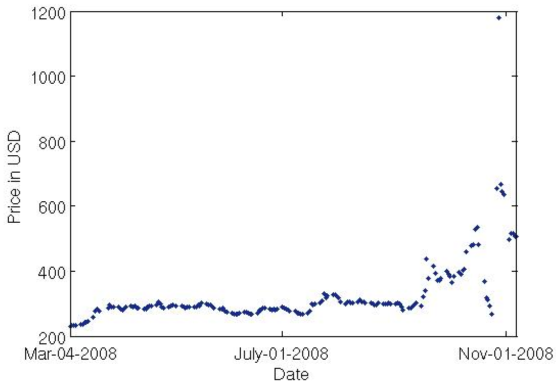

Figure 1.1: Daily Price Data for Volkswagen

Ignoring fat tails and skew leads to a model which ignores important risks associated with the underlying price processes; risks that should be incorporated into the option price. Moreover, the volatility of prices is not constant, as assumed by the Black-Scholes model. Thus researchers began to look for an options pricing model that could capture the volatility smile. In 1976, Robert Merton introduced an options pricing model that included jumps [20]. To achieve this, he used a geometric Brownian motion and a Poisson process. This marked the birth of jump-diffusion models. It was found that adding jumps increased the kurtosis of the underlying distribution. Recently, Lévy processes and additive processes have been explored as more realistic models for stock prices. These processes include Brownian motion and jump diffusion models as special cases and allow for discontinuous sample paths. Jump diffusion models are Lévy models with finite variation. Lévy models assume independent increments, whereas additive processes can have time-inhomogeneous increments. Lévy processes are quite flexible and are successful in capturing the stylized facts which a geometric Brownian motion cannot. Table 1 summarizes the stylized facts observed in the market and compares how well Lévy processes and Brownian motion capture these effects [10].

	Brownian Motion	Lévy Processes
Heavy/Semi-heavy Tails	No	Yes
Discontinuous Trajectories	No	Yes
Incomplete Markets	No	Yes
Asymmetric Distribution of Increments	No	Yes
Clustering of Large Increments	No	No
Positive Autocorrelation in Absolute Returns	No	No

Table 1: Stylized Facts Captured by Brownian Motion and Lévy Processes

Many parametric Lévy models have been proposed that parsimoniously capture the stylized properties of stock returns, among these the CGMY model of Carr, Geman, Madan and Yor [8], which uses the tempered stable Lévy process as a driver for price dynamics. Other parametric Lévy processes include the Meixner process, proposed by Schoutens and Teugels [23], the Normal Inverse Gaussian process, introduced to finance by Barndorff-Nielsen [5], Kou's double exponential model, which is a jump-diffusion model similar to Merton, proposed by Kou [16], and the Generalized Hyperbolic model introduced by Eberlein and Prause [14]. The Generalized Hyperbolic distribution contains as special cases or limiting cases the normal, exponential, gamma, variance gamma, hyperbolic and Normal Inverse Gaussian distributions [15].

In this study, we will examine the small jumps of Lévy processes. Specifically, we will remove the small jumps by truncating the Lévy density. To compensate for this, a diffusion component will be added based on the level of truncation. A Lévy process where the small jumps have been removed is merely a compound Poisson process. It is known that for European options, the prices achieved in the truncated model converge to the prices of the associated Lévy process [24]. However, when working with exotic options, such as Barrier and American contracts, the truncated model will not produce prices that converge to that of the associated Lévy processes. To show this, we will use a tempered stable Lévy process and examine the partial integro-differential equation (PIDE) associated with the expected value of the discounted payoff for Barrier and American options. Finite element methods will be used for the numerical solution of the PIDE, with linear hat functions as a basis. The organization will be as follows: Section 2 will give a brief introduction to Lévy models. Section 3 will cover option pricing using Lévy processes–the PIDE, variational formulation, and localization for European, Barrier and American-style contracts. Section 4 will present the Galerkin discretization of the pricing problems developed in Section 3. Section 5 will discuss the small jump truncation and regularization, including convergence estimates for European options. Section 6 will present results about the convergence of the truncated prices for European, Barrier and American options with numerical examples.

2. Lévy Models

2.1. Introduction to Lévy Processes

Lévy processes comprise a generalized class of stochastic processes that includes both Poisson processes and Brownian motion as special cases.

Definition 2.1. (Lévy Process) An adapted process $X = (X_t)_{t\geq 0}$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfying the usual conditions with $X_0 = 0$ a.s. is a *Lévy Process* if

(i) X has independent increments: For $0 \leq s < t < \infty$, $X_t - X_s$ is independent of $\mathcal{F}_s$.

(ii) X has stationary increments: For $0 \leq s < t < \infty$, $X_t - X_s \sim X_{t-s}$.

(iii) X is continuous in probability, $\forall \varepsilon > 0, \ \lim_{s \to t} P\left(|X_t - X_s| > \varepsilon\right) = 0$

Remark 2.1. For any Lévy Process X, there exists a unique càdlàg modification, which is also a Lévy Process. We will always consider this càdlàg modification, and denote it X.

Let $X_1 = \sum_{k=1}^n \left(X_{\frac{j}{n}} - X_{\frac{j-1}{n}}\right)$. By the definition of a Lévy Process, this representation shows that X_1 can be expressed as the sum of n independent random variables, and hence follows an *infinitely divisible distribution.* Its characteristic function can be expressed (for $u \in \mathbb{R}$, $t \geq 0$) as follows:

$$\varphi_{X_t}(u) = \mathbb{E}\left[e^{iuX_t}\right] = \left(\mathbb{E}\left[e^{iuX_1}\right]\right)^t \tag{2.1}$$

Theorem 2.1. (*Lévy Khinchin*) *For any infinitely divisible distribution on $\mathbb{R}$, there exists a unique Lévy process X such that X_1 follows that distribution. For any $u \in \mathbb{R}$ and $t \geq 0$ we have that*

$$\varphi_{X_t}(u) = \mathbb{E}\left[e^{iuX_t}\right] = e^{t\psi(u)} \tag{2.2}$$

where $\psi(u)$ is the characteristic exponent and is given by

$$\psi(u) = -\frac{\sigma^2 u^2}{2} + ibu + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1}\right) \nu(dx) \tag{2.3}$$

for σ, $b \in \mathbb{R}$ and ν a σ-finite measure on $\mathbb{R}$ satisfying

$$\nu(\{0\}) = 0 \tag{2.4}$$

$$\int_{\mathbb{R}} \left(1 \wedge |x|^2\right) \nu(dx) < \infty \tag{2.5}$$

Proof. See [22, Theorem 25.3]. □

Remark 2.2. The triple (b, σ, ν) is known as the *Lévy Characteristic Triple* and leads to a unique Lévy process, as the following theorem will show.

Theorem 2.2. (*Lévy-Itô Decomposition*)

1. *Every Lévy process X can be decomposed in a unique fashion as a sum of three independent Lévy processes $X = X^{(1)}+X^{(2)}+X^{(3)}$ where $X^{(1)}$ is a linear transform of Brownian motion, $X^{(2)}$ is a compound Poisson process containing all jumps of X which are of magnitude greater than 1, and $X^{(3)}$, a pure jump square-integrable martingale containing all jumps of X of magnitude less than 1.*

2. *Given a triple (b, σ, ν) which satisfies the properties of Theorem 2.1, there exists a unique probability measure P on $(\Omega, \mathcal{F})$ under which the process X_t with characteristic exponent ψ as above is a Lévy process. The decomposition of the process corresponds to a decomposition of the characteristic exponent, $\psi = \psi^{(1)}+\psi^{(2)}+\psi^{(3)}$, where*

$$\psi^{(1)}(u) = -\frac{1}{2}\sigma^2 u^2 + ibu \tag{2.6}$$

$$\psi^{(2)}(u) = \int_{\mathbb{R}} \left(e^{iux} - 1\right) \mathbb{1}_{|x|\geq 1} \nu(dx) \tag{2.7}$$

$$\psi^{(3)}(u) = \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\right) \mathbb{1}_{|x|<1} \nu(dx) \tag{2.8}$$

Proof. See [3, Section 2.4]. □

Note that the Lévy measure does not have to be a probability measure in general, as the Lévy process can be of *infinite variation*, i.e., $\int_{\mathbb{R}} \nu(dx) = \infty$. However, if $\nu(dx)$ is of *finite variation*, i.e., $\lambda := \int_{\mathbb{R}} \nu(dx) < \infty$, then ν can be normalized to define a probability measure μ on $\mathbb{R} \setminus \{0\}$. If $\lambda < \infty$, then X is a compound Poisson process with jump intensity–or expected number of jumps per unit of time–λ. Next, we will consider a particular class of Lévy processes, tempered stable processes.

2.2. Tempered Stable Processes

The tempered stable process is a four parameter Lévy process. When considered by Carr, Geman, Madan and Yor (and termed the CGMY process), the tempered stable process could either be a pure jump process, or a diffusion component could be added [8]. The characteristic triple of the pure jump process is given by

$$\left(\gamma^{TS}, 0, \nu^{TS}\right)$$

where

$$\nu^{TS} = k^{TS}(x) = c \begin{cases} |x|^{-1-\alpha} e^{-\beta_-|x|} & \text{if } x < 0 \\ |x|^{-1-\alpha} e^{-\beta_+ x} & \text{if } x > 0 \end{cases} \tag{2.9}$$

Note that since k^{TS} is decaying exponentially, ψ^{TS} can be expressed as

$$\psi^{TS}(u) = i\gamma^{TS}u + \int_{\mathbb{R}} \left(e^{iux} - 1 - iux\right) k^{TS}(x)\, dx \tag{2.10}$$

where γ^{TS} is given such that e^{X_t} is a martingale.

2.3. Empirical Facts

Skew and Kurtosis The tempered stable process provides a parametric model which can capture the features of various types of data (stock returns, index returns, options prices) by calibrating the parameters of the Lévy density, $k^{TS}(x)$. Because it is possible to specify different exponential decays for positive and negative jumps, skew is incorporated (see Figure 2.1). By assigning a smaller rate of decay, β_-, to negative jumps and a larger rate of decay to positive jumps, β_+, the empirical observation that returns exhibit an asymmetric distribution of increments is incorporated into the model (See Table 1). Moreover, because it is a jump model with semi-heavy tails (specified by exponential decay), kurtosis is seen. The model can accommodate finite activity, infinite activity, and finite and infinite variation. Through calibrating the Lévy density $k^{TS}(x)$ to time series and option data, Carr, Geman, Madan and Yor found that market indices lack a diffusion component, which adds to the substantial evidence that a Brownian motion alone cannot adequately model price processes, and more general models such as Lévy models could be more appropriate [8].

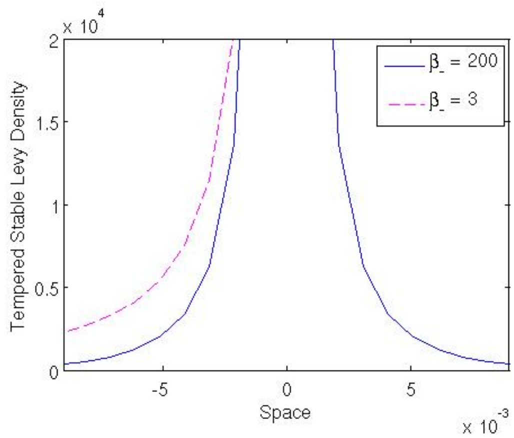

Figure 2.1: Tempered Stable Lévy Density with Skew

Implied Volatility Tempered stable processes are able to capture the volatility smile across strike. Volatility is an important consideration when investing in derivatives. Indeed, options such as straddles and strangles (simple combinations of plain vanilla calls and puts) are used to gain an imperfect exposure to volatility. Option pricing models driven by Lévy processes–including the tempered stable process explained in Section 2.2–are preferable to diffusion-driven models in part because they are able to model the empirically-observed volatility smile across the strike of a given option. Accurately modeling the implied volatility surface over strike and maturity has inspired much research since the Black-Scholes model was first proposed. An implied volatility for a given strike and maturity is calculated by inverting the Black-Scholes formula (i.e., finding the volatility that would correctly price the option of a given strike and maturity using the Black-Scholes model). The Merton model was the first model proposed to try to capture the volatility smile (among other empirical observations) [20]. Given an options pricing model, parameters are calibrated to plain vanilla option data and the calibrated model is then used to price exotic options. In this sense it is important to gauge the level to which a given pricing model captures the volatility surface across strike and maturity. For Lévy models, the empirical skew and smile observed across strikes is captured relatively well. However, Lévy models as well as stochastic volatility models decay too quickly across maturity (both have an order of decay of around $1/T$, whereas the implied volatility surface approximately decays like $1/\sqrt{T})$ [10]. In the case of Lévy models, this fast decay is due to the assumption of independent, stationary increments.

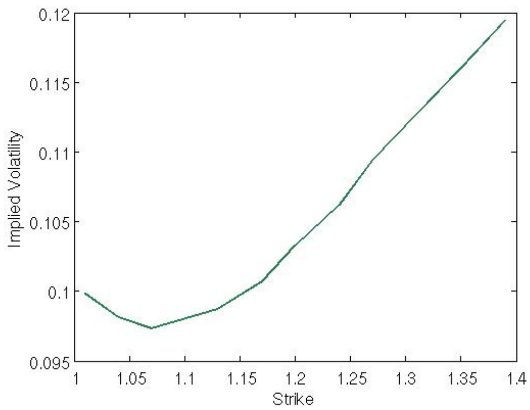

Figure 2.2: Implied Volatility across Strike for Tempered Stable Lévy Process Calibrated to S&P 500 Options

However, the calibration procedure is ill-posed: there may be many combinations of parameters that describe the option price data. Further, by assuming a specific Lévy model for calibration, a set of parameters that exactly represents the data may no longer exist. To compensate for this, a relative entropy problem has been developed where the distance from a set of parameters to a pre-specified prior is penalized. This method gives the parameters that best explain the option prices which are the closest distance to prior assumptions. The details of the relative entropy calibration procedure as well as empirical results can be found in [11].

To demonstrate how the volatility smile is capture by the tempered stable Lévy process, we use the parameters of the calibration from [9]. Here, for June 2, 2003, options on the S&P 500 with maturity $T = 0.7968$ (in years) were used. The resulting parameters were $\alpha = 0.5839$, $\beta_+ = 19.5587$, $\beta_- = 4.3120$, and $c = 0.3970$. In Figure 2.2 you can see the implied volatility smile.

3. Option Pricing

Given a payoff function for an option at maturity T, the expected discounted payoff will be the options price today. It is advantageous to change from the real-world probability measure $\mathbb{P}$ to an equivalent risk-neutral measure $\mathbb{Q}$, because under $\mathbb{Q}$, all log-price processes are assumed to be martingales (with certain integrability requirements). There a couple of ways to establish $\mathbb{Q}$. The Esscher Transform is one particular type of measure change that can be used to define the relation between $\mathbb{P}$ and $\mathbb{Q}$ when working with Lévy processes (see [21]). We assume the equivalent martingale measure $\mathbb{Q}$ has been chosen.

In Lévy markets, the price process under $\mathbb{Q}$ is given by

$$S_t = S_0 \; e^{\left(r - \frac{\sigma^2}{2} + c_{exp}\right)t + X_t} \tag{3.1}$$

where X_t is a Lévy process. Given $\int_{\mathbb{R}} \min\left(1, x^2\right) v_{\mathbb{Q}}\left(dx\right) < \infty$, by the Lévy Khinchin formula X_t can be decomposed into a Brownian motion (diffusion) component B_t, and Y_t, a quadratic pure jump process independent of B_t. That is, $X_t = \sigma B_t + Y_t$. The parameter c_{exp} is chosen such that the discounted exponential quadratic pure jump process is a martingale, and the mean rate of return on S is risk-neutrally r, the risk-free interest rate. This is achieved through the following equality:

$$e^{-c_{exp}t} = \mathbb{E}_{\mathbb{Q}}\left[e^{Y_t}\right] \tag{3.2}$$

For an explicit formula for c_{exp}, refer to Equation (3.9). To find the dynamics of the price process, define $\mu\left(dx, dt\right)$ to be the integer-valued jump measure that counts the number of jumps of Y_t in space-time. By Itô's formula, S_t solves the following SDE:

$$dS_t = S_{t\_} dX_t + S_{t\_} \int_{\mathbb{R}} \left(e^y - 1 - y\right) \mu\left(dy, dt\right) + \left(r + c_{exp}\right) dt \tag{3.3}$$

Because Lévy processes are time homogeneous (stationary increments), the jump measure can be decomposed further as follows: $\mu\left(dx, dt\right) = v_{\mathbb{Q}}\left(dx\right) \times dt$ where dt is the Lebesgue measure. We assume that the Lévy measure has a density under $\mathbb{Q}$: $v_{\mathbb{Q}}\left(dx\right) = k\left(x\right) dx$, where $k\left(x\right)$ describes the jumps of size x in Y_t.

3.1. European Options

A European option is a contract where the option holder can exercise at a fixed point in the future. This date is called the maturity of the option and is denoted T. For a European option on a single underlying asset, the payoff is a function of the asset price at maturity. If the price process of the underlying asset is denoted $\{S_t\}_{t=0}^{T}$, then the payoff of a European option at T can be expressed as $g(S_T)$. For a call option, $g(S_T) = (S_T - K)_+$.

