Isotropic Random Fields on the Sphere

Stochastic Heat Equation and Regularity of Random Elliptic PDEs

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2-weakly isotropic spherical random fields are defined and analyzed, especially 2-weakly isotropic Gaussian spherical random fields. The connection between the angular power spectrum, the path regularity and the integrability of these fields is in particular investigated. Expansions of realizations of these fields into spherical harmonics are a key tool in the analysis. One application of this is the discussion of the stochastic heat equation with additive 2-weakly isotropic Q-Wiener noise. Unique solvability and Hölder regularity of second order, elliptic partial differential equations on the sphere with log-normal, 2-weakly isotropic coefficients will be discussed. Here, existence, uniqueness and integrability of the random solution and Hölder norms of its realizations with respect to the Gaussian measure are established by tracking the constants through the Schauder estimates and employing Fernique's theorem. Here, the connection between the angular power spectrum and path regularity of the solution will be seen.

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Contents

1. Introduction

In this project we introduce the notion of a 2-weakly isotropic spherical random field which is in some sense invariant under the action of $SO(3)$. We briefly summarize the spherical harmonics, its real version and some useful properties of these as orthonormal basis of $L^2(S^2, \mathbb{R})$ and as diagonalizing sequence of the spherical Laplace operator Δ_{S^2}, where S^2 denotes the unit sphere as a subset of $\mathbb{R}^3$. The discussion of the spherical harmonics enables us to establish a spherical expansion of 2-weakly isotropic spherical random fields which exists in $L^2_{P\otimes \mathrm{d}\sigma}(\Omega \times S^2, \mathbb{R})$ and in $L^2_P(\Omega, \mathbb{R})$ sense, where $(\Omega, \mathcal{A}, P)$ is a probability space and $\mathrm{d}\sigma$ denotes the Lebesgue measure on S^2. This is the first main result.

Theorem 1.1. *For a 2-weakly isotropic spherical random field T there exists an expansion in the spherical harmonics $(Y_{l,m} : l \geq 0, m = -l, ..., l)$ in $L^2_{P\otimes \mathrm{d}\sigma}(\Omega \times S^2, \mathbb{R})$ and in $L^2_P(\Omega, \mathbb{R})$*

$$T = \sum_{l\geq 0} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}.$$

For a 2-weakly isotropic spherical random field the coefficients of the spherical expansion $(a_{l,m} : l \geq 0, m = -l, ..., l)$ are pairwise uncorrelated and define the positive sequence $(C_l : l \geq 0)$, which is called the *angular power spectrum* in the following way:

$$C_l \delta_{l,l'} \delta_{m,m'} = E[a_{l,m} \overline{a_{l',m'}}].$$

The notion of a 2-weakly isotropic spherical random field can be combined with the Gaussian distribution on $L^2(S^2, \mathbb{R})$ to form a Q-Wiener process $W = (W(t) : t \geq 0)$ with given angular power spectrum. Then we can formulate the stochastic heat equation with 2-weakly isotropic Q-Wiener noise W

$$X(t) = X(0) + \int_0^t \Delta_{S^2} X(s) \mathrm{d}s + W(t). \tag{1.1}$$

The property that the noise is 2-weakly isotropic will be the key to a solution formula in the discussion of this problem. An implementation with MATLAB of the obtained solution to Equation (1.1) can be found in Appendix B.
With the first result Theorem 1.1 at hand, we develop further properties of 2-weakly isotropic Gaussian spherical random fields. After a slight generalization of the Kolmogorov–Čentsov continuity theorem to random fields, we provide sufficient conditions for 2-weakly isotropic Gaussian spherical random fields to have a Hölder continuous modification. This is the second main result.

Theorem 1.2. *If T is a 2-weakly isotropic Gaussian spherical random field, such that the angular power spectrum fulfills that $(C_l l^{1+\delta} : l \geq 0)$ is summable for $\delta \in (0, 2]$, then for all $\gamma \in (0, \frac{\delta}{2})$ there exists a γ-Hölder continuous modification of T.*

Since a modification of a 2-weakly isotropic spherical random field is also 2-weakly isotropic, Theorem 1.1 and Theorem 1.2 provide the existence of a continuous 2-weakly isotropic Gaussian spherical random field. The first five chapter of this exposition were developed in parallel to a recent paper by Lang and Schwab [20]. Up to this point these results can also be found in this paper by Lang and Schwab. However the development of the results in this exposition follows a different, independent and more elementary approach.

In the following we are interested in higher order regularity of a continuous 2-weakly isotropic Gaussian spherical random field T. In Chapter 6 we develop conditions on the angular power spectrum of T such that T is P-a.s. a member of a Sobolev space, i.e. weak partial derivatives of T up to ι^{th} order are P-a.s. in $L^2(S^2, \mathbb{R})$ if the angular power spectrum satisfies that $(C_l l^{1+2\iota} : l \geq 0)$ is summable for an integer $\iota \geq 0$. In the case of Hölder continuity, we observe that 2-weakly isotropic spherical random fields are generally P-a.s. in $L^2(S^2, \mathbb{R})$ and the angular power spectrum satisfies that $(C_l l : l \geq 0)$ is summable. When this condition can be strengthen by adding $\delta \in (0, 2]$ to the exponent of the weight, i.e. $(C_l l^{1+\delta} : l \geq 0)$ is summable, we achieve Hölder continuity. This concept can be generalized to weak derivatives as the next theorem shows.

Theorem 1.3. *If T is a continuous 2-weakly isotropic Gaussian spherical random field, such that the angular power spectrum satisfies that $(C_l l^{1+2\iota+\delta} : l \geq 0)$ is summable for $\delta \in (0, 2]$ and some integer $\iota \geq 0$, then $T \in L^p_P(\Omega, C^{\iota,\gamma}(S^2))$ for all $p \in (0, \infty)$ and all $\gamma \in (0, \frac{\delta}{2})$.*

We will give an elementary proof of this result and will find a second approach, which relies on successive application of elliptic regularity.
In engineering and scientific applications log-normally distributed spherical random fields are important and are introduced in Chapter 7. They are denoted by A and will be defined through Gaussian spherical random fields, i.e. for a Gaussian spherical random field T we define $A = \exp(T)$. We are able to transfer the regularity results on Gaussian spherical random fields to the log-normal case. The regularity of log-normally distributed spherical random fields is the important ingredient to consider in Chapter 8 elliptic partial differential equations on the sphere with a 2-weakly isotropic log-normally distributed coefficient, i.e. we consider the problem to find u such that

$$-\nabla_{S^2} \cdot (A \nabla_{S^2} u) = f \tag{1.2}$$

for a given deterministic and sufficiently smooth right hand side f. We recapitulate the Schauder theory and analyze the precise constants in the Schauder estimates. With these estimates we will be able to deduce higher regularity of the random solution u. The regularity of u is governed by the regularity of the 2-weakly isotropic log-normal spherical random field A, which is implied by the decay of the angular power spectrum of the underlying continuous 2-weakly isotropic Gaussian spherical random field.

Theorem 1.4. *Let A be a 2-weakly isotropic log-normal spherical random field, that results from a continuous 2-weakly isotropic Gaussian spherical random field, whose angular power spectrum satisfies that $(C_l l^{1+2\iota+\delta} : l \geq 0)$ is summable for some $\delta \in (0, 2]$ and some integer $\iota \geq 0$. For all $\gamma \in (0, \frac{\delta}{2})$ there exists a unique solution $u \in L^p_P(\Omega, C^{\iota+1,\gamma}(S^2))$ of Equation (1.2) for all $p \in (0, \infty)$.*

The appropriate setup and formulation of these four theorems will of course be made precise in the following exposition.

2. Review of Peter–Weyl theory on the sphere

In this chapter we discuss some important tools from harmonic analysis. Before we get started we introduce some notation. For $k \in \mathbb{N}$ the Euclidean norm on $\mathbb{R}^k$ is denoted by $\|\cdot\|_{\mathbb{R}^k}$ and the inner product by $\langle\cdot,\cdot\rangle_{\mathbb{R}^k}$. In the case $k = 3$ the subscript index '$\mathbb{R}^3$' will be omitted. For $k \in \mathbb{N}$ the components of elements in $\mathbb{R}^k$ will be denoted by subscript indices, i.e. $x = (x_1, ..., x_k)^\top$. The Kronecker delta for two integers l and m is defined by $\delta_{l,m} = 1$ if $l = m$, $\delta_{l,m} = 0$ else. For a complex number z, the complex conjugate of z is denoted by $\overline{z}$, the real part is denoted by $\mathcal{R}(z)$ and the imaginary part by $\mathcal{I}(z)$. The sphere as a subset of $\mathbb{R}^3$ is denoted by $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$. We will use the following convention of the spherical coordinates on the sphere for $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi)$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix},$$

for $\theta \in \{0, \pi\}$ we set $\varphi = 0$. The metric d on S^2 induced by geodesics is given by

$$d(x, y) = \arccos(\langle x, y\rangle),$$

for x, y in S^2. In our coordinates this reads

$$d(x, y) = \arccos\left(\sin(\theta_x)\sin(\theta_y)\cos(\varphi_x - \varphi_y) + \cos(\theta_x)\cos(\theta_y)\right). \tag{2.1}$$

The Lebesgue measure on the sphere is denoted by $\mathrm{d}\sigma(x) = \sin(\theta)\mathrm{d}\theta\mathrm{d}\varphi$. The function space of square integrable functions from S^2 to the complex or real numbers is denoted by $L^2_{\mathrm{d}\sigma}(S^2, \mathbb{C})$ and $L^2_{\mathrm{d}\sigma}(S^2, \mathbb{R})$. The usual norms in these spaces are denoted by $\|\cdot\|_{L^2_{\mathrm{d}\sigma}(S^2,\mathbb{C})}$ and $\|\cdot\|_{L^2_{\mathrm{d}\sigma}(S^2,\mathbb{R})}$. In cases of other domains or measures the notation will be adapted accordingly. In the case of the Lebesgue measure $\mathrm{d}\sigma$ on the sphere the measure will be omitted. Equality of functions is understood in the L^2 sense, if not otherwise stated. Finally the special orthogonal group is given by $SO(n) = \{g \in \mathbb{R}^{n\times n} : g^\top g = I_n, \det(g) = 1\}$ for $n \in \mathbb{N}$, where $g^\top$ denotes the transpose of g in $\mathbb{R}^{n\times n}$ and I_n the identity matrix in $\mathbb{R}^{n\times n}$. Basic knowledge of functional analysis and probability theory is assumed throughout the whole text.

Lemma 2.1. *For $x, y, z, w \in S^2$ such that $\langle x, y\rangle = \langle z, w\rangle$, there exists $g \in SO(3)$ such that*

$$gx = z \quad \text{and} \quad gy = w.$$

Proof. The proof is done in two steps. First we consider only x and z. Proposition 2.12 of [21] with $x = x, G = SO(3)$ and $X = S^2$ directly says that there exists $h \in SO(3)$ such that $hx = z$. We denote the image of y under the left action of h by $\tilde{y} = hy$.

The second step is to show that there exists $\hat{g} \in SO(3)$, such that $\hat{g}\tilde{y} = w$ and $\hat{g}z = z$, i.e. $\hat{g}$ is in the stabilizer of z. This will give the claim of the lemma with $g = \hat{g}h$.
To show the second step we may assume after rotation, that $z = (1, 0, 0)^\top$. Since action by $SO(3)$ preserves the inner product, the assumption about the inner product of the two pairs gives

$$\langle x, y\rangle = \langle hx, hy\rangle = \langle z, \tilde{y}\rangle = \langle z, w\rangle.$$

The evaluation of the inner products gives that the first components of $\tilde{y}$ and w agree. This implies that $\tilde{y}$ and w lie on a circle with radius $r = \sqrt{1 - w_1^2}$ and are separated by some angle θ. Since the stabilizer of z regarding the left group action under $SO(3)$ is given by

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} : g \in SO(2) \right\},$$

we can take

$$\hat{g} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

This gives the claim. □

The book of Marinucci and Peccati [21] contains a development of representation theory for compact groups including the special case $SO(3)$. They use the Peter–Weyl theorem on $SO(3)$ to deduce spherical Fourier expansions on the sphere S^2. We will summarize the needed material with a few definitions, lemmas and a theorem.

Definition 2.2. *The associated Legendre polynomials P_m^l for integers $l \geq 0$, $m = -l, ..., l$ and $x \in [-1, 1]$ are defined by*

$$P_m^l(x) = \frac{(-1)^m}{2^l l!}(1 - x^2)^{\frac{m}{2}} \frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}(x^2 - 1)^l.$$

For the special case $m = 0$, the functions P_0^l for integers $l \geq 0$ are called Lengendre polynomials. The spherical harmonics $Y_{l,m}$ for integers $l \geq 0$, $m = -l, ..., l$ and (θ, φ) in $[0, \pi] \times [0, 2\pi)$ are defined by

$$Y_{l,m}(\theta, \varphi) = \sqrt{\frac{2l + 1}{4\pi}} \sqrt{\frac{(l - m)!}{(m + l)!}} P_m^l(\cos\theta) e^{im\varphi}.$$

Theorem 2.3. *For any T in $L^2(S^2, \mathbb{C})$ there exists the following Fourier expansion involving the spherical harmonics which converges in the $L^2(S^2, \mathbb{C})$-norm*

$$T = \sum_{l \geq 0} \sum_{m=-l}^{l} a_{l,m} Y_{l,m},$$
$$a_{l,m} = \int_{S^2} T(x) \overline{Y_{l,m}(x)} \mathrm{d}\sigma(x),$$

and for integers $l, l' \geq 0$, $m = -l, ..., l$ and $m' = -l', ..., l'$ it holds that

$$\int_{S^2} Y_{l,m}(x) \overline{Y_{l',m'}(x)} \mathrm{d}\sigma(x) = \delta_{l,l'} \delta_{m,m'}.$$

Proof. The first claim is stated as Proposition 3.29 in [21] and the second claim is stated as Equation (3.39) in [21]. □

For real-valued functions, we wish to have a real version of this expansion. Moreover we want to establish relations between the real and complex coefficients. We will first define the real spherical harmonics and the corresponding real coefficients and then prove the needed properties as a corollary of Theorem 2.3.

Definition 2.4. *The real spherical harmonics $\tilde{Y}_{l,m}$ and the real coefficients $\tilde{a}_{l,m}$ for integers $l \geq 0$, $m = -l, ..., l$ are defined by*

$$\tilde{a}_{l,m} = \begin{cases} \sqrt{2}\mathcal{R}(a_{l,m}) & m > 0, \\ a_{l,0} & m = 0, \\ -\sqrt{2}\mathcal{I}(a_{l,|m|}) & m < 0, \end{cases} \quad \text{and} \quad \tilde{Y}_{l,m} = \begin{cases} \sqrt{2}\mathcal{R}(Y_{l,m}) & m > 0, \\ Y_{l,0} & m = 0, \\ \sqrt{2}\mathcal{I}(Y_{l,|m|}) & m < 0. \end{cases}$$

Corollary 2.5. *For any $T \in L^2(S^2, \mathbb{R})$ there exists the following real Fourier expansion, which converges in the $L^2(S^2, \mathbb{R})$-norm,*

$$T = \sum_{l \geq 0} \sum_{m=-l}^{l} \tilde{a}_{l,m} \tilde{Y}_{l,m},$$
$$\tilde{a}_{l,m} = \int_{S^2} T(x) \tilde{Y}_{l,m}(x) \mathrm{d}\sigma(x),$$

and for integers $l, l' \geq 0$, $m = -l, ..., l$ and $m' = -l', ..., l'$, it holds that

$$\int_{S^2} \tilde{Y}_{l,m}(x) \tilde{Y}_{l',m'}(x) \mathrm{d}\sigma(x) = \delta_{l,l'} \delta_{m,m'}.$$

Proof. The first part of the proof will be to show that $\mathrm{span}\{\tilde{Y}_{l,m} : l \geq 0, m = -l, .., l\}$ is dense in $L^2(S^2, \mathbb{R})$. Since $L^2(S^2, \mathbb{R}) \subset L^2(S^2, \mathbb{C})$ with $\| \cdot \|_{L^2(S^2,\mathbb{C})} = \| \cdot \|_{L^2(S^2,\mathbb{R})}$ for real-valued functions, T can be seen as an element of $L^2(S^2, \mathbb{C})$. For any T in $L^2(S^2, \mathbb{C})$ Theorem 2.3 already yields the complex Fourier expansion

$$T = \sum_{l \geq 0} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}.$$

This means it is sufficient to show that the real Fourier expansion converges against the complex one, i.e.

$$\begin{aligned} 0 &= \lim_{L \to \infty} \left\| T - \sum_{l=0}^{L} \sum_{m=-l}^{l} \tilde{a}_{l,m} \tilde{Y}_{l,m} \right\|^2_{L^2(S^2,\mathbb{C})} \\ &= \lim_{L \to \infty} \left\| \sum_{l \geq 0} \sum_{m=-l}^{l} a_{l,m} Y_{l,m} - \sum_{l=0}^{L} \sum_{m=-l}^{l} \tilde{a}_{l,m} \tilde{Y}_{l,m} \right\|^2_{L^2(S^2,\mathbb{C})}. \end{aligned} \tag{2.2}$$

We attempt this with looking at the sums over $m = -l, ..., l$ for a fixed integer $l \geq 0$ individually and show that the difference vanishes, i.e.

$$\sum_{m=-l}^{l} a_{l,m} Y_{l,m} = \sum_{m=-l}^{l} \tilde{a}_{l,m} \tilde{Y}_{l,m}. \tag{2.3}$$

To show this, we apply two properties of the spherical harmonics. The first one is $\overline{Y_{l,m}} = (-1)^m Y_{l,-m}$, which follows from the definition and the second is $\overline{a_{l,m}} = (-1)^m a_{l,-m}$. We quickly calculate the latter where we use that T is real-valued

$$\begin{aligned}\overline{a_{l,m}} &= \overline{\int_{S^2} T(x)\overline{Y_{l,m}(x)}\mathrm{d}\sigma(x)} = \int_{S^2} \overline{T(x)} Y_{l,m}(x)\mathrm{d}\sigma(x) \\ &= \int_{S^2} T(x) Y_{l,m}(x)\mathrm{d}\sigma(x) = \int_{S^2} T(x)(-1)^m \overline{Y_{l,-m}(x)}\mathrm{d}\sigma(x) = (-1)^m a_{l,-m}. \end{aligned} \tag{2.4}$$

We apply these two properties in the next two steps on the way to prove Equation (2.3),

$$\begin{aligned}\sum_{m=-l}^{l} a_{l,m}Y_{l,m} &= a_{l,0}Y_{l,0} + \sum_{m=1}^{l} (a_{l,m}(\mathcal{R}(Y_{l,m}) + i\mathcal{I}(Y_{l,m})) + a_{l,-m}(\mathcal{R}(Y_{l,-m}) + i\mathcal{I}(Y_{l,-m})) \\ &= a_{l,0}Y_{l,0} + \sum_{m=1}^{l} (a_{l,m}\mathcal{R}(Y_{l,m}) + a_{l,-m}(-1)^m\mathcal{R}(Y_{l,m}) \\ &\qquad + a_{l,m}i\mathcal{I}(Y_{l,m}) - a_{l,-m}i(-1)^m\mathcal{I}(Y_{l,m})) \\ &= a_{l,0}Y_{l,0} + \sum_{m=1}^{l} ((a_{l,m} + \overline{a_{l,m}})\mathcal{R}(Y_{l,m}) + i(a_{l,m} - \overline{a_{l,m}})\mathcal{I}(Y_{l,m})).\end{aligned}$$

We remark that for $x \in \mathbb{C}$, it holds $x + \overline{x} = 2\mathcal{R}(x)$ and $x - \overline{x} = 2i\mathcal{I}(x)$ and leads to

$$\begin{aligned}\sum_{m=-l}^{l} a_{l,m}Y_{l,m} &= a_{l,0}Y_{l,0} + \sum_{m=1}^{l} (\sqrt{2}\mathcal{R}(a_{l,m})\sqrt{2}\mathcal{R}(Y_{l,m}) - \sqrt{2}\mathcal{I}(a_{l,m})\sqrt{2}\mathcal{I}(Y_{l,m})) \\ &= \sum_{m=-l}^{l} \tilde{a}_{m,l}\tilde{Y}_{l,m}.\end{aligned}$$

We can apply Equation (2.3) to Equation (2.2) and then use the Parseval identity. Since we choose T to be in $L^2(S^2, \mathbb{R})$, we know that the sequence of absolute values of Fourier coefficients is square summable, thus the limit vanishes, i.e.

$$\begin{aligned}\lim_{L\to\infty} \left\| \sum_{l\geq 0} \sum_{m=-l}^{l} a_{l,m}Y_{l,m} - \sum_{l=0}^{L} \sum_{m=-l}^{l} \tilde{a}_{l,m}\tilde{Y}_{l,m} \right\|^2_{L^2(S^2,\mathbb{C})} &= \lim_{L\to\infty} \left\| \sum_{l\geq L+1} \sum_{m=-l}^{l} a_{l,m}Y_{l,m} \right\|^2_{L^2(S^2,\mathbb{C})} \\ &= \lim_{L\to\infty} \sum_{l\geq L+1} \sum_{m=-l}^{l} |a_{l,m}|^2 = 0.\end{aligned}$$

This shows that $\mathrm{span}\{\tilde{Y}_{l,m} : l \geq 0, m = -l, .., l\}$ is dense in $L^2(S^2, \mathbb{R})$, since for real-valued functions $\|\cdot\|_{L^2(S^2,\mathbb{C})} = \|\cdot\|_{L^2(S^2,\mathbb{R})}$.
The second part of the proof is to show that the real spherical harmonics $\tilde{Y}_{l,m}$ for integers $l \geq 0$, $m = -l, ..., l$ are orthonormal. We fix $l, l' \geq 0$ and start with the case $m, m' > 0$. We insert the definition of the real spherical harmonics into the expression and expand the brackets to obtain that

$$\int_{S^2} \tilde{Y}_{l,m}(x)\tilde{Y}_{l',m'}(x)\mathrm{d}\sigma(x)s = \frac{1}{2}\int_{S^2} (Y_{l,m}(x) + \overline{Y_{l,m}(x)})(Y_{l',m'}(x) + \overline{Y_{l',m'}(x)})\mathrm{d}\sigma(x)$$

$$= \frac{1}{2}\int_{S^2}(Y_{l,m}(x)Y_{l',m'} + \overline{Y_{l,m}(x)}Y_{l',m'} + Y_{l,m}(x)\overline{Y_{l',m'}}(x) + \overline{Y_{l,m}(x)}\ \overline{Y_{l',m'}(x)})\mathrm{d}\sigma(x).$$

We apply the already used relation $\overline{Y_{l,m}} = (-1)^m Y_{l,-m}$ in order to use the orthogonality of the spherical harmonics, which is stated in Theorem 2.3, on all four combinations. The fact $m > -m'$ then gives

$$\begin{aligned}\int_{S^2}\tilde{Y}_{l,m}(x)\tilde{Y}_{l',m'}(x)\mathrm{d}\sigma(x) &= \frac{1}{2}(\int_{S^2}(-1)^{m'}(Y_{l,m}(x)\overline{Y_{l',-m'}(x)} + \overline{Y_{l,m}(x)}Y_{l',-m'}(x))\mathrm{d}\sigma(x) \\ &\quad + 2\delta_{l,l'}\delta_{m,m'}) \\ &= (-1)^{m'}\delta_{l,l'}\delta_{m,-m'} + \delta_{l,l'}\delta_{m,m'} = \delta_{l,l'}\delta_{m,m'}.\end{aligned}$$

The case for $m, m' < 0$ is treated similarly. We proceed with $m > 0, m' < 0$. The same steps as we performed in the case $m, m' > 0$ give the result here as well. We start with the definition of the real spherical harmonics and expand the brackets to obtain that

$$\begin{aligned}\int_{S^2}\tilde{Y}_{l,m}(x)\tilde{Y}_{l',m'}(x)\mathrm{d}\sigma(x) &= \frac{1}{2i}\int_{S^2}(Y_{l,m}(x) + \overline{Y_{l,m}(x)})(Y_{l',|m'|}(x) - \overline{Y_{l',|m'|}(x)})\mathrm{d}\sigma(x) \\ &= \frac{1}{2i}\int_{S^2}(Y_{l,m}(x)Y_{l',-m'}(x) + \overline{Y_{l,m}(x)}Y_{l',-m'}(x) \\ &\quad - Y_{l,m}(x)\overline{Y_{l',-m'}(x)} - \overline{Y_{l,m}(x)}\ \overline{Y_{l',-m'}(x)})\mathrm{d}\sigma(x).\end{aligned}$$

The relation $\overline{Y_{l,m}} = (-1)^m Y_{l,-m}$ and the orthogonality of the spherical harmonics are used to obtain that

$$\begin{aligned}\int_{S^2}\tilde{Y}_{l,m}(x)\tilde{Y}_{l',m'}(x)\mathrm{d}\sigma(x) &= \frac{1}{2i}(\int_{S^2}(-1)^m(\overline{Y_{l,-m}(x)}Y_{l,-m'}(x) - Y_{l,-m}(x)\overline{Y_{l',-m'}(x)})\mathrm{d}\sigma(x) \\ &\quad + \delta_{l,l'}\delta_{m,-m'} - \delta_{l,l'}\delta_{m,-m'}) \\ &= \frac{1}{2i}(\delta_{l,l'}\delta_{-m,-m'} - \delta_{l,l'}\delta_{-m,-m'}) = 0.\end{aligned}$$

This gives the claim. In the case $m = 0$ the real and spherical harmonics agree, that implies the normalization. The orthogonality is also clear by Definition 2.4 and Theorem 2.3. □

For a given expansion in the real spherical harmonics, it is easier to perform manipulations on the corresponding expansion in the spherical harmonics than on the real ones. That is why it is useful to know how the complex coefficients $a_{l,m}$ are expressed in terms of the real ones $\tilde{a}_{l,m}$.

Lemma 2.6. *For T in $L^2(S^2, \mathbb{R})$ the complex coefficients $a_{l,m}$ for integers $l \geq 0$, $m = -l, ..., l$ can be obtained from the real ones $\tilde{a}_{l,m}$ in the following way*

$$a_{l,m} = \begin{cases} \frac{1}{\sqrt{2}}(\tilde{a}_{l,m} - i\tilde{a}_{l,-m}) & m > 0, \\ \tilde{a}_{l,0} & m = 0, \\ (-1)^m\frac{1}{\sqrt{2}}(\tilde{a}_{l,-m} + i\tilde{a}_{l,m}) & m < 0. \end{cases}$$

Proof. The proof is similar to the arguments before. We simply fix an integer $l \geq 0$ and look at the sums over $m = -l, ..., l$ individually. We start with the expansion in the real spherical harmonics and insert their definition in terms of the spherical harmonics,

$$\sum_{m=-l}^{l}\tilde{a}_{l,m}\tilde{Y}_{l,m} = \sum_{m=1}^{l}\tilde{a}_{l,m}\sqrt{2}\mathcal{R}(Y_{l,m}) + \tilde{a}_{l,0}Y_{l,0} + \sum_{m=-l}^{-1}\tilde{a}_{l,m}\sqrt{2}\mathcal{I}(Y_{l,|m|})$$

$$= \sum_{m=1}^{l} \tilde{a}_{l,m} \frac{1}{\sqrt{2}} (Y_{l,m} + \overline{Y_{l,m}}) + \tilde{a}_{l,0} Y_{l,0} + \sum_{m=-l}^{-1} \tilde{a}_{l,m} \frac{1}{\sqrt{2}i} (Y_{l,|m|} - \overline{Y_{l,|m|}}).$$

After we eliminated all complex conjugates of spherical harmonics due to the relation $\overline{Y_{l,m}} = (-1)^m Y_{l,-m}$, we reorder the terms and obtain the claim:

$$\begin{aligned}
\sum_{m=-l}^{l} \tilde{a}_{l,m} \tilde{Y}_{l,m} &= \sum_{m=1}^{l} \tilde{a}_{l,m} \frac{1}{\sqrt{2}} (Y_{l,m} + (-1)^m Y_{l,-m}) + \tilde{a}_{l,0} Y_{l,0} \\
&\quad + \sum_{m=-l}^{-1} \tilde{a}_{l,m} \frac{1}{\sqrt{2}i} (Y_{l,|m|} - (-1)^m Y_{l,m}) \\
&= \sum_{m=1}^{l} \frac{1}{\sqrt{2}} (\tilde{a}_{l,m} - i\tilde{a}_{l,-m}) Y_{l,m} + \tilde{a}_{l,0} Y_{l,0} \\
&\quad + \sum_{m=-l}^{-1} \frac{1}{\sqrt{2}} (-1)^m (\tilde{a}_{l,-m} + i\tilde{a}_{l,m}) Y_{l,m}. \qquad \square
\end{aligned}$$

In the next chapter we introduce isotropy, which is in some sense invariance under the action of $SO(3)$. The next lemma provides a decent tool for the next chapter, it discusses a group action of $SO(3)$ on $L^2(S^2, \mathbb{C})$, the so called *left regular representation.*

Lemma 2.7. *For $T \in L^2(S^2, \mathbb{C})$ and $g \in SO(3)$, the left regular representation D of $SO(3)$ on $L^2(S^2, \mathbb{C})$ is defined by*

$$D(g)T = T(g^{-1} \cdot).$$

This representation D is unitary. Furthermore D can be characterized on finite dimensional subspaces of $L^2(S^2, \mathbb{C})$ spanned by the spherical harmonics $Y_{l,m}$ for some fixed integers $l \geq 0$ and varying $m = -l, ..., l$, i.e.

$$D(g)Y_{l,m} = Y_{l,m}(g^{-1} \cdot) = \sum_{m'=-l}^{l} D^l_{m',m}(g) Y_{l,m'}.$$

D^l denotes a unitary matrix on the vector space $\mathbb{C}^{2l+1}$.

Proof. To get familiar with representations, we check that this definition of D is actually a representation. For g_1, $g_2 \in SO(3)$ and $T \in L^2(S^2, \mathbb{C})$ we obtain

$$D(g_1 g_2)T = T((g_1 g_2)^{-1} \cdot) = T(g_2^{-1} g_1^{-1} \cdot) = D(g_1) T(g_2^{-1} \cdot) = D(g_1) D(g_2) T.$$

For the rest of the proof we fix T in $L^2(S^2, \mathbb{C})$ and g in $SO(3)$. To finish the proof of the first claim, we check that the condition for D to be unitary is satisfied

$$\|D(g)T\|^2_{L^2(S^2,\mathbb{C})} = \|T(g^{-1} \cdot)\|^2_{L^2(S^2,\mathbb{C})} = \int_{S^2} |T(g^{-1}x)|^2 \mathrm{d}\sigma(x).$$

The transformation formula from calculus yields the claim, since the determinant of g is one, i.e. $\det(g) = 1$,

$$\|D(g)T(\cdot)\|^2_{L^2(S^2,\mathbb{C})} = \int_{S^2} |T(x)|^2 |\det(g)| \mathrm{d}\sigma(x) = \int_{S^2} |T(x)|^2 \mathrm{d}\sigma(x) = \|T\|^2_{L^2(S^2,\mathbb{C})}.$$

Thus D is a unitary representation of $SO(3)$ on $L^2(S^2, \mathbb{C})$.
For the second claim of the lemma we reference an auxiliary result. For any fixed integer $l \geq 0$ and $m = -l, ..., l$, [21] contains the following relation as equation (3.44) for a unitary matrix $D^l \in \mathbb{C}^{(2l+1)\times(2l+1)}$

$$Y_{l,m}(g^{-1}\cdot) = \sum_{m'=-l}^{l} D^l_{m',m}(g) Y_{l,m'}.$$

The matrix D^l is unitary due to proposition 3.6 in [21]. This formula gives the claim, since for the same l and m by definition of D, it holds that

$$D(g)Y_{l,m}(\cdot) = Y_{l,m}(g^{-1}\cdot). \qquad \square$$

There is another important non-trivial relation about the spherical harmonics.

Lemma 2.8. *For $x, y \in S^2$ and an integer $l \geq 0$, it holds that*

$$P_0^l(\langle x, y\rangle) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{l,m}(x)\overline{Y_{l,m}(y)} = \sum_{m=-l}^{l} \tilde{Y}_{l,m}(x)\tilde{Y}_{l,m}(y).$$

In the special case $x = y \in S^2$ this yields

$$\sum_{m=-l}^{l} Y_{l,m}(x)\overline{Y_{l,m}(x)} = \frac{2l+1}{4\pi}.$$

Proof. The first claim is stated as equation (3.42) in [21] with a proof below. They use the notation P_l, which is equal to P_0^l in this manuscript. For the second claim, we have to check that $P_0^l(1) = 1$. We start with the definition of the Legendre polynomials and compute the first two derivatives in the definition

$$\begin{aligned} P_0^l(1) &= \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l\bigg|_{x=1} = \frac{1}{2^l l!}\frac{\mathrm{d}^{l-1}}{\mathrm{d}x^{l-1}} 2xl(x^2-1)^{l-1}\bigg|_{x=1} \\ &= \frac{1}{2^l l!}\frac{\mathrm{d}^{l-2}}{\mathrm{d}x^{l-2}} 2(l(x^2-1)^{l-1} + 2x^2 l(l-1)(x^2-1)^{l-2})\bigg|_{x=1}. \end{aligned}$$

We can already see a certain pattern. When we compute the next $l-2$ derivatives, the first term of the above expression with the exponent $l-1$ will decompose into a sum of terms, but each term will have (x^2-1) as a factor. They vanish after we inserted $x = 1$. A similar argument applies to the term with exponent $l-2$. After we computed the $l-2$ derivatives the only term without (x^2-1) as a factor looks like

$$P_0^l(1) = \frac{1}{2^l l!} 2^l x^l l!\bigg|_{x=1} = 1.$$

For the second equality in the first claim we apply Definition 2.4 and the relation $\overline{Y_{l,m}} = (-1)^m Y_{l,-m}$ to obtain that

$$\sum_{m=-l}^{l} \tilde{Y}_{l,m}(x)\tilde{Y}_{l,m}(y) = \sum_{m=1}^{l} 2(\mathcal{R}(Y_{l,m}(x))\mathcal{R}(Y_{l,m}(y)) + \mathcal{I}(Y_{l,m}(x))\mathcal{I}(Y_{l,m}(y))) + Y_{l,0}(x)Y_{l,0}(y)$$

$$
\begin{aligned}
&= \sum_{m=1}^{l} (\mathcal{R}(Y_{l,m}(x))\mathcal{R}(Y_{l,m}(y)) + \mathcal{I}(Y_{l,m}(x))\mathcal{I}(Y_{l,m}(y))) + Y_{l,0}(x)Y_{l,0}(y) \\
&\quad + \sum_{m=-l}^{-1} ((-1)^m \mathcal{R}(Y_{l,m}(x))(-1)^m \mathcal{R}(Y_{l,m}(y)) \\
&\qquad\qquad + (-1)^{m+1}\mathcal{I}(Y_{l,m}(x))(-1)^{m+1}\mathcal{I}(Y_{l,m}(y))) \\
&= \sum_{m=-l}^{l} (\mathcal{R}(Y_{l,m}(x))\mathcal{R}(Y_{l,m}(y)) + \mathcal{I}(Y_{l,m}(x))\mathcal{I}(Y_{l,m}(y))). \qquad (2.5)
\end{aligned}
$$

Also we obtain that

$$
\begin{aligned}
\sum_{m=-l}^{l} Y_{l,m}(x)\overline{Y_{l,m}(y)} &= \sum_{m=-l}^{l} (\mathcal{R}(Y_{l,m}(x))\mathcal{R}(Y_{l,m}(y)) + \mathcal{I}(Y_{l,m}(x))\mathcal{I}(Y_{l,m}(y))) \qquad (2.6) \\
&\quad + i \sum_{m=-l}^{l} (\mathcal{I}(Y_{l,m}(x))\mathcal{R}(Y_{l,m}(y)) - \mathcal{R}(Y_{l,m}(x))\mathcal{I}(Y_{l,m}(y))).
\end{aligned}
$$

We already proved that $\sum_{m=-l}^{l} Y_{l,m}(x)\overline{Y_{l,m}(y)}$ is real-valued and therefore

$$
\sum_{m=-l}^{l} (\mathcal{I}(Y_{l,m}(x))\mathcal{R}(Y_{l,m}(y)) - \mathcal{R}(Y_{l,m}(x))\mathcal{I}(Y_{l,m}(y))) = 0. \qquad (2.7)
$$

We combine Equalities (2.5), (2.6) and (2.7) and conclude that

$$
\sum_{m=-l}^{l} Y_{l,m}(x)\overline{Y_{l,m}(y)} = \sum_{m=-l}^{l} \tilde{Y}_{l,m}(x)\tilde{Y}_{l,m}(y). \qquad \square
$$

We introduce the spherical Laplace operator, which is also known as the Laplace–Beltrami operator, as an operator on $C^\infty(S^2)$ in terms of our coordinates represented by $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi)$

$$
\Delta_{S^2} = \frac{1}{\sin(\theta)} \frac{\partial}{\partial\theta}(\sin(\theta)\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2(\theta)} \frac{\partial^2}{\partial\varphi^2}.
$$

Equation (3.51) in [21] together with the proof of Proposition 3.33 also in [21] yield that the spherical harmonics fulfill the following eigenvalue relation with the spherical Laplace operator

$$
\Delta_{S^2} Y_{l,m} = -l(l+1)Y_{l,m}.
$$

Since the introduced real spherical harmonics are in either case a linear combination of the spherical harmonics, they fulfill the same eigenvalue relation with the spherical Laplace operator

$$
\Delta_{S^2} \tilde{Y}_{l,m} = -l(l+1)\tilde{Y}_{l,m}.
$$

We introduce the spherical gradient ∇_{S^2}, which is also known as the Beltrami operator, on $C^\infty(S^2)$. In our coordinates it reads

$$
\nabla_{S^2} = \hat{\theta}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{\sin(\theta)}\frac{\partial}{\partial\varphi},
$$

where $\hat{\theta}$ and $\hat{\varphi}$ form an orthonormal basis of the tangent space at each point $x \in S^2$ represented by (θ, φ). In our coordinates they are given by

$$\hat{\theta} = \begin{pmatrix} \cos(\theta)\cos(\varphi) \\ \cos(\theta)\sin(\varphi) \\ -\sin(\theta) \end{pmatrix} \qquad \text{and} \qquad \hat{\varphi} = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix}$$

Note that the spherical divergence is also denoted by ∇_{S^2}. For a smooth vector field X, it reads in our coordinates

$$\nabla_{S^2} \cdot X = \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta}(\sin(\theta) X_\theta) + \frac{1}{\sin(\theta)} \frac{\partial}{\partial \varphi} X_\varphi,$$

where X_θ denotes the $\hat{\theta}$-component and X_φ denotes the $\hat{\varphi}$-component. This finishes the discussion about the real spherical harmonics.

3. Isotropic random fields on the sphere

In this chapter, we introduce the notion of spherical random fields and 2-weakly isotropic spherical random fields. We also establish results on spherical expansions. For our analysis, we introduce the probability space $(\Omega, \mathcal{A}, P)$. For a random variable X on $(\Omega, \mathcal{A}, P)$ the expectation of X will be denoted by $E[X]$, whenever this is well-defined.

Definition 3.1. *A set of real-valued random variables $\{T(x) : x \in S^2\}$ on $(\Omega, \mathcal{A}, P)$ is called spherical random field if*

$$T : \Omega \times S^2 \to \mathbb{R}$$

is measurable with respect to the product σ-algebra $\mathcal{A} \otimes \mathcal{B}(S^2)$, where $\mathcal{B}(S^2)$ denotes the Borel σ-algebra of S^2.

The index set of a spherical random field might be omitted whenever this is convenient and we will simply say spherical random field T without explicitly referring to the probability space $(\Omega, \mathcal{A}, P)$. In the case of a different index set than S^2 which is also a Borel set and such that the measurability property in the definition of spherical random fields is satisfied, we will say random field.

3.1. Expansions of spherical random fields in the spherical harmonics

The property of a random field to be n-weakly isotropic will connect the random variables, which form a spherical random field.

Definition 3.2. *For $n \in \mathbb{N}$, a spherical random field T is called n-weakly isotropic if the following two properties hold. First, for all positive $k \leq n$ and for all $x \in S^2$ it holds that $E[|T(x)|^k] < \infty$, and second, that for all $g \in SO(3)$ and for all $x_1, ..., x_k \in S^2$ it holds that*

$$E[T(gx_1) \cdots T(gx_k)] = E[T(x_1) \cdots T(x_k)].$$

We will consider the case $n = 2$. Our first aim is to prove one main result about 2-weakly isotropic spherical random fields, which was mentioned in the introduction. To prepare ourselves we need a few lemmas about 2-weakly isotropic spherical random fields.

Lemma 3.3. *A 2-weakly isotropic spherical random field T is an element of $L^2(S^2, \mathbb{R})$ P-a.s. and an element of $L^2_{\mathrm{d}\sigma \otimes P}(\Omega \times S^2, \mathbb{R})$, i.e.*

$$E\left[\int_{S^2} |T(x)|^2 \mathrm{d}\sigma(x)\right] < \infty.$$

Proof. Since T is 2-weakly isotropic, for a fixed $x_0 \in S^2$ the following mapping from $SO(3)$ to $\mathbb{R}$ is constant

$$g \mapsto E[|T(gx_0)|^2].$$

According to Lemma 2.1, for every y in S^2 we can find a g in $SO(3)$ such that $y = gx_0$. This means that the following map is also constant:

$$y \mapsto E[|T(y)|^2].$$

Now we can simply fix x_0 arbitrarily in S^2, and because $|T|$ is a non-negative function we can apply Tonelli's theorem and the property that S^2 has finite measure under $\mathrm{d}\sigma$:

$$E\left[\int_{S^2} |T(x)|^2 \mathrm{d}\sigma(x)\right] = \int_{S^2} E[|T(x)|^2]\mathrm{d}\sigma(x) = E[|T(x_0)|^2] \int_{S^2} \mathrm{d}\sigma(x) < \infty.$$

This shows the second statement of the lemma. To see the first statement, we look at the random variable $\int_{S^2} |T(x)|^2 \mathrm{d}\sigma(x)$. Assuming that this random variable is infinite with positive probability, would imply that the expectation is also infinite, since this random variable is non-negative. This is a contradiction to the second statement which we just proved before. □

Remark 3.4. *The previous lemma implies in particular that the following random variables are P-a.s. well-defined for integers $l \geq 0$ and $m = -l, ..., l$:*

$$a_{l,m} = \int_{S^2} T(x)\overline{Y_{l,m}(x)}\mathrm{d}\sigma(x).$$

Lemma 3.5. *For a 2-weakly isotropic spherical random field T the random variables $a_{l,m}$ are uncorrelated for all $l \geq 0$ and all $m = -l, ..., l$, i.e.*

$$E[a_{l,m}\overline{a_{l',m'}}] = C_l \delta_{l,l'} \delta_{m,m'},$$

where $(C_l : l \geq 0)$ is a sequence of positive numbers.

Proof. The main part of the following argument can be found in Chapter 5.2 of [21]. The key idea is to use that the Legendre polynomials $(P_0^l : l \geq 0)$ form an orthonormal basis of $L^2([-1,1], \mathbb{R})$.
Since $E[T(x)T(y)] = E[T(gx)T(gy)]$ for all $g \in SO(3)$, we have the motivation to define the real-valued function $\Gamma(\langle x, y\rangle) = E[T(x)T(y)]$. Γ depends only on the angle between x and y. This is determined by the inner product $\langle x, y\rangle$ which lies in $[-1,1]$. To see that Γ is well-defined as a function from $[-1,1]$ to $\mathbb{R}$, we take the points $x, y, z, w \in S^2$ and pair them such that $\langle x, y\rangle = \langle z, w\rangle$. We need to show that $\Gamma(\langle x, y\rangle) = \Gamma(\langle z, w\rangle)$. Lemma 2.1 says that there exists $g \in SO(3)$ such that $gx = z$ and $gy = w$. So we can manipulate $\Gamma(\langle x, y\rangle)$ using the 2-weakly isotropy property of T to obtain that

$$\Gamma(\langle x, y\rangle) = E[T(x)T(y)] = E[T(gx)T(gy)] = E[T(z)T(w)] = \Gamma(\langle z, w\rangle).$$

The next claim is that $\Gamma \in L^2([-1,1], \mathbb{R})$. We introduce a reparameterization h of the interval $[-1,1]$ as mapping from $[0,\pi]$ to $[-1,1]$. It is defined by $h(\theta) = \langle x, y(\theta)\rangle$, where $x = (0,0,-1)$ and $y(\theta) = (\sin(\theta), 0, \cos(\theta))$. Note that the derivative of h is $h' = \langle x, y(\theta)\rangle' =$

$\sin(\theta)$. We insert this reparameterization h into Γ and obtain with the Cauchy–Schwarz inequality

$$\begin{aligned}\int_{-1}^{1}\Gamma(t)^2\mathrm{d}t &= \int_0^\pi \Gamma(h(\theta))^2 h'(\theta)\mathrm{d}\theta = \int_0^\pi \Gamma(\langle x, y(\theta)\rangle)^2\sin(\theta)\mathrm{d}\theta \\ &= \int_0^\pi E[T(x)T(y(\theta))]^2\sin(\theta)\mathrm{d}\theta \le E[|T(x)|^2]\int_0^\pi E[|T(y(\theta))|^2]\sin(\theta)\mathrm{d}\theta. \quad (3.1)\end{aligned}$$

To exploit the 2-weakly isotropy, we introduce the following matrix which is in $SO(3)$ and has the real parameter φ

$$g(\varphi) = \begin{pmatrix}\cos(\varphi) & -\sin(\varphi) & 0\\ \sin(\varphi) & \cos(\varphi) & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Note that

$$g(\varphi)y(\theta) = \begin{pmatrix}\sin(\theta)\cos(\varphi)\\ \sin(\theta)\sin(\varphi)\\ \cos(\theta)\end{pmatrix}.$$

We insert $g(\varphi)$ into the second spherical random field which is inside the integral in the right hand side of Equation (3.1) and integrate over the parameter φ to obtain that

$$\begin{aligned}\int_{-1}^{1}\Gamma(t)^2\mathrm{d}t &\le \frac{1}{2\pi}E[|T(x)|^2]\int_0^{2\pi}\int_0^\pi E[|T(g(\varphi)y(\theta))|^2]\sin(\theta)\mathrm{d}\theta\,\mathrm{d}\varphi \\ &= \frac{1}{2\pi}E[|T(x)|^2]\int_{S^2}E[|T(y)|^2]\mathrm{d}\sigma(y) < \infty.\end{aligned}$$

The last quantity is finite due to the 2-weakly isotropy of T and Lemma 3.3.
Now it is save to apply the fact that the Legendre polynomials are an orthonormal basis of $L^2([-1,1],\mathbb{R})$. We can expand Γ in the usual way

$$\Gamma(\langle x, y\rangle) = \sum_{l\ge 0} a_l P_0^l(\langle x, y\rangle), \qquad \text{where} \qquad a_l = \int_{-1}^{1}\Gamma(t)P_0^l(t)\mathrm{d}t.$$

For positive integers l, we define $C_l = \frac{4\pi}{2l+1}a_l$ and rewrite the expansion of Γ in this way

$$\Gamma(\langle x, y\rangle) = \sum_{l\ge 0}\frac{2l+1}{4\pi}C_l P_0^l(\langle x, y\rangle).$$

This expansion is now in the right form to apply Lemma 2.8 and obtain that

$$\Gamma(\langle x, y\rangle) = \sum_{l\ge 0}\sum_{m=-l}^{l} C_l Y_{l,m}(x)\overline{Y_{l,m}(y)}.$$

Then we manipulate applying Fubini's theorem to change the order of sums and integrals:

$$E[a_{l,m}\overline{a_{l',m'}}] = E\left[\int_{S^2}T(x)\overline{Y_{l,m}(x)}\mathrm{d}\sigma(x)\int_{S^2}T(y)Y_{l',m'}(y)\mathrm{d}\sigma(y)\right]$$

$$
\begin{aligned}
&= \int_{S^2}\int_{S^2} E[T(x)T(y)]\overline{Y_{l,m}(x)}Y_{l',m'}(y)\mathrm{d}\sigma(x)\mathrm{d}\sigma(y)\\
&= \int_{S^2}\int_{S^2} \Gamma(\langle x,y\rangle)\overline{Y_{l,m}(x)}Y_{l',m'}(y)\mathrm{d}\sigma(x)\mathrm{d}\sigma(y)\\
&= \int_{S^2}\int_{S^2} \sum_{l_0\geq 0}\sum_{m_0=-l_0}^{l_0} C_{l_0}Y_{l_0,m_0}(x)\overline{Y_{l_0,m_0}(y)}\ \overline{Y_{l,m}(x)}Y_{l',m'}(y)\mathrm{d}\sigma(x)\mathrm{d}\sigma(y)\\
&= \sum_{l_0\geq 0}\sum_{m_0=-l_0}^{l_0} C_{l_0}\int_{S^2} Y_{l_0,m_0}(x)\overline{Y_{l,m}(x)}\mathrm{d}\sigma(x)\int_{S^2}\overline{Y_{l_0,m_0}(y)}Y_{l',m'}(y)\mathrm{d}\sigma(y)\\
&= \sum_{l_0\geq 0}\sum_{m_0=-l_0}^{l_0} C_{l_0}\delta_{l_0,l}\delta_{m_0,m}\delta_{l_0,l'}\delta_{m_0,m'} = C_l\delta_{l,l'}\delta_{m,m'}. \qquad \square
\end{aligned}
$$

Definition 3.6. *For a spherical random field T, the positive sequence $(C_l : l \geq 0)$, which we introduced in the previous lemma, is called the* angular power spectrum *of T.*

The previous lemma also shows that it is non-negative and well-defined through the following relation, for $l \geq 0$ and $m = -l, ..., l$

$$E[a_{l,m}\overline{a_{l',m'}}] = C_l\delta_{l,l'}\delta_{m,m'}.$$

Lemma 3.7. *For a 2-weakly isotropic spherical random field T, the following spherical random fields*

$$T^L = \sum_{l=0}^{L}\sum_{m=-l}^{l} a_{l,m}Y_{l,m} \qquad \textit{and} \qquad T - T^L$$

are also 2-weakly isotropic.