The price process S_t evolves under the real world probability measure $\mathbb{P}$. Under an equivalent martingale measure $\mathbb{Q}$, however, price processes become martingales (under $\mathbb{Q}$, there is no advantage to assuming additional risk, so the expected future rate of return for all assets is the risk-free interest rate, r).

In Lévy markets, we assume the price process S_t is driven by a Lévy process as in Equation (3.1). Under the chosen risk neutral equivalent measure $\mathbb{Q}$, the price $f(t,s)$, of a European option with $\mathbb{Q}$-integrable payoff $g(s)$ can be written

$$f(t,s) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)} g(S_T) \,|S_t = s\right]$$

To compute the option value deterministically, we need a generalization of the Feynman-Kac Formula to relate the expectation to a PIDE. First we convert to log price and time to maturity. Let $X_t = \ln\left(e^{r(T-t)} S_t\right)$; $\tau = T - t$. Then $f(t,s) = u(\tau, x)$.

Theorem 3.1. (*Extended Feynman Kac*) *Given a Lévy process X on $\mathbb{R}$ with characteristic triple $\left(\gamma, \sigma^2, \nu\right)$ where $\sigma \geq 0$, and the Lévy measure ν satisfies $\nu(dx) = k(x)\,dx$ and $\int_{\mathbb{R}} \left(1 \wedge |x|^2\right) \nu(dx) < \infty$ and the following three conditions:*

> **[FK1]** *(Activity of Small Jumps): There exist constants $c_1, C_+ > 0$ and $\alpha < 2$ such that*
>
> $$|k(z)| \leq C_+ \frac{1}{|z|^{\alpha+1}} \qquad 0 \leq |z| \leq 1 \tag{3.4}$$
>
> **[FK2]** *(Semi-heavy Tails): There are constants $C > 0, \beta_- > 0$ and $\beta_+ > 1$ such that*
>
> $$k(z) \leq C \begin{cases} e^{-\beta_-|z|} & \text{if } z < 0 \\ e^{-\beta_+|z|} & \text{if } z > 0 \end{cases} \qquad \forall |z| > 1 \tag{3.5}$$
>
> *If $\sigma = 0$, we assume in addition that $0 < \alpha < 2$ and*
>
> **[FK3]** *(Boundedness from below of $k(z)$): There is $C_- > 0$ such that*
>
> $$\frac{1}{2}\left(k(-z) + k(z)\right) \geq \frac{C_-}{|z|^{1+\alpha}} \qquad \forall 0 < |z| < 1 \tag{3.6}$$

Assume that $u(\tau, x)$ in $C^{1,2}((0,T) \times \mathbb{R}) \cap C^0([0,T] \times \mathbb{R})$ solves the PIDE

$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + \left(\frac{\sigma^2}{2} - r + c_{exp}\right)\frac{\partial u}{\partial x} + A[u] + ru = 0 \tag{3.7}$$

in $(0,T)\times\mathbb{R}$, where A denotes the integro-differential operator defined for $\varphi\in C^2(\mathbb{R})$ by

$$A[\varphi](x) := -\int_{\mathbb{R}}\left\{\varphi(x+y)-\varphi(x)-\varphi'(x)\,y\right\}k(y)\,dy \tag{3.8}$$

and $c_{exp}\in\mathbb{R}$ is given by

$$c_{exp} = \int_{\mathbb{R}}(e^y-1-y)\,k(y)\,dy \tag{3.9}$$

with the initial condition,

$$u_0 = u(0,x) := g(e^x) \tag{3.10}$$

Then $f(t,s)=u(T-t,log(s))$ satisfies

$$f(t,s)=\mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)}g(S_T)\,\middle|S_t=s\right].$$

Conversely, if $f(t,s)$ above is sufficiently regular, the function $u(\tau,x)=f(T-\tau,e^x)$ solves the given PIDE.

Proof. See [21, Section 1.5]. □

Let X be a Lévy process with state space $\mathbb{R}$ and characteristic triplet (c_{exp},σ,ν) such that ν satisfies [FK2] ("Semi-heavy Tails"). The interest rate in the PIDE can be set to zero by the transformation $u(\tau,x)=e^{-r\tau}\breve{u}(\tau,x+r\tau)$. Therefore the PIDE is reduced to:

$$\frac{\partial u}{\partial\tau}+A^{BS}[u]+A^J[u]=0 \tag{3.11}$$

where

$$A^J[u] = -\int_{\mathbb{R}}\left\{u(x+y)-u(x)-u'(x)\,y\right\}k(y)\,dy \tag{3.12}$$

$$A^{BS}[u] = -\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}+\left(c_{exp}+\frac{\sigma^2}{2}\right)\frac{\partial u}{\partial x} \tag{3.13}$$

The drift can be removed via the transformation $\breve{u}(\tau,x)=v\left(\tau,x-\left(c_{exp}+\frac{\sigma^2}{2}\right)\tau\right)$.

Therefore the strong form of the PIDE can be expressed as follows:
Find $u(\tau,x)\in C^{1,2}([0,T]\times\mathbb{R})\cap C^0([0,T]\times\mathbb{R})$ such that

$$\begin{aligned}\frac{\partial u}{\partial\tau}-\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}+A^J[u]=0 &\quad \text{in } (0,T)\times\mathbb{R}\\ u(0,x)=u_0 &\quad \text{in } \mathbb{R}\end{aligned} \tag{3.14}$$

3.1.1. Variational Formulation

To apply the finite element method and find a numerical solution of the above PIDE, we consider the variational formulation of (3.14). Formally, we take test functions v in $C_0^\infty(\mathbb{R})$ and integrate over $\mathbb{R}$:

$$\underbrace{\frac{\partial}{\partial\tau}\int_{\mathbb{R}} u(x)\,v(x)\,dx}_{I_1} - \underbrace{\frac{\sigma^2}{2}\int_{\mathbb{R}} u''(x)\,v(x)\,dx}_{I_2} - \tag{3.15}$$
$$-\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}}\left(u(x+y)-u(x)-\frac{\partial u}{\partial x}(x)\,y\right)k(y)\,dy\;v(x)\,dx}_{I_3} = 0$$

We denote the L^2-inner product by $(u,v) := \int_{\mathbb{R}} u(x)\,v(x)\,dx$, therefore I_1 can be written $\frac{\partial}{\partial\tau}(u,v)$. By integration by parts we have for I_2:

$$\begin{aligned} I_2 &= -\frac{\sigma^2}{2}\int_{\mathbb{R}} u''(x)\,v(x)\,dx \\ &= -\frac{\sigma^2}{2}\left(\underbrace{v(x)\,u'(x)\,|_{x=\pm\infty}}_{=0 \text{ if } v|_{\pm\infty}=0} - \int_{\mathbb{R}} u'(x)\,v'(x)\,dx\right) \\ &= \frac{\sigma^2}{2}\int_{\mathbb{R}} u'(x)\,v'(x)\,dx \end{aligned} \tag{3.16}$$

Now consider I_3. First, we look at the order of the integrand in I_3. In the discretization step, we would like to use piecewise linear hat functions. However, by computing the Taylor expansion of $u(x+y)$ about $y=0$ we see that it is $O\left(y^2\right)$ if $u \in C^2(\mathbb{R})$:

$$\begin{aligned} u(x+y) &= u(x) + yu'(x) + \frac{1}{2}y^2u''(x) + O\left(y^3\right) \\ \Rightarrow \quad u(x+y) - u(x) - yu'(x) &= \frac{1}{2}y^2u''(x) + O\left(y^3\right) \\ \Rightarrow \quad |u(x+y) - u(x) - yu'(x)| &= O\left(y^2\right) \end{aligned} \tag{3.17}$$

Therefore it is not clear that linear hat functions are applicable. We integrate by parts with respect to the integral in y:

$$\begin{aligned} I_3 &= -\int_{\mathbb{R}}\int_{\mathbb{R}}\big(u(x+y)-u(x)-u'(x)\,y\big)\,k(y)\,dy\;v(x)\,dx \\ &= -\int_{\mathbb{R}}\left(\underbrace{\big(u(x+y)-u(x)-u'(x)\,y\big)}_{u\in C^2(\mathbb{R})}\underbrace{k^{-1}(y)}_{[FK2]}\,|_{-\infty}^{\infty}\right)v(x)\,dx \\ &\quad + \int_{\mathbb{R}}\int_{\mathbb{R}}\big(u'(x+y)-u'(x)\big)\,k^{-1}(y)\,dy\;v(x)\,dx \end{aligned} \tag{3.18}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left(u'(x+y) - u'(x) \right) k^{-1}(y)\, dy\; v(x)\, dx \\
&= \int_{\mathbb{R}} \left(\underbrace{k^{-2}(y)}_{[FK2]} \underbrace{\left(u'(x+y) - u'(x) \right)}_{u \,\in\, C^2(\mathbb{R})} |_{-\infty}^{\infty} \right) v(x)\, dx \qquad (3.19) \\
&\qquad - \int_{\mathbb{R}} \int_{\mathbb{R}} u''(x+y)\, k^{-2}(y)\, dy\; v(x)\, dx \\
&= - \int_{\mathbb{R}} \int_{\mathbb{R}} u''(x+y)\, k^{-2}(y)\, dy\; v(x)\, dx
\end{aligned}
$$

In Equation (3.18) and (3.19), we must have a Lévy process which satisfies [FK2]. Then the Lévy density $k(x)$ decays at least exponentially fast, hence its first and second antiderivatives as well. Since $u \in C^2(\mathbb{R})$, the antiderivatives will decay faster than u, u' and the integral will vanish. By again performing integration by parts on I_3 we obtain:

$$
\begin{aligned}
I_3 &= - \int_{\mathbb{R}} \int_{\mathbb{R}} u''(x+y)\, k^{-2}(y)\, dy\; v(x)\, dx \\
&\overset{Fubini}{=} - \int_{\mathbb{R}} k^{-2}(y) \int_{\mathbb{R}} u''(x+y)\, v(x)\, dxdy \\
&= - \int_{\mathbb{R}} k^{-2}(y) \left(\underbrace{v(x)}_{v(\pm\infty)=0} \; \underbrace{u'(x+y)}_{u \,\in\, C^2(\mathbb{R})} |_{x=-\infty}^{\infty} \right) dy \qquad (3.20) \\
&\qquad + \int_{\mathbb{R}} k^{-2}(y) \int_{\mathbb{R}} u'(x+y)\, v'(x)\, dxdy \\
&\overset{z=x+y}{=} \int_{\mathbb{R}} \int_{\mathbb{R}} u'(z)\, v'(x)\, k^{-2}(z-x)\, dxdz \\
&\overset{Fubini}{=} \int_{\mathbb{R}} \int_{\mathbb{R}} u'(z)\, v'(x)\, k^{-2}(z-x)\, dzdx
\end{aligned}
$$

In Equation (3.20), we choose test functions which vanish at $\pm\infty$ so that the first integral will equal zero. In order for the resulting integral to be well-defined, $v' \in L^2(\mathbb{R})$. Now (3.15) can be written as: Given $u_0 \in L^2(\mathbb{R})$

$$
\frac{\partial}{\partial \tau}(u, v) + a^{BS}(u, v) + a^J(u, v) = 0 \qquad (3.21)
$$

with

$$
a^{BS}(u, v) = \frac{\sigma^2}{2} \int_{\mathbb{R}} u'(x)\, v'(x)\, dx \qquad (3.22)
$$

$$
a^J(u, v) = \int_{\mathbb{R}} \int_{\mathbb{R}} u'(z)\, v'(x)\, k^{-2}(z-x)\, dzdx \qquad (3.23)
$$

Here, a^{BS} and a^J are well defined for piecewise linear hat functions. In the following sections we denote $a(u,v) := a^{BS}(u,v) + a^J(u,v)$ for simplicity of notation.

At this point there remain two important details to discuss: first, does the assumption that $u_0 \in L^2(\mathbb{R})$ make sense in the context of options pricing; second, does the functional setup described above admit a unique solution to the pricing problem? Concerning the first question, in the case of call options, $u_0 = g$ grows exponentially at infinity, $g(e^x) = (e^x - K)_+$, which is not in $L^2(\mathbb{R})$. However, to compute the solution of the PIDE problem numerically, we must truncate the infinite domain $\mathbb{R}$ to a finite domain, which will be denoted $\Omega_R = (-R, R)$. In the context of the finite computational domain, even a payoff function with exponential growth is square integrable, e.g., for a call option, $g \in L^2(\Omega_R)$. For the functional setup on the entire domain $\mathbb{R}$ where the payoff function g can grow exponentially, see [19]. Our second concern is whether the weak formulation admits a unique solution.

3.1.2. Localization

To evaluate the weak formulation of the pricing problem numerically, we must truncate $\mathbb{R}$ to a finite domain $\Omega_R := (-R, R)$ and impose artificial zero Dirichlet boundary conditions, $u(\tau,\cdot)|_{\Omega_R^c} = 0$. The localized problem reads as follows:

Find $u \in L^2(0,T;V) \cap H^1(0,T;V^*)$ such that

$$\begin{aligned} \frac{\partial}{\partial \tau}(u,v)_{L^2(\Omega_R)} + a_R(u,v) &= 0 \quad \text{in} \quad (0,T)\times\Omega_R \\ u(\tau,x) &= 0 \quad \text{on} \quad (0,T)\times\Omega_R^c \\ u(0,x) &= u_0 \quad \text{in} \quad \Omega_R \end{aligned} \tag{3.24}$$

Here $a_R(u,v) := a(\widetilde{u},\widetilde{v})$, where u, v have support in Ω_R, and $\widetilde{u}, \widetilde{v}$ denote the extension of u, v by zero to all of $\mathbb{R}$.

Proposition 3.2. *Given a Lévy process X_t which satisfies* [FK1], [FK2], *and* [FK3], *the localized problem admits a unique solutions $u \in L^2(0,T;V) \cap H^1(0,T;V^*)$, where*

$$V = \begin{cases} H_0^{\alpha/2}(\Omega_R) & \text{if } \sigma = 0 \\ H_0^1(\Omega_R) & \text{if } \sigma > 0 \end{cases} \tag{3.25}$$

Proof. See [19, Theorem 3.4]. □

By first converting to excess-to-transformed-payoff ($U := u - e^{-rt} g(e^{x+rt})$), it is possible to show that the localization error decays exponentially (see [19, Theorem 4.1]).

3.2. Barrier Options

Barrier options are exotic, path dependent financial contracts that become worthless or valuable if the underlying price process S_t passes above or below a barrier level. We consider Barrier options with a single exercise date, the maturity T. This is known as a European-style Barrier contract. There are many types of Barrier options:

Down-and-Out: The spot price S_0 starts above the barrier B, and if $S_t < B$ for $0 \leq t \leq T$, the option expires worthless.

Up-and-Out: The spot price S_0 starts below the barrier B, and if $S_t \geq B$ for $0 \leq t \leq T$, the option expires worthless.

Up-and-In: The spot price S_0 starts below the barrier, B, and if $S_t \geq B$ for $0 \leq t \leq T$, the option becomes valuable.

Down-and-In: The spot price S_0 starts above the barrier, B, and if $S_t \leq B$ for $0 \leq t \leq T$, the option becomes valuable.

There are also Barrier contracts that have more than one barrier, which are generally combinations of the above basic Barrier contracts. Barrier options offer more flexibility than plain vanilla European options in the following way: say you would like to purchase a European call on an asset S where you expect that the asset price will never fall below a certain level before maturity, T. In this case, you could purchase a Down-and-Out Barrier call option for a lower premium than the plain vanilla contract.

The path dependency of Barrier options is expressed mathematically as an indicator function in the expected payoff. For example, in the case of a Down-and-Out Barrier option,

$$f(t,s) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)} g\left(S_T\right) \mathbb{1}_{\{\tau_B > T\}} | S_t = s\right]$$

where τ_B is the hitting time of $[0, B]$ of the process S_t. For Down-and-In options,

$$f(t,s) = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)} g\left(S_T\right) \mathbb{1}_{\{\tau_B < T\}} | S_t = s\right]$$

Barrier options are European options, except they have extra features depending on the path of the price process, S_t. Therefore we can use the same PIDE as in section 3.1, and merely impose the appropriate boundary conditions at the barriers. Note that the barriers automatically bring a localization to the pricing problem.

The strong form of the PIDE for Down-and-Out Barrier options can be expressed as follows: Define $\Omega_B := (\ln(B), \infty)$ and denote by $\Omega_B^c = \mathbb{R} \setminus \Omega_B$ its complement.
Find $u(\tau, x) \in C^{1,2}([0,T] \times \mathbb{R}) \cap C^0([0,T] \times \mathbb{R})$ such that

$$\begin{aligned} \frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + A^J[u] = &\ 0 \quad \text{in } (0,T) \times \Omega_B \\ u(\tau, x) = &\ 0 \quad \text{in } (0,T) \times \Omega_B^c \end{aligned} \tag{3.26}$$

$$u(0,x) = u_0 \quad \text{in} \quad \Omega_B$$

The strong form of the PIDE for Down-and-In Barrier options can be expressed as follows:
Find $u(\tau,x) \in C^{1,2}([0,T]\times\mathbb{R}) \cap C^0([0,T]\times\mathbb{R})$ such that

$$\begin{aligned} \frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} + A^J[u] &= 0 && \text{in } (0,T)\times\Omega_B \\ u(\tau,x) &= u^{\mathrm{Eur}}(\tau,x) && \text{in } (0,T)\times\Omega_B^c \\ u(0,x) &= 0 && \text{in } \Omega_B \end{aligned} \tag{3.27}$$

where u^{Eur} is the value of the European option of the same payoff.

3.2.1. Variational Formulation

Analogous to the European case, the variational formulation is derived by integrating the PIDEs above against suitable test functions. In the case of Barrier options, we immediately see a localization of the domain, because the PIDE is valid for a subinterval of $\mathbb{R}$.

For Down-and-Out Barrier options the variational formulation reads as follows:

Find $u \in L^2(0,T;V) \cap H^1(0,T;V^*)$ such that

$$\begin{aligned} \frac{\partial}{\partial \tau}(u,v)_{L^2(\Omega_B)} + a_B(u,v) &= 0 \quad \text{in } (0,T)\times\Omega_B \\ u(\tau,x) &= 0 \quad \text{in } (0,T)\times\Omega_B^c \\ u(0,x) &= u_0 \text{ in } \quad \Omega_B \end{aligned} \tag{3.28}$$

Here $a_B(u,v)$ is defined as in the European localized problem (see 3.24), where for $u,v \in V$ we denote $\widetilde{u},\widetilde{v}$ the extension of u,v by zero to all of $\mathbb{R}$. Then $a(\widetilde{u},\widetilde{v}) = a_B(u,v)$. The space V is given by:

$$V = \begin{cases} H_0^{\alpha/2}(\Omega_B) & \text{if } \sigma = 0 \\ H_0^1(\Omega_B) & \text{if } \sigma > 0 \end{cases} \tag{3.29}$$

The same continuity and coercivity results for European options guarantee a unique solution to the weak pricing problem for Down-and-Out and Down-and-In Barrier contracts (see Proposition 3.2). No error is induced at this step because we have not artificially truncated the domain of the PIDE–the truncation was specified by the nature of the contract.

For Down-and-In Barrier options the variational formulation is exactly the same, though the boundary conditions change to match the corresponding strong formulation above:

Find $u \in L^2(0,T;V) \cap H^1(0,T;V)^*)$ such that

$$\frac{\partial}{\partial \tau}(u,v)_{L^2(\Omega_B)} + a_B(u,v) = 0 \qquad \text{in } (0,T)\times\Omega_B$$

$$
\begin{aligned}
u(\tau, x) &= u^{\mathrm{Eur}}(\tau, x) && \text{in } (0,T) \times \Omega_B^c \\
u(0, x) &= 0 && \text{in } \Omega_B
\end{aligned}
\tag{3.30}
$$

The Hilbert space V is given as in Equation (3.29).

3.2.2. Localization

For Barrier options with two barriers, the localization is already specified. This is the case with No Touch Barrier options. In a No Touch contract, the spot price is between two barriers, $B_1 < S_0 < B_2$. If the price process hits B_1 or B_2 before maturity T, the option expires worthless. In this case the pricing problem is defined on a bounded domain, $\Omega_R = (\ln(B_1), \ln(B_2))$, no localization errors result, and no artificial boundary conditions must be specified (zero Dirichlet boundary conditions arise from the contract itself).

For the case of Down-and-Out Barrier options, we must truncate the upper region of the domain Ω_B to a finite computational domain $\Omega_R := (\ln B, R)$, and impose an artificial zero Dirichlet boundary condition at $\{R\}$. The localized problem reads as follows:

Find $u \in L^2(0,T;V) \cap H^1(0,T;V^*)$ such that

$$
\begin{aligned}
\frac{\partial}{\partial \tau}(u,v)_{L^2(\Omega_R)} + a_R(u,v) =&\ 0 \text{ in } && (0,T) \times \Omega_R \\
u(\tau, x) =&\ 0 \text{ on } && (0,T) \times \Omega_R^c \\
u(0, x) =&u_0 \text{ in } && \Omega_R
\end{aligned}
\tag{3.31}
$$

The bilinear form is as defined for the European localized problem. The space V is given by Equation (3.25).

For Down-and-In Barrier options, Ω_R remains unchanged, but the boundary conditions must be changed. Let $\Omega_R^c = \mathcal{D}_- \cup \mathcal{D}_+$ where $\mathcal{D}_- = (-\infty, \ln(B)]$ and $\mathcal{D}_+ = [R, \infty)$. Then the problem reads: Find $u \in L^2(0,T;V) \cap H^1(0,T;V^*)$ such that

$$
\begin{aligned}
\frac{\partial}{\partial \tau}(u,v)_{L^2(\Omega_R)} + a_R(u,v) &= 0 && \text{in } (0,T) \times \Omega_R \\
u(\tau, x) &= u^{\mathrm{Eur}}(\tau, x) && \text{in } (0,T) \times \mathcal{D}_- \\
u(\tau, x) &= 0 && \text{in } (0,T) \times \mathcal{D}_+ \\
u(0, x) &= 0 && \text{in } \Omega_R
\end{aligned}
\tag{3.32}
$$

This localized problem induces no error at $\ln(B)$, because the localization and boundary conditions at the barrier were a result of the contract specification. Where the domain was truncated for computational purposes, if one first would transform to excess to payoff, the localization error is precisely the error in the European case, which decays exponentially with increasing R [19, Theorem 4.1]. Moreover, the localized problem for Barrier options also admits a unique solution under the Proposition 3.2

3.3. American Options

American options are contracts which can be exercised at any time until maturity, T for a payoff $g(s)$. Options of this type can be formulated as an optimal stopping problem:

$$f(t,s) = \sup_{t\leq\tau\leq T}\mathbb{E}_{\mathbb{Q}}\left[e^{-r(\tau-t)}g(S_\tau)\,|S_t = s\right]$$

where $\mathbb{Q}$ is fixed such that $e^{-rt}S_t$ is a $\mathbb{Q}$-martingale. The supremum above is taken over all stopping times τ on the σ-field generated by $\{S_t\}_{0\leq t\leq T}$. This formulation makes intuitive sense: you would like to exercise the option precisely at the time which maximizes the payoff. However, you can only make this decision based on the information set generated by S_t. In our modeling assumptions, S_t is a càdlàg process, which means that at t, the information set consists of all information up to t and an infinitesimal amount of future information. The times which are measurable with respect to this information set are precisely the stopping times as defined above.

American options are characterized by a continuation region and a stopping region: In the continuation region, the value of an American option is greater than the payoff, so it is more valuable to hold the option. In the stopping region, the value of the American option equals the payoff. As soon as the price process S_t enters the stopping region, one should exercise the option, receiving the payoff and the time-value of the remaining time until maturity. A priori, the boundary between the exercise region and the continuation region is not known. In the continuation region, the price of the American option solves the pricing PIDE. In the stopping region, the price of the American option is equal to the payoff.

The solution of the optimal stopping problem can be formulated as the solution of a parabolic integro-differential inequality.

Theorem 3.3. *Let u_0 be a sufficiently regular payoff function on $\mathbb{R}$ and let $\sigma > 0$. Then the solution $u(\tau,x) = f(T-t,e^x)$ of the above optimal stopping problem is given by the following integro-differential inequality:*

$$\begin{aligned}
\frac{\partial u}{\partial \tau} + A^{BS}[u] + A^J[u] &\geq 0 \; in \quad (0,T)\times\mathbb{R} && (3.33)\\
u(\tau,x) &\geq u_0 \; in \quad [0,T]\times\mathbb{R} && (3.34)\\
(u(\tau,x)-u_0)\left(\frac{\partial u}{\partial \tau} + A^{BS}[u] + A^J[u]\right) &= 0 \; in \quad (0,T)\times\mathbb{R} && (3.35)\\
u(0,x) &= u_0 \; in \quad \mathbb{R} && (3.36)
\end{aligned}$$

Proof. See [6]. □

Denote $\mathcal{C}$ the continuation region and $\mathcal{E}$ the stopping (exercise) region. In the continuation region, u satisfies the PIDE for a European option, therefore

$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - \int_{\mathbb{R}} u''(x+y)\,k^{-2}(y)\,dy = 0 \qquad \text{in } \mathcal{C} \tag{3.37}$$

In the stopping region, the value of the American option is equal to the payoff. Inserting the payoff into the above PIDE will result in a positive value, therefore

$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - \int_{\mathbb{R}} u''(x+y)\,k^{-2}(y)\,dy > 0 \qquad \text{in } \mathcal{E} \tag{3.38}$$

Together, Equations (3.37) and (3.38) justify (3.33). The inequality (3.34) holds by no arbitrage. For (3.35), note the following "complementarity":

$$u(\tau,x) > u_0 \text{ and } \frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - \int_{\mathbb{R}} u''(x+y)\,k^{-2}(y)\,dy = 0 \text{ in } \mathcal{C} \tag{3.39}$$

$$u(\tau,x) = u_0 \text{ and } \frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - \int_{\mathbb{R}} u''(x+y)\,k^{-2}(y)\,dy > 0 \text{ in } \mathcal{E} \tag{3.40}$$

Therefore it must hold in $\mathcal{E} \cup \mathcal{C} = (0,T) \times \mathbb{R}$ that:

$$(u(\tau,x) - u_0)\left(\frac{\partial u}{\partial \tau} - \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} - \int_{\mathbb{R}} u''(x+y)\,k^{-2}(y)\,dy\right) = 0 \tag{3.41}$$

Finally, (3.36) holds because at maturity, the American option is equivalent to a European option (there is no longer any early exercise premium).

3.3.1. Variational Formulation

We define the cone of admissible solutions based on the second inequality above: $K = \{v \in C_0^\infty | v \geq u_0\}$. To derive the variational form of the parabolic inequality problem, we multiply (3.33) by $v - u_0$ and integrate over the domain $\mathbb{R}$:

$$\begin{aligned}
\int_{\mathbb{R}}\left(\frac{\partial u}{\partial \tau}(x) - \frac{\sigma^2}{2}u''(x) - \int_{\mathbb{R}} u''(x+y)\,k^{-2}(y)\,dy\right)(v(x) - u_0)\,dx &\geq 0 \\
\int_{\mathbb{R}}\left(\frac{\partial u}{\partial \tau}(x) - \frac{\sigma^2}{2}u''(x) - \int_{\mathbb{R}} u''(x+y)\,k^{-2}(y)\,dy\right)\left(v(x) - u(x) + \underbrace{u(x) - u_0}_{=0^1}\right)dx &\geq 0 \\
\int_{\mathbb{R}}\left(\frac{\partial u}{\partial \tau}(x) - \frac{\sigma^2}{2}u''(x) - \int_{\mathbb{R}} u''(x+y)\,k^{-2}(y)\,dy\right)(v(x) - u(x))\,dx &\geq 0 \\
\underbrace{\int_{\mathbb{R}}\frac{\partial u}{\partial \tau}(x)(v(x) - u(x))\,dx}_{:=I_1} - \underbrace{\frac{\sigma^2}{2}\int_{\mathbb{R}} u''(x)(v(x) - u(x))\,dx}_{:=I_2} - & \\
\underbrace{\int_{\mathbb{R}}\int_{\mathbb{R}} u''(x+y)\,k^{-2}(y)\,dy\,(v(x) - u(x))\,dx}_{:=I_3} &\geq 0
\end{aligned}$$

We need to examine I_1, I_2, and I_3 to specify the variational formulation for the finite element method. $I_1 = \left(\frac{\partial}{\partial \tau}u, v - u\right)_{L^2(\mathbb{R})}$. By performing integration by parts on I_2 we

[1] By Inequality (3.35) in Theorem 3.3

find:

$$
\begin{aligned}
I_2 &= -\frac{\sigma^2}{2}\int_{\mathbb{R}} u''(x)\,(v(x)-u(x))\,dx \qquad (3.42)\\
&= -\frac{\sigma^2}{2}\left(\underbrace{(v(x)-u(x))}_{v(\pm\infty)=0,\ u(\pm\infty)=0}\ u'(x)\,|_{x=-\infty}^{\infty} - \int_{\mathbb{R}} u'(x)\big(v'(x)-u'(x\ )\big)\,dx\right)\\
&= \frac{\sigma^2}{2}\int_{\mathbb{R}} u'(x)\big(v'(x)-u'(x)\big)\,dx
\end{aligned}
$$

Given u, v which vanish at $\pm\infty$, the equality above makes sense. Further, v must be integrable and have a first weak derivative. Now examine I_3:

$$
\begin{aligned}
I_3 &= -\int_{\mathbb{R}}\int_{\mathbb{R}} u''(x+y)\,k^{-2}(y)\,dy\,(v(x)-u(x))\,dx \qquad (3.43)\\
&\overset{\text{Fubini, } z=x+y}{=} -\int_{\mathbb{R}} k^{-2}(y)\int_{\mathbb{R}} u''(z)\,(v(z-y)-u(z-y))\,dzdy\\
&= -\int_{\mathbb{R}}\left(k^{-2}(y)\underbrace{(v(z-y)-u(z-y))}_{u(\pm\infty)=0,\ v(\pm\infty)=0}u'(z)\,|_{z=-\infty}^{\infty}\right)dy\\
&\quad + \int_{\mathbb{R}}\left(k^{-2}(y)\int_{\mathbb{R}} u'(z)\big(v'(z-y)-u'(z-y)\big)\,dz\right)dy\\
&= \int_{\mathbb{R}} k^{-2}(y)\int_{\mathbb{R}} u'(z)\big(v'(z-y)-u'(z-y)\big)\,dzdy\\
&\overset{\text{Fubini, } x+y=z}{=} \int_{\mathbb{R}}\int_{\mathbb{R}} u'(x+y)\big(v'(x)-u'(x)\big)\,k^{-2}(y)\,dydx
\end{aligned}
$$

Because u, v vanish at $\pm\infty$, $I_2+I_3 = a(u, v-u)$, where $a = a^{BS}+a^{J}$ as before. Define $K := \{v \in V | v \geq u_0 \text{ a.e. in } \mathbb{R}\}$. Then the variational formulation reads as follows: Given $u_0 \in L^2(\mathbb{R})$,

$$
\left(\frac{\partial}{\partial \tau}u, v-u\right)_{L^2(\mathbb{R})} + a^{BS}(u, v-u) + a^{J}(u, v-u) \geq 0 \text{ in } (0,T)\times\mathbb{R} \qquad (3.44)
$$

The bilinear forms a^{BS}, a^{J} are as in (3.22), (3.23), respectively. Note that this functional setup only allows for square-integrable payoff functions, u_0. As in the European case, the localized problem allows for payoff functions with exponential growth.

For our problem, V has the following form:

$$
V = \begin{cases} H_0^{\alpha/2}(\mathbb{R}) & \text{if } \sigma = 0\\ H_0^{1}(\mathbb{R}) & \text{if } \sigma > 0 \end{cases} \qquad (3.45)
$$

For the variational formulation for American options in a more general functional setting see [18].

3.3.2. Localization

For numerical computations, we truncate to a finite domain $\Omega_R = (-R, R)$ and define $K := \{v \in V | v \geq u_0 \text{ a.e. in } \Omega_R\}$. The variational formulation reads as follows:
Find $u \in L^2(0, T; V)$, $\frac{\partial u}{\partial \tau} \in L^2\left(0, T; L^2(\Omega_R)\right)$, such that $u(\tau, x) \geq u_0$ in $(0, T)$ and such that for all $v \in K$:

$$\begin{aligned} \left(\frac{\partial}{\partial \tau} u, v - u\right)_{L^2(\Omega_R)} + a_R(u, v - u) \geq&\ 0 \quad \text{in } (0, T) \times \Omega_R \\ u(0, x) =&\ u_0 \quad \text{in } \Omega_R \\ u(\tau, x) =&\ 0 \quad \text{in } \Omega_R^c \end{aligned} \tag{3.46}$$

Here u, v have support in Ω_R, $a_R(u, v) := a(\widetilde{u}, \widetilde{v})$, where $\widetilde{u}, \widetilde{v}$ denote the extension of u, v by zero to all of $\mathbb{R}$. The Hilbert space V is given by (3.25).