Proof. The second moment of both random fields exists by the triangle inequality and the Parseval identity. The first moment can be estimated with the second moment using Hölder's inequality. So the following is left to show, that for any x, y in S^2 and any g in $SO(3)$

$$
\begin{aligned}
E[T^L(x)T^L(y)] &= E[T^L(gx)T^L(gy)]\\
E[(T(x)-T^L(x))(T(y)-T^L(y))] &= E[(T(gx)-T^L(gx))(T(gy)-T^L(gy))].
\end{aligned}
$$

If we expand the second equation, we see, that to show the two equations above it suffices to show

$$E[T^L(x)T^L(y)] = E[T^L(gx)T^L(gy)], \tag{3.2}$$

$$E[T(x)T^L(y)] = E[(T(gx)T^L(gy)]. \tag{3.3}$$

We start with the proof of Equation (3.3) and expand it

$$E[T(x)T^L(y)] = \sum_{l=0}^{L} E[T(x)\sum_{m=-l}^{l} a_{l,m}Y_{l,m}(y)].$$

So it suffices to show for integers $l \geq 0$ that it holds that

$$E[T(x) \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(y)] = E[T(gx) \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(gy)].$$

In the proof of this claim, we apply our knowledge of the representations of $SO(3)$, which we introduced in the previous chapter. First we have to manipulate a little to put the terms into the right shape. We start with inserting the definition of the coefficients of the expansion in the spherical harmonics. In the second step we tacitly do an interchange of the integral and the expectation due to Fubini's theorem, to be able to apply the 2-weakly isotropy property of T as stated in Definition 3.2 namely to put in g. In the third step we do a coordinate change to obtain that

$$\begin{aligned}
E[T(x) \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(y)] &= \sum_{m=-l}^{l} E[T(x) \int_{S^2} T(z) \overline{Y_{l,m}(z)} \mathrm{d}\sigma(z)] Y_{l,m}(y) \\
&= \sum_{m=-l}^{l} E[T(gx) \int_{S^2} T(gz) \overline{Y_{l,m}(z)} \mathrm{d}\sigma(z)] Y_{l,m}(y) \\
&= \sum_{m=-l}^{l} E[T(gx) \int_{S^2} T(z) \overline{Y_{l,m}(g^{-1}z)} \mathrm{d}\sigma(z)] Y_{l,m}(y).
\end{aligned}$$

Now we can apply the announced result of Lemma 2.7. This is the first step of the second part of the calculation. We remember that, $D^l(g)$ is the matrix of a unitary finite dimensional representation, in particular the Hermitian is the inverse. This way, we can apply Lemma 2.7 again in a second step to shift the left action of g to the spherical harmonic outside the integral. The claim follows after putting in the definition of the coefficients,

$$\begin{aligned}
&\sum_{m=-l}^{l} E[T(gx) \int_{S^2} T(z) \overline{Y_{l,m}(g^{-1}z)} \mathrm{d}\sigma(z)] Y_{l,m}(y) \\
&\quad = \sum_{m=-l}^{l} \sum_{m_1=-l}^{l} E[T(gx) \int_{S^2} T(z) \overline{D^l_{m_1,m}(g)}\, \overline{Y_{l,m_1}(z)} \mathrm{d}\sigma(z)] Y_{l,m}(y) \\
&\quad = \sum_{m_1=-l}^{l} E[T(gx) \int_{S^2} T(z) \overline{Y_{l,m_1}(z)} \mathrm{d}\sigma(z)] \sum_{m=-l}^{l} D^l_{m,m_1}(g^{-1}) Y_{l,m}(y) \\
&\quad = \sum_{m_1=-l}^{l} E[T(gx) \int_{S^2} T(z) \overline{Y_{l,m_1}(z)} \mathrm{d}\sigma(z)] Y_{l,m_1}(gy) = E[T(gx) \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(gy)].
\end{aligned}$$

Equation (3.2) is proven similarly using Equation (3.3). We expand the left hand side of Equation (3.2) in the following way by directly inserting the definition of the coefficients and interchange the integral and the expectation. In the second step we insert Equation (3.3) to obtain

$$E[T^L(x) T^L(y)] = \sum_{l,l' \geq 0}^{L} \sum_{m=-l}^{l} \int_{S^2} E[T(z) \sum_{m'=-l'}^{l'} a_{l',m'} Y_{l',m'}(y)] \overline{Y_{l,m}(z)} \mathrm{d}\sigma(z) Y_{l,m}(x)$$

$$= \sum_{l,l'\geq 0}^{L} \sum_{m=-l}^{l} \int_{S^2} E[T(gz) \sum_{m'=-l'}^{l'} a_{l',m'} Y_{l',m'}(gy)] \overline{Y_{l,m}(z)} \mathrm{d}\sigma(z) Y_{l,m}(x)$$

$$= \sum_{l,l'\geq 0}^{L} \sum_{m=-l}^{l} \int_{S^2} E[a_{l,m} \sum_{m'=-l'}^{l'} a_{l',m'} Y_{l',m'}(gy)] Y_{l,m}(gx) = E[T^L(gx) T^L(gy)].$$

The last step of the above calculation is proven with a similar argument as we just did to prove Equation (3.3), which would also rely on Lemma 2.7. We omit the details. This finishes the proof of the lemma. □

Now we are in good shape to prove the first main result, which was already mentioned in the introduction. We formulate Theorem 1.1 precisely.

Theorem 3.8. *For a 2-weakly isotropic spherical random field T, it holds that the following spherical expansion*

$$T = \sum_{l\geq 0} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}, \qquad a_{l,m} = \int_{S^2} T(x) \overline{Y_{l,m}(x)} \mathrm{d}\sigma(x),$$

converges in $L^2_{P\otimes \mathrm{d}\sigma}(\Omega \times S^2, \mathbb{R})$ and in $L^2_P(\Omega, \mathbb{R})$, i.e.

$$\lim_{L\to\infty} E\left[\int_{S^2} \left| T(x) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(x) \right|^2 \mathrm{d}\sigma(x) \right] = 0 \tag{3.4}$$

and for all $x \in S^2$

$$\lim_{L\to\infty} E\left[\left| T(x) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(x) \right|^2 \right] = 0. \tag{3.5}$$

Proof. We already know from Remark 3.4, that the coefficients $a_{l,m}$ of T are P-a.s. well-defined complex random variables and Lemma 3.3 implies that the sequence of coefficients converges P-a.s. Hence the expansion of T is well-defined and converges P-a.s. in $L^2(S^2, \mathbb{R})$ due to Theorem 2.3, i.e.

$$\lim_{L\to\infty} \int_{S^2} \left| T(x) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(x) \right|^2 \mathrm{d}\sigma(x) = 0 \quad P\text{-a.s.} \tag{3.6}$$

So it is left to show the convergence in $L^2_{P\otimes \mathrm{d}\sigma}(\Omega \times S^2, \mathbb{R})$ and in $L^2_P(\Omega, \mathbb{R})$. Let $\Omega^* \subset \Omega$ be of full probability such that $T(\omega) \in L^2(S^2, \mathbb{R})$ for all $\omega \in \Omega^*$.The orthogonality of the spherical harmonics gives a dominating function using the Parseval identity twice. Explicitly for any $\omega \in \Omega^*$, it holds that

$$\int_{S^2} \left| T(\omega, x) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m}(\omega) Y_{l,m}(x) \right|^2 \mathrm{d}\sigma(x) = \int_{S^2} \left| \sum_{l>L} \sum_{m=-l}^{l} a_{l,m}(\omega) Y_{l,m}(x) \right|^2 \mathrm{d}\sigma(x)$$

$$= \sum_{l>L} \sum_{m=-l}^{l} |a_{l,m}(\omega)|^2 \leq \sum_{l\geq 0} \sum_{m=-l}^{l} |a_{l,m}(\omega)|^2 = \int_{S^2} |T(\omega, x)|^2 \mathrm{d}\sigma(x).$$

To make the setup explicit, where we want to apply the dominated convergence theorem, we define the sequence of functions, $\varphi_L = \|T - T^L\|^2_{L^2(S^2,\mathbb{R})}$ and $h = \|T\|^2_{L^2(S^2,\mathbb{R})}$. These functions are integrable mappings from Ω to $\mathbb{R}$ due to Lemma 3.3 and Lemma 3.7. Note that we simply set them to zero on the complement of Ω^*. We know from Equation (3.6), that φ_L converges to zero P-a.s. and we just showed that h is a dominating function of the sequence of functions φ_L. This means by the dominated convergence theorem we can interchange the following limit:

$$0 = E[\lim_{L\to\infty} \|T - T^L\|^2_{L^2(S^2,\mathbb{R})}] = E[\lim_{L\to\infty} \varphi_L] = \lim_{L\to\infty} E[\varphi_L] = \lim_{L\to\infty} E[\|T - T^L\|^2_{L^2(S^2,\mathbb{R})}].$$

Thus we proved the first assertion of the theorem.

To prove the second assertion we apply Lemma 3.7 and Definition 3.2 to see that for any $x \in S^2$ and for any $g \in SO(3)$

$$g \mapsto E[|T(gx) - T^L(gx)|^2] = E[|T(gx) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(gx)|^2]$$

is constant. We fix x_0 arbitrarily in S^2. Because of Lemma 2.1 for any $y \in S^2$ there is $g \in SO(3)$ such that $gy = x_0$. Hence, for all $x \in S^2$ the mapping

$$x \mapsto E[|T(x) - T^L(x)|^2] = E[|T(x) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(x)|^2]$$

is also constant. So we are in the pleasant position to integrate over the expectation and to change the order of the integrals applying Tonelli's theorem:

$$\begin{aligned} E[|T(x) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(x)|^2] &= \frac{1}{4\pi} \int_{S^2} E[|T(x) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(x)|^2] \mathrm{d}\sigma(x) \\ &= \frac{1}{4\pi} E\left[\int_{S^2} \left| T(x) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{l,m} Y_{l,m}(x) \right|^2 \mathrm{d}\sigma(x) \right]. \end{aligned}$$

The last expression converges to zero because Equation (3.4) has already been proven. This finishes the proof of Equation (3.5). □

3.2. An example given in the real spherical harmonics

Throughout our preceding analysis we were always working with expansions in the spherical harmonics. In applications it will be relevant to analyze spherical random fields, which are given in the real spherical harmonics. We would like to establish conditions on a spherical random field such that it fulfills the 2-weakly isotropic property of Definition 3.2.
For a 2-weakly isotropic spherical random field T, we have shown in Lemma 3.5, that the coefficients are uncorrelated. So it seems natural that the corresponding real coefficients are also uncorrelated, which will be proven in the next lemma. Complex and real coefficients are related through Definition 2.4 and Lemma 2.6, the latter are also introduced in Definition 2.4.

Lemma 3.9. *For a 2-weakly isotropic spherical random field T the corresponding real coefficients $\tilde{a}_{l,m}$ for integers $l \geq 0$ and $m = -l, ..., l$ are uncorrelated. Moreover for integers $l, l' \geq 0,\ m = -l, ..., l$ and $m' = -l', ..., l'$ it holds that*

$$E[\tilde{a}_{l,m}\tilde{a}_{l',m'}] = C_l \delta_{l,l'} \delta_{m,m'}. \tag{3.7}$$

The sequence $(C_l : l \geq 0)$ is the angular power spectrum of T.

Proof. From Lemma 3.5 we already know that the complex coefficients are uncorrelated in the sense, that for any $l, l' \geq 0,\ m = -l, ..., l$ and $m' = -l', ..., l'$, it holds that

$$E[a_{l,m}\overline{a_{l',m'}}] = C_l \delta_{l,l'} \delta_{m,m'}. \tag{3.8}$$

This means we can set $l = l'$, because in Definition 2.4 the index l of the real coefficients is the same as the index for complex coefficients and Equation (3.8) yields that Equation (3.7) vanishes for positive integers $l \neq l'$. We can now fix the index l and calculate the different cases for integers $m, m' = -l, ..., l$.
We start with the case $m, m' > 0$. We insert Definition 2.4 and manipulate with Equation (2.4) to be able to apply Equation (3.8) to conclude the claim in Equation (3.7):

$$\begin{aligned}
E[\tilde{a}_{l,m}\tilde{a}_{l,m'}] &= \frac{1}{2}E[(a_{l,m} + \overline{a_{l,m}})(a_{l,m'} + \overline{a_{l,m'}})] \\
&= \frac{1}{2}E[a_{l,m}a_{l,m'} + a_{l,m}\overline{a_{l,m'}} + \overline{a_{l,m}}a_{l,m'} + \overline{a_{l,m}}\ \overline{a_{l,m'}}] \\
&= \frac{1}{2}((-1)^{m'}E[a_{l,m}\overline{a_{l,-m'}} + \overline{a_{l,m}}a_{l,-m'}] + 2C_l\delta_{m,m'}) = C_l\delta_{m,m'}.
\end{aligned}$$

The case $m, m' < 0$ is done in a similar way as well as the case $m > 0, m' < 0$. For the latter we write down the computation. It uses the same tools as the first case, which we computed in detail. We obtain that

$$\begin{aligned}
E[\tilde{a}_{l,m}\tilde{a}_{l,m'}] &= \frac{1}{2i}E[(a_{l,m} + \overline{a_{l,m}})(a_{l,m'} - \overline{a_{l,m'}})] \\
&= \frac{1}{2i}E[a_{l,m}a_{l,m'} - a_{l,m}\overline{a_{l,m'}} + \overline{a_{l,m}}a_{l,m'} - \overline{a_{l,m}}\ \overline{a_{l,m'}}] \\
&= \frac{1}{2i}((-1)^m E[\overline{a_{l,-m}}a_{l,m'} - a_{l,-m}\overline{a_{l,m'}}] - C_l\delta_{m,m'} + C_l\delta_{m,m'}) \\
&= \frac{1}{2i}((-1)^m C_l(\delta_{-m,m'} - \delta_{-m,m'}) = 0.
\end{aligned}$$

□

In the case, that a real-valued sequence of possible coefficients is given, a similar conclusion about the correlation is possible. It is of interest in this case, that a spherical random field is given in an expansion in the real spherical harmonics.

Lemma 3.10. *For a sequence $(\tilde{\beta}_{l,m} : l \geq 0, m = -l, ..., l)$ of uncorrelated real-valued random variables, interpreted as real coefficients of an expansion in the real spherical harmonics, the corresponding complex-valued random variables $(\beta_{l,m} : l \geq 0, m = -l, ..., l)$ are also uncorrelated in the sense that, if*

$$E[\tilde{\beta}_{l,m}\tilde{\beta}_{l',m'}] = C_l \delta_{l,l'} \delta_{m,m'},$$

then

$$E[\beta_{l,m}\overline{\beta_{l',m'}}] = C_l \delta_{l,l'} \delta_{m,m'}.$$

Proof. The proof of this lemma is very similar. It uses the relation between the real and complex coefficients as introduced in Lemma 2.6. We omit the details. □

Now we give sufficient conditions for a spherical random field given in its expansion in the real spherical harmonics to be 2-weakly isotropic.

Proposition 3.11. *Let T be a spherical random field such that the first and second moment exist for all x in S^2. Let T have the expansion*

$$T = \sum_{l\geq 0} \sum_{m=-l}^{l} \tilde{\beta}_{l,m} \tilde{Y}_{l,m},$$

for a sequence of real-valued random variables $(\tilde{\beta}_{l,m} : l \geq 0, m = -l, ..., l)$. If for $l \geq 0$ and $m = -l, ..., l$ $\tilde{\beta}_{l,m}$ has mean zero and for all integers $l, l' \geq 0$, $m = -l, ..., l$ and $m' = -l', ..., l'$

$$E[\tilde{\beta}_{l,m}\tilde{\beta}_{l',m'}] = C_l \delta_{l,l'} \delta_{m,m'},$$

then T is 2-weakly isotropic.

Proof. Since the existence of the first two moments belongs to the conditions of the proposition, the invariance under the action of $SO(3)$ is left to show. The proof will be a calculation using Lemma 2.7 as a main tool. For arbitrary $g \in SO(3)$ and $x_1, x_2 \in S^2$ we look at the following expression and switch to the complex expansion. We can interchange the expectation and the infinite sums due to Fubini's theorem, which is applicable, since the second moment of $T(x)$ is finite for all $x \in S^2$, hence $|T(gx_1)T(gx_2)|$ has finite expectation. First we switch to the complex expansion using Lemma 2.6. In the second step we apply Lemma 3.10. This yields that the corresponding complex coefficients are also uncorrelated with the same angular power spectrum. In addition we apply Lemma 2.7 to separate the g and the spherical harmonics, that yields two extra finite sums over the unitary matrices $D^l(g^{-1})$. We obtain that

$$\begin{aligned}
E[T(gx_1)T(gx_2)] &= \sum_{l,l'\geq 0} \sum_{m=-l}^{l} \sum_{m'=-l'}^{l'} E[\beta_{l,m}\overline{\beta_{l',m'}}] Y_{l,m}(gx_1)\overline{Y_{l',m'}(gx_2)} \\
&= \sum_{l\geq 0} \sum_{m=-l}^{l} C_l \Big(\sum_{m_1=-l}^{l} D^l_{m_1,m}(g^{-1}) Y_{l,m_1}(x_1)\Big)\Big(\sum_{m_2=-l}^{l} \overline{D^l_{m_2,m}}(g^{-1})\overline{Y_{l,m_2}(x_2)}\Big) \\
&= \sum_{l\geq 0} C_l \sum_{m_1=-l}^{l} \sum_{m_2=-l}^{l} \Big(\sum_{m=-l}^{l} D^l_{m_1,m}(g^{-1})\overline{D^l_{m_2,m}(g^{-1})}\Big) Y_{l,m_1}(x_1)\overline{Y_{l,m_2}(x_2)}.
\end{aligned}$$

We apply the fact that the matrices $D^l(g^{-1})$ are unitary, which holds due to Lemma 2.7, i.e., for all integers $l \geq 0, m_1, m_2 = -l, ..., l$ and $g \in SO(3)$ it holds that

$$\sum_{m'=-l}^{l} D^l_{m_1,m}(g)\overline{D^l_{m_2,m}(g)} = \delta_{m_1,m_2}.$$

This relation of the matrices $D^l(g^{-1})$ and the evaluation of the Kronecker delta will eliminate the g-dependence and afterwards we perform all manipulations backwards to obtain that

$$\sum_{l\geq 0} C_l \sum_{m_1=-l}^{l} \sum_{m_2=-l}^{l} \Big(\sum_{m=-l}^{l} D^l_{m_1,m}(g^{-1})\overline{D^l_{m_2,m}(g^{-1})}\Big) Y_{l,m_1}(x_1)\overline{Y_{l,m_2}(x_2)}$$

$$= \sum_{l\geq 0} C_l \sum_{m_1=-l}^{l} \sum_{m_2=-l}^{l} \delta_{m_1,m_2} Y_{l,m_1}(x_1)\overline{Y_{l,m_2}(x_2)}$$
$$= \sum_{l\geq 0} C_l \sum_{m=-l}^{l} Y_{l,m}(x_1)\overline{Y_{l,m}(x_2)} = \cdots = E[T(x_1)T(x_2)].$$

It remains to check the invariance of the expectation under the action of an arbitrary $g \in SO(3)$ for arbitrary $x \in S^2$. Since the first moment of $T(x)$ exists by assumption, we can apply Fubini's theorem to interchange expectation and the infinite sum similar to the argument before. The assumption, that all coefficients but $\beta_{0,0}$ have mean zero, will give the claim with the fact that $Y_{0,0}$ is constant:

$$E[T(gx)] = \sum_{l\geq 0} \sum_{m=-l}^{l} E[\tilde{\beta}_{l,m}]\tilde{Y}_{l,m}(gx) = E[\tilde{\beta}_{0,0}]\tilde{Y}_{0,0}(gx) = E[\tilde{\beta}_{0,0}]\tilde{Y}_{0,0}(x) = E[T(x)]. \quad \square$$

In the following we want to discuss an example, which will lead us to the next chapter. We define the spherical random field

$$T = \sum_{l\geq 0} \sum_{m=-l}^{l} \sqrt{C_l}\tilde{\beta}_{l,m}\tilde{Y}_{l,m} \tag{3.9}$$

where $(\tilde{\beta}_{l,m} : l \geq 0, m = -l, ..., l)$ is an i.i.d. sequence of standard normally distributed random variables and $(C_l : l \geq 0)$ is a positive sequence, such that $(C_l l : l \geq 0)$ is summable.

Proposition 3.12. *The spherical random field T defined in Equation* (3.9) *is 2-weakly isotropic.*

Proof. We check that the first two moments are finite first. In the first step, with Hölder's inequality we can bound the first moment by the second moment and switch to the complex expansion in the spherical harmonics using Lemma 2.6. In the second step we apply Fubini's theorem to interchange the expectation and the infinite sum. Due to Lemma 3.10 the corresponding complex coefficients are also uncorrelated. So we obtain that

$$E[|T(x)|] \leq E[|T(x)|^2] = E[|\sum_{l\geq 0} \sqrt{C_l} \sum_{m=-l}^{l} \tilde{\beta}_{l,m}\tilde{Y}_{l,m}(x)|^2] = E[|\sum_{l\geq 0} \sqrt{C_l} \sum_{m=-l}^{l} \beta_{l,m}Y_{l,m}(x)|^2]$$
$$= \sum_{l,l'\geq 0} \sqrt{C_l}\sqrt{C_{l'}} \sum_{m,m'=-l,l'}^{l,l'} E[\beta_{l,m}\overline{\beta_{l',m'}}]Y_{l,m}(x)\overline{Y_{l',m'}(x)} = \sum_{l\geq 0} C_l \sum_{m=-l}^{l} Y_{l,m}(x)\overline{Y_{l,m}(x)}.$$

In the last step we apply Lemma 2.8 and see that this infinite sum has a finite value by our assumption on $(C_l : l \geq 0)$. So we obtain that

$$E[|T(x)|] \leq E[|T(x)|^2] = \sum_{l\geq 0} C_l \frac{2l+1}{4\pi} < \infty.$$

We just proved that the condition of Proposition 3.11 are satisfied, this proposition gives the claim, that T is 2-weakly isotropic. $\square$

3.3. Truncation error estimation

Another topic of interest is the error, which we make if we truncate the expansion of a 2-weakly isotropic spherical random field T. This is interesting, when we wish to implement an approximate realization of T. We will do this analysis in the $L^2_{P\otimes \mathrm{d}\sigma}(\Omega \times S^2, \mathbb{R})$-norm and in the $L^2_P(\Omega, \mathbb{R})$-norm. We study the truncation error for a 2-weakly isotropic spherical random field T. The truncation of T is denoted by T^L. We assume for the angular power spectrum, that $C_l \simeq l^{-\alpha}$, i.e. that there exist constants $c, C > 0$ and some $\alpha > 0$ such that $c\, l^{-\alpha} \leq C_l \leq C\, l^{-\alpha}$ for all but finitely many integers $l \geq 0$. The proof of Proposition 3.9 implies that $\alpha > 2$, otherwise the second moment of T would not be finite, which exists due to the definition of a 2-weakly isotropic spherical random field.

Proposition 3.13. *Let T be a 2-weakly isotropic spherical random field such that for the angular power spectrum holds $C_l \simeq l^{-\alpha}$ for $\alpha > 2$. This condition implies, that the truncation error of T converges in the $L^2_{P\otimes \mathrm{d}\sigma}(\Omega \times S^2, \mathbb{R})$-norm and the truncation error of $T(x)$ converges in the $L^2_P(\Omega, \mathbb{R})$-norm for all $x \in S^2$ with order $\frac{\alpha-2}{2}$ in terms of L^{-1} in both cases, i.e.*

$$\|T - T^L\|^2_{L^2_{P\otimes \mathrm{d}\sigma}(\Omega\times S^2,\mathbb{R})} = \mathcal{O}\left((L^{-1})^{\alpha-2}\right) \tag{3.10}$$

and

$$\|T(x) - T^L(x)\|^2_{L^2_P(\Omega,\mathbb{R})} = \mathcal{O}\left((L^{-1})^{\alpha-2}\right). \tag{3.11}$$

Proof. We fix $L \in \mathbb{N}$ and start computing the error. In the first step, we apply Theorem 3.8 and the Parseval identity. In the second step we insert the assumption on the angular power spectrum. We obtain that

$$\begin{aligned}\|T - T^L\|^2_{L^2_{P\otimes \mathrm{d}\sigma}(\Omega\times S^2,\mathbb{R})} &= E\left[\left\|\sum_{l>L}\sum_{m=-l,\dots,l}^{l} \tilde{a}_{l,m}\tilde{Y}_{l,m}\right\|^2_{L^2(S^2,\mathbb{R})}\right] = E\left[\sum_{l>L}\sum_{m=-l}^{l}\tilde{a}^2_{l,m}\right] \\ &= \sum_{l>L}\sum_{m=-l}^{l} E[\tilde{a}^2_{l,m}] = \sum_{l>L} C_l(2l+1) \leq C\sum_{l>0}\left(2(l+L)^{-\alpha+1} + (l+L)^{-\alpha}\right).\end{aligned}$$

The last expression is bounded with a simple calculation:

$$\begin{aligned}C\sum_{l>0}\left(2(l+L)^{-\alpha+1} + (l+L)^{-\alpha}\right) &\leq C\int_0^\infty 2(x+L)^{-\alpha+1} + (x+L)^{-\alpha}\mathrm{d}x \\ &= C\left(\frac{2}{\alpha-2}(L^{-1})^{\alpha-2} + \frac{1}{\alpha-1}(L^{-1})^{\alpha-1}\right) = \mathcal{O}\left((L^{-1})^{\alpha-2}\right).\end{aligned} \tag{3.12}$$

This gives the desired convergence rate in terms of L^{-1} and proves the first claim. For the second claim we again fix $L \in \mathbb{N}$ and compute for an arbitrary $x \in S^2$ the error in the $L^2_P(\Omega, \mathbb{R})$-norm. In the first step we apply Theorem 3.8 and the Parseval identity. In the second step we interchange the expectation and the infinite sum and apply Lemma 3.5. Then we apply Lemma 2.8 to lose the spherical harmonics and the dependence on x to obtain that

$$\|T(x) - T^L(x)\|^2_{L^2_P(\Omega,\mathbb{R})} = E[|T(x) - T^L(x)|^2] = E\left[\left|\sum_{l>L} a_{l,m}Y_{l,m}(x)\right|^2\right]$$

$$= \sum_{l>L} C_l \sum_{m=-l}^{l} Y_{l,m}(x)\overline{Y_{l,m}(x)} = \sum_{l>L} C_l \frac{2l+1}{4\pi}.$$

Now we can insert the assumption on the angular power spectrum and the same calculation, which we just did in Equation (3.12), yields the second claim of the proposition

$$\sum_{l>L} C_l \frac{2l+1}{4\pi} \leq \frac{C}{4\pi} \sum_{l<L} l^{-\alpha}(2l+1) = \mathcal{O}\left((L^{-1})^{\alpha-2}\right). \qquad \square$$

To finish the chapter we show two plots of realizations of the example Equation (3.9), which is a 2-weakly isotropic spherical random field T. The realizations are approximated with truncations T^L for $L = 70$. The MATLAB code of the implementation can be found in Appendix A.

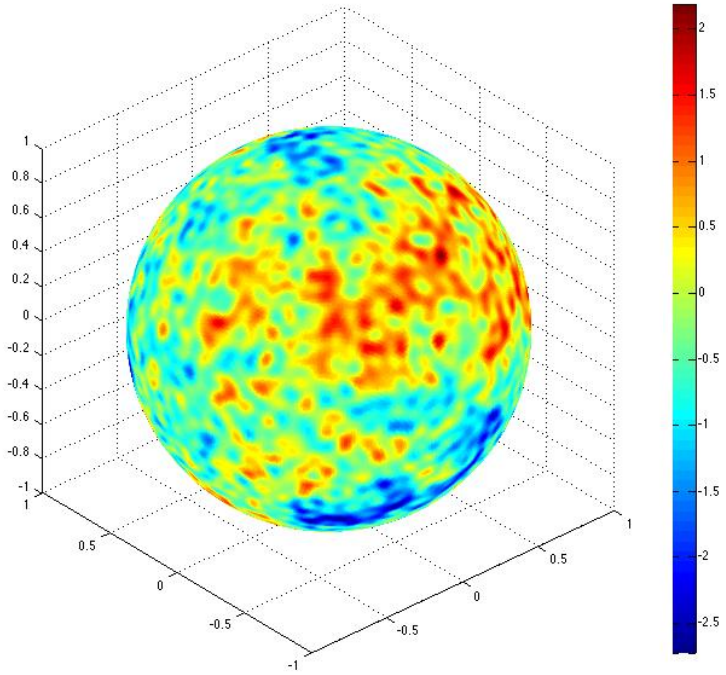

Figure 3.1.: realization of T^L with $C_l = (1+l)^{-2.1}$

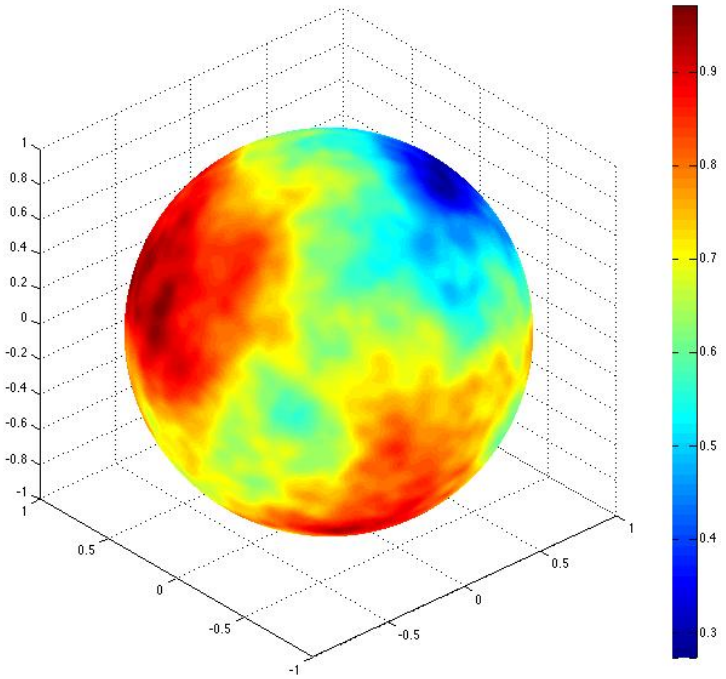

Figure 3.2.: realization of T^L with $C_l = (1+l)^{-4.1}$

4. Stochastic heat equation with isotropic Q-Wiener noise

In this chapter, we interpret the 2-weakly isotropic spherical random field T as a $L^2(S^2, \mathbb{R})$-valued random variable. In particular we choose T to be Gaussian with mean zero and covariance operator Q, which will later form the Q-Wiener process W. Having this we are able to formulate the stochastic heat equation on the sphere with additive 2-weakly isotropic Q-Wiener noise W. For our analysis we introduce the filtered probability space $(\Omega, \mathcal{F}, \mathcal{A}, P)$ such that the filtration $\mathcal{F}$ is complete and right-continuous.

4.1. Hilbert space valued Gaussian random variables

We review some definitions and statements which we will apply here. The definitions are taken from [8]. We define $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ where $\mathcal{H} = L^2(S^2, \mathbb{R})$ and $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is the standard inner product induced by the the Lebesgue measure on the sphere $\mathrm{d}\sigma$.

Definition 4.1. *An $\mathcal{H}$-values random variable T is called Gaussian, if $\langle \phi, T \rangle_{\mathcal{H}}$ is normally distributed with mean m_ϕ and variance σ_ϕ^2 for all $\phi \in \mathcal{H}$.*

This definition implies that

$$E[\langle \phi, T \rangle_{\mathcal{H}}] = \langle \phi, m \rangle_{\mathcal{H}}$$

for some $m \in \mathcal{H}$ and for the same m and any $\phi_1, \phi_2 \in \mathcal{H}$

$$E[\langle \phi_1, T \rangle_{\mathcal{H}} \langle \phi_2, T \rangle_{\mathcal{H}}] - \langle \phi_1, m \rangle_{\mathcal{H}} \langle \phi_2, m \rangle_{\mathcal{H}} = \langle Q\phi_1, \phi_2 \rangle_{\mathcal{H}}$$

defines the symmetric, bounded, positive operator Q.

Definition 4.2. *For a random variable T, m is referred to as the* mean *and Q is called the* covariance operator.

This means for an $\mathcal{H}$-valued Gaussian random variable T, it makes sense to introduce the notation that T is $\mathcal{N}(m, Q)$ distributed.

Proposition 4.3. *For an $\mathcal{H}$-valued Gaussian random variable, the covariance operator Q is of trace class.*

Proof. This is Proposition 2.15 in [8]. □

This means that Q is a trace class, symmetric, positive, bounded operator and therefore the following diagonalization result holds.

Proposition 4.4. *There is an orthonormal basis $(e_i : i \in \mathbb{N})$ and a positive decreasing sequence $(\lambda_i : i \in \mathbb{N})$ such that $\lambda_i \to 0$ and $Qe_i = \lambda_i e_i$ for all $i \in \mathbb{N}$.*

Proof. The conditions imply the conditions in the standard diagonalization theorem for self-adjoint compact operators. □

Proposition 4.5. *(Karhunen–Loève expansion) An $\mathcal{H}$-valued random variable T with mean m and trace class, symmetric, positive, bounded covariance operator Q which is diagonalized with orthonormal eigenbasis $(e_i : i \in \mathbb{N})$ and decreasing positive eigenvalues $(\lambda_i : i \in \mathbb{N})$ is Gaussian if and only if it has the expansion*

$$T = m + \sum_{i \in \mathbb{N}} \sqrt{\lambda_i} \beta_i e_i.$$

The infinite sum converges in $L^2(\Omega, \mathcal{H})$ and $(\beta_i : i \in \mathbb{N})$ is an i.i.d. sequence of standard normally distributed random variables.

Proof. For given mean m and covariance operator Q, the existence of a Gaussian random variable with such an expansion follows by Proposition 2.18 in [8]. We omit the proof of the other direction since the statement will only be applied in one direction. □

The so called Karhunen–Loève expansion will be the right tool to combine the notion of 2-weakly isotropic spherical random fields with the notion of $\mathcal{H}$-valued Gaussian random variables. We already established in Theorem 3.8 and Lemma 3.9 that a 2-weakly isotropic spherical random field has the expansion

$$T = \sum_{l \geq 0} \sqrt{C_l} \sum_{m=-l}^{l} \tilde{b}_{l,m} \tilde{Y}_{l,m}, \tag{4.1}$$

where we introduce $\tilde{b}_{l,m} = \tilde{a}_{l,m} \sqrt{C_l}^{-1}$ for all $l \geq 0$ and $m = -l, ..., l$ with $\tilde{a}_{l,m}$ the usual real coefficients appearing in Lemma 3.9. The random variables of the sequence $(\tilde{b}_{l,m} : l \geq 0, m = -l, ..., l)$ are uncorrelated such that $E[\tilde{b}_{l,m} \tilde{b}_{l',m'}] = \delta_{ll'} \delta_{mm'}$ for integers $l, l' \geq 0$, $m = -l, ..., l$ and $m' = -l', ..., l'$ and $(C_l : l \geq 0)$ is the angular power spectrum introduced in Remark 3.6. Similar to the argument in the proof of Proposition 3.9 we see that $(C_l l : l \geq 0)$ is summable. Theorem 3.8 also ensures that T can be interpreted as an $\mathcal{H}$-valued random variable. Now we wish that T has the extra property to be Gaussian distributed. Our definition of Gaussian distribution and the expansion of T imply that $(\tilde{a}_{l,m} : l \geq 0, m = -l, ..., l)$ has to be a sequence of standard normally distributed random variables. The fact that this sequence also has to be uncorrelated implies that it is i.i.d. The mean of T is zero and the covariance operator is given by

$$Q\tilde{Y}_{l,m} = C_l \tilde{Y}_{l,m} \qquad \text{for } l \geq 0, m = -l, ..., l.$$

Note that Q is a well-defined bounded linear operator on $\mathcal{H}$ since $(\tilde{Y}_{l,m} : l \geq 0, m = -l, ..., l)$ is an orthonormal basis of $\mathcal{H}$. It is also easy to see that Q is of trace class because $(C_l l : l \geq 0)$ is summable. The other properties of Q are also easily verified. Proposition 4.5 says that T is Gaussian with mean zero and covariance operator Q. Thus we obtained the expansion of a 2-weakly isotropic Gaussian spherical random field. Note that our notion of Gaussian distribution differs from Gaussian distribution in [21] which is defined pointwise on S^2. The following lemma will give a statement about the distribution of a 2-weakly isotropic Gaussian random field evaluated at some $x \in S^2$.

Lemma 4.6. *For a 2-weakly isotropic Gaussian spherical random field T with angular power spectrum $(C_l : l \geq 0)$, the random variable $T(x)$ is normally distributed, i.e. $T(x) \sim \mathcal{N}(0, \sigma_T^2)$ for all $x \in S^2$. The variance σ_T^2 of $T(x)$ is independent of $x \in S^2$ and is given by*

$$\sigma_T^2 = \sum_{l \geq 0} C_l \frac{2l+1}{4\pi}.$$

Proof. We fix $x \in S^2$. The strategy in this proof is to interpret $T(x)$ as a limit of normally distributed random variables and to argue via Lévy's continuity theorem and the characteristic functions that the limit is also normally distributed.
We look at the truncated spherical random field

$$T^L(x) = \sum_{l=0}^{L} \sum_{m=-l}^{l} \sqrt{C_l} \beta_{l,m} \tilde{Y}_{l,m}(x).$$

Since $T^L(x)$ is a finite sum of independent normally distributed random variables, $T^L(x)$ is also normally distributed. The fact that $(\beta_{l,m} : l \geq 0, m = -l, ..., l)$ is an i.i.d. sequence of standard normally distributed random variables and Lemma 2.8 imply that $T^L(x) \sim \mathcal{N}(0, \sigma_{T^L}^2)$, where the variance is given by

$$\sigma_{T^L}^2 = \sum_{l=0}^{L} C_l \frac{2l+1}{4\pi}.$$

Note that the variance $\sigma_{T^L}^2$ of $T^L(x)$ is already independent of $x \in S^2$. For a normally distributed random variable $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ its characteristic function is given by

$$E\left[e^{iX\lambda}\right] = \exp\left(i\lambda\mu_X - \frac{\sigma_X^2 \lambda^2}{2}\right).$$

Therefore the characteristic function $\phi_{T^L(x)}$ of $T^L(x)$ is given by

$$\phi_{T^L(x)}(\lambda) = E\left[e^{iT^L(x)\lambda}\right] = \exp\left(-\frac{\sigma_{T^L}^2 \lambda^2}{2}\right).$$

Now we are interested in the limit $(L \to \infty)$ of the function $\phi_{T^L(x)}$. Since the exponential function is continuous we can bring the limit into the exponential function. Furthermore we know that the sequence $(C_l l : l \geq 0)$ is summable due to an argument in Proposition 3.9, which implies that $\lim_{L\to\infty} \sigma_{T^L}^2 = \sigma_T^2$ exists and defines σ_T^2. Therefore for all $\lambda \in \mathbb{R}$ the limit $(L \to \infty)$ of $\phi_{T^L(x)}(\lambda)$ exists and is given by

$$\lim_{L\to\infty} \phi_{T^L(x)}(\lambda) = \exp\left(-\frac{\lim_{L\to\infty} \sigma_{T^L}^2 \lambda^2}{2}\right) = \exp\left(-\frac{\sigma_T^2 \lambda^2}{2}\right).$$

We set $\phi(\lambda) = \exp(-\frac{1}{2}\sigma_T^2 \lambda^2)$. So the sequence of characteristic functions $(\phi_{T^L(x)} : L \geq 0)$ converges pointwise for all $\lambda \in \mathbb{R}$ to a function ϕ, which is continuous. From Theorem 3.8 we now that $T^L(x)$ converges to $T(x)$ in $L_P^2(\Omega, \mathbb{R})$ for $L \to \infty$, so $T^L(x)$ converges also in distribution to $T(x)$. Now all the conditions are satisfied to apply Lévy's continuity theorem, which says that ϕ is the characteristic function of $T(x)$. This is the characteristic

function of normally distributed random variable with mean zero and variance σ_T^2. Since the characteristic function determines the distribution of its random variable, it holds that $T(x)$ is normally distributed, i.e. $T(x) \sim \mathcal{N}(0, \sigma_T^2)$. The implication of Lévy's continuity theorem which we used is proven in [13] as Theorem 15 in Chapter 14. □

The knowledge about the pointwise distribution of a 2-weakly isotropic Gaussian spherical random field T, i.e. the distribution of $T(x)$ for some $x \in S^2$, gives raise to analyze the distribution of random vectors $(T(x_1), T(x_2))$ for some $x_1, x_2 \in S^2$ with the help of the property that T is 2-weakly isotropic and Gaussian.

Lemma 4.7. *Let T be a 2-weakly isotropic Gaussian spherical random field, then for all $x_1, x_2 \in S^2$ and for all $g \in SO(3)$, it holds that $(T(x_1), T(x_2))$ and $(T(gx_1), T(gx_2))$ have the same law, i.e.*

$$(T(x_1), T(x_2)) \overset{law}{=} (T(gx_1), T(gx_2)).$$

Proof. From Lemma 4.6 we know that $T(x_1)$ is normally distributed with mean zero. Therefore the random vector $(T(x_1), T(x_2))$ is multivariate normally distributed with mean $0 \in \mathbb{R}^2$ and covariance matrix $\Sigma \in \mathbb{R}^{2\times 2}$. In the same way the random vector $(T(gx_1), T(gx_2))$ is multivariate normally distributed with mean $0 \in \mathbb{R}^2$ and covariance matrix $\Sigma_g \in \mathbb{R}^{2\times 2}$. The 2-weakly isotropy of T implies that $\Sigma = \Sigma_g$. This implies the claim of the lemma, since the distribution of a multivariate normally distributed random vector is determined by its mean and covariance matrix. □

Remark 4.8. *For a 2-weakly isotropic Gaussian spherical random field T and $x_1, x_2 \in S^2$, the random vector $(T(x_1), T(x_2))$ is multivariate normally distributed, i.e. $(T(x_1), T(x_2)) \sim \mathcal{N}(0, \Sigma)$, where the covariance matrix Σ is given by*

$$\Sigma = \begin{pmatrix} \sum_{l\geq 0} C_l \frac{2l+1}{4\pi} & \sum_{l\geq 0} C_l P_0^l(\langle x_1, x_2\rangle) \frac{2l+1}{4\pi} \\ \sum_{l\geq 0} C_l P_0^l(\langle x_1, x_2\rangle) \frac{2l+1}{4\pi} & \sum_{l\geq 0} C_l \frac{2l+1}{4\pi} \end{pmatrix}.$$

Proof. The proof of Lemma 4.7 says that the random vector is multivariate normally distributed. The covariance matrix can be calculated in the same way as we calculated the variance of $T(x)$ for $x \in S^2$ with Lemma 2.8 in the proof of Lemma 4.6. □

4.2. 2-weakly isotropic Q-Wiener process

The main aim of this chapter is to discuss the heat equation with additive Q-Wiener noise. Therefore we recall the Q-Wiener process.

Definition 4.9. *For a trace class, symmetric, positive, bounded operator Q, an $\mathcal{H}$-valued $\mathcal{F}$-adapted process $(W(t) : t \in [0, T])$ is called Q-Wiener process if*

(i) $W(0) = 0 \quad P$*-a.s.,*
(ii) W *has P-a.s. continuous trajectories,*
(iii) W *has independent increments,*
(iv) $W(t) - W(s) \sim \mathcal{N}(0, (t-s)Q) \quad$ *for* $0 \leq s < t \leq T$.

Remark 4.10. *According to Proposition 4.1 (ii) in [8] the Q-Wiener process W has the following expansion*

$$W(t) = \sum_{i\in\mathbb{N}} \sqrt{\lambda_i}\beta_i(t)e_i \qquad \text{for all } t \in [0, T],$$

where $(\beta_i : i \in \mathbb{N})$ is a sequence of mutually independent real-valued Brownian motions. The sequence of real numbers $(\lambda_i : i \in \mathbb{N})$ consists of the eigenvalues of the covariance operator Q with orthonormal eigenbasis $(e_i : i \in \mathbb{N})$ such that $Qe_i = \lambda_i e_i$ for all $i \in \mathbb{N}$.

For the covariance operator Q of a 2-weakly isotropic Gaussian spherical random field as seen in the previous section, the Q-Wiener process will have the following decomposition for mutually independent Brownian motions $(\beta_{l,m} : l \geq 0, m = -l, ..., l)$

$$W(t) = \sum_{l \geq 0} \sqrt{C_l} \sum_{m=-l}^{l} \beta_{l,m}(t) \tilde{Y}_{l,m} \qquad \text{for all } t \in [0, T].$$

Note that for fixed $t \in [0, T]$ the spherical random field $W(t)$ is Gaussian and 2-weakly isotropic.

4.3. Stochastic heat equation with additive 2-weakly isotropic Q-Wiener noise

So now we are able to formally state the stochastic heat equation, Equation (1.1) which was already mentioned in the introduction. This is of main interest in this chapter and is given by

$$X(t) = X(0) + \int_0^t \Delta_{S^2} X(s) \mathrm{d}s + W(t),$$
$$X(0) \in \mathcal{H} \qquad P\text{-a.s.}$$

Definition 4.11. *A predictable stochastic process $(X(t), t \in [0, T])$ is called a weak solution of Equation (1.1) if its trajectories are P-a.s. Bochner integrable and for all $\phi \in \mathcal{D}(\overline{\Delta_{S^2}}) \subset \mathcal{H}$ it holds that*

$$\langle X(t), \phi \rangle_{\mathcal{H}} = \langle X(0), \phi \rangle_{\mathcal{H}} + \int_0^t \langle X(s), \overline{\Delta_{S^2}} \phi \rangle_{\mathcal{H}} \mathrm{d}s + \langle W(t), \phi \rangle_{\mathcal{H}} \qquad P\text{-a.s.},$$

where $\overline{\Delta_{S^2}}$ denotes the closure of the spherical Laplace operator Δ_{S^2}.

Note that the closure of a symmetric densely defined operator is self-adjoint, for example $\overline{\Delta_{S^2}} = \overline{\Delta_{S^2}}^*$. We try to solve it formally and use the special form of the noise term W. We attempt to expand the expression in its decomposition in spherical harmonics

$$\begin{aligned}
X(t) &= \sum_{l \geq 0} \sum_{m=-l}^{l} \tilde{a}_{l,m}(t) \tilde{Y}_{l,m}, \qquad \text{where } \tilde{a}_{l,m}(t) = \langle X(t), \tilde{Y}_{l,m} \rangle_{\mathcal{H}}, \\
&= \sum_{l \geq 0} \sum_{m=-l}^{l} \tilde{a}_{l,m}(0) \tilde{Y}_{l,m} - \int_0^t \sum_{l \geq 0} \sum_{m=-l}^{l} l(l+1) \tilde{a}_{l,m}(s) \tilde{Y}_{l,m} \mathrm{d}s + \sum_{l \geq 0} \sum_{m=-l}^{l} \sqrt{C_l} \beta_{l,m}(t) \tilde{Y}_{l,m} \\
&= \sum_{l \geq 0} \sum_{m=-l}^{l} \left(\tilde{a}_{l,m}(0) - \int_0^t l(l+1) \tilde{a}_{l,m}(s) \mathrm{d}s + \sqrt{C_l} \beta_{l,m}(t) \right) \tilde{Y}_{l,m}.
\end{aligned}$$

This leads to the infinite system of uncoupled 1-dimensional stochastic differential equations (SDE) for $l \geq 0$ and $m = -l, .., l$

$$\tilde{a}_{l,m}(t) = \tilde{a}_{l,m}(0) - \int_0^t l(l+1) \tilde{a}_{l,m}(s) \mathrm{d}s + \sqrt{C_l} \beta_{l,m}(t).$$

This is a general type of SDE, which is solved using Itô' s formula and the existence and uniqueness theorem for SDE's. We reformulate it for notational convenience, B is a 1-dimensional Brownian motion,

$$Z(t) = Z(0) - \theta \int_0^t Z(s)\mathrm{d}s + \sigma B(t)$$
$$Z(0) \in \mathbb{R} \qquad P\text{-a.s.}$$

The existence theorem for such SDE's, which is Theorem 2.5 and Theorem 2.9 in Chapter 5 of [19], shows that Z exists and is unique. So we attempt to find a formula for Z applying Itô's formula, also found in [19] as Theorem 3.6 in Chapter 3, to the function $f(t, Z(t)) = Z(t)e^{\theta t}$ in the first step and insert the solution of the SDE in the second step. We manipulate further and obtain P-a.s.

$$\begin{aligned}
Z(t)e^{\theta t} &= Z(0) - \int_0^t \theta e^{\theta s} Z(s)\mathrm{d}s + \int_0^t e^{\theta s}\mathrm{d}Z(s) \\
&= Z(0) + \int_0^t \theta e^{\theta s} Z(s)\mathrm{d}s + \int_0^t e^{\theta s}\mathrm{d}(-\int_0^s \theta Z(\hat{s})\mathrm{d}\hat{s} + \sigma B(s)) \\
&= Z(0) + \int_0^t \theta e^{\theta s} Z(s)\mathrm{d}s - \int_0^t e^{\theta s}\theta Z(s)\mathrm{d}s + \sigma \int_0^t e^{\theta s}\mathrm{d}B(s) \\
&= Z(0) + \sigma \int_0^t e^{\theta s}\mathrm{d}B(s).
\end{aligned}$$

Thus the solution has the formula

$$Z(t) = e^{-\theta t} Z(0) + \sigma \int_0^t e^{-\theta(t-s)}\mathrm{d}B(s).$$

The solution is called the Ornstein–Uhlenbeck process. It can be applied to our situation to obtain that

$$\tilde{a}_{l,m}(t) = e^{-l(l+1)t}\tilde{a}_{l,m}(0) + \sqrt{C_l}\int_0^t e^{-l(l+1)(t-s)}\mathrm{d}\beta_{l,m}(s)$$

and

$$X(t) = \sum_{l\geq 0}\sum_{m=-l}^{l}\left(e^{-l(l+1)t}\tilde{a}_{l,m}(0) + \sqrt{C_l}\int_0^t e^{-l(l+1)(t-s)}\mathrm{d}\beta_{l,m}(s)\right)\tilde{Y}_{l,m}. \tag{4.2}$$

This derivation was only formal. To finish the analysis we have to show that the infinite sum in Equation (4.2) converges in $L^2_{\mathrm{d}t\otimes P}([0,T]\times\Omega,\mathcal{H})$ and that this expression is the unique solution of Equation (1.1).
First we prove that the limit exists. We look at a truncation of the expression in Equation (4.2), which we denote with $X^L(t)$. We fix $L_1 < L_2 \in \mathbb{N}$ and try to bound the difference of two truncations in the $L^2_P(\Omega,\mathcal{H})$-norm. In the first step we apply the Parseval identity and the fact that the stochastic integral with respect to a square integrable martingale like a Brownian motion has zero expectation. In the second step we use the Itô isometry. We obtain that

$$E\left[\left\|X^{L_1}(t) - X^{L_2}(t)\right\|^2_{\mathcal{H}}\right]$$

$$
\begin{aligned}
&= \sum_{l=L_1}^{L_2} \sum_{m=-l}^{l} e^{-2l(l+1)t} E[\tilde{a}_{l,m}^2(0)] + \sum_{l=L_1}^{L_2} \sum_{m=-l}^{l} C_l E\left[\left(\int_0^t e^{-l(l+1)(t-s)} \mathrm{d}\beta_{l,m}(s)\right)^2\right] \\
&= \sum_{l=L_1}^{L_2} \sum_{m=-l}^{l} e^{-2l(l+1)t} E[\tilde{a}_{l,m}^2(0)] + \sum_{l=L_1}^{L_2} \sum_{m=-l}^{l} C_l \int_0^t e^{-2l(l+1)(t-s)} \mathrm{d}s \\
&\leq \sum_{l=L_1}^{L_2} \sum_{m=-l}^{l} E[\tilde{a}_{l,m}^2(0)] + \sum_{l=L_1}^{L_2} \sum_{m=-l}^{l} C_l \frac{1}{2l(l+1)} (1 - e^{-2l(l+1)t}).
\end{aligned}
$$

If we take the limit ($L_2 \to \infty$) the first infinite sum exists because P-a.s. $X(0) \in \mathcal{H}$, which means that the sequence $(E[\tilde{a}_{l,m}^2(0)] : l \geq 0, m = -l, ..., l)$ is summable. The second infinite sum exists because the sequence $(C_l l : l \geq 0)$ is summable. Similarly the just mentioned summability also implies that for the limit ($L_1 \to \infty$) the whole expression converges to zero. We observe that integrating the parameter t over the compact interval $[0, T]$ will at most increase the expression by the factor T. Hence the sequence $X^L(t)$ is Cauchy in $L^2_{\mathrm{d}t \otimes P}([0, T] \times \Omega, \mathcal{H})$. Similarly we could take the supremum over the compact interval $[0, T]$. Since $(1 - e^{-2l(l+1)t})$ is bounded by one, the limit of the supremum also vanishes. The Brownian motions in the sequence $(\beta_{l,m} : l \geq 0, m = -l, ..., l)$ are P-a.s. continuous. This implies that X^L is also P-a.s. continuous because it is a finite linear combination. Hence there exists a modification $\tilde{X}^L$ of the sequence X^L, which is Cauchy in $C^0([0, T], L^2_P(\Omega, \mathcal{H}))$. Since $L^2_{\mathrm{d}t \otimes P}([0, T] \times \Omega, \mathcal{H})$ and $C^0([0, T], L^2_P(\Omega, \mathcal{H}))$ are both complete spaces, the sequence $X^L(t)$ converges in $L^2_{\mathrm{d}t \otimes P}([0, T] \times \Omega, \mathcal{H})$ to some X and the sequence $\tilde{X}^L$ converges in $C^0([0, T], L^2_P(\Omega, \mathcal{H}))$ to some $\tilde{X}$, which is a modification of X.