Proposition 3.4. *Given a Lévy process X_t which satisfies* [FK1], [FK2], *and* [FK3], *the localized problem admits a unique solutions $u \in L^2(0, T; V \cap K)$ where V is given by (3.25).*

Proof. See [18, Theorem 5]. □

4. Discretization

4.1. European Options

4.1.1. Discretization in Space

For the space discretization of the weak formulation of the pricing problem for European options, we use the Galerkin method with a finite element subspace $V^N \subset V$ where $V^N = S^1_\tau \cap V$, and S^1_τ denotes the space of piecewise continuous functions on a mesh τ. As a basis for V^N, we use linear hat functions, defined as

$$b_i(x) = \max\left(1 - \frac{|x - x_i|}{h}, 0\right) \tag{4.1}$$

Then $V^N = \operatorname{span}\{b_i(x)\}_{i=1}^N$. We discretize using a uniform mesh with N subintervals of size $h = \frac{2R}{N}$ on the interval $\Omega_R = (-R, R)$. We approximate $u(\tau, x)$ by an element $u^N(\tau, x) \in V^N$. Then $u^N(\tau, x)$ can be written as a linear combination of the basis elements $b_i(x)$:

$$u^N(\tau, x) = \sum_{j=1}^{N} u_j^N(\tau)\, b_j(x) = \left(\mathbf{u}^N(\tau)\right)^T \mathbf{b} \tag{4.2}$$

where $\left(\mathbf{u}^N(\tau)\right)^T = \left(u_1^N(\tau), u_2^N(\tau), \ldots, u_N^N(\tau)\right)$ and $\mathbf{b} = (b_1(x), b_2(x), \ldots, b_N(x))^T$. Here $\mathbf{u}^N(\tau)$ is an unknown vector of coefficient functions. Then the variational formulation for European options can be approximated by:
Find $u^N(\tau, x) \in V^N$ such that

$$\frac{\partial}{\partial \tau}\left(u^N(\tau, x), v^N(x)\right) + a\left(u^N(\tau, x), v^N(x)\right) = 0 \qquad \forall v^N \in V^N \tag{4.3}$$

Substituting the representations of $u^N(\tau, x)$ and $v^N(x) = b_j$ $j = 1, \cdots, N$ in the hat function basis we find:

$$\frac{\partial}{\partial \tau}\left(\left(\mathbf{u}^N\right)^T \mathbf{b}, b_j\right) + a\left(\left(\mathbf{u}^N\right)^T \mathbf{b}, b_j\right) = 0 \qquad j = 1, \ldots, N \tag{4.4}$$

Define the mass matrix $\mathbf{M} := (b_j, b_i)_{1 \le i,j \le N} =$

$$\begin{pmatrix} (b_1(x), b_1(x)) & (b_2(x), b_1(x)) & \ldots & (b_N(x), b_1(x)) \\ (b_1(x), b_2(x)) & (b_2(x), b_2(x)) & \ldots & (b_N(x), b_2(x)) \\ \vdots & \vdots & \ddots & \vdots \\ (b_1(x), b_N(x)) & (b_2(x), b_N(x)) & \ldots & (b_N(x), b_N(x)) \end{pmatrix} \tag{4.5}$$

and the stiffness matrix $\mathbf{A} := a(b_j, b_i)_{1 \le i,j \le N} =$

$$\begin{pmatrix} a(b_1(x), b_1(x)) & a(b_2(x), b_1(x)) & \ldots & a(b_N(x), b_1(x)) \\ a(b_1(x), b_2(x)) & a(b_2(x), b_2(x)) & \ldots & a(b_N(x), b_2(x)) \\ \vdots & \vdots & \ddots & \vdots \\ a(b_1(x), b_N(x)) & a(b_2(x), b_N(x)) & \ldots & a(b_N(x), b_N(x)) \end{pmatrix} \tag{4.6}$$

Because the hat functions have support on $[x_{i-1}, x_{i+1}]$, the mass matrix $\mathbf{M}$ will be tridiagonal. However, due to the jump component of the bilinear form $a(\cdot, \cdot)$, the stiffness matrix $\mathbf{A}$ will be fully populated. At this point we have the following matrix equation:
Find $\mathbf{u}^N(\tau) \in \mathbb{R}^N$ such that

$$\mathbf{M}\frac{\partial}{\partial \tau}\mathbf{u}^N(\tau) + \mathbf{A}\mathbf{u}^N(\tau) = 0 \tag{4.7}$$

For the computation of $\mathbf{A}$ for general Lévy process satisfying [FK1]-[FK3] and also for the special case of tempered stable Lévy processes see Appendices A, B.

4.1.2. Discretization in Time

In time we discretize using the backward Euler scheme with time step defined as $k = \frac{T}{M}$, where $M \in \mathbb{N}$ is the number of time points. The time points can be written as $t_m = m\,k$, $m = 0, \ldots, M$. This gives the fully discretized problem:
Given $\mathbf{u}_0^N \in \mathbb{R}^N$ for $m = 0, \cdots, M-1$ find $\mathbf{u}_{m+1}^N \in \mathbb{R}^N$ such that

$$\frac{1}{k}\mathbf{M}\left(\mathbf{u}_{m+1}^N - \mathbf{u}_m^N\right) + \mathbf{A}\mathbf{u}_{m+1}^N = 0 \tag{4.8}$$

Here $\mathbf{u}_m^N$ is the coefficient vector of $u^N(t_m, x)$ in the hat function basis.

4.2. Barrier Options

The discretization for Barrier options is analogous to discretization for European options elaborated in the previous section, with special considerations for the domain and the boundary conditions (as in the localization, see Section 3.2.2). In space, we employ a Galerkin discretization with finite element subspace $S_1^{\tau} \cap V = V^N$. We use piecewise linear hat functions as a basis for V^N. We discretize on the localized domain $\Omega_R = (\log(B), R)$ using a uniform mesh with N subintervals of size $h = \frac{R-\log(B)}{N}$. In space, we discretize using the backward Euler scheme with time step defined as $k = \frac{T}{M}$, where $M \in \mathbb{N}$ is the number of time points. The time points can be written as $t_m = m\ k$, $m = 0, \ldots, M$. Then for Down-and-Out Barrier options, the discretized problem reads as follows: Given $\mathbf{u}_0^N \in \mathbb{R}^N$ for $m = 0, \cdots, M-1$ find $\mathbf{u}_{m+1}^N \in \mathbb{R}^N$ such that

$$\frac{1}{k}\mathbf{M}\left(\mathbf{u}_{m+1}^N - \mathbf{u}_m^N\right) + \mathbf{A}\mathbf{u}_{m+1}^N \quad = \quad 0 \tag{4.9}$$

and similarly for Down-and-In Barrier options.

4.3. American Options

4.3.1. Discretization in Space

We use the same mesh as in the European case and use the Galerkin discretization based on piecewise continuous hat functions. Approximating $u(\tau, x)$ by an element $u^N(\tau, x) \in V^N$ and the test functions $v \in V$ by $v^N \in V^N$ we can approximate the localized pricing problem for American options as follows:
Find $u^N(\tau, x) \in K = \left\{v^N \in V^N | v^N \geq u_0\right\}$ such that

$$\left(\frac{\partial}{\partial \tau}u^N(\tau), v^N - u^N(\tau)\right) + a\left(u^N(\tau), v^N - u^N(\tau)\right) \geq 0 \tag{4.10}$$

Substituting the representation of u^N and v^N in the hat function basis, we find the equivalent matrix inequality:
Find $\mathbf{u}^N(\tau) \in \mathbf{K} = \left\{\mathbf{v}^N \in \mathbb{R}^N | \mathbf{v}^N \geq \mathbf{u}_0^N\right\}$ such that

$$\left(\mathbf{v}^N - \mathbf{u}^N(\tau)\right)^T \left(\mathbf{M}\frac{\partial}{\partial \tau}\mathbf{u}^N(\tau) + \mathbf{A}\mathbf{u}^N(\tau)\right) \geq 0 \tag{4.11}$$

where $\mathbf{M}$ and $\mathbf{A}$ are defined by (4.5) and (4.6), respectively.

4.3.2. Discretization in Time

In time we discretized using the backward Euler scheme:
Find $\mathbf{u}_{m+1}^N \in \mathbf{K} := \left\{\mathbf{v}^N \in \mathbb{R}^N | \mathbf{v}^N \geq \mathbf{u}_0^N\right\}$ such that

$$\left(\mathbf{v}^N - \mathbf{u}_{m+1}^N\right)^T \left(\frac{1}{k}\mathbf{M}\left(\mathbf{u}_{m+1}^N - \mathbf{u}_m^N\right) + \mathbf{A}\mathbf{u}_{m+1}^N\right) \geq 0 \tag{4.12}$$

Proposition 4.1. *The following formulations for the discretized American option pricing problem are equivalent:*

I. The Discretized Variational Formulation:

(a) $\mathbf{u}_{m+1}^N \in \mathbf{K} := \left\{\mathbf{v}^N \in \mathbb{R}^N | \mathbf{v}^N \geq \mathbf{u}_0^N\right\}$

(b) $\forall\ \mathbf{v}^N \in \mathbf{K},\quad \left(\mathbf{v}^N - \mathbf{u}_{m+1}^N\right)^T \left(\frac{1}{k}\mathbf{M}\left(\mathbf{u}_{m+1}^N - \mathbf{u}_m^N\right) + \mathbf{A}\mathbf{u}_{m+1}^N\right) \geq 0$

II. The Matrix Linear Complementary Problem (LCP):

(c) $\mathbf{u}_{m+1}^N \geq \mathbf{u}_0^N$

(d) $\left(\mathbf{M} + k\mathbf{A}\right)\mathbf{u}_{m+1}^N \geq \mathbf{M}\mathbf{u}_m^N$

(e) $\left(\mathbf{u}_{m+1}^N - \mathbf{u}_0^N\right)\left(\left(\mathbf{M} + k\mathbf{A}\right)\mathbf{u}_{m+1}^N - \mathbf{M}\mathbf{u}_m^N\right) = 0$

Proof.

II ⇒ I
(c) and (a) are equivalent.

Because $k > 0$ and by (d) we have that

$$\frac{1}{k}\mathbf{M}\left(\mathbf{u}_{m+1}^N - \mathbf{u}_m^N\right) + \mathbf{A}\mathbf{u}_{m+1}^N \geq 0$$

Then we have:

$$\begin{aligned}
\left(\mathbf{v}^N - \mathbf{u}_{m+1}^N\right)^T \left(\frac{1}{k}\mathbf{M}\left(\mathbf{u}_{m+1}^N - \mathbf{u}_m^N\right) + \mathbf{A}\mathbf{u}_{m+1}^N\right) &= \\
\left(\mathbf{v}^N - \mathbf{u}_0^N + \mathbf{u}_0^N - \mathbf{u}_{m+1}^N\right)^T \left(\frac{1}{k}\mathbf{M}\left(\mathbf{u}_{m+1}^N - \mathbf{u}_m^N\right) + \mathbf{A}\mathbf{u}_{m+1}^N\right) &= \\
\underbrace{\left(\mathbf{v}^N - \mathbf{u}_0^N\right)^T}_{\geq 0} \underbrace{\left(\frac{1}{k}\mathbf{M}\left(\mathbf{u}_{m+1}^N - \mathbf{u}_m^N\right) + \mathbf{A}\mathbf{u}_{m+1}^N\right)}_{\geq 0} - & \\
\underbrace{\left(\mathbf{u}_{m+1}^N - \mathbf{u}_0^N\right)^T \left(\frac{1}{k}\mathbf{M}\left(\mathbf{u}_{m+1}^N - \mathbf{u}_m^N\right) + \mathbf{A}\mathbf{u}_{m+1}^N\right)}_{=0} &\geq 0
\end{aligned}$$

which implies (b).

I ⇒ II
(a) and (c) are equivalent.

To prove (d), we assume there is a vector component

$$\frac{1}{k}\left(\mathbf{M}\left(\mathbf{u}_{m+1}^N - \mathbf{u}_m^N\right)\right)_i + \left(\mathbf{A}\mathbf{u}_{m+1}^N\right)_i < 0 \tag{4.13}$$

for some $i \in \{1, \cdots, N\}$. By (b) we have that

$$\left(\mathbf{v}^N\right)^T\left(\frac{1}{k}\mathbf{M}\left(\mathbf{u}_{m+1}^N-\mathbf{u}_m^N\right)+\mathbf{A}\mathbf{u}_{m+1}^N\right) \geq \left(\mathbf{u}_{m+1}^N\right)^T\left(\frac{1}{k}\mathbf{M}\left(\mathbf{u}_{m+1}^N-\mathbf{u}_m^N\right)+\mathbf{A}\mathbf{u}_{m+1}^N\right)$$

for all $\mathbf{v}^N$. Note that the right hand side is a constant. Fix all components $j \in \{1, \cdots, N\} \setminus \{i\}$, recall (4.13), and let $v_i \to \infty$. Then the right hand side can become arbitrarily negative, which is a contradiction. Therefore (d) must hold.

In (b), let $\mathbf{v}^N = \mathbf{u}_0^N$.
Then,

$$\begin{gathered}\left(\mathbf{u}_0^N-\mathbf{u}_{m+1}^N\right)^T\left(\tfrac{1}{k}\mathbf{M}\left(\mathbf{u}_{m+1}^N-\mathbf{u}_m^N\right)+\mathbf{A}\mathbf{u}_{m+1}^N\right) \geq 0\\ \Leftrightarrow\\ \left(\mathbf{u}_{m+1}^N-\mathbf{u}_0^N\right)^T\left(\tfrac{1}{k}\mathbf{M}\left(\mathbf{u}_{m+1}^N-\mathbf{u}_m^N\right)+\mathbf{A}\mathbf{u}_{m+1}^N\right) \leq 0\end{gathered}$$

But by (a),

$$\left(\mathbf{u}_{m+1}^N-\mathbf{u}_0^N\right)^T \geq 0$$

and by (d),

$$\frac{1}{k}\mathbf{M}\left(\mathbf{u}_{m+1}^N-\mathbf{u}_m^N\right)+\mathbf{A}\mathbf{u}_{m+1}^N \geq 0$$

Therefore,

$$\left(\mathbf{u}_{m+1}^N-\mathbf{u}_0^N\right)^T\left(\frac{1}{k}\mathbf{M}\left(\mathbf{u}_{m+1}^N-\mathbf{u}_m^N\right)+\mathbf{A}\mathbf{u}_{m+1}^N\right) = 0$$

which is (e).

□

To solve the LCP, we use the PSOR algorithm [13]. Using the substitution $\mathbf{v}_{m+1}^N = \mathbf{u}_{m+1}^N - \mathbf{u}_0^N$, the LCP for the American pricing problem can be posed as follows:

$$\begin{aligned}\mathbf{v}_{m+1}^N &\geq \mathbf{0}\\ \left(\mathbf{M}+k\mathbf{A}\right)\mathbf{v}_{m+1}^N &\geq \mathbf{M}\mathbf{v}_m^N - k\mathbf{A}\mathbf{u}_0^N\\ \left(\mathbf{v}_{m+1}^N\right)^T\left(\left(\mathbf{M}+k\mathbf{A}\right)\mathbf{v}_{m+1}^N - \mathbf{M}\mathbf{v}_m^N + k\mathbf{A}\mathbf{u}_0^N\right) &= 0\end{aligned} \tag{4.14}$$

Once the matrix LCP is solved, one can simply add back the payoff $\mathbf{u}_0^N$ to obtain the solution vector $\mathbf{u}_M^N$.

5. Small Jump Regularization

To examine the error induced into the pricing problem by removing small jumps it is useful to recall the Lévy decomposition. Assume X_t is a non-singular Lévy process. Then X_t can be decomposed into the sum of a drifted Brownian motion, a compound

Poisson process, and a square integrable martingale containing the small jumps of X_t. S. Asmussen and J. Rosiński [4] studied the removal of the small jump component from X_t and found that under certain conditions, the error induced by the truncation converges weakly to a diffusion process. Based on this convergence result, they suggested the small jumps could be compensated for by adding a diffusion component. For European Options, in [12] a finite difference scheme is proposed using the small jump regularization of [4] and error rates are obtained. M. Signahl [24] showed the weak convergence of the truncated Lévy process with various methods of regularization.

5.1. Truncation of Small Jumps

Under the assumptions of Theorem 2.2, a Lévy process $(X_t)_{t\geq 0}$, for $0<\varepsilon<1$ has the following unique decomposition: $X = X^{(1)} + X^{(2)}_\varepsilon + X^{(3)}_\varepsilon$. Here, we take $\sigma = 0$, hence $X^{(1)} = \gamma t$ is the continuous part of X, $X^{(2)}_\varepsilon$ is a compound Poisson process consisting of all jumps of at least size ε, and $X^{(3)}_\varepsilon$ is a square integrable, pure jump martingale part of X containing all jumps of magnitude less than ε. The small jumps (represented by $X^{(3)}_\varepsilon$) can simply be ignored, or they can be accounted for by adding an additional diffusion component to X.

Step 1. Remove small jumps:

$$Z_t^{0,\varepsilon} := X_t - \sum_{s\leq t} \Delta X_s \mathbb{1}_{\{|\Delta X_s|<\varepsilon\}} = X_t - X^{(3)}_{\varepsilon,t} \tag{5.1}$$

Step 2. Replace the small jumps with their expected value:

$$Z_t^{1,\varepsilon} := \gamma_\varepsilon t + N_t^\varepsilon = Z_t^{0,\varepsilon} + \mathbb{E}\left[X^3_{\varepsilon,t}\right] \tag{5.2}$$

where $\gamma_\varepsilon := \gamma - \int_{\varepsilon<|x|<1} xk(x)\,dx$, $N_t^\varepsilon = X^{(2)}_\varepsilon$.

Step 3. Add a diffusion component based on the level of truncation:

$$Z_t^{2,\varepsilon} := \gamma_\varepsilon t + \sigma(\varepsilon) W_t + N_t^\varepsilon \tag{5.3}$$

where $\sigma(\varepsilon)^2 := \int_{|x|<\varepsilon} x^2 k(x)\,dx$.

Remark 5.1. To approximate a truncated Lévy process, the drift is given as above defined. However, when using the approximations in Steps 2-3 in options pricing, we choose the drift so that our process is a martingale (cf. (3.2)).

Let $X = X(1)$ and $\{X_\varepsilon(1)\}_{\varepsilon>0}$ be a family of processes evaluated at time $t=1$.

Definition 5.1. (Weak Convergence) [24] A family of approximations $\{X_\varepsilon\}_{\varepsilon\geq 0}$ to X is said to converge weakly of order $g(\varepsilon)$, $\lim_{\varepsilon\to 0} g(\varepsilon) = 0$, if for every $g \in C_b^\infty$ there exists a positive constant C, which does not depend on ε, such that

$$|\mathbb{E}f(X_\varepsilon(1)) - \mathbb{E}f(X(1))| \leq Cg(\varepsilon) \text{ as } \varepsilon \to 0$$

In this case we write $|\mathbb{E}f(X_\varepsilon(1)) - \mathbb{E}f(X(1))| = O(g(\varepsilon))$.