Now we know that X is a candidate to be a weak solution of Equation (1.1). We have to check, that it fulfills the conditions of Definition 4.11. We fix $t \in [0, T]$ and $\phi \in \mathcal{D}\left(\overline{\Delta_{S^2}}\right)$ and check the conditions directly. We take the expansion of $X(t)$ and insert the SDE which the coefficients satisfy in the first and second step. In the third step we apply Fubini' s theorem to interchange the infinite sum and the inner product and apply the eigenvalue relation of the spherical harmonics and the spherical Laplace operator Δ_{S^2} and obtain that

$$
\begin{aligned}
\langle X(t), \phi \rangle_{\mathcal{H}} &= \langle \sum_{l \geq 0} \sum_{m=-l}^{l} \tilde{a}_{l,m}(t) \tilde{Y}_{l,m}, \phi \rangle_{\mathcal{H}} \\
&= \langle \sum_{l \geq 0} \sum_{m=-l}^{l} (\tilde{a}_{l,m}(0) - \int_0^t l(l+1) \tilde{a}_{l,m}(s) \mathrm{d}s + \sqrt{C_l} \beta_{l,m}(t)) \tilde{Y}_{l,m}, \phi \rangle_{\mathcal{H}} \\
&= \langle X(0), \phi \rangle_{\mathcal{H}} + \langle \sum_{l \geq 0} \sum_{m=-l}^{l} - \int_0^t l(l+1) \tilde{a}_{l,m}(s) \mathrm{d}s \tilde{Y}_{l,m}, \phi \rangle_{\mathcal{H}} + \langle W(t), \phi \rangle_{\mathcal{H}} \\
&= \langle X(0), \phi \rangle_{\mathcal{H}} + \sum_{l \geq 0} \sum_{m=-l}^{l} \int_0^t \tilde{a}_{l,m}(s) \mathrm{d}s \langle \Delta_{S^2} \tilde{Y}_{l,m}, \phi \rangle_{\mathcal{H}} + \langle W(t), \phi \rangle_{\mathcal{H}}.
\end{aligned}
$$

Now we put the adjoint of the closed Δ_{S^2} to ϕ which is possible due to the choice of ϕ. This enable us to interchange the infinite sum and inner product again by Fubini

$$
\langle X(0), \phi \rangle_{\mathcal{H}} + \sum_{l \geq 0} \sum_{m=-l}^{l} \int_0^t \tilde{a}_{l,m}(s) \mathrm{d}s \langle \Delta_{S^2} \tilde{Y}_{l,m}, \phi \rangle_{\mathcal{H}} + \langle W(t), \phi \rangle_{\mathcal{H}}
$$

$$= \langle X(0), \phi\rangle_{\mathcal{H}} + \sum_{l\geq 0}\sum_{m=-l}^{l}\int_0^t \tilde{a}_{l,m}(s)\mathrm{d}s\langle \tilde{Y}_{l,m}, \overline{\Delta_{S^2}}\phi\rangle_{\mathcal{H}} + \langle W(t), \phi\rangle_{\mathcal{H}}$$
$$= \langle X(0), \phi\rangle_{\mathcal{H}} + \int_0^t \langle X(s), \overline{\Delta_{S^2}}\phi\rangle_{\mathcal{H}})\mathrm{d}s + \langle W(t), \phi\rangle_{\mathcal{H}}.$$

The process X is predictable because it is P-a.s. continuous. The P-a.s. Bochner integrability of the trajectories of X is clear since the trajectories are P-a.s. continuous and defined on a compact interval $[0, T]$. Hence we showed that X is a weak solution of Equation (1.1).

If we assume that X and Y are both weak solutions of Equation (1.1). We wish to show that they are equal. We take fixed integers $l \geq 0$ and $m = -l, ..., l,\ \phi = \tilde{Y}_{l,m}$ and apply Definition 4.11 to obtain that P-a.s.

$$\langle X(t) - Y(t), \tilde{Y}_{l,m}\rangle_{\mathcal{H}} = \int_0^t \langle X(s) - Y(s), \overline{\Delta_{S^2}}\tilde{Y}_{l,m}\rangle_{\mathcal{H}}\mathrm{d}s$$
$$= \int_0^t \langle X(s) - Y(s), -l(l+1)\tilde{Y}_{l,m}\rangle_{\mathcal{H}}\mathrm{d}s.$$

We also applied the fact $\overline{\Delta_{S^2}}\tilde{Y}_{l,m} = \Delta_{S^2}\tilde{Y}_{l,m}$ and the eigenvalue relation of Δ_{S^2} with $\tilde{Y}_{l,m}$. We take now the absolute value on both sides and obtain this inequality,

$$|\langle X(t) - Y(t), \tilde{Y}_{l,m}\rangle_{\mathcal{H}}| \leq l(l+1)\int_0^t |\langle X(s) - Y(s), \tilde{Y}_{l,m}\rangle_{\mathcal{H}}|\mathrm{d}s \qquad P\text{-a.s.}$$

Gronwall's inequality implies that P-a.s. $|\langle X(t) - Y(t), \tilde{Y}_{l,m}\rangle_{\mathcal{H}}| = 0$. Since this can be done for all possible choices of l and m, $X(t) - Y(t) \in \mathrm{span}\{\tilde{Y}_{l,m} : l \geq 0, m = -l, ..., l\}^{\perp}$ P-a.s. Thus $X(t) = Y(t)$ P-a.s. for all $t \in [0, T]$ and the solution is unique up to modification and has the formula stated in Equation (4.2)

$$X(t) = \sum_{l\geq 0}\sum_{m=-l}^{l}\left(e^{-l(l+1)t}\tilde{a}_{l,m}(0) + \sqrt{C_l}\int_0^t e^{-l(l+1)(t-s)}\mathrm{d}\beta_{l,m}(s)\right)\tilde{Y}_{l,m}.$$

It was not needed to introduce tools from stochastic calculus in infinite dimensions. The specific form of the Q-Wiener process and the multidimensional Itô formula were sufficient to solve Equation (1.1).

4.4. Truncation error estimation

As in the previous chapter we want to discuss the error of truncations of this expansions. We do this in the $L^2_P(\Omega, \mathcal{H})$-norm. For the error analysis of the solution of Equation (1.1) we want to approximate with a truncation of the infinite sum in the solution formula Equation (4.2)

$$X^L(t) = \sum_{l=0}^{L}\sum_{m=-l}^{l}\left(e^{-l(l+1)t}\tilde{a}_{l,m}(0) + \sqrt{C_l}\int_0^t e^{-l(l+1)(t-s)}\mathrm{d}\beta_{l,m}(s)\right)\tilde{Y}_{l,m}.$$

The error in the $L^2_P(\Omega, \mathcal{H})$-norm can be calculated in several steps. In the first step we apply the Parseval identity to lose the dependence on the real spherical harmonics. In the

second step we apply Tonelli's theorem to interchange expectation and infinite sum to be able to apply the Itô isometry in the third step. We obtain that

$$
\begin{aligned}
&\left\|X(t)-X^L(t)\right\|^2_{L^2_{P\otimes \mathrm{d}\sigma}(\Omega\times S^2,\mathbb{R})} \\
&\quad = E\left[\sum_{l>L}\sum_{m=-l}^{l}\left(e^{-l(l+1)t}\tilde{a}_{l,m}(0)+\sqrt{C_l}\int_0^t e^{-l(l+1)(t-s)}\mathrm{d}\beta_{l,m}(s)\right)^2\right] \\
&\quad = \sum_{l>L}\sum_{m=-l}^{l} e^{-2l(l+1)t}E[\tilde{a}^2_{l,m}(0)]+\sum_{l>L}\sum_{m=-l}^{l} C_l E\left[\left(\int_0^t e^{-l(l+1)(t-s)}\mathrm{d}\beta_{l,m}(s)\right)^2\right] \\
&\quad = \sum_{l>L}\sum_{m=-l}^{l} e^{-2l(l+1)t}E[\tilde{a}^2_{l,m}(0)]+\sum_{l>L}\sum_{m=-l}^{l} C_l \int_0^t e^{-2l(l+1)(t-s)}\mathrm{d}s. \qquad (4.3)
\end{aligned}
$$

The integral inside the second sum is equal to $\frac{1}{2l(l+1)}(1-e^{-l(l+1)t})$. We simply bound this by $\frac{1}{2l(l+1)}$ and assume for the angular power spectrum that $C_l \simeq l^{-\alpha}$ necessarily for some $\alpha > 2$, as we did in the previous chapter. We repeat a similar calculation as we did for the spherical random field at the end of Chapter 3 to bound the second infinite sum in Equation (4.3)

$$
\begin{aligned}
\sum_{l>L}\sum_{m=-l}^{l} C_l \int_0^t e^{-2l(l+1)(t-s)}\mathrm{d}s &\leq \sum_{l>L} C_l \frac{2l+1}{2l(l+1)} \leq \sum_{l>L} l^{-\alpha-1} \leq \sum_{l>0}(l+L)^{-\alpha-1} \\
&\leq \int_0^\infty (x+L)^{-\alpha-1}\mathrm{d}x = \mathcal{O}\left((L^{-1})^{\alpha}\right).
\end{aligned}
$$

This finishes our brief error analysis. Note that we did not take into account the first infinite sum in Equation (4.3), which also encodes how well the initial value is approximated by the real spherical harmonics. The truncation of the solution can be used in principle for simulation purposes. MATLAB code for an implementation of the truncated solution can be found in Appendix B.

5. Continuity properties of isotropic Gaussian spherical random fields

In this chapter we study the continuity properties of 2-weakly isotropic Gaussian spherical random fields as mappings from the sphere to the real numbers. For our analysis we introduce the probability space $(\Omega, \mathcal{A}, P)$. We know from the previous chapter that for a 2-weakly isotropic Gaussian spherical random field, we have an expansion in the real spherical harmonics which we stated in the previous chapter as Equation (4.1):

$$T = \sum_{l \geq 0} \sqrt{C_l} \sum_{m=-l}^{l} \tilde{b}_{l,m} \tilde{Y}_{l,m}. \tag{5.1}$$

The sequence $(C_l : l \geq 0)$ is the angular power spectrum, $(\tilde{b}_{l,m} : l \geq 0, m = -l, .., l)$ is an i.i.d. sequence of standard normally distributed random variables and $(\tilde{Y}_{l,m} : l \geq 0, m = -l, .., l)$ are the real spherical harmonics. We recall that the sequence $(C_l l : l \geq 0)$ has to be summable such that the expansion converges in $L^2_{P \otimes \mathrm{d}\sigma}(\Omega \times S^2, \mathbb{R})$ and in $L^2_P(\Omega, \mathbb{R})$, see Theorem 3.8.
Our goal is now to establish conditions on the angular power spectrum such that the 2-weakly isotropic Gaussian spherical random field T is γ-Hölder continuous for any $\gamma \in (0, 1)$. This is the content of Theorem 1.2, which was mentioned at the end of the introduction.

5.1. Preliminaries for the proof of Theorem 1.2

We start our investigation with the following lemma.

Lemma 5.1. *For the real spherical harmonic function $\tilde{Y}_{l,m}$, with fixed $l \geq 0$ and $m = -l, .., l$, the following Lipschitz continuity relation holds for fixed $\theta = \frac{\pi}{2}$, $\varphi_1, \varphi_2 \in \mathbb{R}$ and for some constant $K > 0$ independent of l, m, φ_1 and φ_2:*

$$|\tilde{Y}_{l,m}(\theta, \varphi_1) - \tilde{Y}_{l,m}(\theta, \varphi_2)| \leq \begin{cases} K l^{\frac{5}{4}} |\varphi_1 - \varphi_2| & \text{if } |m| = l, \\ K \frac{\sqrt{2l+1}}{(l^2 - m^2)^{\frac{1}{4}}} \frac{|m|}{2} |\varphi_1 - \varphi_2| & \text{if } |m| < l. \end{cases} \tag{5.2}$$

Proof. We recall the representation of the real spherical harmonics, which are defined in Definition 2.4. Their explicit expression is given by:

$$\tilde{Y}_{l,m}(\theta, \varphi) = \begin{cases} \sqrt{2} \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P^l_m(\cos(\theta)) \cos(m\varphi) & \text{if } m > 0, \\ \sqrt{\frac{2l+1}{4\pi}} P^l_0(\cos(\theta)) & \text{if } m = 0, \\ -\sqrt{2} \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P^l_{|m|}(\cos(\theta)) \sin(m\varphi) & \text{if } m < 0. \end{cases} \tag{5.3}$$

First we evaluate the associated Legendre polynomial $P^l_m(\cos(\theta))$. Since $\cos(\theta) = 0$, the expression simplifies using the binomial identity for $l + m$ even to the following:

$$
\begin{aligned}
P_m^l(0) &= \frac{(-1)^m}{2^l l!}(1-x^2)^{\frac{m}{2}}\big|_{x=0}\frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}(x^2-1)^l\big|_{x=0} \\
&= \frac{(-1)^m}{2^l l!}\frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}\left(\sum_{k=0}^{l}\binom{l}{k}x^{2k}(-1)^{l-k}\right)\Bigg|_{x=0}.
\end{aligned}
$$

The only term in the sum, which is non-zero after the taking the derivative $(l+m)$-times is the term with $k=\frac{l+m}{2}$. Hence the expression simplifies to

$$
\left|P_m^l(0)\right| = \frac{1}{2^l l!}\binom{l}{\frac{l+m}{2}}(l+m)!\,.
$$

Note that the expression is zero in the case that $l+m$ is odd. This combination of a factorial and binomial coefficient can be dealt with using the constant in the above definition of $\tilde{Y}_{l,m}$ and applying the Stirling formula. We obtain that

$$
\begin{aligned}
&\left|P_m^l(0)\right|\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} \\
&\quad= \frac{1}{2^l l!}\binom{l}{\frac{l+m}{2}}\sqrt{\frac{2l+1}{4\pi}(l-m)!(l+m)!} = \frac{1}{2^l}\sqrt{\frac{2l+1}{4\pi}}\frac{\sqrt{(l-m)!(l+m)!}}{(\frac{l+m}{2})!(\frac{l-m}{2})!} \qquad (5.4)
\end{aligned}
$$

and use the Stirling formula in this form:

$$
\text{for all strictly positive } n\in\mathbb{N} \text{ it holds that,} \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n} \le n! \le en^{n+\frac{1}{2}}e^{-n}. \qquad (5.5)
$$

Inequalities of this type are the objective of two journal articles. The first Inequality (5.5) follows directly from the result of [23] because $1 \le e^{\frac{1}{12n+1}}$ for all $n\in\mathbb{N}$. The second Inequality (5.5) is part of the result of [17]. We now apply the Stirling formula (5.5) and manipulate Equation (5.4) further to obtain for $|m|<l$ and for some $K_1>0$ independent of l and m that

$$
\begin{aligned}
&\frac{1}{2^l}\sqrt{\frac{2l+1}{4\pi}}\frac{\sqrt{(l-m)!(l+m)!}}{(\frac{l+m}{2})!(\frac{l-m}{2})!} \\
&\quad\le \frac{1}{2^l}\sqrt{\frac{2l+1}{4\pi}}\frac{e}{2\pi}(l-m)^{-\frac{1}{4}}(l+m)^{-\frac{1}{4}}2^{l+1} = \frac{e}{\pi\sqrt{4\pi}}\frac{\sqrt{2l+1}}{(l^2-m^2)^{\frac{1}{4}}} = K_1\frac{\sqrt{2l+1}}{(l^2-m^2)^{\frac{1}{4}}}.
\end{aligned}
$$

The case $|m|=l$ has to be treated separately, but we get the result similarly also using the Stirling formula (5.5)

$$
\begin{aligned}
&\frac{1}{2^l}\sqrt{\frac{2l+1}{4\pi}}\frac{\sqrt{(l-m)!(l+m)!}}{(\frac{l+m}{2})!(\frac{l-m}{2})!} \\
&\quad= \frac{1}{2^l l!}\sqrt{\frac{2l+1}{4\pi}}\sqrt{(2l)!} \le \sqrt{\frac{e}{2\pi}}\sqrt{\frac{2l+1}{4\pi}}\frac{2^{l+\frac{1}{4}}}{2^l}l^{-\frac{1}{4}} = \frac{\sqrt{e}}{2^{\frac{5}{4}}\pi}\frac{\sqrt{2l+1}}{l^{\frac{1}{4}}} \le K_2 l^{\frac{1}{4}}
\end{aligned}
$$

for some $K_2>0$ independent of l. We combine these constants and thus have shown for $K=\max(K_1,K_2)>0$ that

$$
\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_m^l(\cos(\theta)) \le \begin{cases} Kl^{\frac{1}{4}} & \text{if } |m|=l, \\ K\frac{\sqrt{2l+1}}{(l^2-m^2)^{\frac{1}{4}}} & \text{if } |m|<l. \end{cases} \qquad (5.6)
$$

In the second part of the proof we have to examine the part of $\tilde{Y}_{l,m}$ which depends on φ. First we apply a well known relation of trigonometric functions

$$\sin(m\varphi_1) - \sin(m\varphi_2) = 2\cos\left(m\frac{\varphi_1+\varphi_2}{2}\right)\sin\left(m\frac{\varphi_1-\varphi_2}{2}\right)$$
$$\cos(m\varphi_1) - \cos(m\varphi_2) = 2\sin\left(m\frac{\varphi_1+\varphi_2}{2}\right)\sin\left(m\frac{\varphi_1-\varphi_2}{2}\right).$$

To finish the proof, it is sufficient to state the following claim

$$\left|\sin\left(m\frac{\varphi_1-\varphi_2}{2}\right)\right| \leq \frac{|m|}{2}|\varphi_1-\varphi_2| \tag{5.7}$$

which is clear by a small argument. After evaluating the derivative of $\sin(x)$ at $x=0$ we obtain with Taylor that $\sin(x) \leq x$ for all $x \in [0,1]$. The point symmetry of sine and the fact that $|\sin(x)| \leq 1$ for all $x \in \mathbb{R}$ gives the claim. Now we can conclude the claim of the lemma combining Equation (5.6) and Equation (5.7)

$$\left|\tilde{Y}_{l,m}(\theta,\varphi_1) - \tilde{Y}_{l,m}(\theta,\varphi_2)\right| \leq \begin{cases} Kl^{\frac{1}{4}}\frac{|m|}{2}|\varphi_1-\varphi_2| & \text{if } |m|=l, \\ K\frac{\sqrt{2l+1}}{(l^2-m^2)^{\frac{1}{4}}}\frac{|m|}{2}|\varphi_1-\varphi_2| & \text{if } |m|<l. \end{cases} \qquad \square$$

This lemma is used to verify a condition on the 2-weakly isotropic Gaussian spherical random field T which will be needed in the proof of Theorem 1.2, the main theorem of this chapter. We do this in the following lemma.

Lemma 5.2. *Let T be a 2-weakly isotropic Gaussian spherical random field, given in Equation* (5.1)*, such that for the angular power spectrum it holds that $(C_l l^{1+\delta} : l \geq 0)$ is summable for $\delta \in [0,2]$, then for any $k \in \mathbb{N}$, $x, y \in S^2$ and for an appropriate constant $K_k > 0$ depending on k it holds that*

$$E\left[|T(x)-T(y)|^{2k}\right] \leq K_k\left(\sum_{l\geq 0} C_l l^{1+\delta}\right)^k d(x,y)^{\delta k}.$$

Proof. First, we give the claim for $k=1$ then the argument will be generalized for an arbitrary $k \in \mathbb{N}$. We can choose $g \in SO(3)$ such that $\theta_{gx} = \theta_{gy} = \frac{\pi}{2}$ due to Lemma 2.1, just choose $z, w \in S^2$ such that $z_3 = w_3 = 0$ and $\langle x, y\rangle = \langle z, w\rangle$ to find $g \in SO(3)$ by the lemma. In the first step, we use the property that T is 2-weakly isotropic to insert g. In the second step, we write T in its spherical expansion due to Theorem 3.8 and apply Lemma 3.3 and the Parseval identity to obtain that

$$\begin{aligned} E\left[|T(x)-T(y)|^2\right] &= E\left[|T(gx)-T(gy)|^2\right] = E\left[|T(\theta_{gx},\varphi_{gx}) - T(\theta_{gy},\varphi_{gy})|^2\right] \\ &= E\left[\left|\sum_{l\geq 0}\sqrt{C_l}\sum_{m=-l}^{l}\tilde{b}_{l,m}\left(\tilde{Y}_{l,m}\left(\frac{\pi}{2},\varphi_{gx}\right) - \tilde{Y}_{l,m}\left(\frac{\pi}{2},\varphi_{gy}\right)\right)\right|^2\right] \\ &= \sum_{l\geq 0} C_l \sum_{m=-l}^{l}\left(\tilde{Y}_{l,m}\left(\frac{\pi}{2},\varphi_{gx}\right) - \tilde{Y}_{l,m}\left(\frac{\pi}{2},\varphi_{gy}\right)\right)^2. \end{aligned} \tag{5.8}$$

We are now in the setting to apply Lemma 5.1 to the last expression in the computation just above in order to single out the desired Hölder exponent δ. This yields that

$$\begin{aligned}
&E\left[|T(x)-T(y)|^2\right] \\
&\quad \leq K\sum_{l\geq 0} C_l \Bigg(\sum_{m=-l+1}^{l-1} \frac{\sqrt{2l+1}^{\delta}|m|^{\delta}}{(l^2-m^2)^{\frac{\delta}{4}}} \left(\tilde{Y}_{l,m}\left(\frac{\pi}{2},\varphi_{gx}\right)-\tilde{Y}_{l,m}\left(\frac{\pi}{2},\varphi_{gy}\right)\right)^{2-\delta} \\
&\qquad\qquad +2l^{\frac{5}{4}\delta}\left(\tilde{Y}_{l,l}\left(\frac{\pi}{2},\varphi_{gx}\right)-\tilde{Y}_{l,l}\left(\frac{\pi}{2},\varphi_{gy}\right)\right)^{2-\delta}\Bigg)|\varphi_{gx}-\varphi_{gy}|^{\delta}.
\end{aligned}$$

We apply Inequality (5.6) to bound the remaining term of the real spherical harmonics with the help of the triangle inequality and obtain that

$$E\left[|T(x)-T(y)|^2\right] \leq K^2\sum_{l\geq 0} C_l \left(l\sum_{m=-l+1}^{l-1}\frac{|m|^{\delta}}{\sqrt{l^2-m^2}}+2l^{\frac{1}{2}+\delta}\right)|\varphi_{gx}-\varphi_{gy}|^{\delta}.$$

The inner sum can be bounded with the following integral, when the inner sum is interpreted as a Riemann sum, i.e.

$$\sum_{m=1}^{l-1}\frac{m^{\delta}}{\sqrt{l^2-m^2}} \leq \int_0^l \frac{x^{\delta}}{\sqrt{l^2-x^2}}\mathrm{d}x \leq l^{\delta}\int_0^l \frac{1}{\sqrt{l^2-x^2}}\mathrm{d}x = l^{\delta}\arctan\left(\frac{x}{\sqrt{l^2-x^2}}\right)\Bigg|_0^l = l^{\delta}\frac{\pi}{2}.$$

We apply the summability condition of the lemma on the angular power spectrum to continue the above calculation to obtain that

$$\begin{aligned}
E\left[|T(x)-T(y)|^2\right] &\leq 4K^2\sum_{l\geq 0} C_l l^{1+\delta}|\varphi_{gx}-\varphi_{gy}|^{\delta} \leq K_1\sum_{l\geq 0} C_l l^{1+\delta}|\varphi_{gx}-\varphi_{gy}|^{\delta} \\
&= K_1\sum_{l\geq 0} C_l l^{1+\delta} d(gx,gy)^{\delta} = K_1\sum_{l\geq 0} C_l l^{1+\delta} d(x,y)^{\delta}.
\end{aligned}$$

We applied that $\theta_{gx}=\theta_{gy}=\frac{\pi}{2}$ and Equation (2.1) to obtain the metric d and the fact that the action under $SO(3)$ preserves the metric d. This gives the claim for $k=1$. Now we treat the general case for all $k\in\mathbb{N}$ and insert the same g. In the first step, we apply that all $\tilde{b}_{l,m}$ have mean zero, are independent and the fact that all odd moments are equal to zero. Also we interchange the infinite sum and the expectation by Tonelli's theorem. In the second and third step, we use that the $2k^{\text{th}}$ moment of a standard normally distributed random variable is equal to $\frac{(2k)!}{2^k k!}$, which is increasing in k, to obtain that

$$\begin{aligned}
&E[|T(x)-T(y)|^{2k}] \\
&\quad = \sum_{l_1,\ldots,l_k\geq 0}\sum_{m_1=-l_1}^{l_1}\cdots\sum_{m_k=-l_k}^{l_k}\prod_{j=1}^{k} C_{l_j}\left|Y_{l_j,m_j}(x)-Y_{l_j,m_j}(y)\right|^2 E\left[\tilde{b}^2_{l_1,m_1}\cdots\tilde{b}^2_{l_k,m_k}\right] \\
&\quad \leq \frac{(2k)!}{2^k k!}\sum_{l_1,\ldots,l_k\geq 0}\sum_{m_1=-l_1}^{l_1}\cdots\sum_{m_k=-l_k}^{l_k}\prod_{j=1}^{k} C_{l_j}\left|Y_{l_j,m_j}(x)-Y_{l_j,m_j}(y)\right|^2 \\
&\quad = \frac{(2k)!}{2^k k!}\left(\sum_{l\geq 0} C_l\sum_{m=-l}^{l}\left|\tilde{Y}_{l,m}(x)-\tilde{Y}_{l,m}(y)\right|^2\right)^k. \qquad (5.9)
\end{aligned}$$

For a fixed integer $l \geq 0$ we single out the sum over $m = -l, ..., l$. In the first step we switch to the spherical harmonics. This can be verified with Definition 2.2 and Definition 2.4. In the second step we apply Lemma 2.7 with the same g which we already used in the beginning of this proof. In the third and fourth step we recognize that this sums correspond to the squared Euclidean norm on $\mathbb{C}^{2l+1}$ denoted by $\|\cdot\|_{\mathbb{C}^{2l+1}}$ of a unitary matrix $D^l(g^{-1})$ on $\mathbb{C}^{2l+1}$ times a vector $Y_l(\frac{\pi}{2}, \varphi_{gx}) - Y_l(\frac{\pi}{2}, \varphi_{gy}) \in \mathbb{C}^{2l+1}$. We obtain that

$$\begin{aligned}
\sum_{m=-l}^{l} \left|\tilde{Y}_{l,m}(x) - \tilde{Y}_{l,m}(y)\right|^2 &= \sum_{m=-l}^{l} |Y_{l,m}(x) - Y_{l,m}(y)|^2 \\
&= \sum_{m=-l}^{l} \left| \sum_{m'=-l}^{l} D^l_{m',m}(g^{-1}) \left(Y_{l,m'}\left(\frac{\pi}{2}, \varphi_{gx}\right) - Y_{l,m'}\left(\frac{\pi}{2}, \varphi_{gy}\right)\right)\right|^2 \\
&= \left\| D^l(g^{-1})^\top \left(Y_l\left(\frac{\pi}{2}, \varphi_{gx}\right) - Y_l\left(\frac{\pi}{2}, \varphi_{gy}\right)\right)\right\|^2_{\mathbb{C}^{2l+1}} \\
&= \left\| Y_l\left(\frac{\pi}{2}, \varphi_{gx}\right) - Y_l\left(\frac{\pi}{2}, \varphi_{gy}\right)\right\|^2_{\mathbb{C}^{2l+1}} \\
&= \sum_{m=-l}^{l} \left|\tilde{Y}_{l,m}\left(\frac{\pi}{2}, \varphi_{gx}\right) - \tilde{Y}_{l,m}\left(\frac{\pi}{2}, \varphi_{gy}\right)\right|^2.
\end{aligned}$$

We insert this equation into Equation (5.9) to obtain that

$$E[|T(x) - T(y)|^{2k}] \leq \frac{(2k)!}{2^k k!} \left(\sum_{l \geq 0} C_l \sum_{m=-l}^{l} \left|\tilde{Y}_{l,m}\left(\frac{\pi}{2}, \varphi_{gx}\right) - \tilde{Y}_{l,m}\left(\frac{\pi}{2}, \varphi_{gy}\right)\right|^2 \right)^k.$$

Note that we obtained the kth power of Equation (5.8). So we can insert, what we derived for Equation (5.8) to obtain that

$$E[|T(x) - T(y)|^{2k}] \leq \frac{(2k)!}{2^k k!} K_1^k \left(\sum_{l \geq 0} C_l l^{1+\delta} \right)^k d(x,y)^{\delta k} = K_k \left(\sum_{l \geq 0} C_l l^{1+\delta} \right)^k d(x,y)^{\delta k}. \quad \square$$

5.2. Kolmogorov–Čentsov continuity theorem

Before we can apply the previous lemma, we prove a generalized version of the Kolmogorov–Čentsov continuity theorem which gives sufficient conditions for stochastic processes to be almost surely locally Hölder continuous. We will generalize this to index sets of cubes in $\mathbb{R}^d$. The Kolmogorov–Čentsov continuity theorem is stated in [19] as Theorem 2.8. We will take the proof given in [19], elaborate some more details and adapt it to prove our more general statement.

Theorem 5.3. *Let T be a random field indexed by the cube $D = [a,b]^d$ for $d \in \mathbb{N}$ and $a < b$ such that for $\alpha, \beta, K > 0$ it holds that*

$$E\left[|T(x) - T(y)|^\alpha\right] \leq K \|x - y\|^{d+\beta}_{\mathbb{R}^d}.$$

Under this conditions for all $\gamma \in (0, \frac{\beta}{\alpha})$ there exists $K' > 0$, a P-a.s. positive random variable h^* and a modification $\tilde{T}$ of T such that $\tilde{T}$ is almost surely locally γ-Hölder continuous,

i.e. there exists a set of full probability Ω^* such that for all $\omega \in \Omega^*$ and $x, y \in D$ with $\|x-y\|_{\mathbb{R}^d} < h^*(\omega)$ it holds that

$$\left|\tilde{T}(x,\omega) - \tilde{T}(y,\omega)\right| \leq K' \|x-y\|_{\mathbb{R}^d}^{\gamma}.$$

Proof. Without loss of generality (w.l.o.g.) we assume that $D = [0,1]^d$. In this proof we take the one norm on $\mathbb{R}^d$, i.e. for $x \in \mathbb{R}^d$ this norm is given by $\|x\|_1 = \sum_{i=1}^d |x_i|$. This is justified because all norms are equivalent on $\mathbb{R}^d$ and the statements are only stated with respect to constants. The main ingredients of this proof are the Borel–Cantelli lemma and the Chebychev inequality. As a starting point we apply the Chebychev inequality and exploit the resulting inequality in order to setup a discretization of the cube, where the inequality holds. We obtain that

$$P\left[|T(x) - T(y)| > \varepsilon\right] \leq \varepsilon^{-\alpha} E\left[|T(x) - T(y)|^{\alpha}\right] \leq \varepsilon^{-\alpha} K \|x-y\|_1^{d+\beta}. \tag{5.10}$$

We choose $\varepsilon = 2^{-\gamma n}$ for $\gamma \in (0, \frac{\beta}{\alpha})$ and obtain for $k_1, ..., k_d \in \{0, 1, ..., 2^n\}$ and $\eta_1, ..., \eta_d \in \{0,1\}$ from Equation (5.10) that

$$\begin{aligned} P\left[\left|T\left(\frac{k_1}{2^n}, ..., \frac{k_d}{2^n}\right) - T\left(\frac{k_1-\eta_1}{2^n}, ..., \frac{k_d-\eta_d}{2^n}\right)\right| > 2^{-\gamma n}\right] &\leq 2^{(-\gamma n)^{-\alpha}} K d 2^{-n^{d+\beta}} \\ &= K d 2^{-n(d+\beta-\gamma\alpha)}. \end{aligned}$$

We can apply this idea to a finite number of points that are equally spread in the cube D and obtain that

$$\begin{aligned} &P\left[\max_{\substack{1\leq k_1,...,k_d\leq 2^n \\ \eta_1,...,\eta_d\in\{0,1\}}} \left|T\left(\frac{k_1}{2^n}, ..., \frac{k_d}{2^n}\right) - T\left(\frac{k_1-\eta_1}{2^n}, ..., \frac{k_d-\eta_d}{2^n}\right)\right| > 2^{-\gamma n}\right] \\ &= P\left[\bigcup_{\substack{1\leq k_1,...,k_d\leq 2^n \\ \eta_1,...,\eta_d\in\{0,1\}}} \left\{\omega \in \Omega : \left|T\left(\omega, \frac{k_1}{2^n}, ..., \frac{k_d}{2^n}\right) - T\left(\omega, \frac{k_1-\eta_1}{2_n}, ..., \frac{k_d-\eta_d}{2_n}\right)\right| > 2^{-\gamma n}\right\}\right] \\ &\leq \sum_{\substack{1\leq k_1,...,k_d\leq 2^n \\ \eta_1,...,\eta_d\in\{0,1\}}} K d 2^{-n(d+\beta-\gamma\alpha)} = K d 2^d 2^{-n(\beta-\alpha\gamma)}. \end{aligned}$$

Since by our choice $\beta - \gamma\alpha > 0$ the last expression is summable and so the Borel–Cantelli lemma is applicable. Instead of quoting the lemma it is more practical to quote its proof which is found in [11] on page 65 and apply it to our situation.
We set

$$\begin{aligned} A_n &= \left\{\omega \in \Omega : \max_{\substack{1\leq k_1,...,k_d\leq 2^n \\ \eta_1,...,\eta_d\in\{0,1\}}} \left|T\left(\omega, \frac{k_1}{2^n}, ..., \frac{k_d}{2^n}\right) - T\left(\omega, \frac{k_1-\eta_1}{2^n}, ..., \frac{k_d-\eta_d}{2^n}\right)\right| > 2^{-\gamma n}\right\}, \\ N(\omega) &= \sum_{n\geq 1} \mathbb{1}_{A_n}(\omega) \qquad \text{for } \omega \in \Omega. \end{aligned}$$

We simply apply the summability which we found before and obtain that

$$E\left[N\right] = \sum_{n\geq 1} P[A_n] \leq \sum_{n\geq 1} K d 2^d 2^{-n(\beta-\alpha\gamma)} < \infty.$$

This implies that N is P-a.s. finite, say on the set $\Omega^* \in \mathcal{A}$, which then has full probability. This implies that for each $\omega \in \Omega^*$ the condition stated in the definition of A_n is violated only for finitely many $n \in \mathbb{N}$. So we can find $n^*(\omega)$ for all $\omega \in \Omega^*$ such that

$$\max_{\substack{1\leq k_1,...,k_d\leq 2^n \\ \eta_1,...,\eta_d\in\{0,1\}}} \left| T\left(\omega, \frac{k_1}{2^n}, ..., \frac{k_d}{2^n}\right) - T\left(\omega, \frac{k_1-\eta_1}{2^n}, ..., \frac{k_d-\eta_d}{2^n}\right)\right| \leq 2^{-\gamma n} \text{ for } n \geq n^*(\omega). \tag{5.11}$$

Having this inequality, we proceed with the next claim, which will provide the desired continuity of a modification of T. We define for $M \in \mathbb{N}$

$$D^M = \left\{(k_1, ..., k_d)^\top / 2^M : k_i \in \left\{0, 1, ..., 2^M\right\}; i = 1, ..., d\right\}.$$

For $n \geq n^*(\omega)$ we claim that for all $M > n \in \mathbb{N}$ it holds that

$$|T(\omega, x) - T(\omega, y)| \leq 2 \sum_{j=n+1}^{M} 2^{-\gamma j} \text{ for all } x, y \in D^M \text{ such that } \|x - y\|_1 < 2^{-n}. \tag{5.12}$$

The statement in Equation (5.12) is proven by induction. We start with the case $M = n+1$. For $x, y \in D^{n+1}$ such that $\|x-y\| < 2^{-n}$ it holds that x and y only differ in one coordinate. In particular in that coordinate the difference has to be less or equal than $2^{-(n+1)}$. So Equation (5.11) can be applied and gives the result Equation (5.12) for $M = n + 1$.
Now we try to conclude that if Equation (5.12) holds for $m = n + 1, ..., M - 1$, then it also holds for M. For $x, y \in D^M$ such that $\|x - y\| < 2^{-n}$, w.l.o.g we assume that $x_i \geq y_i$ for $i = 1, ..., d$. This freedom is clear, since for $y_i > x_i$ the argument could simply be done the other way around, which is briefly explained below. We set $x^*, y^* \in D^{M-1}$ such that for all $i = 1, ..., d$ it holds that

$$\begin{aligned} x_i^* &= \max\{u \in \{0, 1/2^{M-1}, ..., 1\} : x_i \geq u\}, \\ y_i^* &= \min\{u \in \{0, 1/2^{M-1}, ..., 1\} : x_i \leq u\}. \end{aligned}$$

So it holds that $y_i \leq y_i^* \leq x_i^* \leq x_i$ for $i = 1, ..., d$ and the induction hypotheses can be applied to x^* and y^*. So we obtain that

$$|T(\omega, x^*) - T(\omega, y^*)| < 2 \sum_{j=n+1}^{M-1} 2^{-\gamma j}.$$

In the case $y_i > x_i$, x_i^* and y_i^* would be defined the other way around and the inequalities would be $y_i \geq y_i^* > x_i^* \geq x_i$. For x and x^* we know that in every coordinates the difference is at most 2^{-M}. This means we can use Equation (5.11) as an upper bound

$$\begin{aligned} |T(\omega, x) - T(\omega, x^*)| &\leq \max_{\substack{1\leq k_1,...,k_d\leq 2^M \\ \eta_1,...,\eta_d\in\{0,1\}}} \left| T\left(\omega, \frac{k_1}{2^M}, ..., \frac{k_d}{2^M}\right) - T\left(\omega, \frac{k_1-\eta_1}{2^M}, ..., \frac{k_d-\eta_d}{2^M}\right)\right| \\ &\leq 2^{-\gamma M}. \end{aligned}$$

For y and y^* it is done exactly in the same way. Now the triangle inequality will finish the induction argument

$$\begin{aligned}|T(\omega,x)-T(\omega,y)| &\leq |T(\omega,x)-T(\omega,x^*)|+|T(\omega,x^*)-T(\omega,y^*)|+|T(\omega,y^*)-T(\omega,y)| \\ &\leq 2\cdot 2^{-\gamma M}+2\sum_{j=n+1}^{M-1}2^{-\gamma j}=2\sum_{j=n+1}^{M}2^{-\gamma j}.\end{aligned}$$

From here on the proof given for Theorem 2.8 in [19] for stochastic processes instead of random fields fits exactly our situation. Nevertheless, we give it to complete the argument and for the convenience of the reader. To analyze the continuity we introduce the set $D=\bigcup_{m\geq 0}D^m$ and the function $h^*(\omega)=2^{-n^*(\omega)}$ on Ω^*. We now show that the random field T is almost surely locally γ-Hölder continuous on the set D which is dense in the cube $[0,1]^d$. For $\omega\in\Omega^*$ and $x,y\in D$ such that $\|x-y\|_1<h^*(\omega)$ and $x\neq y$, we find $n\geq n^*(\omega)$ such that $2^{-(n+1)}\leq\|x-y\|_1<2^{-n}$. Using Equation (5.12) we find that

$$|T(\omega,x)-T(\omega,y)|\leq 2\sum_{j=n+1}^{\infty}2^{-\gamma j}\leq 2\sum_{j=0}^{\infty}2^{-\gamma j}2^{-(n+1)\gamma}\leq\frac{2}{1-2^{-\gamma}}\|x-y\|_1^{\gamma}. \tag{5.13}$$

On the set Ω^{*c}, we define the random field $\tilde{T}(x)=0$ for all $x\in[0,1]^d$. For $\omega\in\Omega$ and $x\in D$, we define $\tilde{T}(\omega,x)=T(\omega,x)$. For $x\in[0,1]^d\cap D^c$ we choose a sequence $(x_n:n\geq 1)\subset D$ such that x_n converges to x. Using Equation (5.13) we see below that $T(\omega,x_n)$ is a Cauchy sequence, i.e.

$$|T(\omega,x_m)-T(\omega,x_n)|\leq K'\|x_m-x_n\|_1^{\gamma}\longrightarrow 0\qquad\text{for } m,n\to\infty.$$

The limit of this sequence exists by completeness of the real numbers. The limit is independent of the chosen sequence x_n, because if we take a second sequence $(\hat{x}_n:n\geq 1)$ which also converges to x we obtain that

$$|T(\omega,x_m)-T(\omega,\hat{x}_m)|\leq K'\|x_m-\hat{x}_n\|_1^{\gamma}\longrightarrow 0\qquad\text{for } m\to\infty.$$

Therefore $\tilde{T}(\omega,x)=\lim_{n\to\infty}T(\omega,x_n)$ is well-defined and satisfies Equation (5.13). Moreover this implies that for any $x\in[0,1]^d$ and some sequence $(x_n:n\geq 1)\subset D$ which converges to x that $T(x_n)$ converges P-a.s. to $\tilde{T}(x)$. Also Equation (5.10) immediately gives that $T(x_n)$ converges to $T(x)$ in probability. This implies $\tilde{T}(x)=T(x)$ P-a.s. □

5.3. Proof of Theorem 1.2

Now we are in a good position to prove the main theorem of this chapter. We formulate Theorem 1.2 precisely.

Theorem 5.4. *Let T be a 2-weakly isotropic Gaussian spherical random field, such that the angular power spectrum satisfies that $(C_l l^{1+\delta}:l\geq 0)$ is summable for $\delta\in(0,2]$. For any $\gamma\in(0,\frac{\delta}{2})$ there exist a modification T^* of T, a P-a.s. positive random variable h^* and a constant $K>0$ such that T^* is almost sure locally γ-Hölder continuous, i.e. there exists a set of full probability Ω^* such that for all $\omega\in\Omega^*$ and $x,y\in S^2$ with $d(x,y)<h^*(\omega)$ it holds that*

$$|T^*(x,\omega)-T^*(y,\omega)|\leq Kd(x,y)^{\gamma}.$$

Proof. The idea of the proof is to divide T into spherical random subfields, which are only indexed with parts of the sphere. These parts will overlap such that the global local Hölder continuity can be deduced. We will prove the assertion on these parts individually. For the parts, we choose projections of a square which can be seen to lie underneath the specific part of the sphere. The square D and the mapping from D to S^2 are given by

$$D = \left[-\frac{2}{3}, \frac{2}{3}\right]^2, \qquad \Phi : D \to S^2$$
$$(x_1, x_2) \mapsto \left(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}\right).$$

Then we define the random subfield $\hat{T} = T|_{\Phi(D)}$. Now the crucial thing to check is whether distances in D and $\Phi(D)$ are equivalent, i.e. there exist constants $\hat{K}, \hat{k} > 0$ such that for all $x, y \in D$ it holds that

$$\hat{k}\|x - y\|_{\mathbb{R}^2} \leq d(\Phi(x), \Phi(y)) \leq \hat{K}\|x - y\|_{\mathbb{R}^2}. \tag{5.14}$$

It is clear that $\hat{k} = 1$ is sufficient, since distances on the sphere are larger than on an underlying square. The interesting case is the second inequality. We bound $d(.,.)$ from above in two steps. First we bound it in the Euclidean norm of $\mathbb{R}^3$ and then in the Euclidean norm of $\mathbb{R}^2$ restricted to the square D. As a start we want to remove the inverse trigonometric function arccos. We claim that for all $t \in [-1, 1]$ it holds that

$$\arccos(t) \leq \pi\sqrt{1 - t}. \tag{5.15}$$

For $t \in [-1, 0]$ this is obvious since both functions are decreasing and the function $\pi\sqrt{1 - \cdot}$ attains the maximum π of the function arccos already at 0. For the other case $t \in [0, 1]$ we use that $\frac{\mathrm{d}}{\mathrm{d}t}\arccos(t) = -\frac{1}{\sqrt{1-t^2}}$ and the following inequality that is valid on at $[0, 1]$:

$$\frac{1}{\sqrt{1 - s^2}} \leq \frac{\pi}{2}\frac{1}{\sqrt{1 - s}}.$$

We write arccos in an integral form to obtain Inequality (5.15)

$$\arccos(t) = \int_t^1 \frac{1}{\sqrt{1 - s^2}}\mathrm{d}s \leq \frac{\pi}{2}\int_t^1 \frac{1}{\sqrt{1 - s}}\mathrm{d}s = -\pi\sqrt{1 - s}\Big|_t^1 = \pi\sqrt{1 - t}.$$

Now we can show the first step that for $x, y \in \mathbb{R}^2$ it holds that

$$d(\Phi(x), \Phi(y))^2 \leq \frac{\pi^2}{2}\|\Phi(x) - \Phi(y)\|^2. \tag{5.16}$$

Since both sides of Inequality (5.15) are positive, the inequality also holds squared. We apply this inequality after we inserted the expression of the metric $d(.,.)$ in terms of arccos. Then a null addition, the definition of Φ and some manipulations give

$$\begin{aligned} d(\Phi(x), \Phi(y))^2 &= \arccos^2(\langle\Phi(x), \Phi(y)\rangle) \leq \pi^2(1 - \langle\Phi(x), \Phi(y)\rangle) \\ &= \pi^2\left(1 - x_1y_1 - x_2y_2 - \sqrt{(1 - x_1^2 - x_2^2)(1 - y_1^2 - y_2^2)}\right) \\ &= \frac{\pi^2}{2}\Big(x_1^2 - 2x_1y_1 + y_1^2 + x_1^2 - 2x_1y_1 + y_1^2 + 1 - x_1^2 - x_2^2 \end{aligned}$$

$$
\begin{aligned}
&\qquad -2\sqrt{(1-x_1^2-x_2^2)(1-y_1^2-y_2^2)}+1-y_1^2-y_2^2\Big)\\
&= \frac{\pi^2}{2}\left((x_1-y_1)^2+(x_2-y_2)^2+\left(\sqrt{1-x_1^2-x_2^2}-\sqrt{1-y_1^2-y_2^2}\right)^2\right)\\
&= \frac{\pi^2}{2}\|\Phi(x)-\Phi(y)\|^2.
\end{aligned}
$$

For the second step in the proof of the upper bound $\hat{K}$ in Inequality (5.14) we apply a fact from real calculus: for a real-valued continuously differentiable function f on a convex open subset $O \subset \mathbb{R}^2$ it holds that

$$
|f(x)-f(y)| \leq \sup_{z\in O}\|Df(z)\|_{\mathbb{R}^2}\|x-y\|_{\mathbb{R}^2},
$$

where Df denotes the Jacobian of f. This fact can be easily shown by an elementary calculation involving the fundamental theorem of calculus and the Cauchy–Schwarz inequality. In our case f is the third component of Φ, i.e. $f = \Phi_3$, which is extendable to an open domain containing D such that the extension is continuously differentiable. It is readily verified that $\sup_{z\in D}\|D\Phi_3(z)\|_{\mathbb{R}^2} = 2\sqrt{2}$, since the supremum is attained at each corner of D and $D\Phi_3(z) = (\frac{-z_1}{\sqrt{1-z_1^2-z_2^2}}, \frac{-z_2}{\sqrt{1-z_1^2-z_2^2}})$. We apply what we just found and obtain for $x, y \in D$ that

$$
\begin{aligned}
\|\Phi(x)-\Phi(y)\|^2 &= \|x-y\|_{\mathbb{R}^2}^2 + |\Phi_3(x)-\Phi_3(y)|^2 \\
&\leq \|x-y\|_{\mathbb{R}^2}^2 + \left(2\sqrt{2}\right)^2\|x-y\|_{\mathbb{R}^2}^2 = 9\|x-y\|_{\mathbb{R}^2}^2.
\end{aligned} \tag{5.17}
$$

We combine Inequality (5.16) and Inequality (5.17) to obtain Inequality (5.14), which we wanted to prove in the first place: for $x, y \in D$ it holds that

$$
d(\Phi(x),\Phi(y)) \leq \frac{\pi}{\sqrt{2}}\|\Phi(x)-\Phi(y)\| \leq \frac{3\pi}{\sqrt{2}}\|x-y\|_{\mathbb{R}^2}.
$$

So we set $\hat{K} = \frac{3\pi}{\sqrt{2}}$.

The second ingredient of the proof of the theorem is Lemma 5.2. We fix $\gamma \in (0, \frac{\delta}{2})$ for the rest of the proof. Then there is $k \in \mathbb{N}$ such that $\gamma \in (0, \frac{\delta k-2}{2k})$. According to Lemma 5.2 there exists $K_k > 0$ which depends on k such that

$$
E[|\hat{T}(\Phi(x)) - \hat{T}(\Phi(y))|^{2k}] \leq K_k d(\Phi(x),\Phi(y))^{\delta k}.
$$

We combine this with the equivalence of distances on D and $\Phi(D) \subset S^2$ which is expressed in Inequality (5.14) and obtain that

$$
E[|\hat{T}(\Phi(x)) - \hat{T}(\Phi(y))|^{2k}] \leq \hat{K}K_k\|x-y\|_{\mathbb{R}^2}^{\delta k}.
$$

Theorem 5.3 implies that there exist a modification $\tilde{T}$ of $\hat{T}$, a P-a.s. positive random variable h^* and a constant $\tilde{K} > 0$ such that for all $x, y \in D$ that fulfill $\|x-y\|_{\mathbb{R}^2} < h^*$ P-a.s. it holds that

$$
|\tilde{T}(\Phi(x)) - \tilde{T}(\Phi(y))| \leq \tilde{K}\|x-y\|_{\mathbb{R}^2}^{\gamma}.
$$

Note that by Theorem 5.3 we actually obtained a modification of $\hat{T} \circ \Phi$ on D. But since Φ is a bijective, measurable mapping we tacitly pre-composed it with Φ and interpreted it as random field on $\Phi(D)$ and as a modification of $\hat{T}$. This statement can be reformulated using Equation (5.14) and the fact that Φ is a bijective mapping. For all $\hat{x}, \hat{y} \in \Phi(D)$ that fulfill $d(\hat{x}, \hat{y}) < h^*$ P-a.s. it holds that

$$|\tilde{T}(\hat{x}) - \tilde{T}(\hat{y})| \leq \tilde{K} d(\hat{x}, \hat{y})^\gamma. \tag{5.18}$$

So we have shown that $\hat{T}$ is P-a.s. locally γ-Hölder continuous for all $\gamma \in (0, \frac{\delta}{2})$.

This argument can be repeated with other choices of Φ_i for $i = 1, ..., 6$ to cover the whole sphere such that some of the $\Phi_i(D)$ overlap, i.e. $\Phi_i(D) \cap \Phi_j(D) \neq \emptyset$ for at least some $j \neq i$. These choices of mappings are listed at the end of the proof. This gives us six random fields $(\tilde{T}_i : i = 1, ..., 6)$, which are modifications of $(T|_{\Phi_i(D)} : i = 1, ..., 6)$ respectively. Also there are measurable sets $(\Omega_i^* : i = 1, ..., 6)$ of full probability such that $\tilde{T}_i(\omega)$ is locally γ-Hölder continuous for all $\omega \in \Omega_i^*$. The interior of D is denoted by D°. The relatively open cover $(\Phi_i(D^\circ) : i = 1, ..., 6)$ fulfills the conditions of Theorem 1.11 in [26] for the existence of a smooth partition of unity $(\Psi_i : i = 1, ..., 6)$ for this relatively open cover of S^2. In particular it holds that for all i, Ψ_i is compactly supported in $\Phi_i(D^\circ)$. The support of a function Ψ is denoted by $\text{supp}(\Psi)$. We can use it to define

$$T^* = \begin{cases} \sum_{i=1}^6 \Psi_i \tilde{T}_i & \text{on } \bigcap_{i=1}^6 \Omega_i^*, \\ 0 & \text{else.} \end{cases}$$

We have to check that T^* is a modification of T and that it is γ-Hölder continuous. Since $\bigcap_{i=1}^6 \Omega_i^*$ has full probability T^* is a modification of T by construction.
Note that there are also six different constants $(\tilde{K}_i : i = 1, ..., 6)$ and random variables $(h_i^* : i = 1, ..., 6)$ which belong to the $(\tilde{T}_i : i = 1, ..., 6)$. We choose the global constant K and global h^* in the following way taking into account the overlap of the domains:

$$\begin{aligned} \rho_{i,j} &= \text{radius of maximal inscribed circle with respect to the metric } d(.,.) \\ &\quad \text{in } \Phi_i(D) \cap \Phi_j(D), \text{ whenever } \Phi_i(D) \cap \Phi_j(D) \neq \emptyset \text{ and } i, j = 1, ..., 6, \\ K &= \max_{i=1,..,6} \{\tilde{K}_i\}, \\ h^* &= \min_{i,j:\Phi_i(D)\cap\Phi_j(D)\neq\emptyset} \{h_i^*, \rho_{i,j}\}. \end{aligned}$$

It is clear that $\min_{i,j:\Phi_i(D)\cap\Phi_j(D)\neq\emptyset}\{\rho_{i,j}\} > 0$, therefore also h^* is a P-a.s. strictly positive random variable. Note that if $i, j \in \{1, ..., 6\}$ such that $\Phi_i(D) \cap \Phi_j(D) \neq \emptyset$, then for $x \in \Phi_i(D) \cap \Phi_j(D)$ it holds that P-a.s. $\tilde{T}_i(x) = T|_{\Phi_i(D)\cap\Phi_j(D)}(x) = \tilde{T}_j(x)$. Furthermore, because countable intersections of sets of full probability still have full probability it holds that P-a.s. $\tilde{T}_i(x) = \tilde{T}_j(x)$ for all $x \in \Phi_i(D) \cap \Phi_j(D) \cap \mathbb{Q}^3$. Now we know that $\tilde{T}_i$ and $\tilde{T}_j$ are continuous, so they are already determined on a dense subset of their domain. This implies that P-a.s. $\tilde{T}_i(x) = \tilde{T}_j(x)$ for all $x \in \Phi_i(D) \cap \Phi_j(D)$. Therefore, we can introduce the set of full probability $\tilde{\Omega}$ such that for $i, j \in \{1, ..., 6\}$ with $\Phi_i(D) \cap \Phi_j(D) \neq \emptyset$ it holds that $\tilde{T}_i|_{\Phi_j(D)} = \tilde{T}_j|_{\Phi_i(D)}$ on $\tilde{\Omega}$.
We check the local γ-Hölder continuity. For $\omega \in (\bigcap_{i=1}^6 \Omega_i^*) \cap \tilde{\Omega}$ we choose $x \neq y \in S^2$ such that $d(x, y) < h^*(\omega)$, then there is $j \in \{1, ..., 6\}$ such that $x, y \in \Phi_j(D)$. In the first step we insert the definition of T^*. In the second step we use the property of the supports of

the partition of unity. In the third step we apply that for $x \in \Phi_i(D) \cap \Phi_j(D)$ it holds that $\tilde{T}_i(x) = \tilde{T}_j(x)$ on $\tilde{\Omega}$. On the complement of $\tilde{\Omega}$ we additionally set $T^* = 0$. So we obtain using Inequality (5.18) and the property that the partition of unity sums up to one that

$$\begin{aligned}
|T^*(x,\omega) - T^*(y,\omega)| &= \left| \sum_{i=1}^{6} \Psi_i(x)\tilde{T}_i(x,\omega) - \Psi_i(y)\tilde{T}_i(y,\omega) \right| \\
&= \left| \sum_{i:x\in\mathrm{supp}(\Psi_i)} \Psi_i(x)\tilde{T}_i(x,\omega) - \sum_{i:y\in\mathrm{supp}(\Psi_i)} \Psi_i(y)\tilde{T}_i(y,\omega) \right| \\
&= \left| \sum_{i:x\in\mathrm{supp}(\Psi_i)} \Psi_i(x)\tilde{T}_j(x,\omega) - \sum_{i:y\in\mathrm{supp}(\Psi_i)} \Psi_i(y)\tilde{T}_j(y,\omega) \right| \\
&= |\tilde{T}_j(x,\omega) - \tilde{T}_j(y,\omega)| \leq K d(x,y)^\gamma.
\end{aligned}$$

Hence the local γ-Hölder continuity has been proven. Note that for the set of full probability Ω^*, which was announced in the theorem, we can take $\Omega^* = (\bigcap_{i=1}^{6} \Omega_i^*) \cap \tilde{\Omega}$. To complete the argument we state the other Φ's, note that in the proof $\Phi = \Phi_1$,

$$\begin{aligned}
&\Phi_1 : D \to S^2 && \Phi_2 : D \to S^2 \\
&\quad (x_1,x_2) \mapsto \left(x_1, x_2, \sqrt{1-x_1^2-x_2^2}\right), && \quad (x_1,x_2) \mapsto \left(x_1, x_2, -\sqrt{1-x_1^2-x_2^2}\right), \\
&\Phi_3 : D \to S^2 && \Phi_4 : D \to S^2 \\
&\quad (x_1,x_2) \mapsto \left(x_1, \sqrt{1-x_1^2-x_2^2}, x_2\right), && \quad (x_1,x_2) \mapsto \left(x_1, -\sqrt{1-x_1^2-x_2^2}, x_2\right), \\
&\Phi_5 : D \to S^2 && \Phi_6 : D \to S^2 \\
&\quad (x_1,x_2) \mapsto \left(\sqrt{1-x_1^2-x_2^2}, x_1, x_2\right), && \quad (x_1,x_2) \mapsto \left(-\sqrt{1-x_1^2-x_2^2}, x_1, x_2\right). \quad \square
\end{aligned}$$

Let $\Phi = \Phi_1$ be the function from the proof of the previous theorem. We observe that $\Phi(\mathbb{R}^2) = S^2 \backslash \{(0,0,1)^\top\}$. Then Inequality (5.16) implies that for all $\hat{x}, \hat{y} \in S^2 \backslash \{(0,0,1)^\top\}$ it holds that

$$d(\hat{x},\hat{y}) \leq \frac{\pi}{\sqrt{2}} \|\hat{x} - \hat{y}\|.$$

Since $S^2 \backslash \{(0,0,1)^\top\}$ is dense in S^2, the previous inequality holds for all $\hat{x}, \hat{y} \in S^2$. Therefore, we obtain that for all $x, y \in S^2$ it holds that

$$\|\hat{x} - \hat{y}\| \leq d(\hat{x},\hat{y}) \leq \frac{\pi}{\sqrt{2}} \|\hat{x} - \hat{y}\|. \tag{5.19}$$

Remark 5.5. *Let $(V_i, \alpha_i : i \in \mathcal{I})$ be a finite smooth atlas of S^2. Let $i \in \mathcal{I}$ be arbitrary. There are constants $\hat{k}$ and $\hat{K}$ such that for all $x, y \in \alpha_i^{-1}(V_i)$ it holds that*

$$\hat{k}\|x - y\|_{\mathbb{R}^2} \leq d(\alpha_i(x), \alpha_i(y)) \leq \hat{K}\|x - y\|_{R^2} \tag{5.20}$$

Proof. Without loss of generality we assume that for all $i \in \mathcal{I}$, every two points in $\alpha_i^{-1}(V_i)$ and every two points in V_i are connected by a continuously differentiable curve. Since α_i

is a diffeomorphism, α_i and α_i^{-1} are both Lipschitz continuous, i.e. there exists constants k and K such that for all $x, y \in \alpha_i^{-1}(V_i)$ it holds that

$$k\|x - y\|_{\mathbb{R}^2} \leq \|\alpha(x) - \alpha(y)\| \leq K\|x - y\|_{\mathbb{R}^2}.$$

Together with Inequality (5.19) we obtain the claim. $\square$

6. Differentiability of isotropic Gaussian spherical random fields

In this chapter we want to analyze differentiability of 2-weakly isotropic Gaussian spherical random fields. We remind that a 2-weakly isotropic Gaussian spherical random field T has the following expansion

$$T = \sum_{l\geq 0} \sum_{m=-l}^{l} \sqrt{C_l}\tilde{\beta}_{l,m}\tilde{Y}_{l,m}, \tag{6.1}$$

where the sequence $(C_l : l \geq 0)$ is the angular power spectrum of T, $(\tilde{\beta}_{l,m} : l \geq 0, m = -l, ..., l)$ is an i.i.d. sequence of standard normally distributed random variables and $(\tilde{Y}_{l,m} : l \geq 0, m = -l, ..., l)$ are the real spherical harmonics. The reader is referred to Equation (4.1) and Equation (5.1).

In the previous chapter we showed that under certain conditions on the angular power spectrum $(C_l : l \geq 0)$ of a 2-weakly isotropic Gaussian spherical random field T there exists a modification that is Hölder continuous, i.e. Theorem 5.4 implies that if $(C_l l^{1+\delta} : l \geq 0)$ is summable for some $\delta > 0$, then there exists a continuous modification $\hat{T}$ of T. $\hat{T}$ is also a 2-weakly isotropic random field with an expansion in the spherical harmonics due to Theorem 3.8. We recall that Theorem 3.8 implies that for all $x \in S^2$ the expansion of T in Equation (6.1) evaluated at x converges to $T(x)$ in $L^2_P(\Omega, \mathbb{R})$ and the expansion of T converges to T in $L^2_P(\Omega, L^2(S^2, \mathbb{R}))$. Since for all $x \in S^2$ it holds that P-a.s. $\hat{T}(x) = T(x)$, we conclude that for all $x \in S^2$

$$\begin{aligned}\lim_{L\to\infty} E[|\hat{T}(x) - \sum_{l=0}^{L}\sum_{m=-l}^{l} \sqrt{C_l}\tilde{\beta}_{l,m}\tilde{Y}_{l,m}(x)|^2] &= \lim_{L\to\infty} E[|T(x) - \sum_{l=0}^{L}\sum_{m=-l}^{l} \sqrt{C_l}\tilde{\beta}_{l,m}\tilde{Y}_{l,m}(x)|^2] \\ &= 0.\end{aligned}$$

We further exploit that for all $x \in S^2$ it holds that P-a.s. $\hat{T}(x) = T(x)$ and obtain with Tonelli's theorem that

$$\begin{aligned}&\lim_{L\to\infty} E[\int_{S^2} |\hat{T}(x) - \sum_{l=0}^{L}\sum_{m=-l}^{l} \sqrt{C_l}\tilde{\beta}_{l,m}\tilde{Y}_{l,m}(x)|^2 \mathrm{d}\sigma(x)] \\ &\quad = \lim_{L\to\infty} \int_{S^2} E[|T(x) - \sum_{l=0}^{L}\sum_{m=-l}^{l} \sqrt{C_l}\tilde{\beta}_{l,m}\tilde{Y}_{l,m}(x)|^2]\mathrm{d}\sigma(x) \\ &\quad = \lim_{L\to\infty} E[\int_{S^2} |T(x) - \sum_{l=0}^{L}\sum_{m=-l}^{l} \sqrt{C_l}\tilde{\beta}_{l,m}\tilde{Y}_{l,m}(x)|^2 \mathrm{d}\sigma(x)] = 0.\end{aligned}$$

We have shown that the expansion of T also converges to $\hat{T}$ in $L^2_P(\Omega, L^2(S^2, \mathbb{R}))$ and for all $x \in S^2$ the expansion of T evaluated at $x \in S^2$ converges to $\hat{T}(x)$ in $L^2_P(\Omega, \mathbb{R})$. In the

following chapters we will always consider the continuous modification $\hat{T}$ instead of T and denote the modification also by T. This is justified, since we will always consider 2-weakly isotropic Gaussian spherical random fields, whose angular power spectrum satisfies that $(C_l l^{1+\delta} : l \geq 0)$ is summable for some $\delta > 0$ such that Theorem 5.4 is applicable. We just showed that we can still argue with the same expansion in the real spherical harmonics.

T can be interpreted as a mapping from Ω to $C^0(S^2)$. For all $x \in S^2$ we observe that $T(x)$ is $\mathcal{A} - \mathcal{B}(\mathbb{R})$ measurable, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of $\mathbb{R}$. We introduce the canonical coordinates $(X_x : x \in S^2)$ on $C^0(S^2)$ as mappings from $C^0(S^2)$ to $\mathbb{R}$, i.e. for all $x \in S^2$ and all $w \in C^0(S^2)$ we define

$$X_x(w) = w(x).$$

We use them to define the σ-algebra $\mathcal{F}$ on $C^0(S^2)$ that is induced by the canonical coordinates $(X_x : x \in S^2)$, i.e. we set

$$\mathcal{F} = \sigma(X_x : x \in S^2).$$

Since for all $x \in S^2$ it holds that $X_x \circ T$ is $\mathcal{A} - \mathcal{B}(\mathbb{R})$ measurable, we observe that T is $\mathcal{A} - \mathcal{F}$ measurable.

Lemma 6.1. *Let $\mathcal{B}(C^0(S^2))$ be the Borel σ-algebra of $C^0(S^2)$. It holds that $\mathcal{B}(C^0(S^2)) = \mathcal{F}$.*

Proof. This is analyzed in an abstract framework in the paper of Yan in [28]. Since $C^0(S^2)$ is a separable Banach space, it is also a Polish space. Therefore, we can directly apply Theorem 3 in [28]. We set $X = C^0(S^2)$, $(f_\alpha : \alpha \in \mathcal{I}) = (X_x : x \in S^2)$ and $Y_\alpha = \mathbb{R}$. The mapping $w \mapsto (w(x) : x \in \mathcal{I})$ from $C^0(S^2)$ to $\prod_{\alpha \in \mathcal{I}} Y_\alpha$ is bijective to its image. This property also holds if we take the index set $\mathcal{J} = \mathbb{Q}^3 \cap S^2$ instead of $\mathcal{I}$. Note that $\mathcal{J}$ is a countable subset of $\mathcal{I}$. With this setup Theorem 3 in [28] explicitly implies the claim. □

Therefore we conclude that $P \circ T^{-1}$, which is the law of T, is a probability measure on $(C^0(S^2), \mathcal{B}(C^0(S^2))$. Also the mapping $\omega \mapsto T(\omega)$ from Ω to $C^0(S^2)$ is $\mathcal{A} - \mathcal{B}(C^0(S^2))$ measurable.

For the subsequent analysis we introduce Bochner spaces. Let $\mathcal{X}$ be a separable Banach space with norm $\|\cdot\|_{\mathcal{X}}$, for all $p \in (0, \infty)$ we define the Bochner space $L^p_P(\Omega, \mathcal{X})$ as all strongly measurable, $\mathcal{X}$-valued functions X on $(\Omega, \mathcal{A})$ such that

$$\|X\|_{L^p_P(\Omega,\mathcal{X})} = E\left[\|X\|^p_{\mathcal{X}}\right]^{\frac{1}{p}} < \infty.$$

For the definition of strong measurability we refer to Definition 1 and 2 in Section V.4 in [29] and for the Bochner integral we refer to Section V.5 in [29]. In the following we will mostly encounter the case that X is an $\mathcal{X}$-valued function on Ω such that X is $\mathcal{A} - \mathcal{B}(\mathcal{X})$ measurable, where $\mathcal{B}(\mathcal{X})$ denotes the Borel σ-algebra of the separable Banach space $\mathcal{X}$. We briefly argue that separability of $\mathcal{X}$ implies that X is strongly measurable. For all $\ell \in \mathcal{X}^*$ we observe that $\ell(X)$ is $\mathcal{A} - \mathcal{B}(\mathbb{R})$ measurable. Since the space $\mathcal{X}$ is separable we obtain with Pettis' theorem, which is the main theorem in Section V.4 in [29], that X is strongly measurable. Note that $L^p_P(\Omega, \mathcal{X})$ is a Banach space with norm $\|\cdot\|_{L^p_P(\Omega,\mathcal{X})}$ in the case that $p \in [1, \infty)$.

We observe that in general the realizations of T are P-a.s. in $L^2(S^2, \mathbb{R})$ according to Lemma 3.3 because any 2-weakly isotropic spherical random field satisfies that $(C_l l : l \geq 0)$ is summable. Once we added $\delta \in (0, 2]$ to the exponent, i.e. $(C_l l^{1+\delta} : l \geq 0)$ has to be summable, we observe that due to Theorem 5.4 there exists a Hölder continuous modification. Since we assume that the field T is continuous, it follows that realizations of T are P-a.s. Hölder continuous. A similar approach can be used to show that weak derivatives of T are Hölder continuous. First we prove under which decay of the angular power spectrum weak derivatives of realizations of T are P-a.s. in $L^2(S^2, \mathbb{R})$. In the second step we add $\delta \in (0, 2]$ to the exponent and prove P-a.s. Hölder continuity.

6.1. Sobolev and Hölder spaces on the sphere

In this chapter we will use the interpretation of S^2 as a smooth manifold. An atlas of S^2 is denoted by $(V_i, \beta_i : i \in \mathcal{I})$ such that

$$\bigcup_{i \in \mathcal{I}} V_i = S^2 \qquad \text{and} \qquad \beta_i : \beta_i^{-1}(V_i) \to V_i,$$

for all $i \in \mathcal{I}$. We will always consider smooth atlases in this exposition. For a mapping f on S^2, we introduce the notation that for all $i \in \mathcal{I}$

$$f_{\beta_i} = f \circ \beta_i.$$

For basic properties of an atlas, chart domains and coordinate charts the reader is referred to [26]. Let $(V_i, \beta_i : i \in \mathcal{I})$ be an atlas of S^2. Since $(V_i : i \in \mathcal{I})$ is an open cover of S^2, Theorem 1.11 in [26] implies that there exists a partition of unity subordinate to $(V_i : i \in \mathcal{I})$. Sometimes it is useful to be able to choose a partition of unity with additional properties. The following lemma is motivated by Corollary 1.11 in [26].

Lemma 6.2. *Let $(V_i, \beta_i : i \in \mathcal{I})$ be an atlas of S^2. If $A \subset\subset V_j$ is a relatively closed subset of S^2 for some fixed $j \in \mathcal{I}$, then there exists a partition of unity Ψ subordinate to the open cover $(V_i : i \in \mathcal{I})$ such that $\Psi_j = 1$ on A.*

Proof. The proof is essentially the proof of Corollary 1.11 in [26]. Since $A \subset\subset V_j$ is relatively closed in S^2, it holds that $(V_j, V_i \backslash A : i \in \mathcal{I} \backslash \{j\})$ is an open cover of S^2. We apply Theorem 1.11 in [26] to the open cover $(V_j, V_i \backslash A : i \in \mathcal{I} \backslash \{j\})$ and obtain the partition of unity Ψ subordinate to $(V_j, V_i \backslash A : i \in \mathcal{I} \backslash \{j\})$. For all $i \in \mathcal{I} \backslash \{j\}$ it holds that $supp(\Psi_i) \subset\subset V_i \backslash A$, which implies that $\Psi_i(x) = 0$ for all $x \in A$. Since $\sum_{i \in \mathcal{I}} \Psi_i = 1$ on S^2, it must hold that $\Psi_j(x) = 1$ for all $x \in A$. □

For the Sobolev spaces on the sphere we take Definition 3.23 from [3].

Definition 6.3. *For $s \in \mathbb{R}$ the Sobolev space $H^s(S^2)$ is the completion of $C^\infty(S^2)$ with respect to the following norm. For $f \in C^\infty(S^2)$ the $H^s(S^2)$-norm is defined by*

$$\|f\|_{H^s(S^2)} = \left\| \left(-\Delta_{S^2} + \frac{1}{4} \right)^{\frac{s}{2}} f \right\|_{L^2(S^2, \mathbb{R})}.$$

Note that the operator $(-\Delta_{S^2}+\frac{1}{4})^{\frac{s}{2}}$ is formally evaluated on the real spherical harmonics such that for $l \geq 0$ and $m \in \{-l, ..., l\}$ it holds that

$$\left(-\Delta_{S^2}+\frac{1}{4}\right)^{\frac{s}{2}} \tilde{Y}_{l,m} = \left(l+\frac{1}{2}\right)^{s} \tilde{Y}_{l,m}.$$

Another definition of Sobolev spaces on the sphere is taken from Wloka's book [27]. There it is Definition 4.4 and it applies to sufficiently smooth compact manifolds. It uses an atlas of the manifold, which is the sphere in our case. Note that by shrinking the domain of our coordinates, i.e. $[0,\pi] \times [0, 2\pi)$, to a suitable open subset B, one can construct a finite smooth atlas by rotating our coordinates with suitable elements $g_1, ..., g_6 \in SO(3)$, where we set $\eta_i = g_i x$ and $U_i = \eta_i(B)$ for $i = 1, ...6$, where x denotes the coordinates, which we defined at the very beginning of Chapter 2. We obtain the atlas $(U_i, \eta_i : i = 1, ..., 6)$. A partition of unity $\Psi = (\Psi_i : i = 1, ..., 6)$ subordinate to the open cover $(U_i : i = 1, ..., 6)$ of S^2 exists due to Theorem 1.11 in [26] as at the end of Chapter 5. We will refer to this atlas as our standard or usual atlas on S^2.

Definition 6.4. *For $k \in \mathbb{N}$ and $p \in [1, \infty)$ we say that $f \in L^2(S^2, \mathbb{R})$ belongs to $W^{k,p}(S^2)$ if the functions*

$$(f\Psi_i)_{\eta_i} : \eta_i^{-1}(U_i) \to \mathbb{R},$$

belong to the Sobolev spaces $W_0^{k,p}(\eta_i^{-1}(U_i))$ for all $i \in \{1, .., 6\}$. With $W_0^{k,p}(O)$ we denote the closure of $C_0^\infty(O)$ with respect to the $W^{k,p}(O)$-norm for an open set O. The norm on $W^{k,p}(S^2)$ is given by

$$\|f\|_{W^{k,p}(S^2)} = \left(\sum_{i=1}^{6} \|(f\Psi_i)_{\eta_i}\|^p_{W^{k,p}(\eta_i^{-1}(U_i))}\right)^{\frac{1}{p}}.$$

$W^{k,2}(S^2)$ is a Hilbert space with an inner product. For $f, g \in W^{k,2}(S^2)$ it is defined by

$$(f, g)_k = \sum_{i=1}^{6} ((f\Psi_i)_{\eta_i}, (g\Psi_i)_{\eta_i})_{W^{k,2}(\eta_i^{-1}(U_i))}.$$

For Sobolev spaces over bounded domains in Euclidean space we refer to the book of Adams and Fournier [1] or of Triebel [25]. We did not include the atlas in our notation. This is justified by Satz 4.2 of [27] for the case $p = 2$, which says that the norms for different atlases are equivalent. The general case $p \in [1, \infty)$ of this same statement is Theorem 48.19 in [10]. Situations may arise where one of the two definitions for Sobolev spaces on the sphere is more useful. That is why we seek to prove that one is continuously embedded in the other in the integer case.

Proposition 6.5. *For $k \in \mathbb{N}$ it holds with continuous embedding that $H^k(S^2) \subset W^{k,2}(S^2)$.*

The proof of this proposition will be conducted at the end of this chapter. The following notation is helpful to treat higher order derivatives. For a multi-index $\boldsymbol{\beta} \in \mathbb{N}_0^n$ we write for the partial derivative

$$\partial_{\boldsymbol{\beta}} = \frac{\partial^{|\boldsymbol{\beta}|}}{\partial x_1^{\beta_1} ... \partial x_n^{\beta_n}},$$

where $|\boldsymbol{\beta}| = \sum_{i=1}^n \beta_i$. For two multi-indices $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^n$ we introduce the partial order $\leq$ such that $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ if $\alpha_i \leq \beta_i$ for all $i \in \{1, ..., n\}$. Moreover for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^n$ satisfying $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$

we introduce the binomial coefficient $\binom{\boldsymbol{\beta}}{\boldsymbol{\alpha}} = \prod_{i=1}^{n} \binom{\beta_i}{\alpha_i}$. In the following greek letters in bold font will refer to multi-indices of the respective dimension.
We introduce the Sobolev–Slobodeckij spaces on a bounded domain $D \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$. For an integer $k \geq 0$ and $t \in (0,1)$ that satisfy that $s = k + t$ and $p \in [1, \infty)$, we define for $f \in W^{k,p}(D)$

$$|f|_{W^{s,p}(D)} = \sum_{|\boldsymbol{\beta}|=k} \left(\int_{D \times D} \frac{|\partial_{\boldsymbol{\beta}} f(x) - \partial_{\boldsymbol{\beta}} f(y)|^p}{\|x-y\|_{\mathbb{R}^n}^{pt+n}} \, \mathrm{d}x\mathrm{d}y \right)^{\frac{1}{p}}.$$

Definition 6.6. *Let $s > 0$ be not an integer and satisfies that $s = k + t$ for an integer $k \geq 0$ and $t \in (0,1)$ and let $p \in [1, \infty)$. For a bounded domain $D \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$ the Sobolev–Slobodeckij space $W^{s,p}(D)$ is defined by*

$$W^{s,p}(D) = \left\{ f \in W^{k,p}(D) : |f|_{W^{s,p}(D)} < \infty \right\}$$

and the norm on $W^{s,p}(D)$ for $f \in W^{s,p}(D)$ is given by

$$\|f\|_{W^{s,p}(D)} = \left(\|f\|_{W^{k,p}(D)}^p + |f|_{W^{s,p}(D)}^p \right)^{\frac{1}{p}}.$$

The dual space of $W^{s,p}(D)$ is denoted by $W^{-s,p'}(D)$, where p' satisfies that $\frac{1}{p} + \frac{1}{p'} = 1$.

We introduce general notation for the Hölder norms and spaces on a bounded domain $D \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$. By $C^k(D)$ we denote the real-valued k-times differentiable functions with domain D. Functions in $C^k(D)$ whose partial derivatives can be continuously extended to the boundary ∂D form the space $C^k(\overline{D})$

Definition 6.7. *If $f \in C^k(\overline{D})$ and k is a positive integer, we define the semi-norm*

$$|f|_{k,D} = \sup_{\substack{x \in D \\ |\boldsymbol{\beta}|=k}} |\partial_{\boldsymbol{\beta}} f(x)|,$$

and the norm

$$\|f\|_{C^k(\overline{D})} = \sup_{x \in D} |f(x)| + \sum_{j=1}^{k} |f|_{j,D}.$$

If $f \in C^k(\overline{D})$ and k is a positive integer, we define for $\gamma \in (0,1]$ the semi-norm

$$|f|_{\gamma,k,D} = \sup_{\substack{x,y \in D,\, x \neq y \\ |\boldsymbol{\beta}|=k}} \frac{|\partial_{\boldsymbol{\beta}} f(x) - \partial_{\boldsymbol{\beta}} f(y)|}{\|x-y\|_{\mathbb{R}^n}^{\gamma}},$$

and the norm

$$\|f\|_{C^{k,\gamma}(\overline{D})} = \|f\|_{C^k(\overline{D})} + |f|_{\gamma,k,D}.$$

For an integer $k \geq 0$ and $\gamma \in (0,1]$ we define the Hölder space

$$C^{k,\gamma}(\overline{D}) = \{ f \in C^k(\overline{D}) : \|f\|_{C^{k,\gamma}(D)} < \infty \}.$$

Note that vector-valued Hölder functions and spaces are defined accordingly, i.e $C^{k,\gamma}(\overline{D},\overline{O})$ denotes the space of functions whose components are in $C^{k,\gamma}(\overline{D})$ and whose range is included in the closure of a domain $O \subset \mathbb{R}^m$ for some $m \in \mathbb{N}$. Also we introduce the notation $C^k(\overline{D}) = C^{k,0}(\overline{D})$. If D is a bounded Lipschitz domain Remark 16.3 (ii) in Section 16.1.1 in [7] implies that the following norms on $C^k(\overline{D})$ and $C^{k,\gamma}(\overline{D})$ are equivalent: for $f \in C^k(\overline{D})$

$$\|f\|_{C^k(\overline{D})} \simeq \|f\|_{C^0(\overline{D})} + |f|_{k,D} \tag{6.2}$$

and for $f \in C^{k,\gamma}(D)$

$$\|f\|_{C^{k,\gamma}(\overline{D})} \simeq \|f\|_{C^0(\overline{D})} + |f|_{\gamma,k,D}. \tag{6.3}$$

The product and composition of two Hölder functions result again a Hölder function. If D is a bounded Lipschitz domain, and $g, h \in C^{k,\alpha}(\overline{D})$ for an integer $k \geq 0$ and $\alpha \in (0,1]$ then Theorem 16.28 in Section 16.5.2 of [7] states that there exists a constant $K > 0$ depending on k and D such that

$$\|gh\|_{C^{k,\alpha}(\overline{D})} \leq K \left(\|g\|_{C^{k,\alpha}(\overline{D})} \|h\|_{C^0(\overline{D})} + \|g\|_{C^0(\overline{D})} \|h\|_{C^{k,\alpha}(\overline{D})} \right). \tag{6.4}$$

For the statement about the composition of two Hölder functions let $D \subset \mathbb{R}^n$ and $O \subset \mathbb{R}^m$ be two bounded open Lipschitz domains, $\alpha, \beta \in [0,1]$ and $g \in C^{k,\alpha}(\overline{D})$ for some integer $k \geq 0$. The following two statements are due to Theorem 16.31 in Section 16.6.1 of [7]. If $k = 0$ and $h \in C^{k,\beta}(\overline{O},\overline{D})$, then

$$\|g \circ h\|_{C^{0,\alpha\beta}(\overline{O})} \leq \|g\|_{C^{0,\alpha}(\overline{D})} \|h\|^{\alpha}_{C^{0,\beta}(\overline{O},\overline{D})} + \|g\|_{C^0(\overline{D})}. \tag{6.5}$$

If $k \geq 1$ and $h \in C^{k,\alpha}(\overline{O},\overline{D})$, then there exists a constant $K > 0$ depending on k, O and D such that

$$\|g \circ h\|_{C^{k,\alpha}(\overline{O})} \leq K \left(\|g\|_{C^{k,\alpha}(\overline{D})} \|h\|^{k+\alpha}_{C^{1,\beta}(\overline{O},\overline{D})} + \|g\|_{C^1(\overline{D})} \|h\|_{C^{k+\alpha}(\overline{O},\overline{D})} + \|g\|_{C^0(\overline{D})} \right). \tag{6.6}$$

The Hölder spaces on the sphere are defined similarly to the Sobolev spaces with respect to an atlas.

Definition 6.8. *For $k \in \mathbb{N}$, $\alpha \in (0,1)$ and a finite, smooth atlas $(V_i, \beta_i : i \in \mathcal{I})$ of S^2 with respective partition of unity Ψ the Hölder space $C^{k,\alpha}_\beta(S^2)$ is defined as the set of functions $f \in C^k(S^2)$ such that*

$$\|f\|_{C^{k,\alpha}_\beta(S^2)} = \max_{i \in \mathcal{I}} \|(f\Psi_i)_{\beta_i}\|_{C^{k,\alpha}(\overline{\beta_i^{-1}(V_i)})} < \infty.$$

The following proposition justifies this choice of norm and shows that for different atlases the same space results.

Proposition 6.9. *For $k \in \mathbb{N}$, $\alpha \in (0,1)$ and two finite atlases $(V_i, \beta_i : i \in \mathcal{I})$ and $(W_j, \tau_j : j \in \mathcal{J})$ the spaces $C^{k,\alpha}_\beta(S^2)$ and $C^{k,\alpha}_\tau(S^2)$ are equal with equivalent norms.*

Proof. Let Ψ and Φ be partitions of unity subordinate to the open cover $(V_i : i \in \mathcal{I})$ and $(W_j : j \in \mathcal{J})$. It is sufficient to prove that there exists a constant $K > 0$ such that for all function $f \in C^{k,\alpha}_\tau(S^2)$ and for all $i \in \mathcal{I}$ it holds that

$$\|(f\Psi_i)_{\beta_i}\|_{C^{k,\alpha}(\overline{\beta_i^{-1}(V_i)})} \leq K \|f\|_{C^{k,\alpha}_\tau(S^2)}.$$

We fix $i \in \mathcal{I}$. From the definition of a smooth atlas we obtain that the functions $\tau_j^{-1} \circ \beta_i$ are smooth diffeomorphisms from $\beta_i^{-1}(V_i \cap W_j)$ to $\tau_j^{-1}(V_i \cap W_j)$ for all $j \in \mathcal{J}$. In particular these functions are in $C^{l,\alpha}(\beta_i^{-1}(V_i \cap W_j), \tau_j^{-1}(V_i \cap W_j))$ for all $l \in \mathbb{N}$ and their respective norms can be bounded uniformly due to the finite index set $\mathcal{J}$ with a constant $K_l > 0$. Since also $(f\Psi_i\Phi_j)_{\tau_j} \in C^{k,\alpha}(A)$ for some compact set A that satisfies that $supp((\Phi_j)_{\tau_j}) \subset A \subset \tau_j^{-1}(W_j)$ we can apply Inequality (6.5) and Inequality (6.6) to obtain that for a constant $\tilde{K} > 0$ depending on k and the domains it holds that

$$\|(f\Psi_i\Phi_j) \circ \tau_j \circ (\tau_j^{-1} \circ \beta_i)\|_{C^{k,\alpha}(\overline{\beta_i^{-1}(V_i \cap W_j)})} \leq \tilde{K}(K_{\max\{1,k\}}^{k+1} + 1)\|(f\Psi_i\Phi_j) \circ \tau_j\|_{C^{k,\alpha}(A)},$$

where we also used Inequality (6.3) to simplify the right hand side. We use this inequality together with the basic property of the partition of unity to obtain that

$$\begin{aligned}
\|(f\Psi_i)_{\beta_i}\|_{C^{k,\alpha}(\overline{\beta_i^{-1}(V_i)})} &\leq \sum_{j \in \mathcal{J}} \|(f\Psi_i\Phi_j)_{\beta_i}\|_{C^{k,\alpha}(\overline{\beta_i^{-1}(V_i \cap W_j)})} \\
&= \sum_{j \in \mathcal{J}} \|(f\Psi_i\Phi_j) \circ \tau_j \circ (\tau_j^{-1} \circ \beta_i)\|_{C^{k,\alpha}(\overline{\beta_i^{-1}(V_i \cap W_j)})} \\
&\leq \sum_{j \in \mathcal{J}} \tilde{K}(K_{\max\{1,k\}}^{k+1} + 1)\|(f\Psi_i\Phi_j)_{\tau_j}\|_{C^{k,\alpha}(A)}.
\end{aligned}$$

We apply Inequality (6.4) and Inequality (6.3) to the product $(f\Phi_j)_{\tau_j} \cdot (\Psi_i)_{\tau_j}$ and insert the bound into the above expression to obtain that

$$\|(f\Psi_i)_{\beta_i}\|_{C^{k,\alpha}(\overline{\beta_i^{-1}(U_i)})} \leq \sum_{j \in \mathcal{J}} K\|(f\Phi_j)_{\tau_j}\|_{C^{k,\alpha}(\overline{\tau_j^{-1}(W_j)})} \leq |\mathcal{J}|K\|f\|_{C_\tau^{k,\alpha}(S^2)},$$

where we tacitly combined the former constants together with the respective norms of the smooth functions $(\Psi_i)_{\tau_j}$ for $i \in \mathcal{I}$ and $j \in \mathcal{J}$ to a new constant K. □

For these Hölder and Sobolev spaces on S^2 we want to prove a Sobolev embedding theorem.

Theorem 6.10. *Let $k, \iota \in \mathbb{N}_0$, $p \in (1, \infty)$ and $\gamma \in [0, 1)$. If $k - \frac{2}{p} > \iota + \gamma$, then $W^{k,p}(S^2) \subset C^{\iota,\gamma}(S^2)$ with continuous embedding. If $\gamma \in (0, 1)$, then the continuous embedding also holds if $k - \frac{2}{p} = \iota + \gamma$.*

Proof. Since the norms of the Hölder and Sobolev spaces with respect to two different atlases are equivalent, we can prove the claim with our usual finite atlas $(U_i, \eta_i : i = 1, ..., 6)$ with partition of unity Ψ. For $f \in W^{k,p}(S^2)$ we obtain that

$$f = \sum_{i=1}^{6} f\Psi_i.$$

We fix $i \in \{1, ..., 6\}$ and observe that $(f\Psi_i)_{\eta_i} \in W^{k,p}(\eta_i^{-1}(U_i))$. The Sobolev embedding theorem for domains in Euclidean space, which is Theorem 4.6.1.(c) in [25] implies that $(f\Psi_i)_{\eta_i}$ has a continuous representative in $C^{\iota,\gamma}(\overline{\eta_i^{-1}(U_i)})$, which we also denote by $(f\Psi_i)_{\eta_i}$. Also it implies that for a constant K, which is independent of f, it holds that

$$\|(f\Psi_i)_{\eta_i}\|_{C^{\iota,\gamma}(\overline{\eta_i^{-1}(U_i)})} \leq K\|(f\Psi_i)_{\eta_i}\|_{W^{k,p}(\eta_i^{-1}(U_i))} \leq K\|f\|_{W^{k,p}(S^2)},$$

where the last inequality is due to Definition 6.4. Since $i \in \{1, ..., 6\}$ was arbitrarily chosen, we obtain that

$$\|f\|_{C^{\iota,\gamma}(S^2)} \leq K\|f\|_{W^{k,p}(S^2)}.$$

□

Now the Hölder spaces on S^2 are well-defined. We finish this preliminary section with a remark on the Borel σ-algebras of the Hölder spaces.

Remark 6.11. *We can apply the proof of Lemma 6.1 in the case that we analyze the Hölder space $C^{k,\gamma}(S^2)$ for an integer $k \geq 0$ and $\gamma \in (0,1)$. We conclude that the Borel σ-algebra of $C^{k,\gamma}(S^2)$ is equal to the σ-algebra that is induced by the canonical coordinates $(X_x : x \in S^2)$, i.e. $\mathcal{B}(C^{k,\gamma}(S^2)) = \mathcal{F}$. Therefore we obtain that $\mathcal{B}(C^{k,\gamma}(S^2)) = \mathcal{B}(C^0(S^2))$ for every integer $k \geq 0$ and $\gamma \in (0,1)$.*

6.2. First order derivatives of isotropic Gaussian spherical random fields

For a 2-weakly isotropic Gaussian spherical random field the next lemma states how fast its angular power spectrum has to decay such that its expansion converges in the $L_P^{2k}(\Omega, H^s(S^2))$-norm for all $k \in \mathbb{N}$. The case $k = 1$ is the first part of the proof of Theorem 4.5 in [20].

Lemma 6.12. *Let T be a 2-weakly isotropic Gaussian spherical random field such that its angular power spectrum satisfies that $(C_l l^{1+2s} : l \geq 0)$ is summable for some $s > 0$, then T is an element of $L_P^{2k}(\Omega, H^s(S^2))$ for all $k \in \mathbb{N}$ and its expansion in the real spherical harmonics converges in the respective norm and there exists a constant K_k independently of T and $(C_l : l \geq 0)$ such that*

$$\|T\|_{L_P^{2k}(\Omega, H^s(S^2))} \leq K_k \left(\sum_{l \geq 0} C_l l^{1+2s} \right)^{\frac{1}{2}}.$$

Proof. We denote with T^L the truncated expansion of T for $L \in \mathbb{N}_0$ as we did in the previous chapters. For $L_1 > L_2 \in \mathbb{N}_0$ we look at the difference of two truncations of T. With a similar argument as used to show Inequality (5.9) we obtain that

$$\begin{aligned}
\|T^{L_1} - T^{L_2}\|^{2k}_{L_P^{2k}(\Omega, H^s(S^2))} &= E\left[\left(\sum_{l=L_2+1}^{L_1} \sum_{m=-l}^{l} C_l \tilde{\beta}^2_{l,m} \left(l + \frac{1}{2} \right)^{2s} \right)^k \right] \\
&\leq \frac{(2k)!}{2^k k!} \left(\sum_{l=L_2+1}^{L_1} \sum_{m=-l}^{l} C_l \left(l + \frac{1}{2} \right)^{2s} \right)^k \\
&= \frac{(2k)!}{2^k k!} \left(\sum_{l=L_2+1}^{L_1} C_l (2l+1) \left(l + \frac{1}{2} \right)^{2s} \right)^k \\
&\leq K_k \left(\sum_{l=L_2+1}^{L_1} C_l l^{1+2s} \right)^k. \qquad (6.7)
\end{aligned}$$

Since the sequence $(C_l l^{1+2s} : l \geq 0)$ is summable this calculation implies that $(T^L : L \geq 0)$ is a Cauchy sequence in the space $L_P^{2k}(\Omega, H^s(S^2))$. The completeness of this space implies that T^L converges to T in $L_P^{2k}(\Omega, H^s(S^2))$ as $L \to \infty$.
The second claim of the lemma is proven in the same way as we proved Inequality (6.7). □

An important ingredient of this discussion are partial derivatives of the real spherical harmonics, which is the content of the following lemma.

Lemma 6.13. *For $l \geq 0$ and $m \in \{-l, ..., l\}$ the components of the partial derivatives of the real spherical harmonics have the following form:*

$$\partial_\varphi \tilde{Y}_{l,m}(x) = -m\tilde{Y}_{l,-m}(x)$$

and

$$\begin{aligned}\partial_\theta \tilde{Y}_{l,m}(x) = m&\sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}}\frac{\tilde{Y}_{l+1,m}(x)}{\sin(\theta)} \\ &+ m\sqrt{\frac{l^2 - m^2}{(2l+1)(2l-1)}}\frac{\tilde{Y}_{l-1,m}(x)}{\sin(\theta)} \\ &+ \begin{cases}\sqrt{(l+m+1)(l-m)}\tilde{Y}_{l,m+1}(x) & \text{if } m \geq 0, \\ \sqrt{(l+|m|+1)(l-|m|)}\tilde{Y}_{l,m-1}(x) & \text{if } m < 0.\end{cases}\end{aligned}$$

Proof. The expression of the real spherical harmonics, i.e. Equation (5.3), implies the first claim after a simple computation of the partial derivative with respect to φ.
The second claim of the lemma is a bit more involved. We start to compute $\partial_\theta \tilde{Y}_{l,m}(x)$. The part of $\tilde{Y}_{l,m}(x)$ that depends on θ is equal to $\sin^m(\theta)\frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}(x^2-1)^l|_{x=\cos(\theta)}$. For the partial derivative we obtain that

$$\begin{aligned}\partial_\theta \sin^m(\theta)\frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}(x^2-1)^l|_{x=\cos(\theta)} &= m\sin^{m-1}(\theta)\cos(\theta)\frac{\mathrm{d}^{l+m}}{\mathrm{d}x^{l+m}}(x^2-1)^l\Big|_{x=\cos(\theta)} \\ &\quad - \sin^{m+1}(\theta)\frac{\mathrm{d}^{l+m+1}}{\mathrm{d}x^{l+m+1}}(x^2-1)^l\Big|_{x=\cos(\theta)}.\end{aligned}$$

If we regard the full expression of the real spherical harmonics, that is Equation (5.3), we obtain the partial derivative with respect to θ, i.e.

$$\partial_\theta \tilde{Y}_{l,m}(x) = m\frac{\cos(\theta)}{\sin(\theta)}\tilde{Y}_{l,m}(x) + \begin{cases}\sqrt{(l+m+1)(l-m)}\tilde{Y}_{l,m+1}(x) & \text{if } m \geq 0, \\ \sqrt{(l+|m|+1)(l-|m|)}\tilde{Y}_{l,m-1}(x) & \text{if } m < 0.\end{cases}$$

Note that the ambiguity for the case $|m| = l$ is tacitly dealt with the factor $\sqrt{l-|m|}$, that becomes equal to zero in these cases. This is correct since the partial derivative vanishes. Ambiguities of this kind will be tacitly dealt with in this manner throughout this proof. The appendix of [21] in particular Chapter 13.1.3 in [21] contains the following recurrence relation for the associated Legendre polynomials just below Equation (13.7) in [21]: for $l \geq 0$ and $m \in \{-l, ..., l\}$ it holds that

$$(2l+1)xP_m^l = (l-m+1)P_m^{l+1} + (l+m)P_m^{l-1}.$$

This recurrence relation is our approach to simplify $\cos(\theta)\tilde{Y}_{l,m}$. We use Equation (5.3) and manipulate the constants to obtain that

$$\begin{aligned}&\cos(\theta)\tilde{Y}_{l,m} \\ &= \sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-|m|)!}{(l+|m|)!}}\frac{(-1)^m\sin^{|m|}(\theta)}{2l+1}((l-|m|+1)P_{|m|}^{l+1}(\cos(\theta)) + (l+|m|)P_{|m|}^{l-1}(\cos(\theta)))\end{aligned}$$

$$= \sqrt{\frac{2l+3}{4\pi}}\sqrt{\frac{(l+1-|m|)!}{(l+1+|m|)!}}\sqrt{\frac{(l+1+|m|)(l+1-|m|)}{(2l+1)(2l+3)}}(-1)^m \sin^{|m|}(\theta)P_m^{l+1}(\cos(\theta)) + \dots$$

$$= \sqrt{\frac{(l+1+|m|)(l+1-|m|)}{(2l+1)(2l+3)}}\tilde{Y}_{l+1,m} + \sqrt{\frac{(l+|m|)(l-|m|)}{(2l+1)(2l-1)}}\tilde{Y}_{l-1,m}, \tag{6.8}$$

where we omitted the similar derivation of $\tilde{Y}_{l-1,m}$. Equation (6.8) enables us to obtain the expression of the partial derivative with respect to θ, that we were looking for. □

With the help of the previous lemma we arrive at the version of Lemma 5.2 for the first order partial derivatives of a 2-weakly isotropic Gaussian spherical random field.