Theorem 5.1. *(Normal Approximation for the Small Jumps of a Lévy Process) Define $X_\varepsilon := X - Z^{1,\varepsilon}$. Then $\sigma(\varepsilon)^{-1} X_\varepsilon$ converges weakly to a diffusion process as $\varepsilon \to 0$ if and only if for each $\kappa > 0$,*

$$\sigma(\kappa\sigma(\varepsilon) \wedge \varepsilon) \sim \sigma(\varepsilon), \text{ as } \varepsilon \to 0.$$

Proof. See [4, Theorem 2.1]. □

Proposition 5.2. *If $\lim_{\varepsilon\to 0} \frac{\sigma(\varepsilon)}{\varepsilon} = \infty$ then for each $\kappa > 0$,*

$$\sigma(\kappa\sigma(\varepsilon) \wedge \varepsilon) \sim \sigma(\varepsilon), \text{ as } \varepsilon \to 0.$$

Proof. See [4, Proposition 2.1]. □

Example 5.1. Assume X is a Lévy process that satisfies [FK3]. To verify that the small jumps can be approximated with a diffusion component we approximate $\sigma(\varepsilon)$ and apply the above proposition. We assume as before $0 < \alpha < 2$.

$$\begin{aligned}
\sigma^2(\varepsilon) &= \int_{-\varepsilon}^{\varepsilon} x^2 k(x)\,dx \\
&= \int_0^{\varepsilon} x^2 k(x)\,dx + \int_0^{\varepsilon} x^2 k(-x)\,dx \\
&= \int_0^{\varepsilon} x^2 (k(x) + k(-x))\,dx \\
&\overset{[FK3]}{\geq} 2\int_0^{\varepsilon} x^2 C_- |x|^{-1-\alpha} dx \\
&= 2C_- \frac{x^{2-\alpha}}{2-\alpha}\Big|_0^{\varepsilon} \\
&= 2C_- \frac{\varepsilon^{2-\alpha}}{2-\alpha}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{\sigma(\varepsilon)}{\varepsilon} &\geq \lim_{\varepsilon\to 0} \sqrt{\frac{2C_-}{2-\alpha}} \frac{\varepsilon^{1-\alpha/2}}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \sqrt{\frac{2C_-}{2-\alpha}} \frac{1}{\varepsilon^{\alpha/2}} = \infty
\end{aligned}$$

By Proposition 5.2, the small jumps of any Lévy process satisfying [FK3] can be approximated with a diffusion.

5.2. Pricing of European Options

In [12], R. Cont and E. Voltchkova propose a finite difference scheme where the small jumps are replaced by a diffusion component. Small jumps are accounted for when the Lévy measure diverges at zero. They employ the method to calculate European and Barrier options. An advantage of the method is that no explicit form for the characteristic function is needed and boundary value problems can be accommodated (as opposed to Fourier methods). For sufficiently smooth functions, [24] shows the rates of weak convergence of this small jump approximation. Even though the payoff function for European Options is not smooth at the strike, we adopt these convergence rates and show them numerically in later sections.

For options pricing under truncation and regularization, we define the approximated option price as follows:

$$u_\varepsilon(t,x) = \mathbb{E}\left[e^{-r(T-t)} g\left(\exp\left(\tilde{X}_T\right)\right) |\tilde{X}_t = x\right] \tag{5.4}$$

where $\tilde{X}$ denotes the approximated process, either truncated or regularized with a Brownian motion (i.e. $\tilde{X} = Z_t^{1,\varepsilon}$ or $Z_t^{2,\varepsilon}$, depending on context; see Steps 2-3). It is now quite clear that weak convergence results apply directly to the European option pricing problem, and demonstrate the error induced into the pricing problem by truncation and regularization. Though the payoff function g does not typically satisfy the smoothness requirements in the following propositions, we show numerically that the results still hold.

In the following propositions, let X be a Lévy process with no diffusion component and finite variation. Define $M_k(\varepsilon) := \int_{-\varepsilon}^{\varepsilon} |x|^k \nu(x)\, dx$.

Proposition 5.3. *For $0 < \alpha < 2$, $Z_t^{1,\varepsilon}$ converges weakly of order $M_2(\varepsilon)$ to X.*

Proof. See [24, Proposition 2.2] □

Remark 5.2. For any Lévy processes satisfying [FK1], $Z_t^{1,\varepsilon}$ converges weakly of order $\varepsilon^{2-\alpha}$. Observe:

$$\begin{aligned}
M_2(\varepsilon) &= \int_{-\varepsilon}^{\varepsilon} |x|^2 k(x)\, dx \\
&\leq c \int_{-\varepsilon}^{\varepsilon} |x|^2 |x|^{-1-\alpha} dx \\
&= 2c \int_0^{\varepsilon} x^{1-\alpha} dx \\
&= 2c \frac{x^{2-\alpha}}{2-\alpha}\Big|_0^{\varepsilon} \\
&= \frac{2c}{2-\alpha} \varepsilon^{2-\alpha}
\end{aligned}$$

Therefore, $|u_\varepsilon(\tau,x) - u(\tau,x)| = O\left(\varepsilon^{2-\alpha}\right)$.

Proposition 5.4. *$Z_t^{2,\varepsilon}$ converges weakly of order $M_3(\varepsilon)$ to X.*

Proof. See [24, Proposition 2.3]. In [12, Theorem 5.1] this weak convergence result was shown to hold using the drift relevant for options pricing, c_{exp}. □

Remark 5.3. For any Lévy processes satisfying [FK1] such that $0 < \alpha < 1$, $Z_t^{2,\varepsilon}$ converges weakly of order $\varepsilon^{3-\alpha}$. Observe:

$$\begin{aligned} M_3(\varepsilon) &= \int_{-\varepsilon}^{\varepsilon} |x|^3 k(x)\, dx \\ &\leq c\int_{-\varepsilon}^{\varepsilon} |x|^3 |x|^{-1-\alpha} dx \\ &= 2c\int_0^{\varepsilon} x^{2-\alpha} dx \\ &= 2c\frac{x^{3-\alpha}}{3-\alpha}\Big|_0^{\varepsilon} \\ &= \frac{2c}{3-\alpha}\varepsilon^{3-\alpha} \end{aligned}$$

Therefore, $|u_\varepsilon(\tau, x) - u(\tau, x)| = O\left(\varepsilon^{3-\alpha}\right)$.

The results of numerical investigations on European, Barrier, and American options will be presented in the next section.

6. Numerical Results

To calculate option prices under the truncation described above, the Lévy density must be modified. Since the range $0 < x < \varepsilon$ for $k(x)$ corresponds to the jumps of size less than ε, we will define the Lévy density under truncation as follows:

$$k_\varepsilon(x) := \begin{cases} k(x) & \text{if } x > \varepsilon \\ 0 & \text{if } x \leq \varepsilon \end{cases} \tag{6.1}$$

For $Z_t^{1,\varepsilon}$, everything is calculated as in the non-truncated case, though the stiffness matrix A is calculated using $k_\varepsilon(x)$ and the drift is calculated using $k_\varepsilon(x)$ as well:

$$c_{exp} = \int_{\mathbb{R}} (e^x - x - 1)\, k_\varepsilon(x)\, dx = \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} (e^x - x - 1)\, k(x)\, dx \tag{6.2}$$

For $Z_t^{2,\varepsilon}$, the stiffness matrix must be calculated using the truncated Lévy density, $k_\varepsilon(x)$, a diffusion component must be added with $\sigma(\varepsilon)^2 := \int_{|x|<\varepsilon} x^2 k(x)\, dx$, and the drift must be such that the martingale condition is preserved,

$$c_{exp} = \frac{\sigma(\varepsilon)^2}{2} + \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} (e^x - x - 1)\, k(x)\, dx \tag{6.3}$$

Besides these changes, the variational formulations, localizations, and discretized problems elaborated in Section 4 remain unchanged. See Appendices A, B for more explicit calculations.

In all of the following examples a mesh-width of $h = 0.0075$ is used.

6.1. European Options

For European Options, we would like to demonstrate the theoretical convergence results presented in Section 5. That is, for various values of α we will show that the pricing error due to truncation and regularizations converges to the true price at a rate of $2 - \alpha$ and $3 - \alpha$, respectively. Here we will use a variety of parameter values for β_+, β_-, and c as well. In all of the following examples we assume $r = 0.1$ and $K = 1$. Below is a picture of a European Call option with $\alpha = 1$ $\beta_+ = 20$, $\beta_- = 3$, $c = 0.5$, and $T = 0.8$.

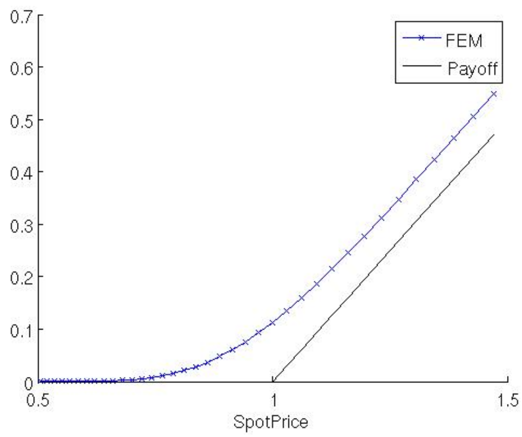

Figure 6.1: European Call Option, $\alpha = 1$

Example 6.1. We first consider a tempered stable process that is positively skewed: $\beta_- > \beta_+$. This would be an ideal stock to invest in, because the probability of positive jumps is high compared to negative jumps. You might expect to make a profit. The following graphs show the convergence rates of a European call where the log-price of the underlying asset is a tempered stable process with the following parameters: $\alpha = 0$ $\beta_+ = 5$, $\beta_- = 25$, and $c = 1$. Further, we assume a maturity $T = 1$ (in years). When $\alpha = 0$ the tempered stable process is also known as the variance gamma process. Because $\alpha = 0$, we expect convergence rates of 2 under truncation and 3 under regularization. This is indeed the case as shown in Figures 6.3 and 6.2.

Example 6.2. Next we consider a series of examples where the tempered stable process is negatively skewed: $\beta_- < \beta_+$ for a variety of values of α. This is a more realistic stock process, because in the markets, the probability of negative jumps is high compared to

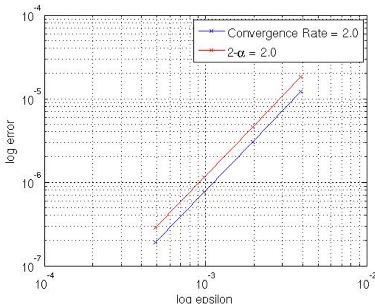

Figure 6.2: $Z^{1,\varepsilon}$ Convergence Rate, $\alpha = 0$

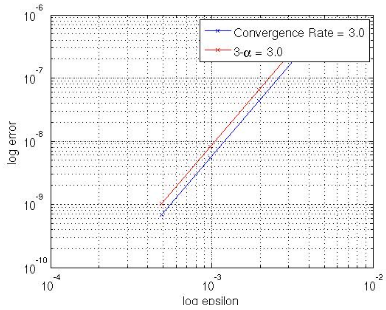

Figure 6.3: $Z^{2,\varepsilon}$ Convergence Rate, $\alpha = 0$

positive jumps. In fact, the parameters used in these examples are very close to the parameters from the calibration of [9]. The following graphs show the convergence of a European call where the log-price of the underlying asset is a tempered stable process with the following parameters: $\alpha = 0.6$ $\beta_+ = 20$, $\beta_- = 3$, and $c = 0.5$. Further, we assume a maturity $T = 0.8$ (in years).

Because $\alpha = 0.6$, we expect convergence rates of 1.4 under truncation and 2.4 under regularization. This is indeed the case as shown in Figures 6.4 and 6.5.

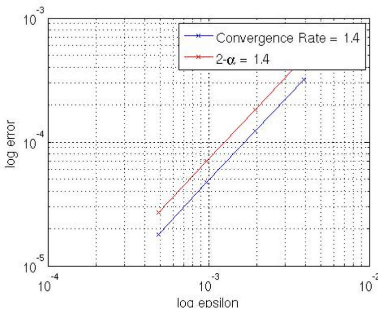

Figure 6.4: $Z^{1,\varepsilon}$ Convergence Rate, $\alpha = 0.6$

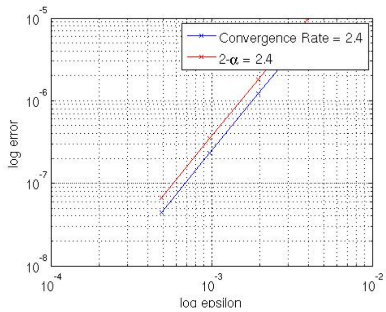

Figure 6.5: $Z^{2,\varepsilon}$ Convergence Rate, $\alpha = 0.6$

Example 6.3. Again we consider a negatively-skewed tempered stable process, but increase α. The following graph shows the price of a European call where the log-price of the underlying asset is a tempered stable process with the following parameters: $\alpha = 1$ $\beta_+ = 20$, $\beta_- = 3$, and $c = 0.5$. Further, we assume a maturity $T = 0.8$ (in years). Because $\alpha = 1$, we expect convergence rates of 1 under truncation and 2 under regularization. This is indeed the case as shown in Figures 6.6 and 6.7.

Example 6.4. Now we present an example for $\alpha > 1$. The following graph shows the price of a European call where the log-price of the underlying asset is a tempered stable

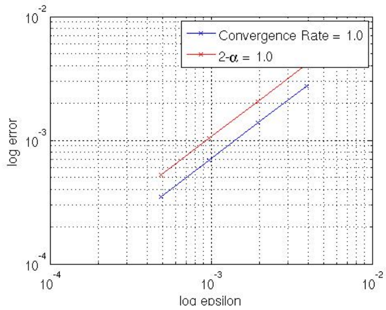

Figure 6.6: $Z^{1,\varepsilon}$ Convergence Rate, $\alpha = 1$

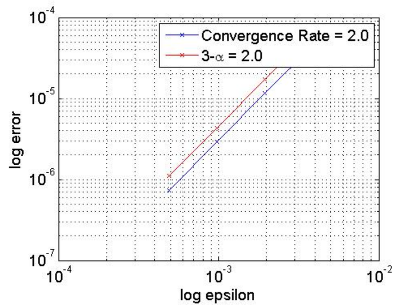

Figure 6.7: $Z^{2,\varepsilon}$ Convergence Rate, $\alpha = 1$

process with the following parameters: $\alpha = 1.4$ $\beta_+ = 20$, $\beta_- = 3$, and $c = 0.5$. Further, we assume a maturity $T = 0.8$ (in years). Because $\alpha = 1.4$, we expect convergence rates of 0.6 under truncation and 1.6 under regularization. This is the case as shown in Figures 6.8 and 6.9.

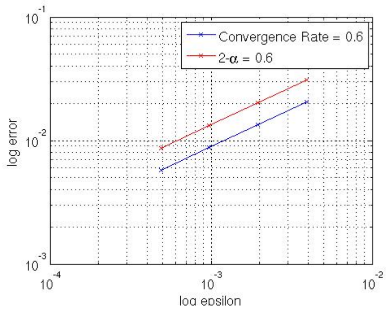

Figure 6.8: $Z^{1,\varepsilon}$ Convergence Rate, $\alpha = 1.4$

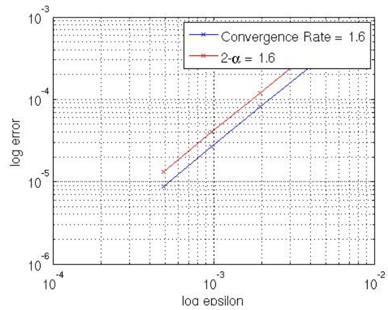

Figure 6.9: $Z^{2,\varepsilon}$ Convergence Rate, $\alpha = 1.4$

Example 6.5. We consider one more example where the tempered stable process is symmetric, $\beta_+ = \beta_-$. These parameters were chosen by [1] when pricing European and American options under the CGMY process (or equivalently, the tempered stable process). The following graph shows the price of a European call where the log-price of the underlying asset is a tempered stable process with the following parameters: $\alpha = 0.5$ $\beta_+ = 5$, $\beta_- = 5$, and $c = 1$. Further, we assume a maturity $T = 1$ (in years). Because $\alpha = 0.5$, we expect convergence rates of 1.5 under truncation and 2.5 under regularization. This is shown in Figure 6.10.

For European options, we have verified numerically that the errors induced into the pricing problem by truncation or regularization converge to the true prices. These convergence rates depend on α. Here, the addition of a diffusion component adds an extra

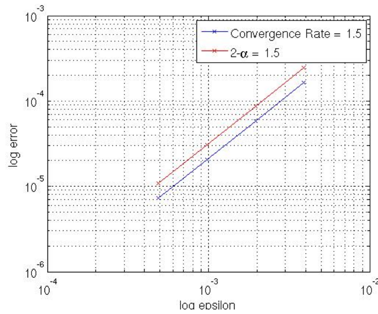

Figure 6.10: $Z^{1,\varepsilon}$ Convergence Rate, $\alpha = 0.5$

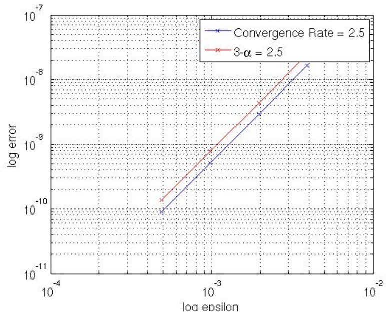

Figure 6.11: $Z^{2,\varepsilon}$ Convergence Rate, $\alpha = 0.5$

order of convergence, and is useful in reducing the errors caused by truncating the Lévy density. For comparison purposes, we include a graph of the relative errors of European options in Figure 6.12.

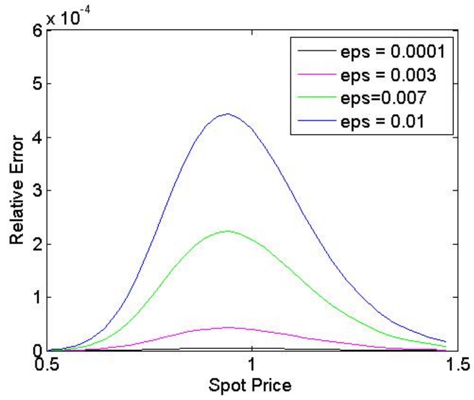

Figure 6.12: European Error under Regularization; $\alpha = 1$ $\beta_+ = 25$, $\beta_- = 2$, $c = 0.5$ and $T = 1$

6.2. Barrier Options

Given that the small jump approximation provides a viable way to use standard numerical methods to account for the small jumps of a Lévy process, it is reasonable to try the same technique in the valuation of exotic contracts. But, as expounded in [17], using the small jump approximation is not appropriate in exotic contracts such as Barrier options. For Barrier options, [17] show that the derivative of the option price with respect to x (log-price) is unbounded as the spot price of the underlying approaches the barrier. Specifically, given a down-and-out put option with barrier B and a tempered stable underlying process, they explain that the option price with respect to x behaves as $|x - \log(B)|^{\alpha/2-1}$ as $x \to \log(B)$.

For Barrier options, we show that the addition of a diffusion component after truncation will induce errors into the option prices which are quite high near the barrier.

Example 6.6. For $\alpha = 1.4$ $\beta_+ = 20$, $\beta_- = 3$, $c = 0.5$ and $T = 0.8$, the Down-and-Out Barrier option has the following price:

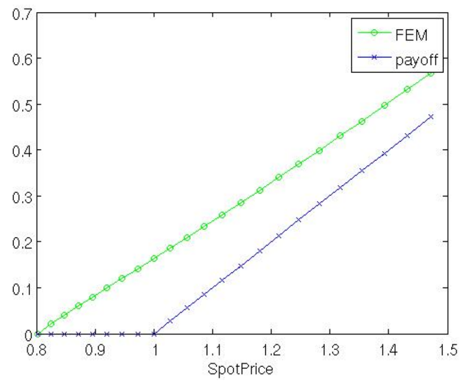

Figure 6.13: Down-and-Out Barrier Call; $\alpha = 1.4$

Under regularization, the error near the barrier is as follows:

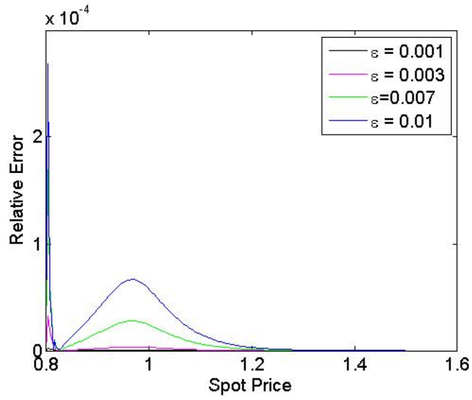

Figure 6.14: Relative Error Of Down-and-Out Call near Barrier; $\alpha = 0.5$, $\beta_+ = 20$, $\beta_- = 5$, $c = 0.5$

It is clear from the above examples that the prices attained at the barrier under regularization is quite far from the original price, and induces a lot of error into the pricing problem. The example of a Down-and-Out Barrier put is even more dramatic.

Example 6.7. For $\alpha = 0$, $\beta_+ = 23$, $\beta_- = 23$, $c = 25$, $T = 0.1$ and barrier at $B = 0.7$, the Down-and-Out Barrier put option has the following price:

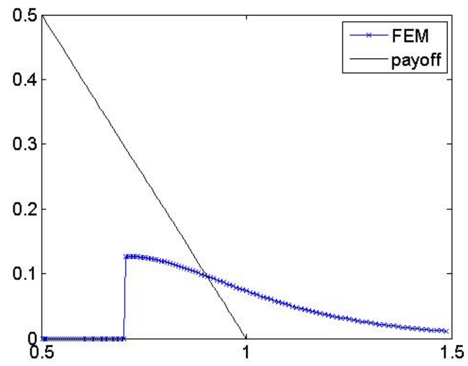

Figure 6.15: Down-and-Out Barrier Put; $\alpha = 0$

Under regularization, the error near the barrier is as follows:

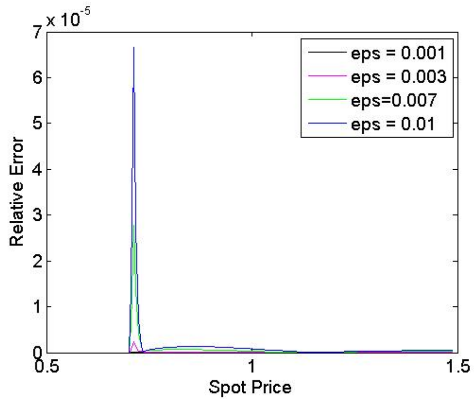

Figure 6.16: Relative Error Of Down-and-Out Put near Barrier; $\alpha = 0$, $\beta_+ = 23$, $\beta_- = 23$, $c = 25$

Due to the steep angle near the boundary, the error introduced into the pricing problem outweighs any other pricing errors. To gain resolution at the barrier, a graded mesh can be used. For the stiffness matrix entries using an arbitrary mesh, see Appendix A. Through both examples it is straightforward to see that the regularization of the small jumps of a Lévy process with a diffusion for pricing Barrier options is not appropriate.

6.3. American Options

Similar high errors can be observed with American option prices as x approaches the exercise boundary.

Example 6.8. For $\alpha = 0.5$, $\beta_+ = 5$, $\beta_- = 5$, $c = 1$ and $T = 1$, the American put option has the following price:

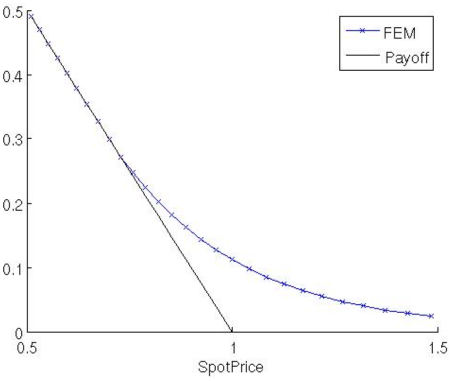

Figure 6.17: American Put Option; $\alpha = 0.5$

Under regularization, the error near the exercise boundary is as follows:

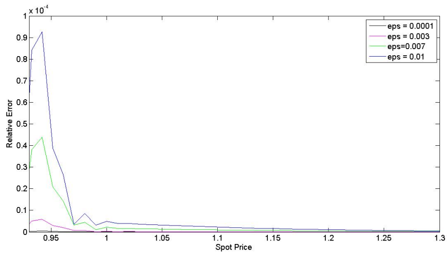

Figure 6.18: Relative Error Of American Put near Exercise Boundary; $\alpha = 0.5$, $\beta_+ = 10$, $\beta_- = 12$, $c = 0.5$ and $T = 1$

Here we have demonstrated that American options which are priced under a regularized model yield prices with very high errors near the exercise boundary. Therefore this type of regularization should not be used for pricing American options. This poses a problem for all methods where truncation of the Lévy denisty is required.

A. Computation of Stiffness Matrix

Equidistant Mesh To find a numerical solution for the PIDE proposed in Section 3, it is necessary to compute the stiffness matrix associated with jumps, A^J $(\sigma = 0)$, where

$$A^J_{ij} = \int_{\Omega_R} \int_{\Omega_R} b'_j(y)\, b'_i(x)\, k^{-2}(y-x)\, dydx.$$

Here b_i corresponds to the linear hat function basis used for the Galerkin discretization. In our numerical investigations, we used tempered stable Lévy processes, which have corresponding Lévy density

$$k(z) = c \begin{cases} e^{(-\beta_+ z)} z^{-1-\alpha} & \text{if } z > 0 \\ e^{(-\beta_- |z|)} |z|^{-1-\alpha} & \text{if } z < 0 \end{cases}$$

By calculating four antiderivatives of the Lévy density, we can write an explicit formula for each entry of the stiffness matrix A^J. Recall the form of the antiderivatives:

$$k^{-i}(z) = \begin{cases} \int_{-\infty}^{z} k^{-i+1}(x)\, dx & \text{if } z < 0 \\ -\int_{z}^{\infty} k^{-i+1}(x)\, dx & \text{if } z > 0 \end{cases}$$

First we examine the diagonal entries of the stiffness matrix, A^J_{ii}:

$$A^J_{ii} = \int_{\Omega_R} \int_{\Omega_R} b'_i(y)\, b'_i(x)\, k^{-2}(y-x)\, dydx$$

Because the elements of the hat function basis have compact support, the integral can be rewritten:

$$A^J_{ii} = \int_{x_{i-1}}^{x_{i+1}} \int_{x_{i-1}}^{x_{i+1}} b'_i(y)\, b'_i(x)\, k^{-2}(y-x)\, dydx$$

To see the relationship between the hat function basis and grid points, see Figure A.1. Recall the first derivative of the linear hat function b_i:

$$b'_i = \frac{1}{h} \begin{cases} 1 & \text{if } x \in (x_{i-1}, x_i] \\ -1 & \text{if } x \ \in (x_i, x_{i+1}] \end{cases}$$

Because we discretize using an equal mesh width, any mesh point x_i can be represented as $x_i = -R + ih$. Define the substitutions $u = x + R - ih$, $v = y + R - ih$. Then the integral reduces as follows:

$$\begin{aligned} A^J_{ii} &= \frac{1}{h^2} \int_{x_{i-1}}^{x_{i+1}} \int_{x_{i-1}}^{x_{i+1}} k^{-2}(y-x)\, dydx \\ &= \frac{1}{h^2} \int_{-R+(i-1)h}^{-R+(i+1)h} \int_{-R+(i-1)h}^{-R+(i+1)h} k^{-2}(y-x)\, dydx \end{aligned}$$

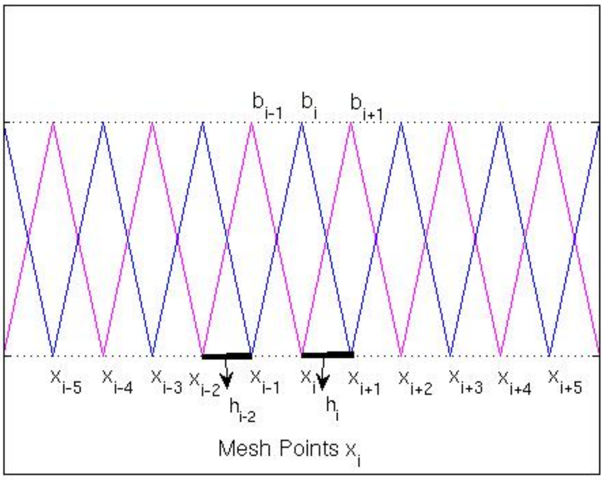

Figure A.1: Hat function basis on uniform grid

$$
\begin{aligned}
&= \frac{1}{h^2}\int_{-R+(i-1)h}^{-R+(i+1)h}\int_{-h}^{h} k^{-2}\left(v-R+ih-x\right)dvdx \\
&= \frac{1}{h^2}\int_{-h}^{h}\int_{-h}^{h} k^{-2}\left(v-R+ih-(u-R+ih)\right)dvdu \\
&= \frac{1}{h^2}\int_{-h}^{h}\int_{-h}^{h} k^{-2}\left(v-u\right)dvdu
\end{aligned}
$$

This nice shifting property can be applied to all of our matrix entries (i.e., $A^J_{11} = A^J_{22} = \ldots = A^J_{nn}$, and similarly, $A^J_{12} = A^J_{23} = \ldots = A^J_{(n-1)n}$, and in general, for $d \in \mathbb{N}$ in a meaningful range, $A^J_{1(1+d)} = A^J_{2(2+d)} = \ldots = A^J_{(n-d)n}$). Therefore, our problem reduces to calculating the components corresponding to the following matrix entries: $A^J_{i(i-d)}, A^J_{i(i-1)}, A^J_{ii}, A^J_{i(i+1)}, A^J_{i(i+d)}$. Let's return to A^J_{ii}. To proceed with the calculation, we must split the cube $[-h,h]^2$ into four sub-regions: $[-h,0]^2$, $[0,h]^2$, $[0,h]\times[-h,0]$, and $[-h,0]\times[0,h]$. By the same shifting argument presented above, the integration over the first two regions reduce to the same calculation.

At this point it is important to recall the definition of b'_i and apply appropriate signs to each square. The weights for each square are calculated in Figure A.2.

$$
\begin{aligned}
A^J_{ii} &= \frac{1}{h^2}\int_{-h}^{h}\int_{-h}^{h} k^{-2}\left(y-x\right)dydx \\
&= \frac{1}{h^2}\int_{-h}^{0}\int_{-h}^{0} k^{-2}\left(y-x\right)dydx + \frac{1}{h^2}\int_{0}^{h}\int_{0}^{h} k^{-2}\left(y-x\right)dydx
\end{aligned}
$$

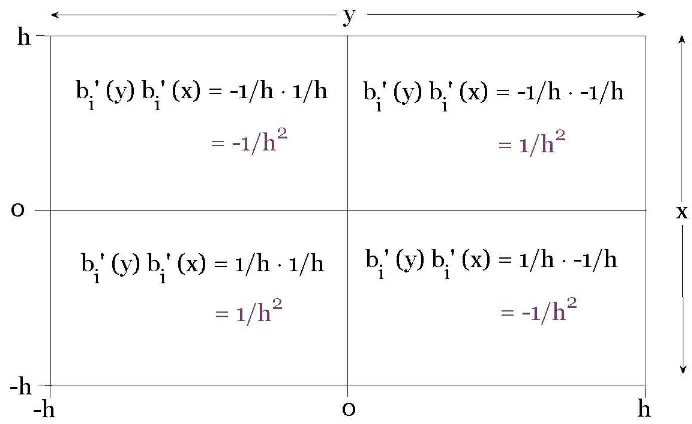

Figure A.2: Integral weightings for A_{ii}

$$
\begin{aligned}
& - \frac{1}{h^2} \int_0^h \int_{-h}^0 k^{-2}(y-x)\, dydx - \frac{1}{h^2} \int_{-h}^0 \int_0^h k^{-2}(y-x)\, dydx \\
& = \frac{2}{h^2} \int_0^h \int_0^h k^{-2}(y-x)\, dydx - \frac{1}{h^2} \int_0^h \int_{-h}^0 k^{-2}(y-x)\, dydx \\
& - \frac{1}{h^2} \int_{-h}^0 \int_0^h k^{-2}(y-x)\, dydx
\end{aligned}
$$

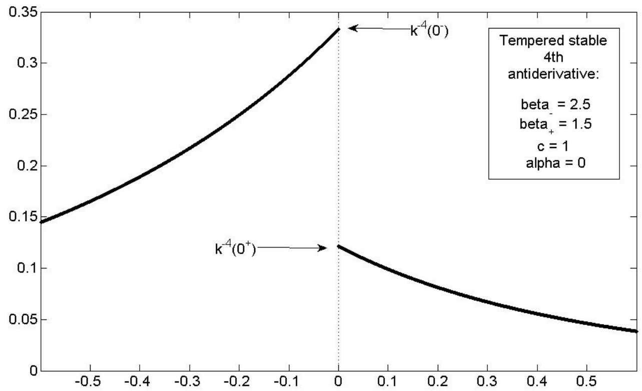

Figure A.3: $k^{-4}(z)$ for $\beta_- \neq \beta_+$

Let us proceed by calculating each integral separately, beginning with $\int_0^h \int_0^h k^{-2}(y-x)\, dydx$. Note that we must be careful when dealing with the second antiderivative–it is only defined for $\mathbb{R} \setminus \{0\}$. Moreover for tempered stable processes where $\beta_- \neq \beta_+$, the limit as z approaches 0 of k^{-2}, k^{-4} will be different from the left

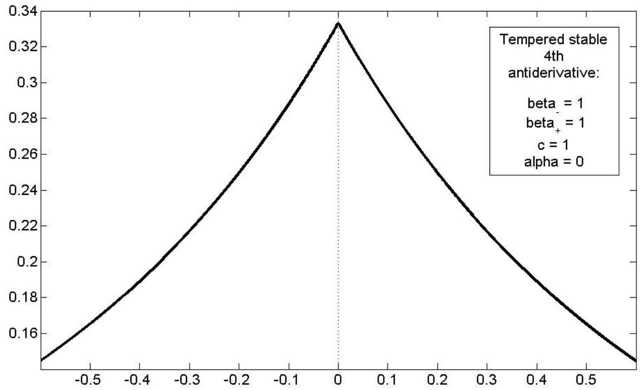

Figure A.4: $k^{-4}(z)$ for $\beta_- = \beta_+$

and the right. See Figures A.3 and A.4 for graphs of $k^{-4}(z)$ for $|z|$ close to 0.