Lemma 6.14. *Let T be a 2-weakly isotropic Gaussian spherical random field such that the angular power spectrum $(C_l : l \geq 0)$ satisfies that $(C_l l^{1+2+\delta} : l \geq 0)$ is summable for some $\delta \in (0, 2]$. For $i \in \{1, ..., 6\}$ and $x, y \in U_i$ and all $k \in \mathbb{N}$ there exists a constant K_k, which depends only on k, such that for $\beta \in \{\theta, \varphi\}$ it holds that*

$$E\left[|\partial_\beta T_{\eta_i}(\theta_x, \varphi_x) - \partial_\beta T_{\eta_i}(\theta_x, \varphi_x)|^{2k}\right] \leq K_k \left(\sum_{l\geq 0} C_l l^{1+2+\delta}\right)^k d(x, y)^{\delta k}.$$

Proof. We start with the case $k = 1$ and $i = 1$, then η_1 is equal to our usual coordinates on the sphere. Therefore we apply the notation $\eta_1(\theta_x, \varphi_x) = x$ and $\eta_1(\theta_y, \varphi_y) = y$. We are allowed to interchange the partial derivative and limit of the expansion of $T(x)$ and $T(y)$ because from Lemma 6.12 we know that $T \in L^2_P(\Omega, H^1(S^2))$ and then Proposition 6.5 implies that $T \in L^2_P(\Omega, W^{1,2}(S^2))$, i.e. the weak derivatives of T are well defined in the L^2_P-sense. This enables us to shift the discussion on how partial derivatives behave on the real spherical harmonics, i.e. we obtain that it holds that

$$\begin{aligned} E\left[|\partial_\beta T(x) - \partial_\beta T(y)|^2\right] &= E\left[\left|\sum_{l\geq 0}\sum_{m=-l}^{l}\sqrt{C_l}\tilde{\beta}_{l,m}(\partial_\beta \tilde{Y}_{l,m}(x) - \partial_\beta \tilde{Y}_{l,m}(y))\right|^2\right] \\ &= \sum_{l\geq 0}\sum_{m=-l}^{l} C_l(\partial_\beta \tilde{Y}_{l,m}(x) - \partial_\beta \tilde{Y}_{l,m}(y))^2. \end{aligned} \tag{6.9}$$

In the following we study the term $(\partial_\beta \tilde{Y}_{l,m}(x) - \partial_\beta \tilde{Y}_{l,m}(y))^2$. We treat the two cases $\beta = \theta, \varphi$ separately and begin with $\beta = \varphi$. We apply Lemma 6.13 to obtain that

$$\begin{aligned} \left(\partial_\varphi \tilde{Y}_{l,m}(x) - \partial_\varphi \tilde{Y}_{l,m}(y)\right)^2 &= m^2(\tilde{Y}_{l,-m}(x) - \tilde{Y}_{l,-m}(y))^2 \\ &\leq l^2(\tilde{Y}_{l,-m}(x) - \tilde{Y}_{l,-m}(y))^2. \end{aligned}$$

We sum this expression multiplied with the angular power spectrum $(C_l : l \geq 0)$ over l and m and obtain with Equation (6.9) that

$$\begin{aligned} E\left[|\partial_\varphi T(x) - \partial_\varphi T(y)|^2\right] &\leq \sum_{l\geq 0}\sum_{m=-l}^{l} C_l l^2(\tilde{Y}_{l,m}(x) - \tilde{Y}_{l,m}(y))^2 \\ &= E\left[|\hat{T}(x) - \hat{T}(y)|^2\right], \end{aligned}$$

where $\hat{T}$ is the 2-weakly isotropic Gaussian spherical random field that results from the angular power spectrum $(\hat{C}_l : l \geq 0) = (C_l l^2 : l \geq 0)$. Note that by assumption the sequence $(\hat{C}_l l^{1+\delta} : l \geq 0)$ is summable. Therefore Lemma 5.2 implies that there exists a constant $\hat{K} > 0$ such that

$$E\left[|\partial_\varphi T(x) - \partial_\varphi T(y)|^2\right] \leq E\left[|\hat{T}(x) - \hat{T}(y)|^2\right] \leq \hat{K} \sum_{l \geq 0} C_l l^{1+2+\delta} d(x,y)^\delta.$$

The fact is noteworthy that for $a_1, ..., a_q \geq 0$ and $h, q \in \mathbb{N}$ it holds that

$$\Big(\sum_{j=1}^{q} a_j\Big)^h \leq q^{h-1}\Big(\sum_{j=1}^{q} a_j^h\Big). \tag{6.10}$$

For the discussion of the case $\beta = \theta$ we note that the factors $\sqrt{\frac{(l+1)^2-m^2}{(2l+1)(2l+3)}}$ and $\sqrt{\frac{l^2-m^2}{(2l+1)(2l-1)}}$ which appear in Lemma 6.13 are bounded independently of l and m, say they are both bounded by $K_0 > 0$. Also $\sqrt{(l+|m|+1)(l-|m|)}$ can be bounded by a constant multiplied by l, say the constant is K_0 as well. According to Lemma 6.13 and Inequality (6.10) the difference of the partial derivatives with respect to θ squared can be bounded in this way:

$$\Big(\partial_\theta \tilde{Y}_{l,m}(x) - \partial_\theta \tilde{Y}_{l,m}(y)\Big)^2 \leq 3K_0^2 m^2 \left(\frac{\tilde{Y}_{l+1,m}(x)}{\sin(\theta_x)} - \frac{\tilde{Y}_{l+1,m}(y)}{\sin(\theta_y)}\right)^2 \tag{6.11}$$

$$+ 3K_0^2 m^2 \left(\frac{\tilde{Y}_{l-1,m}(x)}{\sin(\theta_x)} - \frac{\tilde{Y}_{l-1,m}(y)}{\sin(\theta_y)}\right)^2 (1 - \delta_{l,|m|}) \tag{6.12}$$

$$+ 3K_0^2 l^2 (1 - \delta_{l,|m|}) \begin{cases} (\tilde{Y}_{l,m+1}(x) - \tilde{Y}_{l,m+1}(y))^2 & \text{if } m \geq 0, \\ (\tilde{Y}_{l,m-1}(x) - \tilde{Y}_{l,m-1}(y))^2 & \text{if } m < 0. \end{cases} \tag{6.13}$$

The Expression (6.13) is a difference of real spherical harmonics as in the case $\beta = \varphi$ and can be treated in the same way. Before we begin to bound the Expressions (6.11) and (6.12) , we briefly discuss a needed inequality. Let f and g be two real-valued functions and $a, b > 0$ two constants, then it holds that

$$a(f-g) = (fa - gb) + g(b-a) \qquad \text{and} \qquad b(f-g) = (fa-gb) + f(b-a).$$

the addition of these two equations and the triangle inequality yields that

$$|f-g| \leq \frac{2}{|a+b|}|fa - gb| + \frac{|f+g|}{|a+b|}|a-b|. \tag{6.14}$$

This inequality will be needed to treat the factor $\frac{1}{\sin(\theta)}$, that appears in the partial derivative with respect to θ. Note that on $\eta_1^{-1}(U_1)$ the function $\sin(\theta) \geq \varepsilon$ for some $\varepsilon > 0$. To ease the notation we will seek to bound

$$\sum_{l \geq 0} C_l \sum_{m=-l}^{l} l^2 \left(\frac{\tilde{Y}_{l,m}(x)}{\sin(\theta_x)} - \frac{\tilde{Y}_{l,m}(y)}{\sin(\theta_y)}\right)^2$$

in terms of $d(x,y)^\delta$. Since sine is Lipschitz continuous with Lipschitz constant equal to one, we obtain with Inequalities (6.14) and (6.10) for an integer $l \geq 0$ and $m \in \{-l, ..., l\}$ that

$$l^2 \left(\frac{\tilde{Y}_{l,m}(x)}{\sin(\theta_x)} - \frac{\tilde{Y}_{l,m}(y)}{\sin(\theta_y)} \right)^2 \leq l^2 \frac{4}{\varepsilon^2} (\tilde{Y}_{l,m}(x) - \tilde{Y}_{l,m}(y))^2 \tag{6.15}$$

$$+ 2l^2 \frac{\tilde{Y}_{l,-m}(x)^2 + \tilde{Y}_{l,-m}(y)^2}{\varepsilon^2} |\theta_x - \theta_y|^2. \tag{6.16}$$

When we sum Expression (6.15) multiplied with the angular power spectrum over l and m, we obtain that

$$\sum_{l \geq 0} C_l \sum_{m=-l}^{l} l^2 \frac{4}{\varepsilon^2} (\tilde{Y}_{l,m}(x) - \tilde{Y}_{l,m}(y))^2 = \frac{4}{\varepsilon^2} E[|\hat{T}(x) - \hat{T}(y)|^2].$$

$\hat{T}$ is again the 2-weakly isotropic Gaussian spherical random field that results from the angular power spectrum $(\hat{C}_l : l \geq 0) = (C_l l^2 : l \geq 0)$. Note that by assumption the sequence $(\hat{C}_l l^{1+\delta} :\geq 0)$ is summable. Therefore Lemma 5.2 implies that there exists a constant $\hat{K} > 0$ such that

$$\sum_{l \geq 0} C_l \sum_{m=-l}^{l} l^2 \frac{4}{\varepsilon^2} (\tilde{Y}_{l,m}(x) - \tilde{Y}_{l,m}(y))^2 \leq \frac{4}{\varepsilon^2} \hat{K} \sum_{l \geq 0} C_l l^{1+2+\delta} d(x,y)^\delta.$$

For Expression (6.16) we derive with Lemma 2.8 and Equation (2.1) that

$$\begin{aligned} \sum_{l \geq 0} C_l \sum_{m=-l}^{l} 2l^2 \frac{\tilde{Y}_{l,m}(x)^2 + \tilde{Y}_{l,m}(y)^2}{\varepsilon^2} |\theta_x - \theta_y|^2 &\leq \sum_{l \geq 0} C_l l^2 \frac{2l+1}{\pi \varepsilon^2} |\theta_x - \theta_y|^2 \\ &\leq \frac{3}{\pi \varepsilon^2} \sum_{l \geq 0} C_l l^{1+2+\delta} d(x,y)^\delta \end{aligned}$$

Therefore, when we sum the left hand side of Expression (6.15) multiplied with the angular power spectrum over l and m, we obtain that

$$\sum_{l \geq 0} C_l \sum_{m=-l}^{l} l^2 \left(\frac{\tilde{Y}_{l,m}(x)}{\sin(\theta_x)} - \frac{\tilde{Y}_{l,m}(y)}{\sin(\theta_y)} \right)^2 \leq \left(\frac{4}{\varepsilon^2} \hat{K} + \frac{3}{\pi \varepsilon^2} \right) \sum_{l \geq 0} C_l l^{1+2+\delta} d(x,y)^\delta. \tag{6.17}$$

We now apply Inequality (6.17) to bound the sum of Expressions (6.11) and (6.12) multiplied by the angular power spectrum over l and m and conclude that

$$\sum_{l \geq 0} C_l \sum_{m=-l}^{l} \left(\partial_\theta \tilde{Y}_{l,m}(x) - \partial_\theta \tilde{Y}_{l,m}(y) \right)^2 \leq K_0^2 \left(\frac{12}{\varepsilon^2} \hat{K} + \frac{3}{\pi \varepsilon^2} + \hat{K} \right) \sum_{l \geq 0} C_l l^{1+2+\delta} d(x,y)^\delta.$$

Therefore we can take $K_1 = K_0^2 (\frac{12}{\varepsilon^2} \hat{K} + \frac{9}{\pi \varepsilon^2} + \hat{K})$ and obtain that

$$E \left[\left| \sum_{l \geq 0} \sum_{m=-l}^{l} \sqrt{C_l} \tilde{\beta}_{l,m} (\partial_\beta \tilde{Y}_{l,m}(x) - \partial_\beta \tilde{Y}_{l,m}(y)) \right|^2 \right] \leq K_1 \sum_{l \geq 0} C_l l^{1+2+\delta} d(x,y)^\delta. \tag{6.18}$$

To complete the first step $k = 1$, we have to argue for the other charts as well, i.e. $i \neq 1$. For $i \neq 1$ the chart η_i satisfies $\eta_i = g_i x = \tilde{x}$ for a non-trivial element $g_i \in SO(3)$. In this proof we had to estimate sums over $m = -l, ..., l$ of $(\partial_\beta \tilde{Y}_{l,m}(x) - \partial_\beta \tilde{Y}_{l,m}(y))^2$. We observe with the help of Lemma 2.7 that the expressions are equal in both coordinates, i.e.

$$\begin{aligned}
\sum_{m=-l}^{l} \left(\partial_\beta \tilde{Y}_{l,m}(\tilde{x}) - \partial_\beta \tilde{Y}_{l,m}(\tilde{y})\right)^2 &= \sum_{m=-l}^{l} \left(\partial_\beta \tilde{Y}_{l,m}(g_i x) - \partial_\beta \tilde{Y}_{l,m}(g_i y)\right)^2 \\
&= \sum_{m=-l}^{l} \left(\sum_{m'=-l}^{l} D^l_{m',m}(g_i^{-1})(\partial_\beta \tilde{Y}_{l,m'}(x) - \partial_\beta \tilde{Y}_{l,m'}(y)) \right)^2 \\
&= \sum_{m'=-l}^{l} \left(\partial_\beta \tilde{Y}_{l,m'}(x) - \partial_\beta \tilde{Y}_{l,m'}(y)\right)^2. \qquad (6.19)
\end{aligned}$$

This means we can apply the same argument as we did before to the expression for the other charts. This finishes the proof of the step $k = 1$.

The general case $k \in \mathbb{N}$ will be proven in a similar manner as the case $k \in \mathbb{N}$ was proven in Lemma 5.2. We start with the respective expression and first interchange the weak partial derivative and the limit of the expansion of T, that is justified since due to Lemma 6.12 and the assumption on the angular power spectrum $T \in L^{2k}_P(\Omega, H^1(S^2))$. In the second step we apply the same argument, which was used to prove Inequality (5.9) to obtain that

$$\begin{aligned}
E\left[|\partial_\beta T(x) - \partial_\beta T(y)|^{2k}\right] &= E\left[\left|\sum_{l\geq 0}\sum_{m=-l}^{l} \sqrt{C_l}\tilde{\beta}_{l,m}(\partial_\beta \tilde{Y}_{l,m}(x) - \partial_\beta \tilde{Y}_{l,m}(y))\right|^{2k}\right] \\
&\leq \frac{(2k)!}{2^k k!}\left(\sum_{l\geq 0}\sum_{m=-l}^{l} C_l(\partial_\beta \tilde{Y}_{l,m}(x) - \partial_\beta \tilde{Y}_{l,m}(y))^2\right)^k.
\end{aligned}$$

We see that we derived the kth power of Equation (6.9), which we bounded in Inequality (6.18). We conclude that

$$\begin{aligned}
E\left[|\partial_\beta T(x) - \partial_\beta T(y)|^{2k}\right] &\leq \frac{(2k)!}{2^k k!} K_1^k \left(\sum_{l\geq 0} C_l l^{1+2+\delta}\right)^k d(x,y)^{\delta k} \\
&= K_k \left(\sum_{l\geq 0} C_l l^{1+2+\delta}\right)^k d(x,y)^{\delta k}. \qquad \square
\end{aligned}$$

The previous lemma gives us the respective ingredient in the proof of Theorem 5.4 of the spherical random field for first order partial derivatives. Therefore we expect a similar result to hold for $\partial_\beta T$, where T is a 2-weakly isotropic Gaussian spherical random field. However the following extension of Lemma 6.12 will be needed additionally.

Lemma 6.15. *Let T be a 2-weakly isotropic Gaussian spherical random field such that its angular power spectrum satisfies that $(C_l l^{1+2} : l \geq 0)$ is summable, then T is an element of $L^{2k}_P(\Omega, W^{1,2k}(S^2))$ for all $k \in \mathbb{N}$ and its expansion in the real spherical harmonics converges in the respective norm.*

Proof. The norm on $W^{1,2k}(S^2)$ is defined through the atlas $(U_i, \eta_i : i = 1, .., 6)$ and partition of unity Ψ. We prove the assertion for the chart η_1, the argument for the remaining charts is exactly the same. By the product rule we obtain that $\partial_\theta(T\Psi_1)_{\eta_1} = (\partial_\theta T_{\eta_1})(\Psi_1)_{\eta_1} + T_{\eta_1}\partial_\theta(\Psi_1)_{\eta_1}$. Since Ψ_1 is a smooth compactly supported function, it is sufficient to consider the term $(\partial_\theta T)_{\eta_1}(\Psi_1)_{\eta_1}$. For $L \in \mathbb{N}_0$ we denote the truncation of the expansion of T by T^L. Note that on $\eta_1^{-1}(U_1)$ the function $\sin(\theta) \geq \varepsilon$ for some $\varepsilon > 0$. We apply the formulas for partial derivatives of the real spherical harmonics given in Lemma 6.13, Inequality (6.10) and the same argument, which was used to prove Inequality (5.9) to obtain that for $L_1 \geq L_2 \in \mathbb{N}_0$ it holds that

$$
\begin{aligned}
&E\left[\int_{\eta_1^{-1}(U_1)} |(\partial_\theta T_{\eta_1}^{L_1}(\theta,\varphi) - \partial_\theta T_{\eta_1}^{L_2}(\theta,\varphi))(\Psi_1)_{\eta_1}(\theta,\varphi)|^{2k} \mathrm{d}\varphi \mathrm{d}\theta\right] \\
&\leq \sup_{U_1} |\Psi_1|^{2k} E\left[\int_{\eta_1^{-1}(U_1)} |\partial_\theta T_{\eta_1}^{L_1}(\theta,\varphi) - \partial_\theta T_{\eta_1}^{L_2}(\theta,\varphi)|^{2k} \mathrm{d}\varphi \mathrm{d}\theta\right] \\
&\leq \sup_{U_1} |\Psi_1|^{2k} \frac{(2k)!}{2^k k!} \int_{\eta_1^{-1}(U_1)} \left(\sum_{l=L_2+1}^{L_1} \sum_{m=-l}^{l} C_l (\partial_\theta \tilde{Y}_{l,m}(\theta,\varphi))^2\right)^k \mathrm{d}\theta \mathrm{d}\varphi \\
&\leq K^{2k} \frac{3^k}{\varepsilon^{2k}} \frac{(2k)!}{2^k k!} \sup_{U_1} |\Psi_1|^{2k} \int_{\eta_1^{-1}(U_1)} \left(\sum_{l=L_2}^{L_1+1} \sum_{m=-l}^{l} C_l l^2 \tilde{Y}_{l,m}^2(\theta,\varphi)\right)^k \mathrm{d}\varphi \mathrm{d}\theta,
\end{aligned}
$$

where K is a constant, which is independent of l and m. Due to the pre-factors of the partial derivatives of the real spherical harmonics. We argued in the proof of Lemma 6.14 that these pre-factors can be bounded in this way. In the next step we apply Lemma 2.8 and summarize terms in a constant $K > 0$, which is independent of L_1, L_2, θ, l and m. We obtain that

$$
\begin{aligned}
&E\left[\int_{\eta_1^{-1}(U_1)} |(\partial_\theta T_{\eta_1}^{L_1}(\theta,\varphi) - \partial_\theta T_{\eta_1}^{L_2}(\theta,\varphi))(\Psi_1)_{\eta_1}(\theta,\varphi)|^{2k} \mathrm{d}\varphi \mathrm{d}\theta\right] \\
&\leq K \left(\sum_{l=L_2}^{L_1+1} \sum_{m=-l}^{l} C_l l^{1+2}\right)^k.
\end{aligned}
$$

Since the sequence $(C_l l^{1+2} : l \geq 0)$ is summable and due to the above remarks on the norms in Sobolev spaces $(T^L\Psi_1)_{\eta_1}$ is a Cauchy sequence in $L^{2k}(\Omega, W^{1,2k}(\eta_1^{-1}(U_1))$ and converges to $(T\Psi_1)_{\eta_1} \in L^{2k}(\Omega, W^{1,2k}(\eta_1^{-1}(U_1)))$ in the respective norm.

This argument can be repeated in the same way for the other $i \in \{2, ..., 6\}$ due to Equation (6.19). Also the argument for the partial derivatives with respect to φ can be done in the same way, it includes fewer terms. Since the norm on $W^{1,2k}(S^2)$ is defined as the sum of the $W^{1,2k}(\eta_i^{-1}(U_i))$-norms of T_{η_i} over i, it follows that $T \in L^{2k}(\Omega, W^{1,2k}(S^2))$ as claimed. □

Theorem 6.16. *Let T be a continuous 2-weakly isotropic Gaussian spherical random field, such that the angular power spectrum satisfies that $(C_l l^{1+2+\delta} : l \geq 0)$ is summable for $\delta \in (0, 2]$. For any $\gamma \in (0, \frac{\delta}{2})$ there exists an indistinguishable modification T^* of T such*

that $T^* \subset C^{1,\gamma}(S^2)$ and for all $p \in (0,\infty)$ there exists a constant K_p independently of T and $(C_l : l \geq 0)$ such that

$$\|T\|_{L^p_P(\Omega, C^{1,\gamma}(S^2))} \leq K_p \left(\sum_{l \geq 0} C_l l^{1+2+\delta} \right)^{\frac{1}{2}}.$$

For a spherical random field X we introduce the notation that $X \subset C^{1,\gamma}(S^2)$. We mean by this notation that the function $X(\omega, \cdot) \in C^{1,\gamma}(S^2)$ for all $\omega \in \Omega$. This notation is motivated by interpreting X as a set of function with index set Ω. We will apply this notation in the case of other function spaces over the sphere without further mentioning it. For a spherical random field X we say that a modification X^* of X is indistinguishable of X, if there exists a measurable set Ω^* of full probability such that $X^* \mathbb{1}_{\Omega^*} = X \mathbb{1}_{\Omega^*}$ as functions on S^2, where $\mathbb{1}$ is the indicator function.

Proof of Theorem 6.16. For this proof we consult a recent paper by Andreev and Lang [2], which suits our situation nicely. Lemma 6.14 together with Remark 5.5 and Lemma 6.15 state the essential ingredients of Theorem 3.5 in [2]. We apply Theorem 3.5 in [2] with $d = 1, n = 2$ and $\varepsilon = k\delta - 2$. To obtain the claimed regularity $k \in \mathbb{N}$ has to be chosen such that it satisfies that $1 + \gamma < 1 + \min\{\frac{k\delta - 2}{2k}, \frac{2k-2}{2k}\}$. This is the case for $k > \frac{2}{\delta - 2\gamma}$. We conclude with Theorem 3.5 in [2] that T has a modification T^* such that $T^* \subset C^{1,\gamma}(S^2)$. To prove the indistinguishability it is important that T is already continuous. Since T^* is a modification of T, there exists a measurable set of full probability Ω^* such that $T(\omega, x) = T^*(\omega, x)$ for all $\omega \in \Omega^*$ and all $x \in \mathbb{Q}^3 \cap S^2$. Moreover they are both continuous random fields and the realizations are uniquely determined on a dense subset of S^2. This implies that $T \mathbb{1}_{\Omega^*} = T^* \mathbb{1}_{\Omega^*}$ as functions on S^2.
We fix $i \in \{1, ..., 6\}$ and an arbitrary $p \in (0, \infty)$. We increase k such that $p \leq 2k$. We remind of our standard atlas $(U_i, \eta_i : i = 1, ..., 6)$ on S^2 with partition of unity Ψ. In the development of the proof of Theorem 3.5 in [2] in this particular paper Hölder continuity of the random field is proven with Sobolev spaces and a Sobolev embedding. We observe that for a random field X on $\eta_i^{-1}(U_i)$ that satisfies that $\partial_{\boldsymbol{\alpha}} X \in L^p_{P \otimes \mathrm{d}x}(\Omega \times \eta_i^{-1}(U_i), \mathbb{R})$ for all multi-indices $\boldsymbol{\alpha} \in \mathbb{N}_0^2$ with $|\boldsymbol{\alpha}| = d$ for some $d \in \mathbb{N}$, for $q \in [1, \infty)$ and $\nu \in (0, 1)$ the $L^q_P(\Omega, W^{d+\nu,q}(\eta_i^{-1}(U_i))$-norm of X satisfies that

$$\|X\|^q_{L^q_P(\Omega, W^{d+\nu,q})} = E[\|X\|^q_{W^{d+\nu,q}}] \simeq E[\|X\|^q_{L^q}] + \sum_{|\boldsymbol{\alpha}|=d} E[\int \frac{|\partial_{\boldsymbol{\alpha}} X(x) - \partial_{\boldsymbol{\alpha}} X(y)|^q}{\|x-y\|^{2+\nu q}_{\mathbb{R}^2}} \mathrm{d}x \mathrm{d}y], \tag{6.20}$$

where we tacitly omitted the domain $\eta_i^{-1}(U_i)$ in the Sobolev and L^p space. Also the integral is taken over $\eta_i^{-1}(U_i) \times \eta_i^{-1}(U_i)$. Note that in general the above expression may be infinite. Adapted to our situation it is finite with the choice $\nu \in (\gamma, \frac{\delta}{2})$. This follows because the function $\| \cdot - \cdot \|^{-\kappa}_{\mathbb{R}^2}$ is integrable over $\eta_i^{-1}(U_i) \times \eta_i^{-1}(U_i)$ for all $\kappa \in (2, \infty)$. Also with the choice $\nu \in (\gamma, \frac{\delta}{2})$ the Sobolev embedding theorem, which is Theorem 4.6.1.(e) in [25], Lemma 6.14 and Remark 5.5 imply that

$$E[\|(T\Psi_i)_{\eta_i}\|^{2k}_{C^{1,\gamma}(\overline{\eta_i^{-1}(U_i)})}] \leq CE[\|T_{\eta_i}\|^{2k}_{L^{2k}(\eta_i^{-1}(U_i), \mathbb{R})}] + K_k \left(\sum_{l \geq 0} C_l l^{1+2+\delta} \right)^k, \tag{6.21}$$

where the constant C is due to the Sobolev embedding and the constant K_k is due to Lemma 6.14. Note that the contribution of the partition of unity does not cause problems.

In the paper [2] the boundary of the domain has some regularity. Here we could ignore this because $(\Psi_i)_{\eta_i}$ is compactly supported in $\eta_i^{-1}(U_i)$. We have observed in the proof of Lemma 6.15 that there exists a constant K_k such that

$$E[\|T\|^{2k}_{L^{2k}(S^2,\mathbb{R})}] \leq K_k \left(\sum_{l\geq 0} C_l l \right)^k .$$

The Hölder inequality implies that with continuous embedding $L^{2k}_P(\Omega,\mathbb{R}) \subset L^p_P(\Omega,\mathbb{R})$. We conclude that there exists a constant K_p such that

$$E[\|(T\Psi_i)_{\eta_i}\|^p_{C^{1,\gamma}(\overline{\eta_i^{-1}(U_i)})}]^{\frac{1}{p}} \leq K_p \left(\sum_{l\geq 0} C_l l^{1+2+\delta} \right)^{\frac{1}{2}} .$$

Since $i \in \{1, ..., 6\}$ was arbitrary the claim follows. □

Remark 6.17. *If we assume that the angular power spectrum of a continuous 2-weakly isotropic Gaussian spherical random field T only satisfies that $(C_l l^{1+\delta} : l \geq 0)$ is summable for some $\delta \in (0, 2]$, then the same proof of the previous theorem can be applied. We obtain that for all $\gamma \in (0, \frac{\delta}{2})$ there exists an indistinguishable modification T^* of T such that $T^* \subset C^{0,\gamma}(S^2)$ and for all $p \in (0, \infty)$ there exists a constant K_p independently of T and $(C_l : l \geq 0)$ such that*

$$\|T\|_{L^p_P(\Omega,C^{0,\gamma}(S^2))} \leq K_p \left(\sum_{l\geq 0} C_l l^{1+\delta} \right)^{\frac{1}{2}}$$

Note that the property of the modification to be indistinguishable is due to our decision, that we made at the beginning of this chapter, to always consider the continuous modification of a 2-weakly isotropic Gaussian spherical random field.

6.3. Higher order derivatives of isotropic Gaussian spherical random fields

Now, we prove under which conditions on the angular power spectrum of a 2-weakly isotropic Gaussian spherical random field higher order partial derivatives are P-a.s. Hölder continuous. Therefore we have to find a generalization of Lemma 6.14 for higher order partial derivatives.

Lemma 6.18. *Let T be a 2-weakly isotropic Gaussian spherical random field such that the angular power spectrum $(C_l : l \geq 0)$ satisfies that $(C_l l^{1+2\iota+\delta} : l \geq 0)$ is summable for some $\delta \in (0, 2]$ and some $\iota \in \mathbb{N}$. For all $i \in \{1, ..., 6\}$ and all $x, y \in U_i$, all multi-indices $\boldsymbol{\alpha} \in \mathbb{N}_0^2$ satisfying $|\boldsymbol{\alpha}| = \iota$ and all $k \in \mathbb{N}$ there exists a constant K_k, which only depends on k and ι, such that*

$$E\left[|\partial_{\boldsymbol{\alpha}} T_{\eta_i}(\theta_x, \varphi_x) - \partial_{\boldsymbol{\alpha}} T_{\eta_i}(\theta_y, \varphi_y)|^{2k} \right] \leq K_k \left(\sum_{l\geq 0} C_l l^{1+2\iota+\delta} \right)^k d(x, y)^{\delta k}.$$

Proof. We start with the case $k = 1$ and $i = 1$, then η_1 is equal to our usual coordinates on the sphere. Therefore we apply the notation $\eta_1(\theta_x, \varphi_x) = x$ and $\eta_1(\theta_y, \varphi_y) = y$. We are allowed to interchange the partial derivative and limit of the expansion of $T(x)$ and $T(y)$ because from Lemma 6.12 we know that $T \in L^2_P(\Omega, H^\iota(S^2))$, i.e. the weak partial derivatives of T up to order ι are well defined in the L^2_P-sense. This enables us to shift the discussion on how the partial derivatives behave on the real spherical harmonics, i.e. we obtain that it holds that

$$\begin{aligned} E\left[|\partial_{\boldsymbol{\alpha}} T(x) - \partial_{\boldsymbol{\alpha}} T(y)|^2\right] &= E\left[\left|\sum_{l\geq 0}\sum_{m=-l}^{l} \sqrt{C_l}\tilde{\beta}_{l,m}(\partial_{\boldsymbol{\alpha}}\tilde{Y}_{l,m}(x) - \partial_{\boldsymbol{\alpha}}\tilde{Y}_{l,m}(y))\right|^2\right] \\ &= \sum_{l\geq 0}\sum_{m=-l}^{l} C_l(\partial_{\boldsymbol{\alpha}}\tilde{Y}_{l,m}(x) - \partial_{\boldsymbol{\alpha}}\tilde{Y}_{l,m}(y))^2. \end{aligned} \tag{6.22}$$

The strategy of this proof is to apply the ideas of the proof of Lemma 6.14 and the result of Lemma 6.13, i.e. the formula for the partial derivatives of the real spherical harmonics. For $l \in \mathbb{N}_0$ and $m \in \{-l, ..., l\}$ we examine

$$\partial_{\boldsymbol{\alpha}}\tilde{Y}_{l,m} = \partial_\theta^{\alpha_\theta}(-1)^{\lceil\frac{\alpha_\varphi}{2}\rceil} m^{\alpha_\varphi}\tilde{Y}_{l,(-1)^{\alpha_\varphi}m}$$

$$= (-1)^{\lceil\frac{\alpha_\varphi}{2}\rceil} m^{\alpha_\varphi}\partial_\theta^{\alpha_\theta - 1}\left(m\sqrt{\frac{(l+1)^2 - m^2}{(2l+1)(2l+3)}}\frac{\tilde{Y}_{l+1,(-1)^{\alpha_\varphi}m}(x)}{\sin(\theta)}\right. \tag{6.23}$$

$$+ m\sqrt{\frac{l^2 - m^2}{(2l+1)(2l-1)}}\frac{\tilde{Y}_{l-1,(-1)^{\alpha_\varphi}m}(x)}{\sin(\theta)} \tag{6.24}$$

$$\left. + \begin{cases} \sqrt{(l+m+1)(l-m)}\tilde{Y}_{l,m+1}(x) & \text{if } m \geq 0 \\ \sqrt{(l+|m|+1)(l-|m|)}\tilde{Y}_{l,m-1}(x) & \text{if } m < 0 \end{cases}\right). \tag{6.25}$$

If we evaluate all remaining partial derivatives, we obtain finitely many linear combinations of real spherical harmonics with the first index less or equal to $l + \iota$. We notice that Expressions (6.23) and (6.24) carry the factor $\frac{1}{\sin(\theta)}$, which leads to numerous terms because of the product rules for derivatives. But the number of terms will at most double each time the product rule is applied. Similarly to the proof of Lemma 6.14 it holds that $\sin(\theta) \geq \varepsilon > 0$ on $\eta_1^{-1}(U_1)$. This implies that powers and derivatives of powers of $\frac{1}{\sin(\theta)}$ are smooth and therefore Lipschitz continuous, i.e. there exists a constant K such that for all positive integers $h, q \leq \iota$ it holds that

$$\left|\partial_\theta^q \frac{1}{\sin^h(\theta_x)} - \partial_\theta^q \frac{1}{\sin^h(\theta_y)}\right| \leq K|\theta_x - \theta_y|. \tag{6.26}$$

Also because these functions are smooth, there exists a constant, say also K, such that on $\eta_1^{-1}(U_1)$ for all positive integers $h, q \leq \iota$ it holds that

$$\sup_{(\theta,\varphi)\in\eta_1^{-1}(U_1)}\left|\partial_\theta^q \frac{1}{\sin^h(\theta)}\right| \leq K. \tag{6.27}$$

Since we are only interested in absolute values of differences and we wish to simplify the terms we bound the factors in Expression (6.23) and (6.24), i.e. for all $l' \in \mathbb{N}$ and $m' \in$

$\{-l', ..., l'\}$ it holds that

$$\frac{(l'+1)^2 - m'^2}{(2l'+1)(2l'+3)} \leq \frac{l'^2 + 2l' + 1}{4l'^2 + 8l' + 3} \leq 1,$$
$$\frac{l'^2 - m'^2}{(2l'+1)(2l'-1)} \leq \frac{l'^2}{4l'^2 - 1} = \frac{l'^2 - 4^{-1}}{4l'^2 - 1} + \frac{4^{-1}}{4l'^2 - 1} \leq \frac{1}{4} + \frac{1}{12} \leq 1$$

and

$$\sqrt{(l' + |m'| + 1)(l' - |m'|)} = \sqrt{l'^2 - |m'|^2 + l' - |m'|} \leq \sqrt{(l'^2 + l'} \leq \sqrt{2}l'.$$

We evaluate the remaining partial derivatives and obtain with Inequality (6.10) that

$$(\partial_{\boldsymbol{\alpha}} \tilde{Y}_{l,m}(x) - \partial_{\boldsymbol{\alpha}} \tilde{Y}_{l,m}(y))^2 \leq 6^\iota (l+\iota)^{2\iota} \sum_{j=1}^{6^\iota} (a_j(x) \tilde{Y}_{l_j,m_j}(x) - a_j(y) \tilde{Y}_{l_j,m_j}(y))^2, \tag{6.28}$$

where the coefficients $(a_j : j = 1, ..., 6^\iota)$ are combinations of powers of $\sqrt{2}$ and of powers and derivatives of powers of $\frac{1}{\sin(\theta)}$. Note that the coefficients are clearly functions. It is possible that some of the coefficients are equal to zero. Since every partial derivative of a real spherical harmonic results in at most 3 terms and the product rule applied to the coefficients doubles at most the number, we obtain at most 6^ι terms. Note that $l_j \leq l + \iota$ for all $j \in \{1, ..., 6^\iota\}$ and $m_j \in \{-l_j, ..., l_j\}$. This yields the upper bound $(l+\iota)^\iota$ for powers of different m_j's that result as factors while differentiating the real spherical harmonics. The inequality for $f, g, a, b \in \mathbb{R}$:

$$|af - bg| \leq |a||f - g| + |g||a - b|, \tag{6.29}$$

together with Inequalities (6.26), (6.27) and (6.28) and Inequality (6.10) implies that

$$\begin{aligned}&(\partial_{\boldsymbol{\alpha}} \tilde{Y}_{l,m}(x) - \partial_{\boldsymbol{\alpha}} \tilde{Y}_{l,m}(y))^2 \\ &\quad \leq 2^{2\iota} 6^\iota \iota^{2\iota} l^{2\iota} \sum_{j=1}^{6^\iota} \hat{a}_j K^2 [(\tilde{Y}_{l_j,m_j}(x) - \tilde{Y}_{l_j,m_j}(y))^2 + \tilde{Y}_{l_j,m_j}(y)^2 |\theta_x - \theta_y|^2],\end{aligned} \tag{6.30}$$

where the coefficients $(\hat{a}_j : j = 1, ..., 6^\iota)$ are now only powers of $\sqrt{2}$, with highest exponent 2ι. We can apply the same argument that we used to show Inequality (6.17) in the proof of Lemma 6.14 with the respective assumption on the angular power spectrum to conclude that there exists a constant $\hat{K}$ such that for all $j \in \{1, ..., 6^\iota\}$ it holds that

$$\begin{aligned}&\sum_{l \geq 0} \sum_{m=-l}^{l} C_l l^{2\iota} [(\tilde{Y}_{l_j,m_j}(x) - \tilde{Y}_{l_j,m_j}(y))^2 + \tilde{Y}_{l_j,m_j}(y)^2 |\theta_x - \theta_y|^2] \\ &\quad \leq \sum_{l \geq 0} \sum_{m=-l}^{l} C_l l^{2\iota} [(\tilde{Y}_{l,m}(x) - \tilde{Y}_{l,m}(y))^2 + \tilde{Y}_{l,m}(y)^2 |\theta_x - \theta_y|^2] \leq \hat{K} \sum_{l \geq 0} C_l l^{1+2\iota+\delta} d(x,y)^\delta.\end{aligned} \tag{6.31}$$

Note that the indices l_j and m_j are translations from the original indices l and m by at most ι. The missing terms of the real spherical harmonics in the first sum in Inequality (6.31) will

be added to obtain the second sum in the chain of inequalities. We combine Equation (6.22), Inequality (6.30) and Inequality (6.31) to obtain that

$$E\left[|\partial_{\boldsymbol{\alpha}}T(x)-\partial_{\boldsymbol{\alpha}}T(y)|^2\right]\leq K_1 d(x,y)^{\delta}, \tag{6.32}$$

where we have set $K_1=2^{2\iota}6^{\iota}\iota^{2\iota}6^{\iota}K^2\hat{K}$. This argument can be repeated in the same way for the other $i\in\{1,...,6\}$ due to Equation (6.19).

The general case $k\in\mathbb{N}$ is proven by copying line by line the argument for the general case $k\in\mathbb{N}$ in the proof of Lemma 6.14. Therefore there exist constants K_k such that for all $k\in\mathbb{N}$ and all $i\in\{1,...,6\}$ it holds that

$$E\left[|\partial_{\boldsymbol{\alpha}}T_{\eta_i}(\theta_x,\varphi_x)-\partial_{\boldsymbol{\alpha}}T_{\eta_i}(\theta_y,\varphi_y)|^{2k}\right]\leq K_k\left(\sum_{l\geq 0}C_l l^{1+2\iota+\delta}\right)^k d(x,y)^{\delta k}. \qquad \square$$

The next lemma verifies the membership of weak higher order partial derivatives of a 2-weakly isotropic Gaussian random field in L^p-spaces, similarly to Lemma 6.15.

Lemma 6.19. *Let T be a 2-weakly isotropic Gaussian spherical random field such that its angular power spectrum satisfies that $(C_l l^{1+2\iota}:l\geq 0)$ is summable for some $\iota\in\mathbb{N}$, then T is an element of $L^{2k}_P(\Omega,W^{\iota,2k}(S^2))$ for all $k\in\mathbb{N}$ and its expansion in the real spherical harmonics converges in the respective norm.*

Proof. This proof will benefit from the proofs of Lemma 6.15 and Lemma 6.18. The norm on $W^{\iota,2k}(S^2)$ can be defined through the usual atlas $(U_i,\eta_i:I=1,...,6)$ and the partition of unity Ψ. We start with $i=1$ and notice that for $x\in U_1$, the argument that we used in the proof of Lemma 6.18 to prove Inequality (6.28) also works to estimate squared partial derivatives of a single real spherical harmonic. So we conclude that for an integer $l\geq 0$, $m\in\{-l,...,l\}$ and all multi-indices $\boldsymbol{\alpha}\in\mathbb{N}_0^2$ satisfying $|\boldsymbol{\alpha}|=\iota$ it holds that

$$(\partial_{\boldsymbol{\alpha}}\tilde{Y}_{l,m}(x))^2\leq 6^{\iota}(l+\iota)^{2\iota}\sum_{j=1}^{6^{\iota}}(a_j(x)\tilde{Y}_{l_j,m_j}(x))^2,$$

where the coefficients $(a_j:j=1,...,6^{\iota})$ are combinations of powers of $\sqrt{2}$ and of powers and derivatives of powers of $\frac{1}{\sin(\theta)}$. Note that $l_j\leq l+\iota$ for all $j=1,...,6^{\iota}$ and $m_j\in\{-l_j,...,l_j\}$. We apply Inequality (6.27), which is also part of the proof of Lemma 6.18 to bound the coefficients in the above sum to obtain that for a constant $K>0$ it holds that

$$(\partial_{\boldsymbol{\alpha}}\tilde{Y}_{l,m}(x))^2\leq 2^{2\iota-1}6^{\iota}\iota^{2\iota}l^{2\iota}K\sum_{j=1}^{6^{\iota}}\tilde{Y}^2_{l_j,m_j}(x). \tag{6.33}$$

For $L_1\geq L_2\in\mathbb{N}$ and a multi-index $\boldsymbol{\alpha}\in\mathbb{N}_0^2$ that satisfies $|\boldsymbol{\alpha}|=\iota$ it holds that

$$\begin{aligned}&E\left[\int_{\eta_1^{-1}(U_1)}\left(\partial_{\boldsymbol{\alpha}}T^{L_1}_{\eta_1}(\theta,\varphi)-\partial_{\boldsymbol{\alpha}}T^{L_2}_{\eta_1}(\theta,\varphi)(\Psi_1)_{\eta_1}(\theta,\varphi)\right)^{2k}\mathrm{d}\theta\mathrm{d}\varphi\right]\\&\qquad\leq\sup_{U_1}|\Psi_1|^{2k}\int_{\eta_1^{-1}(U_1)}E\left[\left(\sum_{l=L_2+1}^{L_1}\sum_{m=-l}^{l}\sqrt{C_l}\beta_{l,m}\partial_{\boldsymbol{\alpha}}\tilde{Y}_{l,m}(\theta,\varphi)\right)^{2k}\right]\mathrm{d}\theta\mathrm{d}\varphi\end{aligned}$$

$$\leq \sup_{U_1} |\Psi_1|^{2k} \int_{\eta_1^{-1}(U_1)} \frac{(2k)!}{2^k k!} \left(\sum_{l=L_2+1}^{L_1} \sum_{m=-l}^{l} C_l (\partial_{\boldsymbol{\alpha}} \tilde{Y}_{l,m}(\theta,\varphi))^2 \right)^k \mathrm{d}\theta \mathrm{d}\varphi,$$

where the last step is justified by the same argument, which was used to prove Inequality (5.9). We insert the bound in Inequality (6.33) to obtain that for a constant $\tilde{K} > 0$ it holds that

$$\begin{aligned} E\left[\int_{\eta_1^{-1}(U_1)} \left(\partial_{\boldsymbol{\alpha}} T_{\eta_1}^{L_1}(\theta,\varphi) - \partial_{\boldsymbol{\alpha}} T_{\eta_1}^{L_2}(\theta,\varphi)(\Psi_1)_{\eta_1}(\theta,\varphi) \right)^{2k} \mathrm{d}\theta \mathrm{d}\varphi \right] \\ \leq \tilde{K} \int_{\eta_1^{-1}(U_1)} \left(\sum_{l=L_2+1}^{L_1} \sum_{m=-l}^{l} C_l l^{2\iota} \tilde{Y}_{l,m}^2(\theta,\varphi) \right)^k \mathrm{d}\theta \mathrm{d}\varphi \\ \leq \tilde{K}' \left(\sum_{l=L_2+1}^{L_1} C_l l^{2\iota} \frac{2l+1}{4\pi} \right)^k, \end{aligned} \tag{6.34}$$

where the last step is due to Lemma 2.8. The assumption in the lemma implies that the sequence $(C_l l^{1+2\iota} : l \geq 0)$ is summable, which yields that Expression (6.34) converges to zero as $L_1, L_2 \to \infty$. Since the norm on $W^{\iota,2k}(\eta_1^{-1}(U_1))$ is determined by the L^{2k}-norm of the weak derivatives of highest order and the L^{2k}-norm of the function and Ψ_1 is a smooth, compactly supported function, it is sufficient to consider the term $(\partial_{\boldsymbol{\alpha}} T)\Psi_1$. Since L_P^{2k} and Sobolev spaces are complete, we obtain that $T_{\eta_1}^L$ converges to $T_{\eta_1} \in W^{\iota,2k}(\eta_1^{-1}(U_1)$ in the respective norm.

This argument can be repeated in the same way for the other $i \in \{1, ..., 6\}$ due to Equation (6.19). Since the norm on $W^{\iota,2k}(S^2)$ is defined as the sum of the $W^{\iota,2k}(\eta_i^{-1}(U_i))$-norms of $(T\Psi_i)_{\eta_i}$ over i, it follows that $T \in L_P^{2k}(\Omega, W^{\iota,2k}(S^2))$ as claimed. □

Now we can generalize Theorem 6.16 for arbitrarily high order derivatives. This is the precise version of Theorem 1.3 from the introduction.

Theorem 6.20. *Let T be a continuous 2-weakly isotropic Gaussian spherical random field, such that the angular power spectrum satisfies that $(C_l l^{1+2\iota+\delta} : l \geq 0)$ is summable for $\delta \in (0, 2]$ and $\iota \in \mathbb{N}_0$. For any $\gamma \in (0, \frac{\delta}{2})$ there exists an indistinguishable modification T^* of T such that $T^* \subset C^{\iota,\gamma}(S^2)$ and for all $p \in (0, \infty)$ there exists a constant K_p independently of T and $(C_l : l \geq 0)$ such that*

$$\|T\|_{L_P^p(\Omega, C^{\iota,\gamma}(S^2))} \leq K_p \left(\sum_{l \geq 0} C_l l^{1+2\iota+\delta} \right)^{\frac{1}{2}}.$$

Proof. The cases $\iota = 0, 1$ are implied by Remark 6.17 and Theorem 6.16. Therefore we assume that $\iota \geq 2$. The proof is very similar to the proof of Theorem 6.16 and consults [2]. Lemma 6.18 and Lemma 6.19 state the needed assumptions in Theorem 3.5 in [2]. We apply Theorem 3.5 in [2] with $d = \iota, n = 2$ and $\varepsilon = k\delta - 2$. To obtain the claimed regularity $k \in \mathbb{N}$ has to chosen such that it satisfies that $\iota + \gamma < \iota + \min\{\frac{k\delta-2}{2k}, \frac{2k-2}{2k}\}$. This is again achieved for $k > \frac{2}{\delta - 2\gamma}$. We conclude with Theorem 3.5 in [2] that T has a modification T^*

such that $T^* \subset C^{\iota,\gamma}(S^2)$. The modification is indistinguishable, because T and T^* are both continuous random fields. The same argument as in the proof of Theorem 6.16 applies.
The proof of the bound of the $L^p_P(\Omega, C^{\iota,\gamma}(S^2))$-norm of T follows with the same argument as in the proof of Theorem 6.16. □

Remark 6.21. *Let T be a continuous 2-weakly isotropic Gaussian spherical random field that satisfies the assumptions of the previous theorem. For all $L \in \mathbb{N}_0$ Theorem 6.20 implies an approximation result of T by the truncated random field T^L, since $T - T^L$ is also a continuous 2-weakly isotropic Gaussian spherical random field with angular power spectrum equal to zero in the components from 0 to L. Theorem 6.20 implies that for all $p \in (0, \infty)$ there exists a constant K_p indepedently of L, T, T^L and $(C_l : l \geq 0)$ such that*

$$\|T - T^L\|_{L^p_P(\Omega, C^{\iota,\gamma}(S^2))} \leq K_p \left(\sum_{l>L} C_l l^{1+2\iota+\delta} \right)^{\frac{1}{2}}.$$

6.4. A second approach to prove the differentiability results

In the previous sections we proved differentiability of a continuous 2-weakly isotropic Gaussian spherical random field T by analyzing the behavior of higher order partial derivatives of T. This was technically quite involved. We observe that the spherical Laplace operator applied to T results again a 2-weakly isotropic spherical random field in the case of sufficient decay of the angular power spectrum of T. This is due to fact that the spherical Laplace operator is diagonalized by the real spherical harmonics such that the respective eigenvalues and the angular power spectrum of T are indexed in the same way. Therefore the expansion of $(-\Delta_{S^2} + \frac{1}{4})T$ only differs in the angular power spectrum.
The strategy in this section is to prove Hölder continuity of $(-\Delta_{S^2} + \frac{1}{4})^{\frac{\iota}{2}}T$ for some even integer $\iota \geq 2$ and then conclude Hölder continuity of the partial derivatives of T of order ι with regularity theory of second order elliptic operators. The case of higher odd order partial derivatives will follow with interpolation theory. The same result from Section 6.3 on differentiability of continuous 2-weakly isotropic Gaussian spherical random fields, which is Theorem 6.20, will be achieved.

We will require Hölder regularity of solutions of specific higher order elliptic equations. The following proposition discusses this issue for this particular class of operators.