$$\begin{aligned}
I_1 &:= \frac{2}{h^2}\int_0^h\int_0^h k^{-2}(y-x)\,dydx \\
&= \frac{2}{h^2}\int_0^h\int_0^x k^{-2}(y-x)\,dydx + \frac{2}{h^2}\int_0^h\int_x^h k^{-2}(y-x)\,dydx \\
&= \frac{2}{h^2}\int_0^h \left(k^{-3}\left(0^-\right) - k^{-3}(-x)\right)dx + \frac{2}{h^2}\int_0^h \left(k^{-3}(h-x) - k^{-3}\left(0^+\right)\right)dx \\
&= \frac{2}{h^2}\left(hk^{-3}\left(0^-\right) - k^{-4}\left(0^-\right) + k^{-4}(-h) + k^{-4}(h) - k^{-4}\left(0^+\right) - hk^{-3}\left(0^+\right)\right) \\
&= \frac{2}{h^2}\left(h\left(k^{-3}\left(0^-\right) - k^{-3}\left(0^+\right)\right) - k^{-4}\left(0^-\right) - k^{-4}\left(0^+\right) + k^{-4}(-h) + k^{-4}(h)\right)
\end{aligned}$$

Progressing in order, we denote I_2 as the integral over the region $[0,h]\times[-h,0]$:

$$\begin{aligned}
I_2 &:= -\frac{1}{h^2}\int_0^h\int_{-h}^0 k^{-2}(y-x)\,dydx \\
&= -\frac{1}{h^2}\int_0^h \left(k^{-3}(-x) - k^{-3}(-h-x)\right)dx \\
&= -\frac{1}{h^2}\left(\int_{-h}^0 k^{-3}(u)\,du - \int_{-2h}^{-h} k^{-3}(v)\,dv\right) \\
&= -\frac{1}{h^2}\left(k^{-4}\left(0^-\right) - 2k^{-4}(-h) + k^{-4}(-2h)\right)
\end{aligned}$$

Finally we calculate I_3 which we define as the integral over the region $[-h, 0] \times [0, h]$:

$$
\begin{aligned}
I_3 &:= \frac{1}{h^2} \int_{-h}^{0} \int_{0}^{h} k^{-2} (y - x) \, dy dx \\
&= -\frac{1}{h^2} \int_{-h}^{0} \left(k^{-3} (h - x) - k^{-3} (-x) \right) dx \\
&= -\frac{1}{h^2} \left(\int_{h}^{2h} k^{-3} (u) \, du - \int_{0}^{h} k^{-3} (v) \, dv \right) \\
&= -\frac{1}{h^2} \left(k^{-4} (2h) - 2k^{-4} (h) + k^{-4} \left(0^{+} \right) \right)
\end{aligned}
$$

Adding these results, we find a closed form for the diagonal elements of the stiffness matrix associated with jumps, A^J. Through similar elementary calculations, one can calculate all elements of A^J. The results are as follows:

Let $\mid d \mid = \mid i - j \mid \geq 2$.

$$
\begin{aligned}
A^J_{i,i} = & \frac{1}{h^2} \left(2hk^{-3} \left(0^{-} \right) - 2hk^{-3} \left(0^{+} \right) - 3k^{-4} \left(0^{-} \right) - 3k^{-4} \left(0^{+} \right) + 4k^{-4} (-h) \right) \\
& + \frac{1}{h^2} \left(4k^{-4} (h) - k^{-4} (-2h) - k^{-4} (2h) \right)
\end{aligned}
$$

$$
\begin{aligned}
A^J_{i,i+1} = & \frac{1}{h^2} \left(-hk^{-3} \left(0^{-} \right) + hk^{-3} \left(0^{+} \right) + k^{-4} \left(0^{-} \right) + 3k^{-4} \left(0^{+} \right) - 6k^{-4} (h) \right) \\
& + \frac{1}{h^2} \left(-k^{-4} (-h) + 4k^{-4} (2h) - k^{-4} (3h) \right)
\end{aligned}
$$

$$
A^J_{i,i+d} = \frac{1}{h^2} \sum_{l=-2}^{2} \alpha_m k^{-4} \left((d + l) h \right), \qquad \alpha = \{-1, 4, -6, 4, -1\}
$$

$$
\begin{aligned}
A^J_{i,i-1} = & \frac{1}{h^2} \left(-hk^{-3} \left(0^{-} \right) + hk^{-3} \left(0^{+} \right) + k^{-4} \left(0^{+} \right) + 3k^{-4} \left(0^{-} \right) - 6k^{-4} (-h) \right) \\
& + \frac{1}{h^2} \left(-k^{-4} (h) + 4k^{-4} (-2h) - k^{-4} (-3h) \right)
\end{aligned}
$$

$$
A^J_{i,i-d} = \frac{1}{h^2} \sum_{l=-2}^{2} \alpha_m k^{-4} \left((-d + l) h \right), \qquad \alpha = \{-1, 4, -6, 4, -1\}
$$

Arbitrary Mesh Because of the angle of the price process near the barrier for Barrier options and near the exercise boundary for American options, many grid points are necessary for the linear hat functions to provide a good approximation. However, decreasing the mesh width uniformly causes the method to converge quite slowly. The solution is to place a lot of grid points where the error is large, and fewer where the error is already acceptable. In this section we derive the jump stiffness matrices for a non-uniform discretization grid. For an example of such a grid, see Figure A.5 (and compare with Figure A.1).

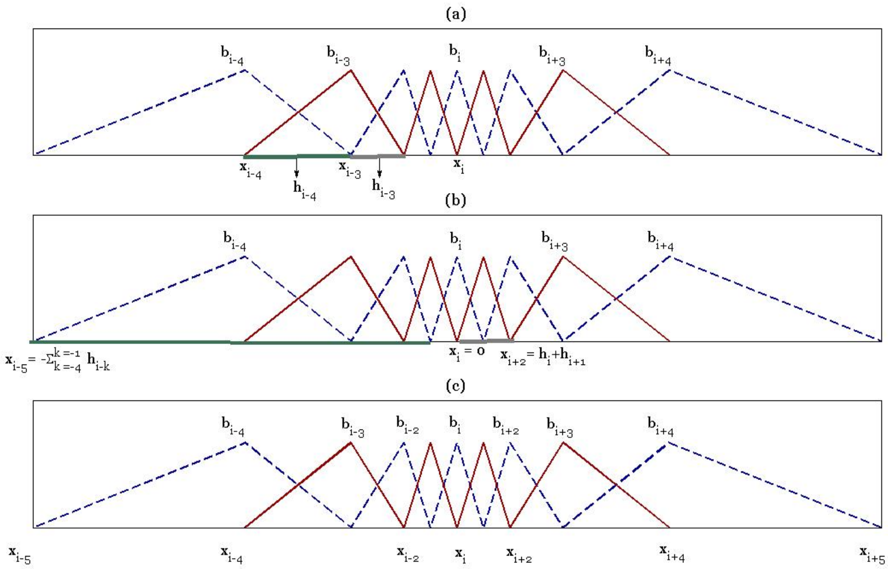

Figure A.5: Non-uniform Discretization Grid

In this setting, we define linear hat functions as follows:

$$b_i(x) = \begin{cases} -\frac{x-x_i}{h_i} & \text{if } x \geq x_i \\ \frac{x-x_i}{h_{i-1}} & \text{if } x < x_i \end{cases}$$

Hence we define $h_i := x_{i+1} - x_i$. This representation can be seen in Figure A.5, part (a). As before, we begin our calculations with the entry A_{ii}^J. Recall

$$A_{ij}^J = \int_{\Omega_R} \int_{\Omega_R} b_j'(y)\, b_i'(x)\, k^{-2}(y-x)\, dydx.$$

The first derivatives of the linear hat functions are now slightly more generalized:

$$b_i' = \begin{cases} \frac{1}{h_{i-1}} & \text{if } x \in (x_{i-1}, x_i] \\ \frac{-1}{h_i} & \text{if } x \ \in (x_i, x_{i+1}] \end{cases}$$

Suppose we consider $x_i = 0$. Then, similar to the uniform grid,

$$A_{ii}^J = \int_{-h_{i-1}}^{h_i} \int_{-h_{i-1}}^{h_i} b_i'(y)\, b_i'(x)\, k^{-2}(y-x)\, dydx$$

If we again subdivide the area of integration into four subareas, we can compute the appropriate weightings (given by $b_i'(y)\, b_i'(x)$) for each sub-rectangle. See Figure A.6.

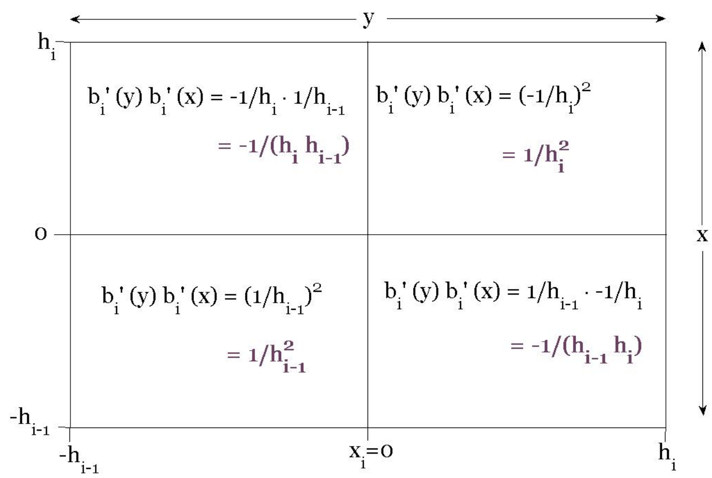

Figure A.6: Integral weightings for A_{ii}^J

There is again a nice shifting property that is useful in computing the matrix entries. Say we still consider $x_i = 0$. Let $x_k = x_{i+j}$. Then $x_k = \sum_{l=0}^{j-1} h_l$. Define $u = x - \sum_{l=0}^{j-1} h_l$, $v = y - \sum_{l=0}^{j-1} h_l$.

$$A_{(i+j)(i+j)}^J = \int_{-h_{j-1}}^{h_j} \int_{-h_{j-1}}^{h_j} b_{(i+j)}'(v)\, b_{(i+j)}'(u)\, k^{-2}(v-u)\, dvdu$$

Because the first derivatives of the hat functions are constants which weight the integrals over sub-rectangles, as in the uniform mesh case, this shifting does not affect their values. Therefore, the stiffness matrices will shift nicely as before. Now, however, the shifting will be sensitive to the weights and the area of integration (if $h_j > h_i$ and $h_{j-1} \geq h_{i-1}$, A_{ii}^J will define an integral over a larger rectangle than A_{jj}^J. At the same time, the weights of A_{ii}^J will be smaller than A_{jj}^J. This is clear in the formulas for the stiffness matrix entries, presented at the end of this section.

Let x_i be a general grid point. We shift the integral defined by A_{ii}^J such that the area of integration is $[-h_{i-1}, h_i] \times [-h_{i-1}, h_i]$. We further subdivide $[-h_{i-1}, 0]^2 \cup [-h_{i-1}, 0] \times [0, h_i] \cup [0, h_i] \times [-h_{i-1}, 0] \cup [0, h_i]^2$. Denote by I_1 the integral over $[-h_{i-1}, 0]^2$.

$$\begin{aligned}
I_1 &:= \frac{1}{h_{i-1}^2} \int_0^{h_{i-1}} \int_0^{h_{i-1}} k^{-2}(y-x)\, dydx \\
&= \frac{1}{h_{i-1}^2} \int_0^{h_{i-1}} \int_0^{x} k^{-2}(y-x)\, dydx + \frac{1}{h_{i-1}^2} \int_0^{h_{i-1}} \int_x^{h_{i-1}} k^{-2}(y-x)\, dydx \\
&= \frac{1}{h_{i-1}^2} \int_0^{h_{i-1}} \left(k^{-3}(0^-) - k^{-3}(-x)\right) dx + \frac{1}{h_{i-1}^2} \int_0^{h_{i-1}} \left(k^{-3}(h_{i-1}-x) - k^{-3}(0^+)\right) dx
\end{aligned}$$

$$= \frac{1}{h_{i-1}^2}\left(h_{i-1}k^{-3}\left(0^-\right) - k^{-4}\left(0^-\right) + k^{-4}\left(-h_{i-1}\right) + k^{-4}\left(h_{i-1}\right) - k^{-4}\left(0^+\right) - h_{i-1}k^{-3}\left(0^+\right)\right)$$
$$= \frac{1}{h_{i-1}^2}\left(h_{i-1}\left(k^{-3}\left(0^-\right) - k^{-3}\left(0^+\right)\right) - k^{-4}\left(0^-\right) - k^{-4}\left(0^+\right) + k^{-4}\left(-h_{i-1}\right) + k^{-4}\left(h_{i-1}\right)\right)$$

It is clear that all calculations will be analogous to the uniform grid case, though the matrix entries now depend highly on which grid point is considered, and how many points are concentrated in that region. The weightings for $A^J_{i(i-d)}$ are presented in Figure A.7.

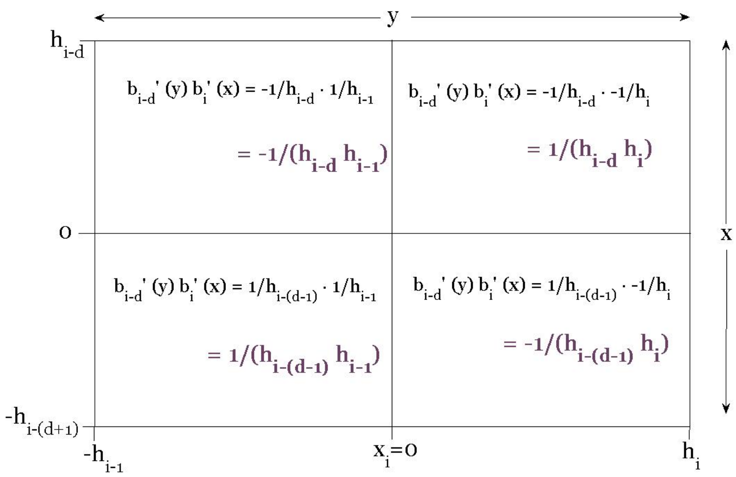

Figure A.7: Integral weightings for $A^J_{i(i-d)}$

The stiffness matrix entries for non-uniform mesh are presented below.
Let $\mid d \mid = \mid i-j \mid \geq 2$ Then the diagonal elements are computed as follows:.

$$\begin{aligned}
A^J_{i,i} = & \frac{1}{h_{i-1}^2}\left(h_{i-1}\left(k^{-3}\left(0^-\right) - k^{-3}\left(0^+\right)\right) - k^{-4}\left(0^-\right) - k^{-4}\left(0^+\right) + k^{-4}\left(-h_{i-1}\right) + k^{-4}\left(h_{i-1}\right)\right) \\
& + \frac{1}{h_i^2}\left(h_i\left(k^{-3}\left(0^-\right) - k^{-3}\left(0^+\right)\right) - k^{-4}\left(0^-\right) - k^{-4}\left(0^+\right) + k^{-4}\left(-h_i\right) + k^{-4}\left(h_i\right)\right) \\
& - \frac{1}{h_{i-1}h_i}\left(k^{-4}\left(h_i + h_{i-1}\right) - k^{-4}\left(h_i\right) - k^{-4}\left(h_{i-1}\right) + k^{-4}\left(0^+\right)\right) \\
& - \frac{1}{h_{i-1}h_i}\left(k^{-4}\left(-h_i - h_{i-1}\right) - k^{-4}\left(h_i\right) - k^{-4}\left(-h_{i-1}\right) + k^{-4}\left(0^-\right)\right)
\end{aligned}$$

The elements adjacent to the diagonal are computed as:

$$\begin{aligned}
A^J_{i,i+1} = & \frac{1}{h_{i-1}h_i}\left(k^{-4}\left(h_i + h_{i-1}\right) - k^{-4}\left(h_i\right) - k^{-4}\left(h_{i-1}\right) + k^{-4}\left(0^+\right)\right) \\
& + \frac{1}{h_{i+1}h_i}\left(k^{-4}\left(h_i + h_{i+1}\right) - k^{-4}\left(h_i\right) - k^{-4}\left(h_{i+1}\right) + k^{-4}\left(0^+\right)\right) \\
& - \frac{1}{h_i^2}\left(h_i\left(k^{-3}\left(0^-\right) - k^{-3}\left(0^+\right)\right) - k^{-4}\left(0^-\right) - k^{-4}\left(0^+\right) + k^{-4}\left(-h_i\right) + k^{-4}\left(h_i\right)\right) \\
& - \frac{1}{h_{i+1}h_{i-1}}\left(k^{-4}\left(h_{i-1} + h_i + h_{i+1}\right) - k^{-4}\left(h_{i+1} + h_i\right) - k^{-4}\left(h_{i-1} + h_i\right) + k^{-4}\left(h_i\right)\right)
\end{aligned}$$

$$
\begin{aligned}
A^J_{i,i-1} = &\frac{1}{h_{i-1}h_{i-2}}\left(k^{-4}\left(-h_{i-2}-h_{i-1}\right)-k^{-4}\left(-h_{i-2}\right)-k^{-4}\left(-h_{i-1}\right)+k^{-4}\left(0^-\right)\right)\\
&+\frac{1}{h_{i-1}h_i}\left(k^{-4}\left(-h_i-h_{i-1}\right)-k^{-4}\left(-h_i\right)-k^{-4}\left(-h_{i-1}\right)+k^{-4}\left(0^-\right)\right)\\
&-\frac{1}{h_{i-1}^2}\left(h_{i-1}\left(k^{-3}\left(0^-\right)-k^{-3}\left(0^+\right)\right)-k^{-4}\left(0^-\right)-k^{-4}\left(0^+\right)+k^{-4}\left(-h_{i-1}\right)+k^{-4}\left(h_{i-1}\right)\right)\\
&-\frac{1}{h_ih_{i-2}}\left(k^{-4}\left(-h_{i-2}-h_{i-1}-h_i\right)-k^{-4}\left(-h_{i-1}-h_{i-2}\right)-k^{-4}\left(-h_{i-1}-h_i\right)+k^{-4}\left(-h_{i-1}\right)\right)
\end{aligned}
$$