Proposition 6.22. *Let $\iota \geq 2$ be an even integer and let $\gamma \in (0, 1)$. If $u \in C^0(S^2) \cap H^\iota(S^2)$ and $f \in C^{0,\gamma}(S^2)$ satisfy that*

$$\left(-\Delta_{S^2} + \frac{1}{4}\right)^{\frac{\iota}{2}} u = f$$

with equality in $L^2(S^2, \mathbb{R})$, then it follows that $u \in C^{\iota,\gamma}(S^2)$ and there exists a constant K independently of u and f such that

$$\|u\|_{C^{\iota,\gamma}(S^2)} \leq K(\|u\|_{H^\iota(S^2)} + \|f\|_{C^{0,\gamma}(S^2)}).$$

Proof. Proposition 6.5 implies that $u \in W^{\iota,2}(S^2)$. Since u is already continuous, the Sobolev embedding theorem, which is Theorem 6.10, implies that $u \in C^{\iota-2,\gamma}(S^2)$ and together with Proposition 6.5 it follows that there exist constants K, K' independently of u such that

$$\|u\|_{C^{\iota-2,\gamma}(S^2)} \leq K\|u\|_{W^{\iota,2}(S^2)} \leq K'\|u\|_{H^\iota(S^2)}. \tag{6.35}$$

We start with the case that $\iota = 2$. Our usual atlas of S^2 is denoted by $(U_i, \eta_i : i = 1, ..., 6)$ and the partition of unity Ψ is subordinate to the open cover $(U_i : i = 1, ..., 6)$ of S^2. Hölder continuity is a local property. Therefore, we multiply u with a cut-off function and aim to pull the problem back to the chart domains and apply regularity theory for elliptic equations in subdomains of Euclidean space. We fix $i \in \{1, ..., 6\}$ and observe with the Leipniz rule that $u\Psi_i$ satisfies that

$$\begin{aligned}\left(-\Delta_{S^2} + \frac{1}{4}\right)(u\Psi_i) &= -\nabla_{S^2} \cdot ((\nabla_{S^2} u)\Psi_i + u\nabla_{S^2}\Psi_i) + \frac{1}{4}u\Psi_i \\ &= -(\Delta_{S^2} u)\Psi_i - \nabla_{S^2} u \cdot \nabla_{S^2}\Psi_i - \nabla_{S^2} \cdot (u\nabla_{S^2}\Psi_i) + \frac{1}{4}u\Psi_i \\ &= f\Psi_i - 2\nabla_{S^2} u \cdot \nabla_{S^2}\Psi_i - u\Delta_{S^2}\Psi_i.\end{aligned}$$

In the chart domain $\eta_i^{-1}(U_i)$ the above equality reads:

$$L(u\Psi_i)_{\eta_i} = G + \partial_\theta F_\theta + \partial_\varphi F_\varphi, \tag{6.36}$$

where

$$L = \partial_\theta^2 + \partial_\varphi^2 \frac{1}{\sin^2(\theta)} + \frac{\cos(\theta)}{\sin(\theta)}\partial_\theta - \frac{1}{4} = \partial_k(a^{k,l}\partial_l) + \frac{\cos(\theta)}{\sin(\theta)}\partial_\theta - \frac{1}{4}, \tag{6.37}$$

$$F = \left(2u_{\eta_i}\partial_\theta(\Psi_i)_{\eta_i},\ 2\frac{u_{\eta_i}}{\sin(\theta)}\partial_\varphi\frac{(\Psi_i)_{\eta_i}}{\sin(\theta)}\right) \tag{6.38}$$

and

$$G = -(f\Psi_i)_{\eta_i} + (u\Delta_{S^2}\Psi_i)_{\eta_i} - 2u_{\eta_i}\partial_\theta^2(\Psi_i)_{\eta_i} - 2\frac{u_{\eta_i}}{\sin(\theta)}\partial_\varphi^2\frac{(\Psi_i)_{\eta_i}}{\sin(\theta)}. \tag{6.39}$$

In Equation (6.37) we employed the summation convention to be able to ease the notation in the following, i.e. for two vectors $x, y \in \mathbb{R}^2$ we define $x^k y_k = \sum_{k=1}^2 x_k y_k$. The order of the subscript and superscript indices may vary and this notation also applies to the product of matrices. The matrix a is given by: $a = diag(1, \sin^{-2}(\theta))$. The fact that $(\Psi_i)_{\eta_i}$ is compactly supported in $\eta_i^{-1}(U_i)$ enables us to consider Equation (6.36) on a subdomain D with smooth boundary that satisfies that $supp((\Psi_i)_{\eta_i}) \subset\subset D \subset\subset \eta_i^{-1}(U_i)$. Hence, $(u\Psi_i)_{\eta_i}$ satisfies the following Dirichlet problem in D:

$$\begin{aligned}L(u\Psi_i)_{\eta_i} &= G + \partial_k F^k, \\ (u\Psi_i)_{\eta_i}\Big|_{\partial D} &= 0.\end{aligned} \tag{6.40}$$

From Definition 6.4 we know that $(u\Psi_i)_{\eta_i} \in W_0^{2,2}(D)$. Hence, the partial derivatives in Equation (6.40) are weak derivatives and we observe that $(u\Psi_i)_{\eta_i}$ satisfies Equation (6.40) in the sense of distributions, i.e.

$$\int_D a^{k,l}\partial_l(u\Psi_i)_{\eta_i}\partial_k v - \frac{\cos(\theta)}{\sin(\theta)}\partial_\theta(u\Psi_i)_{\eta_i} v + \frac{1}{4}(u\Psi_i)_{\eta_i} v \ \mathrm{d}x = \int_D -Gv + F^k\partial_k v \ \mathrm{d}x \tag{6.41}$$

for all $v \in C_0^1(D)$, where we used the notation $(x_1, x_2) = (\theta, \varphi)$. Due to the density of $C_0^\infty(D)$ in $W_0^{1,2}(D)$, we can argue for all $v \in W_0^{1,2}(D)$ with a sequence $(v_l : l \geq 0) \subset C_0^\infty(D)$ converging to v in the $W_0^{1,2}(D)$-norm and the Cauchy–Schwarz inequality that

Equation (6.41) also holds for this v. Since v was arbitrarily chosen, Equation (6.41) holds for all $v \in W_0^{1,2}(D)$. In this case we say that $(u\Psi_i)_{\eta_i}$ is a weak solution of the Dirichlet problem in Equation (6.40). The fact that $\sin^{-2}(\theta)$ is bounded on $\overline{D}$ and the matrix a is symmetric imply that there exist constants $\Lambda, \lambda > 0$ such that

$$\|\xi\|_{\mathbb{R}^2}^2 \Lambda \geq a^{k,l}(x)\xi_k\xi_l \geq \|\xi\|_{\mathbb{R}^2}^2 \lambda$$

for all $\xi \in \mathbb{R}^2$ and all $x \in D$. This property is called strict ellipticity. The fact that G and the components of F are in $C^{0,\gamma}(\overline{D})$, that the coefficients of L are smooth on $\overline{D}$, the strict ellipticity of L and the negative sign in front of $\frac{1}{4}$ in the operator L enable us to apply Theorem 8.34 in [14]. It implies that the Dirichlet problem in Equation (6.40) has a unique, weak solution in $C^{1,\gamma}(\overline{D})$. Hence, we conclude that $(u\Psi_i)_{\eta_i} \in C^{1,\gamma}(\overline{D})$. We apply Theorem 8.32 in [14] with the nested sets $supp((\Psi_i)_{\eta_i})$ and D to obtain a constant K independently of $(u\Psi_i)_{\eta_i}$, G and F such that

$$\|(u\Psi_i)_{\eta_i}\|_{C^{1,\gamma}(supp((\Psi_i)_{\eta_i}))} \leq K(\|(u\Psi_i)_{\eta_i}\|_{C^0(\overline{D})} + \|G\|_{L^\infty(D,\mathbb{R})} + \|F_\theta\|_{C^{0,\gamma}(\overline{D})} + \|F_\varphi\|_{C^{0,\gamma}(\overline{D})}) \tag{6.42}$$

Since $\eta_i(\overline{D})$ is relatively closed in S^2, we can apply Lemma 6.2 and conclude that there exists a partition of unity $\hat{\Psi}$ subordinate to the open cover $(U_j : j = 1, ..., 6)$ such that $(\hat{\Psi}_i)_{\eta_i} = 1$ on $\overline{D}$. We apply this property, Inequality (6.4) and the definition of Hölder norms on S^2 to conclude that there exists a constant K independently of u and f such that

$$\begin{aligned}
\|G\|_{L^\infty(D,\mathbb{R})} &\leq K(\|(f\Psi_i)_{\eta_i}\|_{C^0(\overline{D})} + \|u_{\eta_i}\|_{C^{0,\gamma}(\overline{D})}) \\
&= K(\|(f\Psi_i)_{\eta_i}\|_{C^0(\overline{D})} + \|(u\hat{\Psi}_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{D})}) \\
&\leq K(\|(f\Psi_i)_{\eta_i}\|_{C^0(\overline{\eta_i^{-1}(U_i)})} + \|(u\hat{\Psi}_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})}) \\
&\leq K(\|f\|_{C^{0,\gamma}(S^2)} + \|u\|_{C^{0,\gamma}(S^2)}),
\end{aligned}$$

where the contributions of Ψ_i are included into the constant K. Similarly, there exists a constant K independently of u such that

$$\begin{aligned}
\|F_\theta\|_{C^{0,\gamma}(\overline{D})},\ \ \|F_\varphi\|_{C^{0,\gamma}(\overline{D})} &\leq K\|u_{\eta_i}\|_{C^{0,\gamma}(\overline{D})} \\
&= K\|(u\hat{\Psi}_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{D})} \leq K\|(u\hat{\Psi}_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})} \leq K\|u\|_{C^{0,\gamma}(S^2)}.
\end{aligned}$$

We insert the previous two estimates into Inequality (6.42) and obtain with the fact that $(u\Psi_i)_{\eta_i}$ is equal to zero outside of $supp((\Psi_i)_{\eta_i})$ that there exists a constant K independently of u and f such that

$$\|(u\Psi_i)_{\eta_i}\|_{C^{1,\gamma}(\overline{\eta_i^{-1}(U_i)})} \leq K(\|u\|_{C^{0,\gamma}(S^2)} + \|f\|_{C^{0,\gamma}(S^2)}).$$

This argument can be repeated for all other $j \in \{1, ..., 6\}\backslash\{i\}$. Hence we conclude that $u \in C^{1,\gamma}(S^2)$ and there exists a constant K independently of u and f such that

$$\|u\|_{C^{1,\gamma}(S^2)} \leq K(\|u\|_{C^{0,\gamma}(S^2)} + \|f\|_{C^{0,\gamma}(S^2)}). \tag{6.43}$$

We further investigate the Dirichlet problem in Equation (6.40). We fix the same $i \in \{1, ..., 6\}$ as before and remind of the set D satisfying $supp((\Psi_i)_{\eta_i}) \subset\subset D \subset\subset \eta_i^{-1}(U_i)$ and the partition of unity $\hat{\Psi}$ that satisfies that $(\hat{\Psi}_i)_{\eta_i} = 1$ on $\overline{D}$. Since $u \in C^{1,\gamma}(S^2)$, it follows

that the right hand side in Equation (6.40), i.e. $G+\partial_k F^k$, is now in $C^{0,\gamma}(\overline{D})$ and we do not need to consider F in divergence form anymore. Theorem 6.14 in [14] is applicable and it implies that the Dirichlet problem in Equation (6.40) has a unique solution $\tilde{u} \in C^{2,\gamma}(\overline{D})$. This $\tilde{u}$ is also a weak solution of the Dirichlet problem in Equation (6.40). Since the weak solution was uniquely determined to be $(u\Psi_i)_{\eta_i}$, we obtain that $\tilde{u} = (u\Psi_i)_{\eta_i}$ and therefore $(u\Psi_i)_{\eta_i} \in C^{2,\gamma}(\overline{D})$. We apply Corollary 6.3 of Theorem 6.2 in [14] with the nested sets $supp((\Psi_i)_{\eta_i})$ and D and obtain that there exists a constant K independently of $(u\Psi_i)_{\eta_i}$ and f such that

$$\|(u\Psi_i)_{\eta_i}\|_{C^{2,\gamma}(supp((\Psi_i)_{\eta_i}))} \leq K(\|(u\Psi_i)_{\eta_i}\|_{C^0(\overline{D})} + \|G+\partial_k F^k\|_{C^{0,\gamma}(\overline{D})}). \tag{6.44}$$

We discuss the norm of the right hand side. We argue similarly as in the above discussion and obtain that

$$\begin{aligned}
\|G+\partial_k F^k\|_{C^{0,\gamma}(\overline{D})} &\leq \|G\|_{C^{0,\gamma}(\overline{D})} + \|F_\theta\|_{C^{1,\gamma}(\overline{D})} + \|F_\varphi\|_{C^{1,\gamma}(\overline{D})} \\
&\leq K(\|(f\Psi_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{D})} + \|u_{\eta_i}\|_{C^{0,\gamma}(\overline{D})} + \|u_{\eta_i}\|_{C^{1,\gamma}(\overline{D})}) \\
&= K(\|(f\Psi_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{D})} + \|(u\hat{\Psi}_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{D})} + \|(u\hat{\Psi}_i)_{\eta_i}\|_{C^{1,\gamma}(\overline{D})}) \\
&\leq K(\|(f\Psi_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})} + 2\|(u\hat{\Psi}_i)_{\eta_i}\|_{C^{1,\gamma}(\overline{\eta_i^{-1}(U_i)})}) \\
&\leq K(\|f\|_{C^{0,\gamma}(S^2)} + 2\|u\|_{C^{1,\gamma}(S^2)}),
\end{aligned}$$

where the constant K is independent of u and f. We insert this estimate into Inequality (6.44) and obtain with the fact $(u\Psi_i)_{\eta_i}$ is equal to zero outside of $supp((\Psi_i)_{\eta_i})$ that there exists a constant K independently of u and f such that

$$\|(u\Psi_i)_{\eta_i}\|_{C^{2,\gamma}(\overline{\eta_i^{-1}(U_i)})} \leq K(\|u\|_{C^{1,\gamma}(S^2)} + \|f\|_{C^{0,\gamma}(S^2)}).$$

As in the case of first order differentiability, this argument can be repeated for all other $j \in \{1,...,6\}\backslash\{i\}$. Therefore, we conclude that $u \in C^{2,\gamma}(S^2)$ and together with Inequality (6.43) we obtain that there exists a constant K independently of u and f such that

$$\|u\|_{C^{2,\gamma}(S^2)} \leq K(\|u\|_{C^{0,\gamma}(S^2)} + \|f\|_{C^{0,\gamma}(S^2)}).$$

We insert Inequality (6.35) into the previous estimate and the claim of the proposition in the case that $\iota = 2$ follows, i.e. there exists a constant K independently of u and f such that

$$\|u\|_{C^{2,\gamma}(S^2)} \leq K(\|u\|_{H^2(S^2)} + \|f\|_{C^{0,\gamma}(S^2)}).$$

For the proof of the general case that $\iota \geq 4$, we define for all $\kappa \in \{0,2,...,\iota-2,\iota\}$ the functions

$$u^{(\kappa)} = \left(-\Delta_{S^2} + \frac{1}{4}\right)^{\frac{\kappa}{2}} u.$$

In the case that $\kappa = \iota$ it holds that $u^{(\iota)} = f$. For all $\kappa \in \{0,2,...,\iota-4,\iota-2\}$ it holds that

$$\left(-\Delta_{S^2} + \frac{1}{4}\right) u^{(\kappa)} = u^{(\kappa+2)} \tag{6.45}$$

with equality in $L^2(S^2,\mathbb{R})$. Since $u \in C^{\iota-2,\gamma}(S^2)$, as we showed at the beginning of this proof, we conclude that $u^{(\kappa)} \in C^{\iota-\kappa-2,\gamma}(S^2)$ for all $\kappa \in \{0,2,...,\iota-4,\iota-2\}$. It follows that

$u^{(\iota-2)} \in C^{0,\gamma}(S^2) \cap H^2(S^2)$ as in the proof of the case that $\iota = 2$. We can apply this proof and obtain that $u^{(\iota-2)} \in C^{2,\gamma}(S^2)$ and there exists a constant K independently of u and f such that

$$\begin{aligned}\|u^{(\iota-2)}\|_{C^{2,\gamma}(S^2)} &\leq K(\|u^{(\iota-2)}\|_{H^2(S^2)} + \|f\|_{C^{0,\gamma}(S^2)}) \\ &= K(\|u\|_{H^\iota(S^2)} + \|f\|_{C^{0,\gamma}(S^2)}),\end{aligned}$$

where we applied that $\|u^{(\iota-2)}\|_{H^2(S^2)} = \|u\|_{H^\iota(S^2)}$, which holds by definition of the norm on Sobolev spaces that we introduced in Section 6.1. The strategy is to proof by induction that for all $\kappa \in \{0, 2, ..., \iota-4, \iota-2\}$ it holds that $u^{(\kappa)} \in C^{\iota-\kappa,\gamma}(S^2)$ and there exists a constant K independently of u and f such that

$$\|u^{(\kappa)}\|_{C^{\iota-\kappa,\gamma}(S^2)} \leq K(\|u\|_{H^\iota(S^2)} + \|f\|_{C^{0,\gamma}(S^2)}), \tag{6.46}$$

where the constant K may depend on the index κ. We have already shown this for $\kappa = \iota-2$ and now assume that it holds for some $\kappa \in \{2, 4, ..., \iota-4, \iota-2\}$ and want to prove that this implies the claim for $\kappa-2$, i.e. $u^{(\kappa-2)} \in C^{\iota-\kappa+2,\gamma}(S^2)$ and Inequality (6.46) holds for $\kappa-2$ instead of κ.

As in the proof of the case $\iota = 2$ we have to localize Equation (6.45) and pull it back to the chart domains to be able to apply regularity theory on subdomains of Euclidean space and interior estimates. Now we have higher regularity of the right hand side. We fix the same $i \in \{1, ..., 6\}$ as in the proof of the case $\iota = 2$ and remind of our usual atlas $(U_j, \eta_j : j = 1, ..., 6)$ with partition of unity Ψ subordinate to the open cover $(U_j : j = 1, ..., 6)$. From the proof of the case $\iota = 2$, we will also use the subdomain D with smooth boundary satisfying $supp((\Psi_i)_{\eta_i}) \subset\subset D \subset\subset \eta_i^{-1}(U_i)$ and the partition of unity $\hat{\Psi}$ subordinate to the open cover $(U_j : j = 1, ..., 6)$ that satisfies that $(\hat{\Psi}_i)_{\eta_i} = 1$ on $\overline{D}$. Since Hölder continuity is a local property, we are interested in the behavior of $(u^{(\kappa-2)}\Psi_i)_{\eta_i}$ in the chart domain $\eta_i^{-1}(U_i)$. The fact that $\kappa-2 \leq \iota-4$ implies that $u^{(\kappa-2)} \in C^{\iota-(\kappa-2)-2,\gamma}(S^2) \subset C^{2,\gamma}(S^2)$. Therefore, we observe that $(u^{(\kappa-2)}\Psi_i)_{\eta_i} \in C^{2,\gamma}(\overline{\eta_i^{-1}(U_i)})$ is a classical solution of the Dirichlet problem in D, i.e.

$$\begin{aligned}L(u^{(\kappa-2)}\Psi_i)_{\eta_i} &= F^{(\kappa-2)} \\ (u^{(\kappa-2)}\Psi_i)_{\eta_i}\Big|_{\partial D} &= 0.\end{aligned}$$

The operator L is given in Equation (6.37), whereas the right hand side $F^{(\kappa-2)}$ is given by

$$F^{(\kappa-2)} = -(u^{(\kappa)}\Psi_i)_{\eta_i} + 2\partial_\theta u_{\eta_i}^{(\kappa-2)}\partial_\theta(\Psi_i)_{\eta_i} + \frac{2}{\sin^2(\theta)}\partial_\varphi u_{\eta_i}^{(\kappa-2)}\partial_\varphi(\Psi_i)_{\eta_i} + (u^{(\kappa-2)}\Delta_{S^2}\Psi_i)_{\eta_i}. \tag{6.47}$$

We will prove a successive increase of the regularity of the right hand side. This could be observed in the proof of the case $\iota = 2$, where the right hand side was first given partly in divergence form and was then improved to be Hölder continuous. Note that by the induction hypothesis it holds that $(u^{(\kappa)}\Psi_i)_{\eta_i} \in C^{\iota-\kappa}(\overline{\eta_i^{-1}(U_i)})$. Since $u^{(\kappa-2)} \in C^{\iota-(\kappa-2)-2,\gamma}(S^2)$, we obtain that $F^{(\kappa-2)} \in C^{\iota-\kappa-1}(\overline{\eta_i^{-1}(U_i)})$. Since the operator L is the same as in the proof of the case $\iota = 2$, it satisfies the conditions to apply regularity theory, i.e. L is strictly elliptic and the coefficients are smooth on $\overline{D}$. Theorem 6.19 in [14] implies that $(u^{(\kappa-2)}\Psi_i)_{\eta_i} \in C^{\iota-\kappa+1,\gamma}(\overline{D})$, where we tacitly applied that $(u^{(\kappa-2)}\Psi_i)_{\eta_i}$ is compactly supported in D. Now,

we apply the interior estimate for higher order derivatives with the nested sets $supp((\Psi_i)_{\eta_i})$ and D, which is Problem 6.1 in [14], which will be proven as a part of Theorem 8.10 in Section 8.1.1, and obtain that there exists a constant K independently of u such that

$$\|(u^{(\kappa-2)}\Psi_i)_{\eta_i}\|_{C^{\iota-\kappa+1,\gamma}(supp((\Psi_i)_{\eta_i}))} \leq K(\|(u^{(\kappa-2)}\Psi_i)_{\eta_i}\|_{C^0(\overline{D})}+\|F^{(\kappa-2)}\|_{C^{\iota-\kappa-1,\gamma}(\overline{D})}). \quad (6.48)$$

Similarly as in the proof of the case $\iota = 2$, we observe that there exists a constant K independently of u such that

$$\begin{aligned}\|F^{(\kappa-2)}\|_{C^{\iota-\kappa-1,\gamma}(\overline{D})}) &\leq K(\|(u^{(\kappa)}\Psi_i)_{\eta_i}\|_{C^{\iota-\kappa-1,\gamma}(\overline{D})} + \|u^{(\kappa-2)}_{\eta_i}\|_{C^{\iota-\kappa,\gamma}(\overline{D})})\\ &= K(\|(u^{(\kappa)}\Psi_i)_{\eta_i}\|_{C^{\iota-\kappa-1,\gamma}(\overline{D})} + \|(u^{(\kappa-2)}\hat{\Psi}_i)_{\eta_i}\|_{C^{\iota-\kappa,\gamma}(\overline{D})})\\ &\leq K(\|(u^{(\kappa)}\Psi_i)_{\eta_i}\|_{C^{\iota-\kappa-1,\gamma}(\overline{\eta_i^{-1}(U_i)})} + \|(u^{(\kappa-2)}\hat{\Psi}_i)_{\eta_i}\|_{C^{\iota-\kappa,\gamma}(\overline{\eta_i^{-1}(U_i)})})\\ &\leq K(\|u^{(\kappa)}\|_{C^{\iota-\kappa-1,\gamma}(S^2)} + \|u^{(\kappa-2)}\|_{C^{\iota-\kappa,\gamma}(S^2)}),\end{aligned}$$

where the contributions of Ψ_i are included into the constant K due to Inequality (6.4). Also, we applied the fact that $(\hat{\Psi}_i)_{\eta_i} = 1$ on $\overline{D}$ and the equivalence of Hölder norms with respect to different partitions of unity, which is Proposition 6.9. We insert this estimate into Inequality (6.48) and obtain with the fact that $(u^{(\kappa-2)}\Psi_i)_{\eta_i}$ is equal to zero outside of $supp((\Psi_i)_{\eta_i})$ that there exists a constant K independently of u such that

$$\|(u^{(\kappa-2)}\Psi_i)_{\eta_i}\|_{C^{\iota-\kappa+1,\gamma}(\overline{\eta_i^{-1}(U_i)})} \leq K(\|u^{(\kappa-2)}\|_{C^{\iota-\kappa,\gamma}(S^2)} + \|u^{(\kappa)}\|_{C^{\iota-\kappa-1,\gamma}(S^2)}).$$

This argument can be repeated for all other $j \in \{1, ..., 6\}\backslash\{i\}$. Therefore, we conclude that $u^{(\kappa-2)} \in C^{\iota-\kappa+1,\gamma}(S^2)$ and that there exists a constant K independently of u such that

$$\|u^{(\kappa-2)}\|_{C^{\iota-\kappa+1,\gamma}(S^2)} \leq K(\|u^{(\kappa-2)}\|_{C^{\iota-\kappa,\gamma}(S^2)} + \|u^{(\kappa)}\|_{C^{\iota-\kappa-1,\gamma}(S^2)}). \quad (6.49)$$

We keep the same $i \in \{1, ..., 6\}$ fixed. Since we have shown that $u^{(\kappa-2)} \in C^{\iota-\kappa+1,\gamma}(S^2)$, it follows that the right hand side $F^{(\kappa-2)}$ in Equation (6.47) is now in $C^{\iota-\kappa}(\overline{\eta_i^{-1}(U_i)})$, which is one order more than before. The same argument as before applies and we obtain that $u^{(\kappa-2)} \in C^{\iota-\kappa+2,\gamma}(S^2)$ and there exists a constant K independently of u such that

$$\|u^{(\kappa-2)}\|_{C^{\iota-\kappa+2,\gamma}(S^2)} \leq K(\|u^{(\kappa-2)}\|_{C^{\iota-\kappa+1,\gamma}(S^2)} + \|u^{(\kappa)}\|_{C^{\iota-\kappa,\gamma}(S^2)}).$$

We insert Inequality (6.49) into the previous estimate and obtain that there exists a constant K independently of u such that

$$\|u^{(\kappa-2)}\|_{C^{\iota-\kappa+2,\gamma}(S^2)} \leq K(\|u^{(\kappa-2)}\|_{C^{\iota-\kappa,\gamma}(S^2)} + \|u^{(\kappa)}\|_{C^{\iota-\kappa,\gamma}(S^2)}). \quad (6.50)$$

We observe that $u^{(\kappa-2)} = (-\Delta_{S^2}+\frac{1}{4})^{\frac{\kappa-2}{2}}u$ is a finite linear combination of partial derivatives of u with order smaller or equal to $\kappa-2$. Therefore, there exists a constant K independently of u such that

$$\|u^{(\kappa-2)}\|_{C^{\iota-\kappa,\gamma}(S^2)} \leq K\|u\|_{C^{\iota-2,\gamma}(S^2)}.$$

We insert the previous estimate, Inequality (6.35) and the induction hypothesis into Inequality (6.50) and obtain that there exists a constant K independently of u and f such that

$$\|u^{(\kappa-2)}\|_{C^{\iota-\kappa+2,\gamma}(S^2)} \leq K(\|u\|_{H^\iota(S^2)} + \|f\|_{C^{0,\gamma}(S^2)}).$$

This finishes the induction argument and the proof of the proposition, since we can take $\kappa = 2$ and it holds that $u^{(0)} = u$. □

Corollary 6.23. *For an even integer $\iota \geq 2$ and $\gamma \in (0,1)$ and all $v \in C^{\iota,\gamma}(S^2)$*

$$\|v\|_{\Delta_{S^2},C^{\iota,\gamma}(S^2)} = \left\| \left(-\Delta_{S^2} + \frac{1}{4}\right)^{\frac{\iota}{2}} v \right\|_{C^{0,\gamma}(S^2)}$$

defines a norm on $C^{\iota,\gamma}(S^2)$, which is equivalent to the usual norm on $C^{\iota,\gamma}(S^2)$.

Proof. We arbitrarily fix $v \in C^{\iota,\gamma}(S^2)$. We observe that $C^{\iota,\gamma}(S^2) \subset H^{\iota}(S^2)$. Since $v \in C^{\iota,\gamma}(S^2)$, it holds that $(-\Delta_{S^2} + \frac{1}{4})^{\frac{\iota}{2}} v \in C^{0,\gamma}(S^2)$. Therefore, we can apply the previous proposition with right hand side $(-\Delta_{S^2} + \frac{1}{4})^{\frac{\iota}{2}} v$ and obtain that there exists a constant K independently of v such that

$$\|v\|_{C^{\iota,\gamma}(S^2)} \leq K \left(\|v\|_{H^{\iota}(S^2)} + \left\| \left(-\Delta_{S^2} + \frac{1}{4}\right)^{\frac{\iota}{2}} v \right\|_{C^{0,\gamma}(S^2)} \right).$$

We apply the definition of Sobolev spaces on S^2 to obtain that

$$\|v\|_{H^{\iota}(S^2)} = \left\| \left(-\Delta_{S^2} + \frac{1}{4}\right)^{\frac{\iota}{2}} v \right\|_{L^2(S^2,\mathbb{R})} \leq 2\sqrt{\pi} \left\| \left(-\Delta_{S^2} + \frac{1}{4}\right)^{\frac{\iota}{2}} v \right\|_{C^0(S^2)}.$$

We conclude that there exists a constant K independently of v such that

$$\|v\|_{C^{\iota,\gamma}(S^2)} \leq K \left\| \left(-\Delta_{S^2} + \frac{1}{4}\right)^{\frac{\iota}{2}} v \right\|_{C^{0,\gamma}(S^2)} = K\|v\|_{\Delta_{S^2},C^{\iota,\gamma}(S^2)}.$$

Besides the estimate, this also implies the positivity of $\|\cdot\|_{\Delta_{S^2},C^{\iota,\gamma}(S^2)}$, because $\|\cdot\|_{C^{\iota,\gamma}(S^2)}$ is a norm. The other properties of a norm are clear for $\|\cdot\|_{\Delta_{S^2},C^{\iota,\gamma}(S^2)}$.
For the proof of the other direction, we observe that $(-\Delta_{S^2} + \frac{1}{4})^{\frac{\iota}{2}} v$ is a finite linear combination of partial derivatives with order smaller or equal to ι. Therefore, there exists a constant K' independently of v such that

$$\|v\|_{\Delta_{S^2},C^{\iota,\gamma}(S^2)} = \left\| \left(-\Delta_{S^2} + \frac{1}{4}\right)^{\frac{\iota}{2}} v \right\|_{C^{0,\gamma}(S^2)} \leq K'\|v\|_{C^{\iota,\gamma}(S^2)}. \qquad \square$$

For our discussion of the regularity of continuous 2-weakly isotropic Gaussian spherical random fields, we will discover that this proposition is in particular well-suited, because it provides upper bounds of higher order Hölder norms in terms of powers of the spherical Laplace operator. The effect of the spherical Laplace operator on 2-weakly isotropic spherical random fields is sufficiently well understood, since the spherical Laplace operator is diagonalized by the real spherical harmonics and we have proved that those random fields can be expanded in the real spherical harmonics.

We begin to explore the Hölder continuity of continuous 2-weakly isotropic Gaussian spherical random fields. The following result follows with a version of the Kolmogorov–Čentsov continuity theorem on manifolds from the paper [2] by Andreev and Lang. Additionally, we will apply some related considerations to characterize the L^p_P integrability of Hölder norms. These considerations can be cited from Section 6.2.

Theorem 6.24. *Let T be a continuous 2-weakly isotropic Gaussian spherical random field such that the angular power spectrum satisfies that $(C_l l^{1+\delta} : l \geq 0)$ is summable for some $\delta \in (0, 2]$. For all $\gamma \in (0, \frac{\delta}{2})$ there exists an indistinguishable modification T^* of T such that $T^* \subset C^{0,\gamma}(S^2)$ and for all $p \in (0, \infty)$ there exists a constant K_p independently of T and $(C_l : l \geq 0)$ such that*

$$\|T\|_{L^p_P(\Omega, C^{0,\gamma}(S^2))} \leq K_p \left(\sum_{l \geq 0} C_l l^{1+\delta} \right)^{\frac{1}{2}}.$$

Proof. We check that $T \in L^{2k}_{P \otimes \mathrm{d}\sigma}(\Omega \times S^2, \mathbb{R})$ for all $k \in \mathbb{N}$. This is clear for $k = 1$ by Lemma 3.3. We conclude with the same argument that we used to prove Inequality (5.9) and Lemma 2.8 that

$$\begin{aligned}
\|T\|^{2k}_{L^{2k}_{P \otimes \mathrm{d}\sigma}(\Omega \times S^2, \mathbb{R})} &= E\left[\int_{S^2} \left| \sum_{l \geq 0} \sum_{m=-l}^{l} \sqrt{C_l} \tilde{\beta}_{l,m} \tilde{Y}_{l,m}(x) \right|^{2k} \mathrm{d}\sigma(x) \right] \\
&\leq \frac{(2k)!}{2^k k!} \int_{S^2} \left(\sum_{l \geq 0} \sum_{m=-l}^{l} C_l \tilde{Y}_{l,m}(x) \right)^k \mathrm{d}\sigma(x) \\
&= \frac{(2k)!}{2^k k!} \int_{S^2} \left(\sum_{l \geq 0} C_l \frac{2l+1}{4\pi} \right)^k \mathrm{d}\sigma(x).
\end{aligned}$$

Since $(C_l l : l \geq 0)$ is summable and the sphere has finite volume, $T \in L^{2k}_{P \otimes \mathrm{d}\sigma}(\Omega \times S^2, \mathbb{R})$. Lemma 5.2 provides the other needed condition that we can apply Theorem 3.5 in [2] with $n = 2$, $d = 0$, $\varepsilon = \delta k - 2$ and we choose $k > \frac{2}{\delta - 2\gamma}$, then we conclude that there exists a modification T^* of T that is in $C^{0,\gamma}(S^2)$. Since T is already continuous, T and T^* are indistinguishable. We can apply Equation (6.20) in the same way as we did in the proof of Theorem 6.16 and obtain that for every $p \in (0, \infty)$ there exists a constant K_p independently of T and $(C_l : l \geq 0)$ such that

$$E\left[\|T\|^p_{C^{0,\gamma}(S^2)} \right]^{\frac{1}{p}} \leq K_p \left(\sum_{l \geq 0} C_l l^{1+\delta} \right)^{\frac{1}{2}}. \qquad \square$$

Remark 6.25. *In the case that T is a 2-weakly isotropic Gaussian spherical random field with the same summability condition on the angular power spectrum as in the previous theorem, that is not necessarily continuous, the previous theorem still holds. But the modification and the field are not indistinguishable and the bound of the $L^p_P(\Omega, C^{0,\gamma}(S^2))$-norm is still valid, but it holds for the modification and not for the field.*

Since we understand Hölder continuity, we can apply in the next theorem the deterministic regularity result in Proposition 6.22 pathwise on a continuous 2-weakly isotropic Gaussian spherical random field T to conclude Hölder regularity of order ι, where $\iota \geq 0$ is an even integer. It is convenient here, that the verification of the membership of realizations of T in $H^\iota(S^2)$ is relatively unproblematic, due to the specific definition of the norm on $H^\iota(S^2)$ and the known specific expansion of T.

Proposition 6.26. *Let T be a continuous 2-weakly isotropic Gaussian spherical random field such that the angular power spectrum satisfies that $(C_l l^{1+2\iota+\delta} : l \geq 0)$ is summable for some $\delta \in (0,2]$ and some even integer $\iota \geq 0$. For all $\gamma \in (0, \frac{\delta}{2})$ there exists an indistinguishable modification T^* of T such that $T^* \subset C^{\iota,\gamma}(S^2)$ and for all $p \in (0,\infty)$ there exists a constant K_p independently of T and $(C_l : l \geq 0)$ such that*

$$\|T\|_{L^p_P(\Omega, C^{\iota,\gamma}(S^2))} \leq K_p \left(\sum_{l \geq 0} C_l l^{1+2\iota+\delta} \right)^{\frac{1}{2}}.$$

Proof. The case $\iota = 0$ is already proven in Theorem 6.24. Hence, we can assume that $\iota \geq 2$. We define the 2-weakly isotropic Gaussian spherical random field $\hat{T}$:

$$\hat{T} = \left(-\Delta_{S^2} + \frac{1}{4} \right)^{\frac{\iota}{2}} T.$$

Note that $\hat{T}$ is not necessarily continuous. The angular power spectrum $(\hat{C}_l : l \geq 0)$ of $\hat{T}$ is given by $(\hat{C}_l : l \geq 0) = (C_l (l + \frac{1}{2})^{2\iota} : l \geq 0)$. It is evident that $(\hat{C}_l l^{1+\delta} : l \geq 0)$ is summable. Theorem 6.24 is applicable and together with Remark 6.25 we conclude that there exists a continuous modification $\hat{T}^*$ of $\hat{T}$ such that $\hat{T}^* \subset C^{0,\gamma}(S^2)$. Moreover for all $p' \in (0,\infty)$ there exists a constant $K_{p'}$ independently of $\hat{T}^*$ and $(\hat{C}_l : l \geq 0)$ such that

$$\|\hat{T}^*\|_{L^{p'}_P(\Omega, C^{0,\gamma}(S^2))} \leq K_{p'} \left(\sum_{l \geq 0} C_l l^{1+\delta} \right)^{\frac{1}{2}}. \tag{6.51}$$

The modification $\hat{T}^*$ is again a 2-weakly isotropic spherical random field and the expansion of $\hat{T}$ in the real spherical harmonics also converges to $\hat{T}^*$ in the $L^2_P(\Omega, L^2(S^2, \mathbb{R}))$-norm as already mentioned at the beginning of this chapter. Since the convergence is in L^2_P it is also in probability. Itô and Nisio analyzed the convergence of sums of independent Banach space valued random variables in [18]. Theorem 3.1 in [18] implies that the expansion in the real spherical harmonics of $\hat{T}$ converges P-a.s to $\hat{T}$ and to $\hat{T}^*$ in $L^2(S^2, \mathbb{R})$. Therefore, there exists a measurable set Ω^* of full probability such that $\hat{T}(\omega) = \hat{T}^*(\omega)$ with equality in $L^2(S^2, \mathbb{R})$ for all $\omega \in \Omega^*$.
From Lemma 6.12 we know that $T \in L^{2k}_P(\Omega, H^\iota(S^2))$ for all $k \in \mathbb{N}$ and that the truncated expansion T^L converges to T in the $L^{2k}_P(\Omega, H^\iota(S^2))$-norm as $L \to \infty$. Hence, the convergence is also in probability. An application of Theorem 3.1 in the paper of Itô and Nisio, i.e. [18], implies that P-a.s. T^L converges to T as $L \to \infty$ in the $H^\iota(S^2)$-norm. This implies that there exists a measurable set of full probability Ω^{**} such that $T(\omega) \in H^\iota(S^2)$ for all $\omega \in \Omega^{**}$. The intersection of Ω^* and Ω^{**} is still a measurable set with full probability, which we also denote by Ω^* to limit the used notation. Moreover, Lemma 6.12 implies with the continuous embedding $L^{2(\lceil p' \rceil)}_P(\Omega, \mathbb{R}) \subset L^{p'}_P(\Omega, \mathbb{R})$ that for all $p' \in (0,\infty)$ there exists a constant $K_{p'}$ independently of T and $(C_l : l \geq 0)$ such that

$$\|T\|_{L^{p'}_P(\Omega, H^\iota(S^2))} \leq K_{p'} \left(\sum_{l \geq 0} C_l l^{1+2\iota} \right)^{\frac{1}{2}}. \tag{6.52}$$

We obtain that for all $\omega \in \Omega^*$ it holds that

$$\left(-\Delta_{S^2} + \frac{1}{4} \right)^{\frac{\iota}{2}} T(\omega) = \hat{T}^*(\omega)$$

with equality in $L^2(S^2,\mathbb{R})$. Recall that T is continuous by assumption. We arbitrarily fix $\omega \in \Omega^*$. Since $T(\omega) \in C^0(S^2) \cap H^\iota(S^2)$ and $\hat{T}^*(\omega) \in C^{0,\gamma}(S^2)$, Proposition 6.22 is applicable and it implies that $T(\omega) \in C^{\iota,\gamma}(S^2)$ and that there exists a constant K independently of $T(\omega)$ and $\hat{T}^*(\omega)$ such that

$$\|T(\omega)\|_{C^{\iota,\gamma}(S^2)} \leq K(\|T(\omega)\|_{H^\iota(S^2)} + \|\hat{T}^*(\omega)\|_{C^{0,\gamma}(S^2)}).$$

Since $\omega \in \Omega^*$ was arbitrarily chosen and Ω^* is a measurable set with full probability, we obtain with the last estimate, Inequalities (6.51) and (6.52) and the triangle inequality that for all $p \in [1,\infty)$ there exists a constant K_p independently of T and $(C_l : l \geq 0)$ such that

$$\begin{aligned}\|T\|_{L^p_P(\Omega,C^{\iota,\gamma}(S^2))} = E\left[\|T\|^p_{C^{\iota,\gamma}(S^2)}\right]^{\frac{1}{p}} &\leq K\left(E\left[\|T\|^p_{H^\iota(S^2)}\right]^{\frac{1}{p}} + E\left[\|\hat{T}^*\|^p_{C^{0,\gamma}(S^2)}\right]^{\frac{1}{p}}\right)\\ &\leq K_p\left(\sum_{l\geq 0} C_l l^{1+2\iota+\delta}\right)^{\frac{1}{2}},\end{aligned}$$

where we tacitly used that $\sum_{l\geq 0}\hat{C}_l l^{1+\delta}$ can be bounded with $\sum_{l\geq 0} C_l l^{1+2\iota+\delta}$. The claim for $p \in (0,1)$ follows with the embedding $L^1_P(\Omega,\mathbb{R}) \subset L^p_P(\Omega,\mathbb{R})$, which is due to the Hölder inequality. The indistinguishable modification of T is given by $T^* = T\mathbb{1}_{\Omega^*}$. □

The statement of the previous proposition for an arbitrarily chosen $\iota \in \mathbb{N}_0$ will follow with interpolation theory. For the discussion of the derivatives of odd integer order, we introduce a new interpretation of a 2-weakly isotropic Gaussian spherical random field. We define weighted sequence spaces.

Definition 6.27. *For an integer $k \geq 0$ and $\sigma \in (0,2]$ we introduce the normed sequence space*

$$\ell^2_{k,\sigma}(\mathbb{N}) = \left\{(a_l : l \geq 1) \in \ell^2(\mathbb{N}) : \sum_{l\geq 1} a_l^2 l^{1+2k+\sigma} < \infty\right\}.$$

For $(a_l : l \geq 1) \in \ell^2_{k,\sigma}(\mathbb{N})$ the norm is given by

$$\|(a_l : l \geq 1)\|_{\ell^2_{k,\sigma}(\mathbb{N})} = \left(\sum_{l\geq 1} a_l^2 l^{1+2k+\sigma}\right)^{\frac{1}{2}}.$$

We can interpret a 2-weakly isotropic Gaussian spherical random field as a linear mapping on these sequence spaces. For all even integers $\iota \geq 0$ and $\delta \in (0,2]$ Proposition 6.26 implies that for all $\gamma \in (0,\frac{\delta}{2})$ and $p \in (0,\infty)$ the following mapping:

$$T : \ell^2_{\iota,\delta}(\mathbb{N}) \to L^p_P(\Omega, C^{\iota,\gamma}(S^2)) \tag{6.53}$$

that is defined by

$$(a_l : l \geq 1) \mapsto \sum_{l\geq 1}\sum_{m=-l}^{l} a_l \tilde{\beta}_{l,m}\tilde{Y}_{l,m}, \tag{6.54}$$

is linear and bounded, since by Proposition 6.26 for every $p \in (0,\infty)$ the operator norm of T is bounded by the constant K_p from this proposition. Note that we tacitly used the

same notation for this operator to emphasize the connection to 2-weakly isotropic Gaussian spherical random fields. They are of course different mathematical objects. Also note that as in previous chapters $(\tilde{\beta}_{l,m} : l \geq 1, m = -l, ..., l)$ is an i.i.d. sequence of standard normally distributed random variables and $(\tilde{Y}_{l,m} : l \geq 0, m = -l, ..., l)$ are the real spherical harmonics. We can now use our knowledge of interpolation theory to obtain the general result.

Theorem 6.28. *Let T be a continuous 2-weakly isotropic Gaussian spherical random field such that the angular power spectrum satisfies that $(C_l l^{1+2\iota+\delta} : l \geq 0)$ is summable for some $\delta \in (0, 2]$ and some integer $\iota \geq 0$. For all $\gamma \in (0, \frac{\delta}{2})$ there exists an indistinguishable modification T^* of T such that $T^* \subset C^{\iota,\gamma}(S^2)$ and for all $p \in (0, \infty)$ there exists a constant K_p independently of T and $(C_l : l \geq 0)$ such that*

$$\|T\|_{L^p_P(\Omega, C^{\iota,\gamma}(S^2))} \leq K_p \left(\sum_{l \geq 0} C_l l^{1+2\iota+\delta} \right)^{\frac{1}{2}}.$$

Proof. In the case that ι is even Proposition 6.26 implies the claim. Therefore we assume that $\iota \geq 1$ is odd. Proposition 6.26 implies that the claim is already established for $\iota - 1$ and $\iota + 1$. We see that for all $p \in (0, \infty)$ the 2-weakly isotropic Gaussian spherical random field T can be interpreted as a bounded linear mapping from $\ell^2_{\iota-1,\delta}(\mathbb{N})$ to $L^p_P(\Omega, C^{\iota-1,\gamma}(S^2))$ and from $\ell^2_{\iota+1,\delta}(\mathbb{N})$ to $L^p_P(\Omega, C^{\iota+1,\gamma}(S^2))$. This notion was introduced in Equation (6.53) and Equation (6.54).
We consult now Appendix C for a summary of relevant interpolation theory. Lemma C.3 implies that

$$T : \left(\ell^2_{\iota-1,\delta}(\mathbb{N}), \ell^2_{\iota+1,\delta}(\mathbb{N})\right)_{\frac{1}{2},2} \to \left(L^p_P(\Omega, C^{\iota-1,\gamma}(S^2)), L^p_P(\Omega, C^{\iota+1,\gamma}(S^2))\right)_{\frac{1}{2},2}$$

is also a linear bounded mapping. Lemma C.4 implies that with equivalent norms

$$\left(\ell^2_{\iota-1,\delta}(\mathbb{N}), \ell^2_{\iota+1,\delta}(\mathbb{N})\right)_{\frac{1}{2},2} = \ell^2_{\iota,\delta}(\mathbb{N}).$$

For $p \in [2, \infty)$ Lemma C.5 implies that

$$\left(L^p_P(\Omega, C^{\iota-1,\gamma}(S^2)), L^p_P(\Omega, C^{\iota+1,\gamma}(S^2))\right)_{\frac{1}{2},2} \subset L^p_P(\Omega, C^{\iota,\gamma}(S^2)).$$

Therefore we have established that for all $p \in [2, \infty)$ it holds that

$$\begin{aligned} \|T\|_{L^p_P(\Omega, C^{\iota,\gamma}(S^2))} &= \left\| T(\sqrt{C_l} : l \geq 1) \right\|_{L^p_P(\Omega, C^{\iota,\gamma}(S^2))} \\ &\leq K_p \left\| (\sqrt{C_l} : l \geq 1) \right\|_{\ell^2_{\iota,\delta}(\mathbb{N})} = K_p \left(\sum_{l \geq 1} C_l l^{1+2\iota+\delta} \right)^{\frac{1}{2}}. \end{aligned}$$

The statement follows for $p \in (0, 2)$ with the embedding $L^2_P(\Omega, \mathbb{R}) \subset L^p_P(\Omega, \mathbb{R})$ due to the Hölder inequality. Note that we tacitly proved the case that $C_0 = 0$. Since the term $\sqrt{C_0}\tilde{\beta}_{0,0}\tilde{Y}_{0,0}$ is constant in S^2, it does not cause difficulties and does not contribute, when we consider derivatives.
Since the $L^2_P(\Omega, C^{\iota,\gamma}(S^2))$-norm of T is finite, it follows that there exists a measurable set Ω^* of full probability such that $\|T(\omega)\|_{C^{\iota,\gamma}(S^2)}$ is finite for all $\omega \in \Omega^*$. Hence, $T(\omega) \in C^{\iota,\gamma}(S^2)$ for all $\omega \in \Omega^*$. The indistinguishable modification of T is given by $T^* = T\mathbb{1}_{\Omega^*}$. □

6.5. Notes on Sobolev spaces on the sphere

In this section we will give the proof of Proposition 6.5. Since the norms of $W^{k,p}(S^2)$ are equivalent for all smooth atlases, we will perform the proof with our usual atlas $(U_i, \eta_i : i = 1, ..., 6)$ of S^2 with partition of unity Ψ.

Proof of Proposition 6.5. We use the notation, which we introduced at the beginning of this chapter. For $f \in W^{k,2}(S^2)$ and $i = 1, ..., p$ we define

$$f_i = f\Psi_i : U_i \to \mathbb{R}.$$

Note that $\sum_{i=1}^{6} f_i = f$, which holds by definition of a partition of unity. Also it holds that $f, f_1, ..., f_6 \in L^2(S^2, \mathbb{R})$. Therefore they all obey an expansion in the real spherical harmonics, i.e.

$$f = \sum_{l\geq 0} \sum_{m=-l}^{l} \tilde{a}_{l,m} \tilde{Y}_{l,m} \qquad \text{and} \qquad f_i = \sum_{l\geq 0} \sum_{m=-l}^{l} \tilde{a}^{(i)}_{l,m} \tilde{Y}_{l,m} \qquad \text{for } i = 1, ..., 6.$$

We know that for $i = 1, ..., 6$ the functions $(f_i)_{\eta_i} \in W_0^{k,2}(\eta_i^{-1}(U_i))$. For an arbitrary multi-index $\boldsymbol{\alpha} \in \mathbb{N}_0^2$ with $|\boldsymbol{\alpha}| = k$ we compute the $L^2(\eta_i^{-1}(U_i), \mathbb{R})$-norm of the $\boldsymbol{\alpha}$-weak derivative of the functions $(f_i : i = 1, ..., 6)$. We can interchange the sum and the $\boldsymbol{\alpha}$-weak derivative and evaluate the respective partial derivative of the real spherical harmonics.

$$\begin{aligned}\int_{\eta_i^{-1}(U_i)} (\partial_{\boldsymbol{\alpha}} (f_i)_{\eta_i}(y))^2 \,\mathrm{d}y &\simeq \int_{U_i} (\partial_{\boldsymbol{\alpha}} f_i(x))^2 \,\mathrm{d}\sigma(x) \\ &= \int_{S^2} \left(\sum_{l\geq 0} \sum_{m=-l}^{l} \tilde{a}^{(i)}_{l,m} \partial_{\boldsymbol{\alpha}} \tilde{Y}_{l,m}(x) \right)^2 \mathrm{d}\sigma(x). \end{aligned} \tag{6.55}$$

Note that the image measure of $\mathrm{d}y$ under η_i and the measure $\mathrm{d}\sigma(x)$ are equivalent on U_i for $i = 1, ..., 6$. In the proof of Lemma 6.18 we established that the partial derivatives of real spherical harmonics results in finitely many linear combinations of real spherical harmonics with coefficients of the following type: powers of $m \in \{-l-k, ..., l+k\}$ with exponents at most equal to k and functions $\partial_\theta^q \frac{1}{\sin^h(\theta)}$ for $h, q \leq k$, which are smooth and therefore bounded. This argument led in Lemma 6.19 to Inequality (6.33), i.e.

$$(\partial_{\boldsymbol{\alpha}} \tilde{Y}_{l,m}(x))^2 \leq 2^{2k-1} 6^k k^{2k} l^{2k} K \sum_{j=1}^{6^k} \tilde{Y}^2_{l_j,m_j}(x), \tag{6.56}$$

where $l_j \leq l + k$ and $m_j \in \{-l_j, ..., l_j\}$ for all $j = 1, ..., 6^k$. K is a constant, that is independent of l and m. With Inequalities (6.55) and (6.56) we conclude that

$$\begin{aligned}\|\partial_{\boldsymbol{\alpha}} (f_i)_{\eta_i}\|^2_{L^2(\eta_i^{-1}(U_i),\mathbb{R})} + \|(f_i)_{\eta_i}\|^2_{L^2(\eta_i^{-1}(U_i),\mathbb{R})} &\leq K \sum_{l\geq 0} \sum_{m=-l}^{l} (\tilde{a}^{(i)}_{l,m})^2 (l^{2k} + 1) \\ &\leq K \sum_{l\geq 0} \sum_{m=-l}^{l} (\tilde{a}_{l,m})^2 \left(l + \frac{1}{2}\right)^{2k} \\ &= K \|f\|_{H^k(S^2)},\end{aligned}$$

where we applied the orthonormality of the real spherical harmonics and tacitly included the other pre factors in Inequality (6.56) into the constant K apart from the dependency on l. We also used that $f = \sum_{i=1}^{p} f_i$ and the property that the partition of unity takes values between zero and one to obtain that for all $i \in \{1, ..., 6\}$ it holds that $\|f\|^2_{L^2(S^2,\mathbb{R})} \geq \|f_i\|^2_{L^2(U_i,\mathbb{R})}$. The above inequality can be achieved for all $i \in \{1, ..., 6\}$ and all multi-indices $\boldsymbol{\alpha} \in \mathbb{N}_0^2$ that satisfy that $|\boldsymbol{\alpha}| = k$. It was sufficient to consider the highest weak derivative and the L^2-norm of the function due to Theorem 4.2.4 in [25]. $\square$

We do not discuss whether or not the spaces $H^k(S^2)$ and $W^{k,2}(S^2)$ are actually equal with equivalent norms for $k \in \mathbb{N}$, because this is not necessary in this project. The reader is referred to a recent paper by Dai and Xu [9]. The two authors work with similar definitions of Sobolev spaces on the sphere. In particular Lemma 3.9, Remark 3.1 and Equation 3.17 in [9] could lead to further results.

We close the chapter with a brief example. We wonder, under which decay of the angular power spectrum would a continuous 2-weakly isotropic Gaussian spherical random field be P-a.s. smooth. We mean that derivatives of arbitrary order must have a continuous modification. Theorem 6.20 gives us a condition on the needed decay of the angular power spectrum, i.e. $(C_l l^k : l \geq 0)$ has to be summable for all $k \in \mathbb{N}$. A class of angular power spectra $(C_l : l \geq 0)$ that meets this condition would be that for $\alpha > 0$ and all integers $l \geq 0$ it holds that

$$C_l \lesssim \exp(-l^\alpha).$$

To prove this, we introduce $n \in \mathbb{N}$ such that $\frac{1}{n} < \alpha$. The resulting infinite sum is bounded by an integral, which we manipulate with a transformation and partial integration such that for all $k \in \mathbb{N}$ it holds that

$$\begin{aligned}\sum_{l\geq 0} C_l l^k &\lesssim \sum_{l\geq 0} \exp(-l^\alpha) l^k \lesssim \sum_{l\geq 0} \exp(-l^{\frac{1}{n}}) l^k \\ &\lesssim \int_0^\infty \exp(-x^{\frac{1}{n}}) x^k \mathrm{d}x \simeq n \int_0^\infty \exp(-y) y^{k+n-1} \mathrm{d}y \simeq n(k+n-1)! \,.\end{aligned}$$

With the same Matlab code from Appendix A which we used for the two plots at the end of Chapter 3 we produce plots of truncated expansions of 2-weakly isotropic Gaussian spherical random fields with $L = 200$. For the angular power spectrum we take the one we defined just above with $\alpha = 1$ and $\alpha = \frac{3}{4}$.

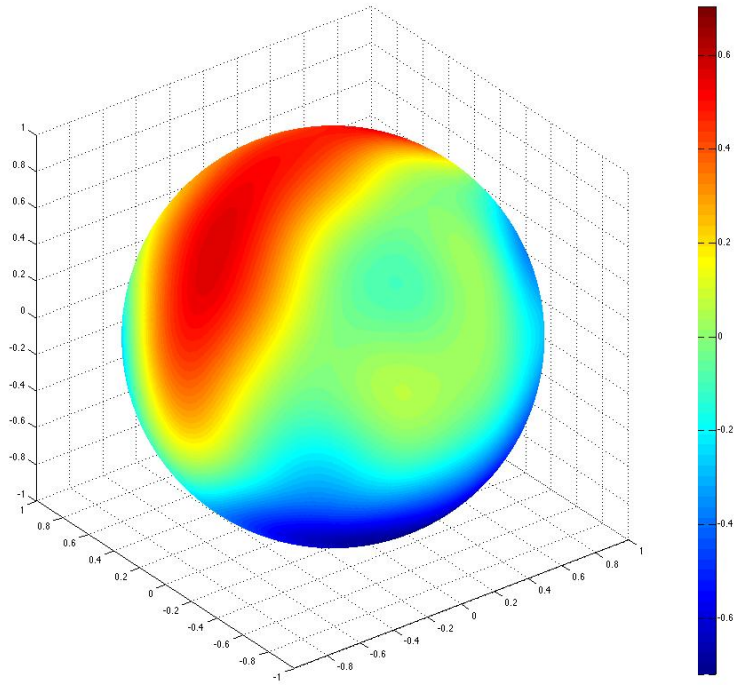

Figure 6.1.: realization of T^L with $C_l = \exp(-l)$

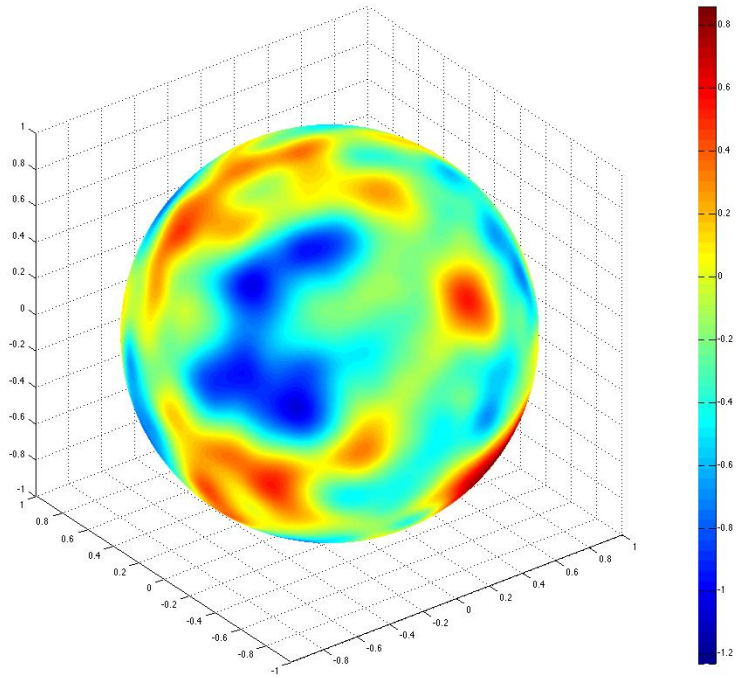

Figure 6.2.: realization of T^L with $C_l = \exp(-l^{\frac{3}{4}})$

7. Log-normally distributed spherical random fields

Random fields which are log-normally distributed play an important role in engineering applications. Isotropic log-normal spherical random field were introduced in the paper [20] of Lang and Schwab. Also they proved P-a.s. Hölder continuity. We recapitulate their statement, which is Corollary 6.2 in [20] in the first section and then investigate higher regularity similar to isotropic Gaussian spherical random fields in the previous chapter.

7.1. Basic properties of log-normally distributed spherical random fields

Definition 7.1. *For a Gaussian spherical random field T we define the log-normal spherical random field to be*

$$A = \exp(T).$$

A log-normal spherical random field is a well-defined spherical random field, since the exponential function is measurable.

Lemma 7.2. *The log-normal spherical random field A, which results from a 2-weakly isotropic Gaussian spherical random field T, is also 2-weakly isotropic.*

Proof. For $x \in S^2$ Lemma 4.6 implies that $T(x)$ is normally distributed with mean zero and finite variance σ_T^2. Consequently $A(x)$ is log-normally distributed with mean $\exp(\frac{1}{2}\sigma_T^2)$ and second moment $\exp(2\sigma_T^2)$. Note that σ_T^2 is independent of x.
We fix $x_1, x_2 \in S^2$ and $g \in SO(3)$ to show the invariance under the action of $SO(3)$ of the covariance of $(A(x_1), A(x_2))$. Lemma 4.7 implies that $(T(x_1), T(x_2))$ and $(T(gx_1), T(gx_2))$ have the same multivariate normal distribution. In particular their probability density functions agree, which are denoted by $f_{T(x_1),T(x_2)}$ and $f_{T(gx_1),T(gx_2)}$. The invariance can now be calculated. We obtain that

$$\begin{aligned}
E[A(x_1)A(x_2)] &= E[\exp(T(x_1))\exp(T(x_2))] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(x)\exp(y) f_{T(x_1),T(x_2)}(x,y)\mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp(x)\exp(y) f_{T(gx_1),T(gx_2)}(x,y)\mathrm{d}x\mathrm{d}y \\
&= E[\exp(T(gx_1))\exp(T(gx_2))] = E[A(gx_1)A(gx_2)].
\end{aligned}$$

In the same way, we obtain that for all $x \in S^2$ and $g \in SO(3)$ it holds that $E[A(x)] = E[A(gx)]$. □

In Chapter 5 we developed sufficient conditions on a 2-weakly isotropic Gaussian spherical random field such that it has a Hölder continuous modification. Now we adapt the strategy to obtain a similar result for a 2-weakly isotropic log-normal spherical random field.

Lemma 7.3. *Let T be a 2-weakly isotropic Gaussian spherical random field, such that for its angular power it holds that $(C_l l^{1+\delta} : l \geq 0)$ is summable for $\delta \in (0, 2]$, then for any $k \in \mathbb{N}$, $x, y \in S^2$ and a constant $K_k > 0$ depending on k it holds that*

$$E\left[|\exp(T(x)) - \exp(T(y))|^{2k}\right] \leq K_k \left(\sum_{l\geq 0} C_l l^{1+\delta}\right)^k d(x,y)^{\delta k}.$$

Proof. A small application of the fundamental theorem of calculus and the monotonicity of the exponential function yield that for $x, y \in \mathbb{R}$ it holds that

$$|e^x - e^y| = |\int_y^x e^s \mathrm{d}s| \leq (e^x + e^y)|x - y|. \tag{7.1}$$

This fact can be applied to the left hand side of the inequality in the claim of the lemma to obtain with the help of the Cauchy–Schwarz inequality and Lemma 5.2 that

$$\begin{aligned}
&E\left[|\exp(T(x)) - \exp(T(y))|^{2k}\right] \\
&\qquad \leq E\left[|T(x) - T(y)|^{2k}(\exp(T(x)) + \exp(T(y)))^{2k}\right] \\
&\qquad \leq E\left[|T(x) - T(y)|^{4k}\right]^{\frac{1}{2}} E\left[(\exp(T(x)) + \exp(T(y)))^{4k}\right]^{\frac{1}{2}} \\
&\qquad \leq \tilde{K}_k \left(\sum_{l\geq 0} C_l l^{1+\delta}\right)^k d(x,y)^{\delta k} E\left[(\exp(T(x)) + \exp(T(y)))^{4k}\right]^{\frac{1}{2}}.
\end{aligned}$$

Note that the summability of the sequence $(C_l l^{1+\delta} : l \geq 0)$ was needed in order to apply Lemma 5.2. If we show that $E[(\exp(T(x)) + \exp(T(y)))^{4k}]^{\frac{1}{2}}$ can be bounded independently of x and y, then the claim of the lemma will be proven. We apply the fact that for all $p > 1$ and $b, c \geq 0$ it holds that

$$(b + c)^p \leq 2^{p-1}(b^p + c^p), \tag{7.2}$$

which follows from the convexity of the function $(x \mapsto x^p)$ for $x \in \mathbb{R}$ and obtain that

$$\begin{aligned}
E\left[(\exp(T(x)) + \exp(T(y)))^{4k}\right]^{\frac{1}{2}} &\leq 2^{\frac{2k-1}{2}} E\left[(\exp(T(x))^{2k} + \exp(T(y))^{2k})^2\right]^{\frac{1}{2}} \\
&\leq 2^{\frac{2k-1}{2}} \left(E\left[(\exp(T(x))^{4k}\right]^{\frac{1}{2}} + E\left[(\exp(T(y))^{4k}\right]^{\frac{1}{2}}\right).
\end{aligned} \tag{7.3}$$

Lemma 4.6 implies that $T(x)$ is normally distributed with mean zero and variance $\sigma_T^2 = \sum_{l\geq 0} C_l \frac{2l+1}{4\pi}$ for all $x \in S^2$. The moments of log-normally distributed random variables are known. For a random variable $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ the moments of $\exp(X)$ are given by

$$E\left[\exp(X)^n\right] = \exp(n\mu_X + \frac{n^2\sigma_X^2}{2})$$

for all $n \in \mathbb{N}$. Therefore the $4k^{\text{th}}$ moment of $\exp(T(x))$ and $\exp(T(y))$ are given by

$$E\left[(\exp(T(x))^{4k}\right] = E\left[(\exp(T(y))^{4k}\right] = \exp(8k^2\sigma_T^2).$$

We insert the value of the $4k^{\text{th}}$ moment into Inequality (7.3) and obtain that

$$E\left[(\exp(T(x))+\exp(T(y)))^{4k}\right]^{\frac{1}{2}} \leq 2^{\frac{2k+1}{2}}\exp(4k^2\sigma_T^2). \tag{7.4}$$

This finishes the proof with the constant $K_k = \tilde{K}_k 2^{\frac{2k+1}{2}}\exp(4k^2\sigma_T^2)$. □

The previous lemma has a very similar content as Lemma 5.2, i.e. the claim is the respective inequality for a 2-weakly isotropic log-normal spherical random field. The difference lies in the fact that Lemma 5.2 deals with a 2-weakly isotropic Gaussian spherical random field T whereas Lemma 7.3 with the resulting 2-weakly isotropic log-normal spherical random field $A = \exp(T)$. Lemma 5.2 was the ingredient in the proof of Theorem 5.4 that relied on the distribution of the spherical random field. Hence we expect Theorem 5.4 also to hold for a 2-weakly isotropic log-normal spherical random field, which results from a 2-weakly isotropic Gaussian spherical random field that satisfies the conditions of Lemma 7.3. This is achieved with the following proposition.

Proposition 7.4. *Let A be a 2-weakly isotropic log-normal spherical random field, which results from the 2-weakly isotropic Gaussian spherical random field T such that for its angular power spectrum it holds that $(C_l l^{1+\delta} : l \geq 0)$ is summable for some $\delta \in (0,2]$. For all $\gamma \in (0, \frac{\delta}{2})$ there exist a modification A^* of A, a P-a.s. positive random variable h^* and a constant $K > 0$ such that A^* is almost surely locally γ-Hölder continuous, i.e. there exists a measurable set of full probability Ω^* such that for all $\omega \in \Omega^*$ and all $x, y \in S^2$ satisfying $d(x,y) < h^*(\omega)$ it holds that*

$$|A^*(x,\omega) - A^*(y,\omega)| \leq K d(x,y)^\gamma.$$

Proof. The proof of Theorem 5.4 can be inserted line by line. At the moment, when Lemma 5.2 is applied in the proof of Theorem 5.4, here we apply Lemma 7.3 instead. □

Note that in the next section, Theorem 7.7 will also partially imply this proposition, but with different characteristics. This is due to the fact that we will use results from Chapter 6 that relied on the version of the Kolmogorov–Čentsov continuity theorem from the paper of Andreev and Lang [2].

7.2. Differentiability of isotropic log-normal spherical random fields

We also wish to transfer the differentiability results for 2-weakly isotropic Gaussian spherical random fields to 2-weakly isotropic log-normal spherical random fields. As a preparation we quote a generalization of the formula of Faà di Bruno for derivatives of compositions of functions in the case of the exponential function from [16]: let $n \in \mathbb{N}$, for a n-times differentiable function $h : \mathbb{R}^n \to \mathbb{R}$ it holds that

$$\frac{\partial^n}{\partial x_1 \cdots \partial x_n}\exp(h(x)) = \exp(h(x))\sum_{\pi}\prod_{B\in\pi}\frac{\partial^{|B|}h(x)}{\prod_{j\in B}\partial x_j}, \tag{7.5}$$

where the sum is taken over all partitions π of the set $\{1, ..., n\}$. The product is taken over the blocks in each partition. This is Equation (4) in [16], which is due to Proposition 1 in

[16]. In our case we would like to discuss partial derivatives that result from multi-indices in $\mathbb{N}_0^2$. Due to Proposition 2 in [16] the variables x_j's in the partial derivative $\frac{\partial^n}{\partial x_1 \cdots \partial x_n}$ can be identical in the case that $h : \mathbb{R}^m \to \mathbb{R}$ with $m \leq n$. The partial derivative that results from a multi-index $\boldsymbol{\alpha} \in \mathbb{N}_0^2$ would be written as $\frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1 \cdots \partial x_{|\boldsymbol{\alpha}|}}$, where $x_1, ..., x_{\alpha_1} = \theta$ and $x_{\alpha_1+1}, ..., x_{\alpha_1+\alpha_2} = \varphi$. This is also explained in Example 1 in [16]. The second ingredient will be the discussion in the paper of Charrier [4], where the author shows the membership of log-normal random fields in $L_P^p(\Omega, C^0)$ for $p \in (0, \infty)$. This is Proposition 2.3 and Proposition 3.10 in [4]. We review the proofs, which rely on a theorem by Fernique, which can be found in [8] as Theorem 2.6. We adapt these statements to our setup in the next proposition.

Proposition 7.5. *Let T be a continuous 2-weakly isotropic Gaussian spherical random field such that its angular power spectrum satisfies that $(C_l l^{1+\delta} : l \geq 0)$ for some $\delta \in (0, 2]$. For all $p \in (0, \infty)$*

$$E\left[\exp(\|T\|_{C^0(S^2)})^p\right] < \infty,$$

and there exists a constant K_p dependent on p such that for all $L \in \mathbb{N}$ it holds that

$$E\left[\exp(\|T^L\|_{C^0(S^2)})^p\right] \leq K_p.$$

Remark 7.6. *The assumptions of the previous proposition imply that the random variables $(\min_{x \in S^2} T(x))^{-1}$ and $\max_{x \in S^2} T(x)$ are P-a.s well defined. And clearly P-a.s. also holds that*

$$(\min_{x \in S^2} \exp(T(x)))^{-1} = \exp(-\min_{x \in S^2} T(x)) \leq \exp(\|T\|_{C^0(S^2)})$$

as well as P-a.s. $\max_{x \in S^2} \exp(T(x)) \leq \exp(\|T\|_{C^0(S^2)})$. The previous proposition implies that for all $p \in (0, \infty)$ it holds that $(\min_{x \in S^2} \exp(T(x)))^{-1}, \max_{x \in S^2} \exp(T(x)) \in L_P^p(\Omega, \mathbb{R})$. Furthermore the $L_P^p(\Omega, \mathbb{R})$-norm of $(\min_{x \in S^2} \exp(T^L(x)))^{-1}$ and $\max_{x \in S^2} \exp(T^L(x))$ can be bounded independently of L for all $p \in (0, \infty)$.

Proof of Proposition 7.5. To be able to apply Fernique's theorem we have to check that $P \circ T^{-1}$ is a Gaussian measure on the Banach space $C^0(S^2)$ in the sense of [8]. From the beginning of Chapter 6 we know that the law of T, i.e. $P \circ T^{-1}$, is a probability measure on $(C^0(S^2), \mathcal{B}(C^0(S^2)))$.
We have to show that $\ell(T)$ is normally distributed for all $\ell \in C^0(S^2)^*$ in order to conclude that $P \circ T^{-1}$ is a Gaussian measure on $C^0(S^2)$. We arbitrarily fix $\ell \in C^0(S^2)^*$. The dual of $C^0(S^2)$ is described by Example 6 in Section 4.9 in [29], i.e. for every linear functional $\ell \in C^0(S^2)^*$ there exists a real-valued signed finite measure μ on the Borel sets $\mathcal{B}(S^2)$ such that $\ell(f) = \int_{S^2} f \mathrm{d}\mu$ for all $f \in C^0(S^2)$. With Hahn's decomposition, which is Theorem 3 in Section 1.3 in [29], we obtain the sets $B_P, B_N \in \mathcal{B}(S^2)$ such that for all $B \in \mathcal{B}(S^2)$ it holds that

$$\mu(B \cap B_P) \geq 0, \qquad \mu(B \cap B_N) \leq 0 \qquad \text{and} \qquad B_P \cup B_N = S^2.$$

We take these sets to define the density g and the finite measure $|\mu|$:

$$g = \mathbb{1}_{B_P} - \mathbb{1}_{B_N} \qquad \text{and} \qquad |\mu|(B) = \mu(B \cap B_P) - \mu(B \cap B_N),$$

for all $B \in \mathcal{B}(S^2)$. We observe that $\mathrm{d}\mu = g \mathrm{d}|\mu|$. We remind that for $L \in \mathbb{N}$ the truncated expansion in the real spherical harmonics, $T^L = \sum_{l=0}^{L} \sum_{m=-l}^{l} \sqrt{C_l} \tilde{\beta}_{l,m} \tilde{Y}_{l,m}$, of T converges

to T in $L^2_P(\Omega, L^2(S^2,\mathbb{R}))$ as $L\to\infty$. Therefore, the convergence is also in probability. Also it is a sum of independent $L^2(S^2,\mathbb{R})$-valued random variables. Theorem 3.1 in [18] implies that T^L converges P-a.s. to T in $L^2(S^2,\mathbb{R})$. Therefore also in $L^1(S^2,\mathbb{R})$ and together with our knowledge about $C^0(S^2)^*$ we obtain that P-a.s. $\lim_{L\to\infty}\ell(T^L)=\ell(T)$. We consider the characteristic function of $\ell(T)$. Since the absolute value of the function $x\mapsto\exp(ix)$ on $\mathbb{R}$ is bounded by one, we obtain with the dominated convergence theorem that for all $\lambda\in\mathbb{R}$ it holds that

$$E[e^{i\lambda\ell(T)}]=E[\lim_{L\to\infty}e^{i\lambda\ell(T^L)}]=\lim_{L\to\infty}E[e^{i\lambda\ell(T^L)}]=\lim_{L\to\infty}\exp\left(-\frac{1}{2}\lambda^2\sum_{l=0}^{L}C_l\sum_{m=-l}^{l}\ell(\tilde{Y}_{l,m})^2\right).$$

We consider the sum over m in the above expression and apply the representation of ℓ in order to obtain with the Cauchy–Schwarz inequality and Lemma 2.8 that

$$\begin{aligned}\sum_{m=-l}^{l}\ell(\tilde{Y}_{l,m})^2=\sum_{m=-l}^{l}\left(\int_{S^2}\tilde{Y}_{l,m}g\mathrm{d}|\mu|\right)^2&\leq\int_{S^2}\sum_{m=-l}^{l}\tilde{Y}_{l,m}^2\mathrm{d}|\mu|\|g\|^2_{L^2_{\mathrm{d}|\mu|}(S^2,\mathbb{R})}\\&=\frac{2l+1}{4\pi}|\mu|(S^2)\|g\|^2_{L^2_{\mathrm{d}|\mu|}(S^2,\mathbb{R})}.\end{aligned}$$

Since $(C_l l : l\geq 0)$ is summable, we obtain with the above inequality that

$$\sigma^2_{\ell(T)}=\sum_{l\geq 0}C_l\sum_{m=-l}^{l}\ell(\tilde{Y}_{l,m})^2\leq\sum_{l\geq 0}C_l\frac{2l+1}{4\pi}|\mu|(S^2)\|g\|^2_{L^2_{\mathrm{d}|\mu|}(S^2,\mathbb{R})}<\infty,$$

where we defined $\sigma^2_{\ell(T)}$ in the above equation. We conclude that

$$E[e^{i\lambda\ell(T)}]=\exp\left(-\frac{1}{2}\lambda^2\sigma^2_{\ell(T)}\right),$$

which implies that $\ell(T)\sim\mathcal{N}(0,\sigma^2_{\ell(T)})$. Therefore, $P\circ T^{-1}$ is a symmetric Gaussian measure on $C^0(S^2)$ in the sense of Section 2.2.1 in [8]. Note that the same argument trivially also applies to the truncated random field T^L. Therefore $P\circ(T^L)^{-1}$ is also a symmetric Gaussian measure on $C^0(S^2)$.

We define the closed balls $\overline{B}_r(0)=\{f\in C^0(S^2):\|f\|_{C^0(S^2)}\leq r\}$. From basic probability theory we conclude that $P\circ T^{-1}[\overline{B}_r(0)]=P[\{\|T\|_{C^0(S^2)}\leq r\}]\to 1$ as $r\to\infty$. Equivalently, for all $\varepsilon\in(0,1)$ there is $r\in(0,\infty)$ such that $P[\{\|T\|_{C^0(S^2)}\leq r\}]>1-\varepsilon$. The strict monotonicity of the logarithm implies that there exists $\varepsilon_0\in(0,1)$ such that $\log(\frac{\varepsilon_0}{1-\varepsilon_0})\leq -2$. For this ε_0 we choose $r_0\in(0,\infty)$ such that $P[\{\|T\|\leq r_0\}]>1-\varepsilon_0$ and we choose $\lambda>0$ such that $32\lambda r_0^2\leq 1$. For these choices of λ and r_0 we obtain that

$$\log\left(\frac{1-P[\{\|T\|_{C^0(S^2)}\leq r_0\}]}{P[\{\|T\|_{C^0(S^2)}\leq r_0\}]}\right)+32\lambda r_0^2\leq\log\left(\frac{\varepsilon_0}{1-\varepsilon_0}\right)+32\lambda r_0^2\leq -1,\tag{7.6}$$

which are the assumptions in Fernique's theorem, which is Theorem 2.6 in [8]. The statement of this theorem adapted to our framework is that

$$E[\exp(\lambda\|T\|^2_{C^0(S^2)})]\leq e^{16\lambda r_0^2}+\frac{e^2}{e^2-1}.\tag{7.7}$$

Furthermore $0 \leq (\sqrt{\lambda}x - \frac{p}{2\sqrt{\lambda}})^2$ implies that $px \leq \lambda x^2 + \frac{p^2}{4\lambda}$. Together with Inequality (7.7) we conclude that

$$E[\exp(\|T\|_{C^0(S^2)})^p] = E[\exp(p\|T\|_{C^0(S^2)})] \leq E[\exp(\lambda\|T\|^2_{C^0(S^2)})]\ e^{\frac{p^2}{4\lambda}}$$
$$\leq e^{16\lambda r_0^2 + \frac{p^2}{4\lambda}} + \frac{e^{2+\frac{p^2}{4\lambda}}}{e^2-1}.$$

The proof of the second claim of the theorem will be closely oriented on the proof of Proposition 3.10 in [4]. The strategy for the proof of the second claim is to choose the values λ and r_0 independently of L such that Inequality (7.6) holds for T^L for all $L \in \mathbb{N}$. Theorem 6.20 applied to the truncated random field T^L implies that there exists a constant $\hat{K}$ such that we can bound the $L^2_P(\Omega, C^0(S^2))$-norm independently of L, i.e.

$$\|T^L\|_{L^2_P(\Omega,C^0(S^2))} \leq \hat{K}\left(\sum_{l\geq 0} C_l l^{1+\delta}\right)^{\frac{1}{2}} < \infty.$$

We define $K = \hat{K}(\sum_{l\geq 0} C_l l^{1+\delta})^{\frac{1}{2}}$. Note that K is finite due to the assumption on the angular power spectrum of T. The Chebychev inequality implies that for $r > 0$ and for all $L \in \mathbb{N}$ it holds that

$$P[\{\|T^L\|_{C^0(S^2)} > r\}] \leq \frac{\|T^L\|^2_{L^2_P(\Omega,C^0(S^2))}}{r^2} \leq \frac{K^2}{r^2}.$$

There exists $x_0 \in (0,1)$ such that $\log(\frac{1-x_0}{x_0}) \leq -2$. We choose $r_0 = \frac{K}{\sqrt{1-x_0}}$. When we also choose $\lambda > 0$ sufficiently small such that $32\lambda r_0^2 \leq 1$, then we obtain for all $L \in \mathbb{N}$ that

$$\log\left(\frac{1 - P[\{\|T^L\|_{C^0(S^2)} \leq r_0\}]}{P[\{\|T^L\|_{C^0(S^2)} \leq r_0\}]}\right) + 32\lambda r_0^2 \leq \log\left(\frac{1-x_0}{x_0}\right) + 32\lambda r_0^2 \leq -1.$$

So we established Inequality (7.6) with choices for λ and r_0 that do not depend on L. Therefore the argument can be completed in the same way as for the first claim. We then take $K_p = e^{16\lambda r_0^2 + \frac{p^2}{4\lambda}} + \frac{e^{2+\frac{p^2}{4\lambda}}}{e^2-1}$. Note that our choices for λ and r_0 in the proof of the first and second claim are in general different. □

Theorem 7.7. *Let A be a 2-weakly isotropic log-normal spherical random field that results from a continuous 2-weakly isotropic Gaussian spherical random field T such that the angular power spectrum of T satisfies that $(C_l l^{1+2\iota+\delta} : l \geq 0)$ is summable for some $\delta \in (0,2]$ and some integer $\iota \geq 0$.*
For all $\gamma \in (0, \frac{\delta}{2})$ there exists an indistinguishable modification A^* of A such that $A^* \subset C^{\iota,\gamma}(S^2)$ and for all $p \in (0,\infty)$ it holds that $A \in L^p_P(\Omega, C^{\iota,\gamma}(S^2))$.
For all $p \in (0,\infty)$ and all $L \in \mathbb{N}_0$ the $L^p_P(\Omega, C^{\iota,\gamma}(S^2))$-norm of $A^L = \exp(T^L)$ is bounded independently of L, i.e. for all $p \in (0,\infty)$ there exists a constant K_p independently of L such that

$$\|A^L\|_{L^p_P(\Omega,C^{\iota,\gamma}(S^2))} < K_p.$$

For all $p \in (0,\infty)$ the sequence $(A^L : L \in \mathbb{N}_0)$ converges to $A = \exp(T)$ in $L^p_P(\Omega, C^{\iota,\gamma}(S^2))$, i.e. for all $p \in (0,\infty)$ there exists a constant K_p independently of L such that

$$\|A - A^L\|_{L^p_P(\Omega,C^{\iota,\gamma}(S^2))} \leq K_p\left(\sum_{l>L} C_l l^{1+2\iota+\delta}\right)^{\frac{1}{2}}.$$

Proof. We start with the case that $\iota = 0$. We know from Theorem 6.20 that there exists an indistinguishable modification $T^* \subset C^{0,\gamma}(S^2)$ of T that results in an indistinguishable modification A^* of A. We remind of our standard atlas $(U_j, \eta_j : j = 1, ..., 6)$ with smooth partition of unity Ψ of S^2. We fix $i \in \{1, ..., 6\}$ and remind of our usual notation that for $x \in U_i$ we set $\eta_i^{-1}(x) = (\theta_x, \varphi_x)$. Since $supp(\Psi_i)$ is relatively closed in S^2, Lemma 6.2 implies that there exists a partition of unity $\hat{\Psi}$ subordinate to the open cover $(U_j : j = 1, ..., 6)$ such that $\hat{\Psi}_i = 1$ on $supp(\Psi_i)$. Let $\hat{\Psi}$ be another partition of unity subordinate to the atlas $(U_i, \eta_i : i = 1, ..., 6)$, such that for all $i \in \{1, ..., 6\}$ on $supp(\Psi_i)$ it holds that $\hat{\Psi}_i = 1$. For $x, y \in U_i$ we obtain with Inequality (7.1) that

$$|A^*(x) - A^*(y)| \leq (A^*(x) + A^*(y))|T^*(x) - T^*(y)|,$$

which implies with the bound of a product of Hölder functions, which is Inequality (6.4), that

$$\begin{aligned}\|(A^*\Psi_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})} &\leq \|(\Psi_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})} \left(\|A^*_{\eta_i}\|_{C^0(supp((\Psi_i)_{\eta_i}))} + |A^*_{\eta_i}|_{\gamma,0,supp((\Psi_i)_{\eta_i})}\right) \\ &\leq K\|(A^*\hat{\Psi}_i)_{\eta_i}\|_{C^0(\overline{\eta_i^{-1}(U_i)})} \left(1 + 2\|(T^*\hat{\Psi}_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})}\right),\end{aligned}$$

where we assume that $\|(\Psi_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})} \leq K$ and applied that $(A^*\Psi_i)_{\eta_i}$ is equal to zero outside of $supp((\Psi_i)_{\eta_i})$. Since the last inequality also holds for all other $j \in \{1, ..., 6\}\backslash\{i\}$ we obtain with the definition of the Hölder norms on S^2 that

$$\|A^*\|_{C^{0,\gamma}(S^2)} \leq K\|A^*\|_{C^0(S^2)} \left(1 + 2\|T^*\|_{C^{0,\gamma}(S^2)}\right).$$

We can now apply Proposition 7.5 and Theorem 6.20 to obtain with the Cauchy–Schwarz inequality that for all $p \in (0, \infty)$ there exists a constant K_p such that

$$E\left[\|A\|^p_{C^{0,\gamma}(S^2)}\right]^{\frac{1}{p}} \leq KE\left[\|A\|^{2p}_{C^0(S^2)}\right]^{\frac{1}{2p}} E\left[\left(1 + 2\|T\|_{C^{0,\gamma}(S^2)}\right)^{2p}\right]^{\frac{1}{2p}} \leq K_p. \tag{7.8}$$

Since the modification T^* and A^* are equal to T and A on measurable set with full probability we can always consider T and A when taking the expectation instead of T^* and A^*. We will do this in the future without mentioning it. The partition of unity Ψ contributed with the factor K. In the future we might tacitly ignore Ψ at some occasions to ease the notation in the proof.
Note that for $L \in \mathbb{N}$ this argument can be repeated for the truncated random field T^L and $A^L = \exp(T^L)$ with the same result. Moreover Proposition 7.5 and Theorem 6.20 will ensure that the constant K_p in Inequality (7.8) will not depend on L.

The proof of the second claim, which is the discussion of the $L^p_P(\Omega, \mathbb{R})$-norm of $\|A^* - A^{L*}\|_{C^{0,\gamma}(S^2)}$ for $L \in \mathbb{N}$ and the desired convergence is computationally more involved. We will mostly apply Inequality (7.1) in a particular way. We look at the difference of the function $A^* - A^{L*}$ at $x, y \in supp(\Psi_i) \subset\subset U_i$ for the same fixed $i \in \{1, ..., 6\}$ and obtain with Inequality (6.29) that

$$\begin{aligned}&|A^*(x) - A^{L*}(x) - (A^*(y) - A^{L*}(y))| \\ &\quad = |e^{T^{L*}(x)}(e^{(T^*-T^{L*})(x)} - 1) - e^{T^{L*}(y)}(e^{(T^*-T^{L*})(y)} - 1)| \\ &\quad \leq e^{T^{L*}(x)}|e^{(T^*-T^{L*})(x)} - e^{(T^*-T^{L*})(y)}| + |e^{(T^*-T^{L*})(y)} - 1||e^{T^{L*}(x)} - e^{T^{L*}(y)}|.\end{aligned} \tag{7.9}$$

We apply Inequality (7.1) to the two differences and the second factor in the last inequality to obtain that

$$\begin{aligned}
&|e^{(T^*-T^{L*})(x)} - e^{(T^*-T^{L*})(y)}| \\
&\qquad \leq (e^{(T^*-T^{L*})(x)} + e^{(T^*-T^{L*})(y)})|(T^* - T^{L*})(x) - (T^* - T^{L*})(y)|, \\
&|e^{T^{L*}(x)} - e^{T^{L*}(y)}| \leq (e^{T^{L*}(x)} + e^{T^{L*}(y)})|T^{L*}(x) - T^{L*}(y)|
\end{aligned}$$

and

$$|e^{(T^*-T^{L*})(y)} - 1| \leq e^{(T^*-T^{L*})(y)}|T^*(y) - T^{L*}(y)|.$$

We insert the last two inequality into Inequality (7.9), divide by $\|(\theta_x, \varphi_x) - (\theta_y, \varphi_y)\|_{\mathbb{R}^2}^{\gamma}$ and take the supremum over all $x, y \in supp(\Psi_i)$ to obtain that

$$\begin{aligned}
&|A^*_{\eta_i} - A^{L*}_{\eta_i}|_{\gamma,0} \\
&\qquad \leq \|A^*_{\eta_i}\|_{C^0}(1 + \|A^{L*}_{\eta_i}\|^2_{C^0})|T^*_{\eta_i} - T^{L*}_{\eta_i}|_{\gamma,0} + \|e^{T^*_{\eta_i} - T^{L*}_{\eta_i}}\|_{C^0}\|T^*_{\eta_i} - T^{L*}_{\eta_i}\|_{C^0}|T^{L*}_{\eta_i}|_{\gamma,0}. \quad (7.10)
\end{aligned}$$

Note that we will sometimes drop the dependence of the domain in the norms and seminorms for notational convenience as we did above. The previous argument of course holds for all $i \in \{1, ..., 6\}$. With the plain application of Inequality (7.1) we obtain that

$$\|A^* - A^{L*}\|_{C^0(S^2)} \leq (\|A^{L*}\|_{C^0(S^2)} + \|A^*\|_{C^0(S^2)})\|T^* - T^{L*}\|_{C^0(S^2)}. \quad (7.11)$$

We combine the Inequalities (7.10) and (7.11) to obtain that

$$\begin{aligned}
\|A^* - A^{L*}\|_{C^{0,\gamma}(S^2)} &\leq \|A^*\|_{C^0}(1 + \|A^{L*}\|^2_{C^0})|T^* - T^{L*}|_{\gamma,0} \\
&\quad + (\|e^{T^*-T^{L*}}\|_{C^0}|T^{L*}|_{\gamma,0} + \|A^{L*}\|_{C^0} + \|A^*\|_{C^0})\|T^* - T^{L*}\|_{C^0} \\
&\leq \Big(\|A^*\|_{C^0}(1 + \|A^{L*}\|^2_{C^0}) + \|e^{T^*-T^{L*}}\|_{C^0}|T^{L*}|_{\gamma,0} \\
&\quad + \|A^{L*}\|_{C^0} + \|A^*\|_{C^0}\Big)\|T^* - T^{L*}\|_{C^{0,\gamma}}.
\end{aligned}$$

We can repeatedly apply the Cauchy–Schwarz inequality, Proposition 7.5 and Remark 6.21 on Theorem 6.20 as above to obtain that for all $p \in (0, \infty)$ there exists a constant K_p independently of L such that

$$E\left[\|A - A^L\|^p_{C^{0,\gamma}(S^2)}\right]^{\frac{1}{p}} \leq K_p \left(\sum_{l>L} C_l l^{1+\delta}\right)^{\frac{1}{2}}.$$

Now we prove the general case $\iota \geq 1$. According to Theorem 6.20 there exists an indistinguishable modification $T^* \subset C^{\iota,\gamma}(S^2)$ of T, then $A^* = \exp(T^*)$ is an indistinguishable modification of A. Again we fix $i \in \{1, ..., 6\}$. To satisfy the definition of the Hölder norms on S^2 we need to analyze partial derivatives of the term $(A^*\Psi_i)_{\eta_i}$. With the multivariate Leipniz rule we obtain that for $\boldsymbol{\beta} \in \mathbb{N}_0^2$ such that $|\boldsymbol{\beta}| \leq \iota$ it holds that

$$\partial_{\boldsymbol{\beta}}(A^*\Psi_i)_{\eta_i} = \sum_{\boldsymbol{\alpha} \leq \boldsymbol{\beta}} \binom{\boldsymbol{\beta}}{\boldsymbol{\alpha}} \partial_{\boldsymbol{\alpha}} A^*_{\eta_i} \partial_{\boldsymbol{\beta}-\boldsymbol{\alpha}}(\Psi_i)_{\eta_i}. \quad (7.12)$$

Since $(\Psi_i)_{\eta_i}$ is smooth and Ψ_i is compactly supported, we can assume that for all $m \leq \iota + 1$ the $C^m(\eta_i^{-1}(U_i))$-norm of $(\Psi_i)_{\eta_i}$ is bounded independently of m and i by a constant K.

The interesting term is where $A^*_{\eta_i}$ is involved. We apply Equation (7.5) to $A^*_{\eta_i}$ to obtain that for $\boldsymbol{\alpha} \in \mathbb{N}_0^2$ with $|\boldsymbol{\alpha}| \leq \iota$ it holds that

$$\partial_{\boldsymbol{\alpha}} A^*_{\eta_i} = A^*_{\eta_i} \sum_{\pi} \prod_{B \in \pi} \frac{\partial^{|B|} T^*_{\eta_i}}{\prod_{j \in B} \partial x_j}, \tag{7.13}$$

where the sum is taken over all partitions π of the set $\{1, ..., |\boldsymbol{\alpha}|\}$ and the product is taken over all blocks of the respective partition. Note that $x_1, ..., x_{\alpha_1} = \theta$ and $x_{\alpha_1+1}, ..., x_{|\boldsymbol{\alpha}|} = \varphi$ as described after we introduced Equation (7.5). For the sake of a convenient notation, we define

$$\mathcal{T}_{\boldsymbol{\alpha}} = \Big(\sum_{\pi} \prod_{B \in \pi} \frac{\partial^{|B|} T^*_{\eta_i}}{\prod_{j \in B} \partial x_j} \Big) \circ \eta_i^{-1}.$$

For $x, y \in U_i$ and $\boldsymbol{\beta} \in \mathbb{N}_0^2$ satisfying $|\boldsymbol{\beta}| = \iota$ we look at the difference of partial derivatives and obtain with Equation (7.12), Equation (7.13) and Inequality (6.29) that

$$\begin{aligned}
|\partial_{\boldsymbol{\beta}}(A^*\Psi_i)(x) - \partial_{\boldsymbol{\beta}}(A^*\Psi_i)(y)| &\leq K \sum_{\boldsymbol{\alpha} \leq \boldsymbol{\beta}} \binom{\boldsymbol{\beta}}{\boldsymbol{\alpha}} |\partial_{\boldsymbol{\alpha}} A^*(x) - \partial_{\boldsymbol{\alpha}} A^*(y)| \\
&= K \sum_{\boldsymbol{\alpha} \leq \boldsymbol{\beta}} \binom{\boldsymbol{\beta}}{\boldsymbol{\alpha}} |A^*(x)\mathcal{T}_{\boldsymbol{\alpha}}(x) - A^*(y)\mathcal{T}_{\boldsymbol{\alpha}}(y)| \\
&\leq K \sum_{\boldsymbol{\alpha} \leq \boldsymbol{\beta}} \binom{\boldsymbol{\beta}}{\boldsymbol{\alpha}} (|A^*(x)||\mathcal{T}_{\boldsymbol{\alpha}}(x) - \mathcal{T}_{\boldsymbol{\alpha}}(y)| \qquad (7.14) \\
&\qquad + |\mathcal{T}_{\boldsymbol{\alpha}}(y)||A^*(x) - A^*(y)|). \qquad (7.15)
\end{aligned}$$

Note that to ease the notation we just made a slight abuse of notation: with $\partial_{\boldsymbol{\beta}}(A^*\Psi_i)(x)$ we mean $\partial_{\boldsymbol{\beta}}(A^*\Psi_i)_{\eta_i}(\eta_i^{-1}(x))$. We will use this notation at some occasions in the remaining part of the proof without mentioning it again.
We will treat the Expressions (7.14) and (7.15) separately and we will start with the first of those. The term $\mathcal{T}_{\boldsymbol{\alpha}}$ consist of finite sums and products of partial derivatives of $T^*_{\eta_i}$ of order at most ι. We need a more general version of Inequality (6.29). For real numbers $a_1, ..., a_n, b_1, ..., b_n \in \mathbb{R}$ we iteratively apply Inequality (6.29) to obtain that

$$\begin{aligned}
|\prod_{i=1}^{n} a_i - \prod_{i=1}^{n} b_i| &\leq |a_n||\prod_{i=1}^{n-1} a_i - \prod_{i=1}^{n-1} b_i| + \prod_{i=1}^{n-1} |b_i||a_n - b_n| \\
&\leq |a_n||a_{n-1}||\prod_{i=1}^{n-2} a_i - \prod_{i=1}^{n-2} b_i| + |a_n| \prod_{i=1}^{n-2} |b_i||a_{n-1} - b_{n-1}| + \prod_{i=1}^{n-1} |b_i||a_n - b_n| \\
&\leq \cdots \leq \sum_{i=1}^{n} |a_i - b_i| \prod_{j=1}^{i-1} |b_j| \prod_{j=i+1}^{n} |a_j|. \qquad (7.16)
\end{aligned}$$

For the difference quotient we obtain with the triangle inequality that

$$\frac{|\mathcal{T}_{\boldsymbol{\alpha}}(x) - \mathcal{T}_{\boldsymbol{\alpha}}(y)|}{\|(\theta_x, \varphi_x) - (\theta_y, \varphi_y)\|^{\gamma}_{\mathbb{R}^2}} \leq \sum_{\pi} \frac{|\prod_{B \in \pi} \frac{\partial^{|B|} T^*_{\eta_i}(\theta_x, \varphi_x)}{\prod_{j \in B} \partial x_j} - \prod_{B \in \pi} \frac{\partial^{|B|} T^*_{\eta_i}(\theta_y, \varphi_y)}{\prod_{j \in B} \partial x_j}|}{\|(\theta_x, \varphi_x) - (\theta_y, \varphi_y)\|^{\gamma}_{\mathbb{R}^2}}.$$

Together with Inequality (7.16) and Equation (6.3) we obtain that for a constant $K > 0$ dependent on the domain $\eta_i^{-1}(U_i)$ and ι it holds that

$$\frac{|\mathcal{T}_{\boldsymbol{\alpha}}(x) - \mathcal{T}_{\boldsymbol{\alpha}}(y)|}{\|(\theta_x, \varphi_x) - (\theta_y, \varphi_y)\|_{\mathbb{R}^2}^{\gamma}} \leq KQ(\|T^*\|_{C^{\iota,\gamma}(S^2)}), \tag{7.17}$$

where $0 \neq Q$ is a polynomial in $\mathbb{P}^{(\iota)}(\mathbb{R})$ with degree smaller or equal than ι, i.e. for $Q(X) = \sum_{j=1}^{\iota} q_j X^j$ such that $q_j \geq 0$ for $j \in \{2, ..., \iota\}$ and $q_1 > 0$. We tacitly applied the fact that $C^m(\overline{\eta_i^{-1}(U_i)}) \subset C^{\iota,\gamma}(\overline{\eta_i^{-1}(U_i)})$ for all integers $0 \leq m \leq \iota$. With a similar argument we obtain that

$$|\mathcal{T}_{\boldsymbol{\alpha}}(x)| \leq KQ(\|T^*\|_{C^{\iota,\gamma}(S^2)}). \tag{7.18}$$

The difference quotient that results from Expression (7.15) can be estimated with Inequality (7.1), i.e.

$$|\mathcal{T}_{\alpha}(y)| \frac{|A^*(x) - A^*(y)|}{\|(\theta_x, \varphi_x) - (\theta_y, \varphi_y)\|_{\mathbb{R}^2}^{\gamma}} \leq 2KQ(\|T^*\|_{C^{\iota,\gamma}(S^2)}) \|A^*\|_{C^0(S^2)} \frac{|T^*(x) - T^*(y)|}{\|(\theta_x, \varphi_x) - (\theta_y, \varphi_y)\|_{\mathbb{R}^2}^{\gamma}}. \tag{7.19}$$

We combine Inequalities (7.17), (7.18) and (7.19) to obtain an estimate for the difference quotient of $\partial_{\boldsymbol{\beta}}(A^*\Psi_i)$, i.e.

$$\begin{aligned}\frac{|\partial_{\boldsymbol{\beta}}(A^*\Psi_i)(x) - \partial_{\boldsymbol{\beta}}(A^*\Psi_i)(y)|}{\|(\theta_x, \varphi_x) - (\theta_y, \varphi_y)\|_{\mathbb{R}^2}^{\gamma}} &\leq KN \left(\|A^*\|_{C^0(S^2)} Q(\|T^*\|_{C^{\iota,\gamma}(S^2)}) \right. \\ &\quad \left. + 2\|A^*\|_{C^0(S^2)} Q(\|T^*\|_{C^{\iota,\gamma}(S^2)}) \|T^*\|_{C^{0,\gamma}(S^2)} \right) \\ &= KN \|A^*\|_{C^0(S^2)} Q(\|T^*\|_{C^{\iota,\gamma}(S^2)})(1 + 2\|T^*\|_{C^{0,\gamma}(S^2)}),\end{aligned}$$

where $N = \sum_{\boldsymbol{\alpha} \leq \boldsymbol{\beta}} \binom{\boldsymbol{\beta}}{\boldsymbol{\alpha}}$. We take the supremum over all $x, y \in supp(\Psi_i)$ and obtain with the definition of the Hölder norms on S^2 and Equation (6.3) that

$$\|A^*\|_{C^{\iota,\gamma}(S^2)} \leq C \left(\|A^*\|_{C^0(S^2)} + KN \|A^*\|_{C^0(S^2)} Q(\|T^*\|_{C^{\iota,\gamma}(S^2)})(1 + 2\|T^*\|_{C^{0,\gamma}(S^2)}) \right), \tag{7.20}$$

where the constant C results from the equivalence of the Hölder norms in Equation (6.3). The right hand side of Inequality (7.20) is in $L_P^p(\Omega, \mathbb{R})$ for all $p \in (0, \infty)$ due to the Cauchy–Schwarz inequality, Theorem 6.20 and Proposition 7.5. Therefore the first claim of the theorem is proven.