Finally, all remaining elements can be calculated in the following way:

$$
\begin{aligned}
A^J_{i,i+d} = &\frac{1}{h_{i-1}h_{i+d-1}}\left(k^{-4}\left(\sum_{j=-1}^{d-1}h_{i+j}\right)-k^{-4}\left(\sum_{j=0}^{d-1}h_{i+j}\right)-k^{-4}\left(\sum_{j=-1}^{d-2}h_{i+j}\right)+k^{-4}\left(\sum_{j=0}^{d-2}h_{i+j}\right)\right)\\
&+\frac{1}{h_ih_{i+d}}\left(k^{-4}\left(\sum_{j=0}^{d}h_{i+j}\right)-k^{-4}\left(\sum_{j=1}^{d}h_{i+j}\right)-k^{-4}\left(\sum_{j=0}^{d-1}h_{i+j}\right)+k^{-4}\left(\sum_{j=1}^{d-1}h_{i+j}\right)\right)\\
&-\frac{1}{h_ih_{i+d-1}}\left(k^{-4}\left(\sum_{j=0}^{d-1}h_{i+j}\right)-k^{-4}\left(\sum_{j=1}^{d-1}h_{i+j}\right)-k^{-4}\left(\sum_{j=0}^{d-2}h_{i+j}\right)+\underbrace{k^{-4}\left(\sum_{j=1}^{d-2}h_{i+j}\right)}_{=k^{-4}\left(0^+\right)\text{ for }d=2}\right)\\
&-\frac{1}{h_ih_{i+d}}\left(k^{-4}\left(\sum_{j=-1}^{d}h_{i+j}\right)-k^{-4}\left(\sum_{j=0}^{d}h_{i+j}\right)-k^{-4}\left(\sum_{j=-1}^{d-1}h_{i+j}\right)+k^{-4}\left(\sum_{j=0}^{d-1}h_{i+j}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
A^J_{i,i-d} = &\frac{1}{h_{i-1}h_{i-d-1}}\left(k^{-4}\left(-\sum_{j=2}^{d}h_{i-j}\right)-k^{-4}\left(-\sum_{j=1}^{d}h_{i-j}\right)-k^{-4}\left(-\sum_{j=2}^{d+1}h_{i-j}\right)+k^{-4}\left(-\sum_{j=1}^{d+1}h_{i-j}\right)\right)\\
&+\frac{1}{h_ih_{i-d}}\left(k^{-4}\left(-\sum_{j=1}^{d-1}h_{i+j}\right)-k^{-4}\left(-\sum_{j=0}^{d-1}h_{i-j}\right)-k^{-4}\left(-\sum_{j=1}^{d}h_{i-j}\right)+k^{-4}\left(-\sum_{j=0}^{d}h_{i-j}\right)\right)\\
&-\frac{1}{h_ih_{i-d}}\left(k^{-4}\left(-\sum_{j=1}^{d}h_{i-j}\right)-k^{-4}\left(-\sum_{j=1}^{d-1}h_{i-j}\right)-k^{-4}\left(-\sum_{j=2}^{d}h_{i-j}\right)+\underbrace{k^{-4}\left(-\sum_{j=2}^{d-1}h_{i-j}\right)}_{=k^{-4}\left(0^-\right)\text{ for }d=2}\right)\\
&-\frac{1}{h_ih_{i-d-1}}\left(k^{-4}\left(-\sum_{j=1}^{d}h_{i-j}\right)-k^{-4}\left(-\sum_{j=0}^{d}h_{i-j}\right)-k^{-4}\left(-\sum_{j=1}^{d+1}h_{i-j}\right)+k^{-4}\left(-\sum_{j=0}^{d+1}h_{i-j}\right)\right)
\end{aligned}
$$

Refer to Appendix B for the antiderivative k^{-4} in the case of tempered stable Lévy processes. Using an arbitrary mesh as derived above allows one to use less mesh points and gain higher accuracy in options pricing calculations.

B. Example: Tempered Stable Processes

Antiderivatives For the case of a tempered stable Lévy process, we are able to analytically compute the first four antiderivatives of the Lévy density. These antiderivatives are needed to calculate the above stiffness matrix entries. To account for singularities, we focus on three cases: $\alpha = 0$, $\alpha = 1$, and $\alpha \in (0,1) \cup (1,2)$.

We define $\Gamma(a,x) := \int_x^\infty t^{a-1}e^{-t}dt$, $\Gamma(a) := \int_0^\infty t^{a-1}e^{-t}dt$, and $E_1(x) := \int_x^\infty t^{-1}e^{-t}dt$. Without truncating the Lévy density (the case where $\varepsilon = 0$), we have the following antiderivatives:

$$k^{-1}(z) = \begin{cases} -E_1(\beta_+ z) & \text{if } \alpha = 0 \\ -e^{-\beta_+ z}z^{-1} + \beta_+ E_1(\beta_+ z) & \text{if } \alpha = 1 \\ \frac{\beta_+^\alpha}{\alpha(1-\alpha)(2-\alpha)}\Gamma(3-\alpha, \beta_+ z) - e^{-\beta_+ z}\left(\frac{z^{2-\alpha}}{\alpha(1-\alpha)(2-\alpha)}\beta_+^2 + \frac{z^{1-\alpha}}{\alpha(1-\alpha)}\beta_+ + \frac{z^{-\alpha}}{\alpha}\right) & \text{else} \end{cases}$$

$$k^{-2}(z) = \begin{cases} k^{-1}(z)\,z + \beta_+^{-1}e^{-\beta_+ z} & \text{if } \alpha = 0 \\ k^{-1}(z)\,z + E_1(\beta_+ z) & \text{if } \alpha = 1 \\ k^{-1}(z)\,z + \frac{\beta_+^{\alpha-1}}{(1-\alpha)(2-\alpha)}\Gamma(3-\alpha, \beta_+ z) - e^{-\beta_+ z}\left(\frac{z^{2-\alpha}}{(1-\alpha)(2-\alpha)}\beta_+ + \frac{z^{1-\alpha}}{(1-\alpha)}\right) & \text{else} \end{cases}$$

$$k^{-3}(z) = \begin{cases} k^{-2}(z)\,z - k^{-1}(z)\,\frac{z^2}{2} - e^{-\beta_+ z}\left(\frac{z}{2}\beta_+^{-1} + \frac{1}{2}\beta_+^{-2}\right) & \text{if } \alpha = 0 \\ k^{-2}(z)\,z - k^{-1}(z)\,\frac{z^2}{2} - \frac{1}{2}\beta_+^{-1}e^{-\beta_+ z} & \text{if } \alpha = 1 \\ k^{-2}(z)\,z - k^{-1}(z)\,\frac{z^2}{2} - \frac{\beta_+^{\alpha-2}}{2(2-\alpha)}\Gamma(3-\alpha, \beta_+ z) - \frac{z^{2-\alpha}}{2(2-\alpha)}e^{-\beta_+ z} & \text{else} \end{cases}$$

$$k^{-4}(z) = \begin{cases} k^{-3}(z)\,z - k^{-2}(z)\,\frac{z^2}{2} + k^{-1}(z)\,\frac{z^3}{6} + e^{-\beta_+ z}\left(\frac{z^2}{6}\beta_+^{-1} + \frac{z}{3}\beta_+^{-2} + \frac{1}{3}\beta_+^{-3}\right) & \text{if } \alpha = 0 \\ k^{-3}(z)\,z - k^{-2}(z)\,\frac{z^2}{2} + k^{-1}(z)\,\frac{z^3}{6} + e^{-\beta_+ z}\left(\frac{z}{6}\beta_+^{-1} + \frac{1}{6}\beta_+^{-2}\right) & \text{if } \alpha = 1 \\ k^{-3}(z)\,z - k^{-2}(z)\,\frac{z^2}{2} + k^{-1}(z)\,\frac{z^3}{6} + \frac{\beta_+^{\alpha-3}}{6}\Gamma(3-\alpha, \beta_+ z) & \text{else} \end{cases}$$

The antiderivatives for $z < 0$ look exactly the same, except that in this case β_- describes the rate of exponential decay.

Under truncation, we deal with a new Lévy density:

$$k_\varepsilon(z) = \begin{cases} k(z) & \text{if} |z| > \varepsilon \\ 0 & \text{if} |z| \le \varepsilon \end{cases}$$

In this case our Lévy density antiderivatives can be expressed in terms of the non-truncated densities as follows ($z > 0$):

$$k_\varepsilon^{-1}(z) = \begin{cases} k^{-1}(z) & \text{if } z > \varepsilon \\ k^{-1}(\varepsilon) & \text{if } z \leq \varepsilon \end{cases}$$

$$k_\varepsilon^{-2}(z) = \begin{cases} k^{-2}(z) & \text{if } z > \varepsilon \\ k^{-2}(\varepsilon) - (\varepsilon - z)\, k^{-1}(\varepsilon) & \text{if } z \leq \varepsilon \end{cases}$$

$$k_\varepsilon^{-3}(z) = \begin{cases} k^{-3}(z) & \text{if } z > \varepsilon \\ k^{-3}(\varepsilon) - (\varepsilon - z)\, k^{-2}(\varepsilon) + \frac{(\varepsilon-z)^2}{2} k^{-1}(\varepsilon) & \text{if } z \leq \varepsilon \end{cases}$$

$$k_\varepsilon^{-4}(z) = \begin{cases} k^{-4}(z) & \text{if } z > \varepsilon \\ k^{-4}(\varepsilon) - (\varepsilon - z)\, k^{-3}(\varepsilon) + \frac{(\varepsilon-z)^2}{2} k^{-2}(\varepsilon) - \frac{(\varepsilon-z)^3}{6} k^{-1}(\varepsilon) & \text{if } z \leq \varepsilon \end{cases}$$

It is necessary to compute the left and right-hand limits as $z \to 0$ for the third and fourth antiderivatives, but these calculations follow easily given the formulas above.

Drift Next we calculate the drift required to make our processes martingales. For $\varepsilon = 0$,

$$\begin{aligned} c_{\exp} &= \int_{-\infty}^{\infty} (e^x - x - 1)\, k(x)\, dx \\ &= \begin{cases} -c\left\{\log\left(1 - \frac{1}{\beta_+} + \frac{1}{\beta_-} - \frac{1}{\beta_+\beta_-}\right) + \frac{1}{\beta_+} - \frac{1}{\beta_-}\right\} & \text{if } \alpha = 0 \\ c\left\{(\beta_+ - 1)\log\left(1 - \frac{1}{\beta_+}\right) + (\beta_- + 1)\log\left(1 + \frac{1}{\beta_-}\right)\right\} & \text{if } \alpha = 1 \\ \frac{c\Gamma(3-\alpha)}{-\alpha(1-\alpha)(2-\alpha)}\left((\beta_+ - 1)^\alpha - \beta_+^\alpha + \alpha\beta_+^{\alpha-1} + (\beta_- + 1)^\alpha - \beta_-^\alpha - \alpha\beta_-^{\alpha-1}\right) & \text{else} \end{cases} \end{aligned}$$

Under truncation,

$$\begin{aligned} c_{\exp} &= \int_{-\infty}^{\infty} (e^x - x - 1)\, k_\varepsilon(x)\, dx \\ &= \int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} (e^x - x - 1)\, k(x)\, dx \end{aligned}$$

For $\alpha = 0$,

$$\begin{aligned} c_{\exp} &= c\left(E_1\left((\beta_+ - 1)\varepsilon\right) - E_1(\beta_+\varepsilon) + E_1\left((\beta_- + 1)\varepsilon\right) - E_1(\beta_-\varepsilon)\right) \\ &- c\left(\beta_+^{-1} e^{-\beta_+\varepsilon} - \beta_-^{-1} e^{-\beta_-\varepsilon}\right) \end{aligned}$$

For $\alpha = 1$,

$$\begin{aligned} c_{\exp} &= c \cdot \varepsilon^{-1} \left(e^{-(\beta_+ - 1)\varepsilon} - e^{-\beta_+ \varepsilon} \right) + c \cdot (\beta_+ - 1) \left(-E_1 \left((\beta_+ - 1) \varepsilon \right) + E_1 \left(\beta_+ \varepsilon \right) \right) \\ &+ c \cdot \varepsilon^{-1} \left(e^{-(\beta_- + 1)\varepsilon} - e^{-\beta_- \varepsilon} \right) + c \cdot (\beta_- + 1) \left(-E_1 \left((\beta_- + 1) \varepsilon \right) + E_1 \left(\beta_- \varepsilon \right) \right) \end{aligned}$$

For $\alpha \in (0,1) \cup (1,2)$,

$$\begin{aligned} c_{\exp} &= c \left(\frac{(\beta_+ - 1)^\alpha}{-\alpha (1-\alpha)(2-\alpha)} \Gamma \left(3-\alpha, (\beta_+ - 1) \varepsilon \right) + \frac{\beta_+^{\alpha - 1}}{-\alpha (1-\alpha)(2-\alpha)} \Gamma \left(3-\alpha, \beta_+ \varepsilon \right) [\alpha - \beta_+] \right) \\ &+ c \left(\frac{(\beta_- + 1)^\alpha}{-\alpha (1-\alpha)(2-\alpha)} \Gamma \left(3-\alpha, (\beta_- + 1) \varepsilon \right) - \frac{\beta_-^{\alpha - 1}}{-\alpha (1-\alpha)(2-\alpha)} \Gamma \left(3-\alpha, \beta_- \varepsilon \right) [\alpha + \beta_-] \right) \\ &+ c e^{-(\beta_+ - 1)\varepsilon} \left(\frac{(\beta_+ - 1)^2 \varepsilon^{2-\alpha}}{\alpha (1-\alpha)(2-\alpha)} + \frac{(\beta_+ - 1) \varepsilon^{1-\alpha}}{\alpha (1-\alpha)} + \frac{\varepsilon^{-\alpha}}{\alpha} \right) \\ &- c e^{-\beta_+ \varepsilon} \left(\frac{\beta_+ \varepsilon^{2-\alpha}}{\alpha (1-\alpha)(2-\alpha)} [\beta_+ - \alpha] + \frac{\varepsilon^{1-\alpha}}{\alpha (1-\alpha)} [\beta_+ - \alpha] + \frac{\varepsilon^{-\alpha}}{\alpha} \right) \\ &+ + c e^{-(\beta_- + 1)\varepsilon} \left(\frac{(\beta_- + 1)^2 \varepsilon^{2-\alpha}}{\alpha (1-\alpha)(2-\alpha)} + \frac{(\beta_- + 1) \varepsilon^{1-\alpha}}{\alpha (1-\alpha)} + \frac{\varepsilon^{-\alpha}}{\alpha} \right) \\ &- c e^{-\beta_- \varepsilon} \left(\frac{\beta_- \varepsilon^{2-\alpha}}{\alpha (1-\alpha)(2-\alpha)} [\beta_- + \alpha] + \frac{\varepsilon^{1-\alpha}}{\alpha (1-\alpha)} [\beta_- + \alpha] + \frac{\varepsilon^{-\alpha}}{\alpha} \right) \end{aligned}$$

Variance for Regularization Finally, we calculate the variance of the diffusion component under small-jump regularization.

$$\sigma(\varepsilon)^2 = \int_{-\varepsilon}^{\varepsilon} x^2 k(x) \, dx$$

For $\alpha = 0$,

$$\sigma(\varepsilon)^2 = -c \left(e^{-\beta_+ \varepsilon} \left(\varepsilon \beta_+^{-1} + \beta_+^{-2} \right) + e^{-\beta_- \varepsilon} \left(\varepsilon \beta_-^{-1} + \beta_-^{-2} \right) - \beta_+^{-2} - \beta_-^{-2} \right)$$

For $\alpha = 1$,

$$\sigma(\varepsilon)^2 = -c \left(e^{-\beta_+ \varepsilon} \beta_+^{-1} + e^{-\beta_- \varepsilon} \beta_-^{-1} - \beta_+^{-1} - \beta_-^{-1} \right)$$

For $\alpha \in (0,1) \cup (1,2)$,

$$\sigma(\varepsilon)^2 = c \left(\frac{\beta_+^{\alpha - 2}}{2-\alpha} \Gamma \left(3-\alpha, \beta_+ \varepsilon \right) - \frac{\varepsilon^{2-\alpha}}{2-\alpha} e^{-\beta_+ \varepsilon} + \frac{\beta_-^{\alpha - 2}}{2-\alpha} \Gamma \left(3-\alpha, \beta_- \varepsilon \right) - \frac{\varepsilon^{2-\alpha}}{2-\alpha} e^{-\beta_- \varepsilon} \right)$$

With these variance formulas, all the calculations needed to numerically compute option prices under no truncation, truncation, or regularization are detailed.

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