For the proof of the second claim of the theorem we can use the same argument as we used to proof Inequality (7.14)/(7.15) to obtain that

$$\begin{aligned}|\partial_{\boldsymbol{\beta}}(A^*\Psi_i)_{\eta_i} - \partial_{\boldsymbol{\beta}}(A^{L*}\Psi_i)_{\eta_i}|_{\gamma,0,\eta_i^{-1}(U_i)} &\leq K \sum_{\boldsymbol{\alpha} \leq \boldsymbol{\beta}} \binom{\boldsymbol{\beta}}{\boldsymbol{\alpha}} |(A^*\mathcal{T}_{\boldsymbol{\alpha}})_{\eta_i} - (A^{L*}\mathcal{T}_{\boldsymbol{\alpha}}^L)_{\eta_i}|_{\gamma,0,supp((\Psi_i)_{\eta_i})} \\ &\leq K \sum_{\boldsymbol{\alpha} \leq \boldsymbol{\beta}} \binom{\boldsymbol{\beta}}{\boldsymbol{\alpha}} \Big(|A_{\eta_i}^{L*}(\mathcal{T}_{\boldsymbol{\alpha}} - \mathcal{T}_{\boldsymbol{\alpha}}^L)_{\eta_i}|_{\gamma,0,supp((\Psi_i)_{\eta_i})} \qquad (7.21) \\ &\qquad + |(\mathcal{T}_{\boldsymbol{\alpha}})_{\eta_i}(A^* - A^{L*})_{\eta_i}|_{\gamma,0,supp((\Psi_i)_{\eta_i})} \Big). \qquad (7.22)\end{aligned}$$

We will discuss the summands individually. For notational convenience we will sometimes drop the dependence of the domain in the norms or semi-norms. We fix $\boldsymbol{\alpha} \leq \boldsymbol{\beta}$ and obtain for Expression (7.21) with Inequalities (7.17) and (7.18) that

$$\begin{aligned}|A^{L*}_{\eta_i}(\mathcal{T}_{\boldsymbol{\alpha}} - \mathcal{T}^L_{\boldsymbol{\alpha}})_{\eta_i}|_{\gamma,0} &\leq \|A^{L*}_{\eta_i}\|_{C^0}|(T_{\boldsymbol{\alpha}})_{\eta_i} - (\mathcal{T}^L_{\boldsymbol{\alpha}})_{\eta_i}|_{\gamma,0} + \|(T_{\boldsymbol{\alpha}})_{\eta_i} - (\mathcal{T}^L_{\boldsymbol{\alpha}})_{\eta_i}\|_{C^0}|A^{L*}|_{\gamma,0} \\ &\leq K\|A^{L*}\|_{C^{0,\gamma}(S^2)}Q(\|T^* - T^{L*}\|_{C^{\iota,\gamma}(S^2)}).\end{aligned}$$

For Expression (7.22) we obtain with Inequality (7.18) that

$$\begin{aligned}|(\mathcal{T}_{\boldsymbol{\alpha}})_{\eta_i}(A^* - A^{L*})_{\eta_i}|_{\gamma,0} &\leq \|(\mathcal{T}_{\boldsymbol{\alpha}})_{\eta_i}\|_{C^0}|A^*_{\eta_i} - A^{L*}_{\eta_i}|_{\gamma,0} + \|A^*_{\eta_i} - A^{L*}_{\eta_i}\|_{C^0}|(\mathcal{T}_{\boldsymbol{\alpha}})_{\eta_i}|_{\gamma,0} \\ &\leq KQ(\|T^*\|_{C^{\iota,\gamma}(S^2)})\|A^* - A^{L*}\|_{C^{0,\gamma}(S^2)}.\end{aligned}$$

We combine the last two inequalities and obtain with the definition of Hölder norms on S^2 and Equation (6.3) that

$$\begin{aligned}\|A^* - A^{L*}\|_{C^{\iota,\gamma}(S^2)} \leq C\big(&\|A^* - A^{L*}\|_{C^0(S^2)} + KN(\|A^{L*}\|_{C^{0,\gamma}(S^2)}Q(\|T^* - T^{L*}\|_{C^{\iota,\gamma}(S^2)}) \\ &+Q(\|T^*\|_{C^{\iota,\gamma}(S^2)})\|A^* - A^{L*}\|_{C^{0,\gamma}(S^2)})\big),\end{aligned} \tag{7.23}$$

where the constant C results from the equivalence relation in Equation (6.3). It remains to show that $\|A^* - A^{L*}\|_{C^{\iota,\gamma}(S^2)}$ converges in $L^p_P(\Omega, \mathbb{R})$ in the desired way. We will discuss the summands in Inequality (7.23) individually and will begin with the first one. The polynomial Q is dominated by the first order term $q_1 x$ for small arguments $x > 0$. Therefore it is sufficient to consider how $\|T^* - T^{L*}\|_{C^{\iota,\gamma}(S^2)}$ converges. Powers of this term due to the polynomial Q will result in powers of the bound of $\|T^* - T^{L*}\|_{C^{\iota,\gamma}(S^2)}$ in L^p_P-sense, which converges to zero faster due to Remark 6.21 on Theorem 6.20. We obtain with the Cauchy–Schwarz inequality and the already proven case $\iota = 0$ that for a constant K_p independent of L it holds that

$$\begin{aligned}E\left[\|A^L\|^p_{C^{0,\gamma}(S^2)}\|T - T^L\|^p_{C^{\iota,\gamma}(S^2)}\right]^{\frac{1}{p}} &\leq E\left[\|A^L\|^{2p}_{C^{0,\gamma}(S^2)}\right]^{\frac{1}{2p}} E\left[\|T - T^L\|^{2p}_{C^{\iota,\gamma}(S^2)}\right]^{\frac{1}{2p}} \\ &\leq K_p E\left[\|T - T^L\|^{2p}_{C^{\iota,\gamma}(S^2)}\right]^{\frac{1}{2p}}.\end{aligned}$$

With Remark 6.21 on Theorem 6.20 we obtain that there exists a $\hat{K}_p$ independently of L such that

$$E\left[\|A^L\|^p_{C^{0,\gamma}(S^2)}\|T - T^L\|^p_{C^{\iota,\gamma}(S^2)}\right]^{\frac{1}{p}} \leq K_p\hat{K}_p\left(\sum_{l>L} C_l l^{1+2\iota+\delta}\right)^{\frac{1}{2}}.$$

Since the second claim of the theorem has already been proven for $\iota = 0$, we obtain for the first and third summand in Inequality (7.23) with a similar argument relying on Theorem 6.20 that

$$E\left[\|A - A^L\|^p_{C^0(S^2)}\right]^{\frac{1}{p}}, \quad E\left[Q(\|T^L\|_{C^{\iota,\gamma}(S^2)})^p\|A - A^L\|^p_{C^{0,\gamma}(S^2)}\right]^{\frac{1}{p}} \leq K_p\left(\sum_{l>L} C_l^{1+\delta}\right)^{\frac{1}{2}},$$

where we tacitly included the constants K, C and N into K_p in a suitable way. Therefore, the second claim in the case that $\iota \geq 1$ is also proven with a threefold application of the triangle inequality. □

7.3. Notes on the proof of the differentiability results

The proof of the differentiability of 2-weakly isotropic log-normal spherical random fields relied on the fact that under our assumptions $A \in L^p_P(\Omega, C^0(S^2))$ for all $p \in (0,\infty)$. This was Proposition 7.5 in the previous section, with Fernique's theorem as the main ingredient. In this section we prove a proposition that will imply Proposition 7.5 and will potentially ease the proof of Theorem 7.7 due to its stronger statement.
In the previous chapter we proved the respective result of Theorem 7.7 for 2-weakly isotropic Gaussian spherical random fields, i.e. Theorem 6.20 and Remark 6.21, and used Lemma 5.2 as a main ingredient. The respective result for the log-normal case is Lemma 7.3. Therefore we expect to obtain Theorem 7.7 in a similar way as in the Gaussian case for Hölder continuity, i.e. $\iota = 0$.

Proposition 7.8. *Let A be a 2-weakly isotropic log-normal spherical random field, which results from a continuous 2-weakly isotropic Gaussian spherical random field T such that its angular power spectrum satisfies that $(C_l l^{1+\delta} : l \geq 0)$ is summable for $\delta \in (0,2]$. For all $\gamma \in (0, \frac{\delta}{2})$ there exists an indistinguishable modification $A^* \subset C^{0,\gamma}(S^2)$ of A and $A \in L^p_P(\Omega, C^{0,\gamma}(S^2))$ for all $p \in (0,\infty)$.*
For all $p \in (0,\infty)$ the sequence $(A^L : L \in \mathbb{N}_0) = (\exp(T^L) : L \in \mathbb{N}_0)$ converges to $A = \exp(T)$ in the $L^p_P(\Omega, C^{0,\gamma}(S^2))$-norm, i.e. for all $p \in (0,\infty)$ there exists a constant K_p independently of L such that

$$\|A - A^L\|_{L^p_P(\Omega, C^{0,\gamma}(S^2))} \leq K_p \left(\sum_{l>L} C_l l^{1+\delta} \right)^{\frac{1}{2}}.$$

Proof. The proof will be very similar to the proof of Theorem 6.16 and Remark 6.17. As in many proofs before we remind of our usual atlas $(U_i, \eta_i : i = 1, ..., 6)$ of S^2 with partition of unity Ψ. Since $T(x) \sim \mathcal{N}(0, \sigma_T^2)$ for all $x \in S^2$ by Lemma 4.6, we obtain with Tonelli's theorem that for all $k \in \mathbb{N}$ $A \in L^{2k}_{P\otimes \mathrm{d}\sigma}(\Omega \times S^2, \mathbb{R})$, i.e. it holds that

$$\begin{aligned} E\left[\int_{S^2} A(x)^{2k} \mathrm{d}\sigma(x) \right] &= \int_{S^2} E\left[A(x)^{2k} \right] \mathrm{d}\sigma(x) \\ &= \int_{S^2} \exp(2k^2\sigma_T^2) d\sigma(x) = 4\pi \exp(2k^2\sigma_T^2). \end{aligned}$$

Lemma 7.3 enables us to apply Theorem 3.5 from [2] with $d = 0$, $n = 2$ and $\varepsilon = k\delta - 2$. We choose $k > \frac{2}{\delta - 2\gamma}$. It follows that there exists a modification A^* of A such that $A^* \subset C^{0,\gamma}(S^2)$. The modification A^* and A are indistinguishable, because both random fields are continuous. The respectively limit argument was given in the proof of Theorem 6.16. Note that Inequality (6.21) can also be established for the 2-weakly isotropic log-normal spherical random field A with Lemma 7.3 instead of Lemma 6.14, i.e.

$$E\left[\|(A\Psi_i)_{\eta_i}\|^{2k}_{C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})} \right] \leq CE\left[\|A_{\eta_i}\|^{2k}_{L^{2k}(\eta_i^{-1}(U_i),\mathbb{R})} \right] + K_k \left(\sum_{l\geq 0} C_l^{1+\delta} \right)^k, \tag{7.24}$$

where the constants K_k is due to Lemma 7.3 and the constant C comes from a Sobolev embedding as in development of Inequality (6.21). Note that Inequality (7.24) and the the embedding $L^{2\lfloor p+1 \rfloor}_P(\Omega, \mathbb{R}) \subset L^p_P(\Omega, \mathbb{R})$ for all $p \in (0,\infty)$ already implies the first claim of the proposition.

For the second claim we have to examine the right hand side of Inequality (7.24) and replace A with $A - A^L$. For the first summand we apply Tonelli's theorem, the Cauchy–Schwarz inequality and the triangle inequality as well as Inequalities (7.1) and (7.4) to obtain that

$$\begin{aligned}
&E\Big[\int_{S^2}(A(x)-A^L(x))^{2k}\mathrm{d}\sigma(x)\Big] \\
&\qquad\le E\Big[\int_{S^2}(A(x)+A^L(x))^{2k}(T(x)-T^L(x))^{2k}\mathrm{d}\sigma(x)\Big] \\
&\qquad\le \int_{S^2} E\left[(A(x)+A^L(x))^{4k}\right]^{\frac{1}{2}} E\left[(T(x)-T^L(x))^{4k}\right]^{\frac{1}{2}}\mathrm{d}\sigma(x) \\
&\qquad\le 4\pi\ 2^{2k}\exp(4k^2\sigma_T^2)\frac{(2k)!}{2^k k!}\left(\sum_{l>L}C_l\frac{2l+1}{4\pi}\right)^k,
\end{aligned}$$

where the second factor is achieved with Inequality (5.9) and Lemma 2.8. For the second summand in Inequality (7.24) we have to prove Lemma 7.3 for $A - A^L$. Note that in the case of a 2-weakly isotropic Gaussian spherical random field, as in Theorem 6.26, this was immediate since the expansion of $T - T^L$ in the real spherical harmonics was known. In our case we have to apply Inequality (7.1) and use a similar argument as in the proof of Theorem 7.7. With Inequality (7.2) we obtain that for $k, L \in \mathbb{N}$ it holds that

$$\begin{aligned}
E\left[|(A-A^L)(x)-(A-A^L)(y)|^{2k}\right] &= E\Big[|A^L(x)(e^{(T-T^L)(x)}-e^{(T-T^L)(y)}) \\
&\qquad +(e^{(T-T^L)(y)}-1)(A^L(x)-A^L(y))|^{2k}\Big] \\
&\le 2^{2k-1}\Big(E\left[(A^L(x))^{2k}|e^{(T-T^L)(x)}-e^{(T-T^L)(y)}|^{2k}\right] \qquad (7.25)\\
&\qquad +E\left[(e^{(T-T^L)(y)}-1)^{2k}|A^L(x)-A^L(y)|^{2k}\right]\Big). \qquad (7.26)
\end{aligned}$$

We will examine the two summands in the above inequality individually. For Expression (7.25) we obtain with Inequality (7.1) and the Cauchy–Schwarz inequality that

$$\begin{aligned}
&E\left[(A^L(x))^{2k}|e^{(T-T^L)(x)}-e^{(T-T^L)(x)}|^{2k}\right] \\
&\le E\left[(A^L(x))^{2k}(e^{(T-T^L)(x)}+e^{(T-T^L)(y)})^{2k}|(T-T^L)(x)-(T-T^L)(y)|^{2k}\right] \\
&\le E\left[(A^L(x))^{8k}\right]^{\frac{1}{4}} E\left[(e^{(T-T^L)(x)}+e^{(T-T^L)(y)})^{8k}\right]^{\frac{1}{4}} E\left[|(T-T^L)(x)-(T-T^L)(y)|^{4k}\right]^{\frac{1}{2}}.
\end{aligned}$$

Since $A^L(x)$ and $e^{(T-T^L)(x)}$ are log-normally distributed with mean and variance independent of x and moments of finite order exists we obtain with Lemma 5.2 that for every $k \in \mathbb{N}$ there exists a constant K_k such that

$$E\left[(A^L(x))^{2k}|e^{(T-T^L)(x)}-e^{(T-T^L)(x)}|^{2k}\right] \le K_k\left(\sum_{l>L}C_l l^{1+\delta}\right)^k d(x,y)^{\delta k}.$$

Expression (7.26) can be treated in a similar way. With Inequality (7.1) and the Cauchy–Schwarz inequality we obtain that

$$E\left[(e^{(T-T^L)(y)}-1)^{2k}|A^L(x)-A^L(y)|^{2k}\right]$$

$$\leq E\left[(e^{(T-T^L)(y)}+1)^{2k}|T(x)-T^L(x)|^{2k}|A^L(x)-A^L(y)|^{2k}\right]$$
$$\leq E\left[(e^{(T-T^L)(y)}+1)^{8k}\right]^{\frac{1}{4}} E\left[|T(x)-T^L(x)|^{8k}\right]^{\frac{1}{4}} E\left[|A^L(x)-A^L(y)|^{4k}\right]^{\frac{1}{2}}.$$

Now we apply that for $\exp((T-T^L)(y))$ and $A^L(x)$ all moments of finite order exists and do not depend on x or y. With the usual argument we bound $E[|T(x)-T^L(x)|^{8k}]$ and obtain that

$$E\left[(e^{(T-T^L)(y)}-1)^{2k}|A^L(x)-A^L(y)|^{2k}\right] \leq K_k\left(\sum_{l>L} C_l \frac{2l+1}{4\pi}\right)^k d(x,y)^{\delta k}.$$

We combine the estimates for Expressions (7.25) and (7.26) and obtain the statement of Lemma 7.3 for $A-A^L$, i.e. there exists a constant K_k independently of x, y and L such that

$$E\left[|(A-A^L)(x)-(A-A^L)(y)|^{2k}\right] \leq K_k\left(\sum_{l>L} C_l l^{1+\delta}\right)^k d(x,y)^{\delta k}.$$

We conclude a respective version of Inequality (7.24), i.e. for all $i \in \{1,...,6\}$ there exists a constant K_k independently of L such that

$$E\left[\|((A-A^L)\Psi_i)_{\eta_i}\|^{2k}_{C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})}\right] \leq K_k\left(\sum_{l>L} C_l l^{1+\delta}\right)^k.$$

The embedding $L_P^{2\lfloor p+1\rfloor}(\Omega,\mathbb{R}) \subset L_P^p(\Omega,\mathbb{R})$ for all $p \in (0,\infty)$ implies the second claim of the proposition. □

We see that the previous proposition does not only include the statement of Proposition 7.5, furthermore it also includes the statement of Theorem 7.7 in the case that $\iota = 0$ without the use of Fernique's theorem.

8. Elliptic partial differential equations on the sphere

In this chapter we want to discuss elliptic partial differential equations on the sphere. The definitions of the differential operators on the sphere will be according to the theory on Riemannian manifolds with respect to a chosen coordinate system and also due to an intrinsic definition valid on the sphere. The latter is discussed in the book of Atkinson and Han [3]. We will consider the following type of equation on S^2:

$$-\nabla_{S^2} \cdot (A\nabla_{S^2} u) = f, \tag{8.1}$$

for given functions A and f. This formal problem will be made precise afterwards. The spherical gradient and the spherical Laplace operator can be also described intrinsically. According to Equation (3.4) in [3] for a function $v \in C^2(S^2)$ it holds that

$$\Delta \hat{v}(x)\Big|_{\|x\|=1} = \Delta_{S^2} v(\theta_x, \varphi_x),$$

where Δ is the usual Laplace operator on three dimensional domains of Euclidean space and $\hat{v}$ denotes the function: $x \mapsto v(\frac{x}{\|x\|})$. Similarly, Equation (3.7) in [3] states for $v \in C^1(S^2)$ that

$$\nabla \hat{v}(x)\Big|_{\|x\|=1} = \nabla_{S^2} v(\theta_x, \varphi_x),$$

where ∇ is the usual gradient on three dimensional domains of Euclidean space. Note that in [3] these Equations are stated for a general atlas of the sphere. Therefore the value of the spherical gradient and the spherical Laplace operator are independent of the atlas and the respective coordinate system.
The following lemma is the divergence theorem on the sphere. It is a slight modification of Proposition 3.3 in [3].

Lemma 8.1. *For $v, A \in C^1(S^2)$ and $w \in C^2(S^2)$ it holds that*

$$\int_{S^2} \nabla_{S^2} v \cdot (A\nabla_{S^2} w) \mathrm{d}\sigma = -\int_{S^2} v \nabla_{S^2} \cdot (A\nabla_{S^2} w) \mathrm{d}\sigma.$$

Proof. This follows directly from Equation (2.4.185) in [22]. □

For the analysis of Equation (8.1) we are interested in the Poincaré inequality for a subspace of $H^1(S^2)$.

Definition 8.2. *Let the equivalence relation $\sim$ be given by: $v \sim w$ for $v, w \in H^1(S^2)$ if and only if there exists $c \in \mathbb{R}$ such that $v = w + c$. We define the quotient space $H^1(S^2)/\sim = H^1(S^2)/\mathbb{R}$.*

If a certain expression is equal for all representatives of an equivalence class in $H^1(S^2)/\mathbb{R}$ it will sometimes be useful to consider the representative v that satisfies that $\int_{S^2} v \mathrm{d}\sigma = 0$. We will see this technique after the following lemma, which is a Poincaré inequality.

Lemma 8.3. *For all $v \in \mathcal{V} = \{w \in H^1(S^2) : \int_{S^2} w \mathrm{d}\sigma = 0\}$ it holds that*

$$\|v\|_{L^2(S^2)} \leq \frac{1}{\sqrt{2}} \|\nabla_{S^2} v\|_{L^2(S^2)}.$$

Proof. Since $\mathcal{V}$ is a closed linear subspace of $H^1(S^2)$, it holds that $C^\infty(S^2) \cap \mathcal{V}$ is dense in $\mathcal{V}$. Therefore it suffices to prove the claimed inequality for functions $\phi \in C^\infty(S^2) \cap \mathcal{V}$, i.e. $\phi \in C^\infty(S^2)$ that satisfies that $\int_{S^2} \phi \mathrm{d}\sigma = 0$. Since $C^\infty(S^2) \subset L^2(S^2, \mathbb{R})$ there exists an expansion of ϕ in the real spherical harmonics. The real spherical harmonic function $\tilde{Y}_{0,0}$ is constant, therefore the coefficient $\phi_{0,0} = \int_{S^2} \phi \tilde{Y}_{0,0} \mathrm{d}\sigma = 0$. We obtain the expansion

$$\phi = \sum_{l \geq 1} \sum_{m=-l}^{l} \phi_{l,m} \tilde{Y}_{l,m}$$

with equality in $L^2(S^2, \mathbb{R})$. Now we consider the Rayleigh quotient and obtain with Lemma 8.1 and the orthonormality of the real spherical harmonics that

$$\begin{aligned} \inf_{\phi \in C^\infty(S^2) \cap \mathcal{V}} \frac{\|\nabla_{S^2} \phi\|^2_{L^2(S^2,\mathbb{R})}}{\|\phi\|^2_{L^2(S^2,\mathbb{R})}} &= \inf_{\phi \in C^\infty(S^2) \cap \mathcal{V}} \frac{-\int_{S^2} \phi \Delta_{S^2} \phi \mathrm{d}\sigma}{\|\phi\|^2_{L^2(S^2,\mathbb{R})}} \\ &= \inf_{0 \neq \{\phi_{l,m}\}_{l \geq 1, m=-l,\dots,l}} \frac{\sum_{l \geq 1} \sum_{m=-l}^{l} l(l+1) \phi^2_{l,m}}{\sum_{l \geq 1} \sum_{m=-l}^{l} \phi^2_{l,m}} = 2. \end{aligned}$$

Note that the infimum will be attained for the sequences that satisfy that $\phi_{l,m} = 0$ for $l \geq 2$. □

For $v \in H^1(S^2)/\mathbb{R}$ the expression $\|\nabla_{S^2} v\|_{L^2(S^2,\mathbb{R})}$ is independent of the representative of the equivalence class v. In the case that $v \neq 0$ in $H^1(S^2)/\mathbb{R}$, we choose the representative $\tilde{v}$ in v that satisfies that $\int_{S^2} \tilde{v} \mathrm{d}\sigma = 0$. Note that we just disregarded the common notation to denote equivalence classes with $[\cdot]$. The Poincaré inequality implies that

$$\|\nabla_{S^2} v\|_{L^2(S^2,\mathbb{R})} = \|\nabla_{S^2} \tilde{v}\|_{L^2(S^2,\mathbb{R})} \geq \sqrt{2} \|\tilde{v}\|_{L^2(S^2,\mathbb{R})} > 0.$$

Therefore, for all $v \in H^1(S^2)/\mathbb{R}$ we define

$$\|v\|_{H^1(S^2)/\mathbb{R}} = \|\nabla_{S^2} v\|_{L^2(S^2,\mathbb{R})},$$

which is a well-defined norm on $H^1(S^2)/\mathbb{R}$, since it is independent of the representative of the respective equivalence class. Moreover $H^1(S^2)/\mathbb{R}$ becomes a Hilbert space with the inner product

$$(v, w)_{H^1(S^2)/\mathbb{R}} = \int_{S^2} \nabla_{S^2} v \cdot \nabla_{S^2} w \mathrm{d}\sigma.$$

We will consider the variational formulation of the problem in Equation (8.1) for a strictly positive continuous function A on S^2: to find a unique $u \in H^1(S^2)/\mathbb{R}$ such that

$$b(u, v) = \int_{S^2} A \nabla_{S^2} u \cdot \nabla_{S^2} v \mathrm{d}\sigma = \int_{S^2} f v \mathrm{d}\sigma = \ell_f(v) \tag{8.2}$$

for all $v \in H^1(S^2)/\mathbb{R}$.

Lemma 8.4. *The bilinear form b in Equation* (8.2) *on $H^1(S^2)/\mathbb{R}\times H^1(S^2)/\mathbb{R}$ is symmetric. Moreover b is also continuous and coercive, i.e. for all $v, w \in H^1(S^2)/\mathbb{R}$ it holds that*

$$|b(w,v)| \leq \|A\|_{C^0(S^2)}\|w\|_{H^1(S^2)/\mathbb{R}}\|v\|_{H^1(S^2)/\mathbb{R}}$$

and for all $w \in H^1(S^2)/\mathbb{R}$ it holds that

$$b(w,w) \geq (\min_{x\in S^2} A(x))\|w\|^2_{H^1(S^2)/\mathbb{R}}.$$

Proof. The continuity of b follows with the Cauchy–Schwarz inequality. Symmetry and coercivity are clear. □

We state the Lax–Milgram lemma as the next theorem. We take the version from Yosida's book [29]. There it is the main theorem in Section 7 in Part 3 of [29].

Theorem 8.5. *Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)$ and induced norm $\|\cdot\|$ and let $\hat{b}$ be a real-valued bilinear functional on the product Hilbert space $\mathcal{H}\times\mathcal{H}$ which satisfies:*

boundedness*, i.e. there exists a constant K_1 such that for all $x, y \in \mathcal{H}$*

$$|\hat{b}(x,y)| \leq K_1\|x\|\cdot\|y\|,$$

and coercivity*, i.e. there exists a constant K_2 such that for all $x \in \mathcal{H}$*

$$\hat{b}(x,x) \geq K_2\|x\|^2.$$

Then there exists a uniquely determined linear bounded operator S with a bounded linear inverse S^{-1} such that

$$(x,y) = \hat{b}(x,Sy)$$

whenever $x, y \in \mathcal{H}$, and $\|S\| \leq K_2^{-1}$, $\|S^{-1}\| \leq K_1$.

Proof. This is explicitly the main theorem in Section 7 in Part 3 of [29] in the case of a real-valued bilinear functional $\hat{b}$. □

Since $H^1(S^2)/\mathbb{R}$ is a Hilbert space the Lax–Milgram lemma, Theorem 8.5, is applicable. First the Riesz representation theorem, which is the main theorem in Section 6 in Part 3 of [29] yields that for every $\ell \in (H^1(S^2)/\mathbb{R})^*$ there exists a unique $u_\ell \in H^1(S^2)/\mathbb{R}$ such that for all $v \in H^1(S^2)/\mathbb{R}$ it holds that

$$(u_\ell, v)_{H^1(S^2)/\mathbb{R}} = \ell(v) \qquad \text{and} \qquad \|u_\ell\|_{H^1(S^2)/\mathbb{R}} = \|\ell\|_{(H^1(S^2)/\mathbb{R})^*}.$$

Now Theorem 8.5 implies that there exists a uniquely determined bounded linear operator S on $H^1(S^2)/\mathbb{R}$ such that

$$b(Su_\ell, v) = (u_\ell, v)_{H^1(S^2)/\mathbb{R}} = \ell(v)$$

for all $v \in H^1(S^2)/\mathbb{R}$. We set $u = Su_\ell$ and u is the unique solution to the problem in Equation (8.2) if we take $\ell \in (H^1(S^2)/\mathbb{R})^*$ as right hand side. Theorem 8.5 and Lemma 8.4 yield the estimate

$$\|u\|_{H^1(S^2)/\mathbb{R}} = \|Su_\ell\|_{H^1(S^2)/\mathbb{R}} \leq \frac{\|u_\ell\|_{H^1(S^2)/\mathbb{R}}}{\min_{x\in S^2} A(x)} = \frac{\|\ell\|_{(H^1(S^2)/\mathbb{R})^*}}{\min_{x\in S^2} A(x)}.$$

In our discussion we want to take a function in $L^2(S^2,\mathbb{R})$ as right hand side of Equation (8.2). If a function $f \in L^2(S^2,\mathbb{R})$ additionally satisfies that $\int_{S^2} f \mathrm{d}\sigma = 0$, then $\ell_f = (v \mapsto \int_{S^2} f v \mathrm{d}\sigma) \in (H^1(S^2)/\mathbb{R})^*$. Therefore we make this assumption on f. For this choice of right hand side $\ell_f \in (H^1(S^2)/\mathbb{R})^*$ we obtain a unique solution $u \in H^1(S^2)/\mathbb{R}$ to the problem in Equation (8.2) with the above argument. The above estimate, Lemma 8.3 and the Cauchy–Schwarz inequality yield the estimate

$$\|u\|_{H^1(S^2)/\mathbb{R}} \leq \frac{\|\ell_f\|_{(H^1(S^2)/\mathbb{R})^*}}{\min_{x\in S^2} A(x)} \leq \sup_{v\in H^1(S^2)/\mathbb{R}} \frac{\|f\|_{L^2(S^2,\mathbb{R})}\|v\|_{L^2(S^2,\mathbb{R})}}{\min_{x\in S^2} A(x)\|v\|_{H^1(S^2)/\mathbb{R}}} \leq \frac{1}{\sqrt{2}} \frac{\|f\|_{L^2(S^2,\mathbb{R})}}{\min_{x\in S^2} A(x)}, \tag{8.3}$$

where we tacitly always selected the representative v that satisfies that $\int_{S^2} v \mathrm{d}\sigma = 0$.

A second continuous function $\hat{A}$ on S^2 such that $\min_{x\in S^2} \hat{A}(x) > 0$ induces the bilinear form $\hat{b}$ and we can consider the respective problem in Equation (8.2) with $\hat{b}$ instead of b and with the same right hand side f. We observe that this problem has a unique solution $\hat{u} \in H^1(S^2)/\mathbb{R}$ that also satisfies Inequality (8.3), i.e.

$$\|\hat{u}\|_{H^1(S^2)/\mathbb{R}} \leq \frac{1}{\sqrt{2}} \frac{\|f\|_{L^2(S^2,\mathbb{R})}}{\min_{x\in S^2} \hat{A}(x)}.$$

With the same argument that is used to prove the first Strang lemma, in Section 4.1 in [6], we will obtain that the mapping $A \mapsto u$ from $C^0(S^2) \cap \{A : \min_{x\in S^2} A(x) > 0\}$ to $H^1(S^2)/\mathbb{R}$ is continuous.

Proposition 8.6. *The unique solution u of Equation* (8.2) *depends continuously on the coefficient A.*
Let $A, \hat{A} \in C^0(S^2) \cap \{\tilde{A} : \min_{x\in S^2} \tilde{A}(x) > 0\}$ be two coefficients that induce bilinear forms b and $\hat{b}$. For the respective solutions u with respect to A and $\hat{u}$ with respect to $\hat{A}$ of Equation (8.2) *with the same right hand side $f \in L^2(S^2,\mathbb{R}) \cap \{\tilde{f} : \int_{S^2} \tilde{f} \mathrm{d}\sigma = 0\}$ it holds that*

$$\|u - \hat{u}\|_{H^1(S^2)/\mathbb{R}} \leq \frac{1}{\sqrt{2}} \frac{\|f\|_{L^2(S^2,\mathbb{R})}}{(\min\limits_{x\in S^2} A(x))(\min\limits_{x\in S^2} \hat{A}(x))} \|A - \hat{A}\|_{C^0(S^2)}. \tag{8.4}$$

Proof. We start with the proof of the second claim We adapt the argument in [6] as announced. We introduce the notation $\mathcal{V} = H^1(S^2)/\mathbb{R}$. We can apply manipulations using that $\hat{u}$ and u solve the respective equations to obtain with the coercivity of the bilinear form that

$$\min_{x\in S^2} \hat{A}(x)\|\hat{u} - u\|_{\mathcal{V}}^2 \leq \hat{b}(\hat{u} - u, \hat{u} - u) = l_f(\hat{u} - u) - \hat{b}(u, \hat{u} - u) = b(u, \hat{u} - u) - \hat{b}(u, \hat{u} - u).$$

We continue this computation and obtain that

$$\min_{x\in S^2} \hat{A}(x)\|\hat{u} - u\|_{\mathcal{V}} \leq \frac{|b(u, \hat{u} - u) - \hat{b}(u, \hat{u} - u)|}{\|\hat{u} - u\|_{\mathcal{V}}} \leq \sup_{w\in\mathcal{V}} \frac{|b(u, w) - \hat{b}(u, w)|}{\|w\|_{\mathcal{V}}}.$$

We obtain with the Cauchy–Schwarz inequality that

$$\|\hat{u} - u\|_{\mathcal{V}} \leq \frac{1}{\min\limits_{x\in S^2} \hat{A}(x)} \sup_{w\in\mathcal{V}} \frac{|b(u, w) - \hat{b}(u, w)|}{\|w\|_{\mathcal{V}}} \leq \frac{1}{\min\limits_{x\in S^2} \hat{A}(x)} \|u\|_{\mathcal{V}} \|A - \hat{A}\|_{C^0(S^2)}.$$

We apply Inequality (8.3) and obtain that

$$\|u-\hat{u}\|_{H^1(S^2)/\mathbb{R}} \leq \frac{1}{\sqrt{2}} \frac{\|f\|_{L^2(S^2,\mathbb{R})}}{(\min\limits_{x\in S^2} A(x))(\min\limits_{x\in S^2} \hat{A}(x))} \|A-\hat{A}\|_{C^0(S^2)}.$$

To show the first claim, we choose a sequence $(A_n : n \geq 0) \subset C^0(S^2) \cap \{\tilde{A} : \min_{x\in S^2} \tilde{A}(x) > 0\}$ that converges to A in $C^0(S^2)$, i.e. $\|A - A_n\|_{C^0(S^2)} = \varepsilon_n \to 0$ as $n \to \infty$. For all $n \in \mathbb{N}$ the solution of Equation (8.2) with respect to A_n is denoted by u_n. We observe that for all $y \in S^2$and $n \in \mathbb{N}$ it holds that $A_n(y) \geq \min_{x\in S^2} A(x) - \varepsilon_n$. There exists $N \in \mathbb{N}$ such that for all integer $n \geq N$ it holds that $\varepsilon_n \leq \frac{1}{2} \min_{x\in S^2} A(x)$ and we obtain that

$$\frac{1}{\min\limits_{x\in S^2} A_n(x)} \leq \frac{2}{\min\limits_{x\in S^2} A(x)}$$

Therefore for all $n \in \mathbb{N}$ $(\min_{x\in S^2} A_n(x))^{-1}$ can be bounded independently of n and with Inequality (8.4) we conclude that u_n converges to u in $H^1(S^2)/\mathbb{R}$. This implies that the mapping $A \mapsto u$ is a continuous mapping from $C^0(S^2) \cap \{\tilde{A} : \min_{x\in S^2} \tilde{A}(x) > 0\}$ to $H^1(S^2)/\mathbb{R}$ and the first claim is proven. □

In the following we want to impose higher regularity on the right hand side f and on the coefficient A that we used to define the bilinear form b. The next section discusses how higher order Hölder regularity on the right hand side and on the coefficients of elliptic operators affects the regularity of the solution in bounded domains of Euclidean space.

8.1. The Schauder interior estimates

Gilbarg and Trudinger elaborate in their book [14] the Schauder estimates of solutions of elliptic partial differential equations of second order in domains of Euclidean space. They present estimates of the Hölder norms of higher order partial derivatives of the solution in terms of the right hand side and the supremum norm of the solution. However, in [14] the dependence on the coefficients of the respective differential operator in the estimates is not emphasized. In this project we are interested in the precise dependence of the constants in the estimates on the coefficients of the respective differential operator. Therefore we will take a close look at the proofs to distinguish the contribution of the coefficients to the constants.

For the discussion of the proof of the Schauder estimates we introduce a set of semi-norms. We fix the bounded domain $D \subset \mathbb{R}^n$ for some $n \in \mathbb{N}$ with diameter $diam(D) = r$. Even though we will be interested in the case $n = 2$, we will discuss the Schauder estimates for an arbitrary dimension $n \in \mathbb{N}$, because the proof will be the same and the concentration on $n = 2$ will not lead to simplifications. For $x, y \in D$ we define

$$d_x = dist(x, \partial D) \qquad \text{and} \qquad d_{x,y} = \min\{d_x, d_y\}.$$

Let $k \in \mathbb{N}$. For $f \in C^k(\overline{D})$, $\gamma \in (0, 1]$ and $\sigma \in \mathbb{R}$ we define the semi-norms

$$[f]^*_{k,D} = \sup_{\substack{x\in D \\ |\boldsymbol{\beta}|=k}} d_x^k |\partial_{\boldsymbol{\beta}} f(x)|,$$

$$[f]^*_{\gamma,k,D} = \sup_{\substack{x,y\in D, x\neq y\\ |\boldsymbol{\beta}|=k}} d_{x,y}^{k+\gamma} \frac{|\partial_{\boldsymbol{\beta}} f(x) - \partial_{\boldsymbol{\beta}} f(y)|}{\|x-y\|^{\gamma}_{\mathbb{R}^n}},$$

$$[f]^{(\sigma)}_{k,D} = \sup_{\substack{x\in D\\ |\boldsymbol{\beta}|=k}} d_x^{\sigma+k} |\partial_{\boldsymbol{\beta}} f(x)|,$$

$$[f]^{(\sigma)}_{\gamma,k,D} = \sup_{\substack{x,y\in D, x\neq y\\ |\boldsymbol{\beta}|=k}} d_{x,y}^{\sigma+k+\gamma} \frac{|\partial_{\boldsymbol{\beta}} f(x) - \partial_{\boldsymbol{\beta}} f(y)|}{\|x-y\|^{\gamma}_{\mathbb{R}^n}},$$

for $f \in C^0(\overline{D})$ we define

$$|f|^{(\sigma)}_{\gamma,0,D} = [f]^{(\sigma)}_{0,D} + [f]^{(\sigma)}_{\gamma,0,D}.$$

For two functions $f, g \in C^{0,\gamma}(\overline{D})$, Inequality (6.11) in [14] states that for the product of f and g and $\sigma + \tau \geq 0$ it holds that

$$|fg|^{(\sigma+\tau)}_{\gamma,0,D} \leq |f|^{(\sigma)}_{\gamma,0,D} |g|^{(\tau)}_{\gamma,0,D}. \tag{8.5}$$

The following lemma states some interpolation inequalities with explicit constants, which hold for these semi-norms. It is Lemma 6.32 in [14]. We will review its proof to discover the precise constants.

Lemma 8.7. *For $\gamma \in (0,1)$, $f \in C^{2,\gamma}(\overline{D})$, $\beta \in (0,1)$ and $\varepsilon > 0$ it holds that*

$$[f]^*_{1,D} \leq 512\ \varepsilon^{-\frac{1}{\gamma+1}} \|f\|_{C^0(\overline{D})} + \varepsilon [f]^*_{\gamma,2,D} \tag{8.6}$$

and

$$[f]^*_{2,D} \leq 64^2 2^{\frac{2}{\gamma}}\ \varepsilon^{-\frac{2}{\gamma}} \|f\|_{C^0(\overline{D})} + \varepsilon [f]^*_{\gamma,2,D}. \tag{8.7}$$

Moreover for a constant K that depends on β and γ it holds that for all $\varepsilon \in (0,1]$

$$[f]^*_{\beta,0,D} \leq K\ \varepsilon^{-\frac{1}{1-\beta}} \|f\|_{C^0(\overline{D})} + \varepsilon [f]^*_{\gamma,2,D} \tag{8.8}$$

and

$$[f]^*_{\beta,1,D} \leq K\ \varepsilon^{-\frac{4}{1-\beta}} \|f\|_{C^0(\overline{D})} + \varepsilon [f]^*_{\gamma,2,D}. \tag{8.9}$$

For $\gamma \in (0,1)$, $f \in C^{1,\gamma}(\overline{D})$, $\beta \in (0,1)$ and $\varepsilon > 0$ it holds that

$$[f]^*_{1,D} \leq 2\ 4^{\frac{1}{\gamma}}\ \varepsilon^{-\frac{1}{\gamma}} \|f\|_{C^0(\overline{D})} + \varepsilon [f]^*_{\gamma,1,D} \tag{8.10}$$

and for a constant K that depends on β and γ it holds that for all $\varepsilon \in (0,1]$

$$[f]^*_{\beta,0,D} \leq K\ \varepsilon^{-\frac{1}{1-\beta}} \|u\|_{C^0(\overline{D})} + \varepsilon [f]^*_{\gamma,1,D}. \tag{8.11}$$

Proof. In this proof we quote some inequalities from the original proof of this lemma given in [14], which can be combined to obtain the constants depending on ε. We state Inequality (6.83) in [14], which says that for $\varepsilon' > 0$ it holds that

$$[f]^*_{1,D} \leq \frac{4}{\varepsilon'} \|f\|_{C^0(\overline{D})} + \varepsilon' [f]^*_{2,D} \tag{8.12}$$

and Inequality (6.85) in [14], which says that for $\varepsilon' > 0$ it holds that

$$[f]^*_{2,D} \leq 16\ \varepsilon'^{-\frac{1}{\gamma}} [f]^*_{1,D} + \varepsilon' [f]^*_{\gamma,2,D}. \tag{8.13}$$

We combine Inequality (8.12) and (8.13) with $\varepsilon_1, \varepsilon_2 > 0$ to obtain that

$$[f]^*_{1,D} \leq \frac{4}{\varepsilon_1}\|f\|_{C^0(\overline{D})} + \varepsilon_1 [f]^*_{2,D} \leq \frac{4}{\varepsilon_1}\|f\|_{C^0(\overline{D})} + 16\varepsilon_1 \varepsilon_2^{-\frac{1}{\gamma}} [f]^*_{1,D} + \varepsilon_1\varepsilon_2 [f]^*_{\gamma,2,D}.$$

We want to achieve that $\frac{\varepsilon}{2} = \varepsilon_1\varepsilon_2$ and $16\varepsilon_1\varepsilon_2^{-\frac{1}{\gamma}} = \frac{1}{2}$. This is obtained with the choice $\varepsilon_1 = 32^{-\frac{\gamma}{\gamma+1}} 2^{-\frac{1}{\gamma+1}} \varepsilon^{\frac{1}{\gamma+1}}$ and $\varepsilon_2 = \frac{\varepsilon}{2\varepsilon_1}$. We insert our choices of ε_1 and ε_2 and obtain what is claimed in the lemma, i.e.

$$[f]^*_{1,D} \leq 512\ \varepsilon^{-\frac{1}{\gamma+1}} \|f\|_{C^0(\overline{D})} + \varepsilon [f]^*_{\gamma,2,D}. \tag{8.14}$$

Now we combine Inequalities (8.12) and (8.13) with $\varepsilon_1, \varepsilon_2 > 0$ in the other order to obtain that

$$[f]^*_{2,D} \leq 16\ \varepsilon_1^{-\frac{1}{\gamma}} [f]^*_{1,D} + \varepsilon_1 [f]^*_{\gamma,2,D} \leq 64\ \varepsilon_1^{-\frac{1}{\gamma}} \varepsilon_2^{-1} \|f\|_{C^0(\overline{D})} + 16\varepsilon_2 \varepsilon_1^{-\frac{1}{\gamma}} [f]^*_{2,D} + \varepsilon_1 [f]^*_{\gamma,2,D}.$$

We want to achieve that $\frac{\varepsilon}{2} = \varepsilon_1$ and $16\varepsilon_2\varepsilon_1^{-\frac{1}{\gamma}} = \frac{1}{2}$. Therefore we choose $\varepsilon_2 = \frac{1}{32} 2^{-\frac{1}{\gamma}} \varepsilon^{\frac{1}{\gamma}}$ and obtain that

$$[f]^*_{2,D} \leq 64^2 2^{\frac{2}{\gamma}}\ \varepsilon^{-\frac{2}{\gamma}} \|f\|_{C^0(\overline{D})} + \varepsilon [f]^*_{\gamma,2,D}. \tag{8.15}$$

We state Inequality (6.88) in [14], which says that for $\varepsilon' > 0$ it holds that

$$[f]^*_{\beta,0,D} \leq 2\ 2^{\frac{\beta}{1-\beta}} \varepsilon'^{-\frac{\beta}{1-\beta}} \|f\|_{C^0(\overline{D})} + \varepsilon' [f]^*_{1,D}. \tag{8.16}$$

From the comments just under Inequality (6.88) in [14] it is evident that for $\varepsilon' > 0$ it holds that

$$[f]^*_{\beta,1,D} \leq 2\ 2^{\frac{\beta}{1-\beta}} \varepsilon'^{-\frac{\beta}{1-\beta}} [f]^*_{1,D} + \varepsilon' [f]^*_{2,D}. \tag{8.17}$$

We combine Inequalities (8.16) and (8.14) with $\varepsilon_1, \varepsilon_2 > 0$ to obtain that

$$[f]^*_{\beta,0,D} \leq 2\ 2^{\frac{\beta}{1-\beta}} \varepsilon_1^{-\frac{\beta}{1-\beta}} \|f\|_{C^0(\overline{D})} + \varepsilon_1 [f]^*_{1,D} \leq \left(\frac{2\ 2^{\frac{\beta}{1-\beta}}}{\varepsilon_1^{\frac{\beta}{1-\beta}}} + \frac{512\varepsilon_1}{\varepsilon_2^{\frac{1}{\gamma+1}}} \right) \|f\|_{C^0(\overline{D})} + \varepsilon_1\varepsilon_2 [f]^*_{\gamma,2,D}.$$

We want to achieve that $\varepsilon = \varepsilon_1\varepsilon_2$ and $\varepsilon_1 = \varepsilon_2^{\frac{1}{\gamma+1}}$. With the choices $\varepsilon_2 = \frac{\varepsilon}{\varepsilon_1}$ and $\varepsilon_1 = \varepsilon^{\frac{1}{\gamma+2}}$ we obtain that

$$[f]^*_{\beta,0,D} \leq \left(2\ 2^{\frac{\beta}{1-\beta}} \varepsilon^{-\frac{\beta}{(1-\beta)(\gamma+2)}} + 512 \right) \|f\|_{C^0(\overline{D})} + \varepsilon [f]^*_{\gamma,2,D}.$$

The fact that $\varepsilon \in (0,1]$ implies that

$$\varepsilon^{-\frac{\beta}{(1-\beta)(\gamma+2)}} \leq \varepsilon^{-\frac{1}{1-\beta}},$$

which implies the third claim of the lemma. For the fourth claim of the lemma we combine Inequalities (8.17), (8.14) and (8.15) with $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ to obtain that

$$[f]^*_{\beta,1,D} \leq 2\ 2^{\frac{\beta}{1-\beta}} \varepsilon_1^{-\frac{\beta}{1-\beta}} [f]^*_{1,D} + \varepsilon_1 [f]^*_{2,D}$$

$$\leq \left(\frac{2\ 2^{\frac{\beta}{1-\beta}}}{\varepsilon_1^{\frac{\beta}{1-\beta}}} \frac{512}{\varepsilon_2^{\frac{1}{\gamma+1}}} + \varepsilon_1 \frac{64^2 2^{\frac{2}{\gamma}}}{\varepsilon_3^{\frac{2}{\gamma}}} \right) \|f\|_{C^0(\overline{D})} + \left(\frac{2\ 2^{\frac{\beta}{1-\beta}}}{\varepsilon_1^{\frac{\beta}{1-\beta}}} \varepsilon_2 + \varepsilon_1 \varepsilon_3 \right) [f]^*_{\gamma,2,D}.$$

We want to achieve that $\varepsilon = 2\ (2\varepsilon_1^{-1})^{\frac{\beta}{1-\beta}} + \varepsilon_1\varepsilon_3$ and $\varepsilon_1 = \varepsilon_3^{\frac{2}{\gamma}}$. The choices $\varepsilon_1 = (2^{-1}\varepsilon)^{\frac{2}{2+\gamma}}$, $\varepsilon_2 = 2^{-\frac{1}{1-\beta}}(2^{-1}\varepsilon)^{\frac{2+(1-\beta)\gamma}{(2+\gamma)(1-\beta)}}$ and $\varepsilon_3 = (2^{-1}\varepsilon)^{\frac{\gamma}{2+\gamma}}$ satisfy that

$$\varepsilon_1^{\frac{\beta}{1-\beta}} \varepsilon_2^{\frac{1}{1+\gamma}} = 2^{-\frac{2+\beta}{(2+\gamma)(1-\beta)}} \varepsilon^{\frac{2\beta(1+\gamma)+\gamma(1-\beta)+2}{(2+\gamma)(1+\gamma)(1-\beta)}}.$$

Then we obtain the estimate

$$[f]^*_{\beta,1,D} \leq \left(2^{\frac{13}{1-\beta}} \varepsilon^{-\frac{2\beta(1+\gamma)+\gamma(1-\beta)+2}{(2+\gamma)(1+\gamma)(1-\beta)}} + 64^2 2^{\frac{2}{\gamma}} \right) \|f\|_{C^0(\overline{D})} + \varepsilon [f]^*_{\gamma,2,D}.$$

The fact that $\varepsilon \in (0,1]$ implies that

$$\varepsilon^{-\frac{2\beta(1+\gamma)+\gamma(1-\beta)+2}{(2+\gamma)(1+\gamma)(1-\beta)}} \leq \varepsilon^{-\frac{4}{(1-\beta)}},$$

which finishes the proof of the fourth claim of the lemma. The fifth inequality in the lemma is stated in Gilbarg and Trudinger [14] as Inequality (6.86). The constant $2\ 4^{\frac{1}{\gamma}} \varepsilon^{-\frac{1}{\gamma}}$ is obtained from the proof of Inequality (8.85) in [14].
For the last inequality we combine Inequalities (8.16) and (8.10) with $\varepsilon_1, \varepsilon_2 > 0$ and obtain that

$$\begin{aligned} [f]^*_{\beta,0,D} &\leq 2\ 2^{\frac{\beta}{1-\beta}} \varepsilon_1^{-\frac{\beta}{1-\beta}} \|f\|_{C^0(\overline{D})} + \varepsilon_1 [f]^*_{1,D} \\ &\leq 2\ 2^{\frac{\beta}{1-\beta}} \varepsilon_1^{-\frac{\beta}{1-\beta}} \|f\|_{C^0(\overline{D})} + 2\ 4^{\frac{1}{\gamma}} \varepsilon_1 \varepsilon_2^{-\frac{1}{\gamma}} \|f\|_{C^0(\overline{D})} + \varepsilon_1\varepsilon_2 [f]^*_{\gamma,1,D}. \end{aligned}$$

We want to achieve that $\varepsilon = \varepsilon_1\varepsilon_2$ and $\varepsilon_1 = \varepsilon_2^{\frac{1}{\gamma}}$. We choose $\varepsilon_1 = \varepsilon^{\frac{1}{1+\gamma}}$, $\varepsilon_2 = \varepsilon^{\frac{\gamma}{1+\gamma}}$ and obtain that

$$[f]^*_{\beta,0,D} \leq \left(2^{\frac{1}{1-\beta}} \varepsilon^{-\frac{\beta}{(1-\beta)(1+\gamma)}} + 2^{\frac{2+\gamma}{\gamma}} \right) \|f\|_{C^0(\overline{D})} + \varepsilon [f]^*_{\gamma,1,D}.$$

The fact that $\varepsilon \in (0,1]$ implies the sixth claim of the lemma, because

$$\varepsilon^{-\frac{\beta}{(1-\beta)(1+\gamma)}} \leq \varepsilon^{-\frac{1}{1-\beta}}.$$

□

8.1.1. The Schauder interior estimates for classical solutions

For sets of real-valued, continuous functions on $\overline{D}$ $(a_{i,j} : a_{i,j} = a_{j,i}, i,j = 1,...,n)$, $(b_i : i = 1,...,n)$ and c we introduce the differential operator

$$L = a^{i,j}\partial_i\partial_j + b^i\partial_i + c \tag{8.18}$$

where we used the usual summation convention to ease the notation, i.e. for two vectors $v, w \in \mathbb{R}^n$ we define $\sum_{i=1}^n v_i w_i = v^i w_i$. We require additionally that for all $x \in D$ and $\xi \in \mathbb{R}^n$ it holds that

$$\Lambda(x)\|\xi\|^2_{\mathbb{R}^n} \geq a^{i,j}(x)\xi_i\xi_j \geq \lambda(x)\|\xi\|^2_{\mathbb{R}^n},$$

for two strictly positive functions Λ and λ on D. If $\inf_{x\in D} \lambda(x) > 0$ we call the operator L in Equation (8.18) strictly elliptic. Without loss of generality we assume that the functions

Λ and λ are continuous on $\overline{D}$ and that $\min_{x\in\overline{D}} \lambda(x) > 0$. This notion of strict ellipticity agrees with [14], which is our main reference for Section 8.1.1 and Section 8.1.2. Schauder developed a theory to estimate the Hölder norms of solutions and derivatives of solutions u to the problem $Lu = f$ in terms of the right hand side f and the supremum norm of the solution. The problem, i.e. $Lu = f$, is only stated formally at this point. It will be made precise afterwards.
The next lemma is Lemma 6.1 in [14] and is a preliminary result for the general statement. In this lemma the coefficients $(a_{i,j} : i, j = 1, ..., n)$ of the operator L are taken to be constant over D and the other coefficients are set to zero. We review its proof to work out the respective constant in the estimate precisely.

Lemma 8.8. *Let $A \in \mathbb{R}^{n\times n}$ be a constant symmetric matrix such that for two constants $\lambda, \Lambda > 0$ and all $\xi \in \mathbb{R}^n$ it holds that*

$$\lambda\|\xi\|^2_{\mathbb{R}^n} \leq A^{i,j}\xi_i\xi_j \leq \Lambda\|\xi\|^2_{\mathbb{R}^n}.$$

If $u \in C^2(\overline{D})$ and $f \in C^{0,\gamma}(\overline{D})$ for $\gamma \in (0, 1)$ satisfy $A^{i,j}\partial_i\partial_j u = f$ in D, then there exists a constant $C > 0$ depending only on n and γ such that

$$[u]^*_{\gamma,2,D} \leq C\Lambda^{2+\frac{\gamma}{2}}\lambda^{-\frac{\gamma}{2}}\left(\|u\|_{C^0(\overline{D})} + (\lambda^{-1} + \Lambda^{\frac{\gamma}{2}}\lambda^{-1-\frac{\gamma}{2}})|f|^{(2)}_{\gamma,0,D}\right). \tag{8.19}$$

Proof. The idea of the proof is to use a suitable coordinate transformation to be able to apply known results for the case of the Poisson equation. For an invertible matrix $R \in \mathbb{R}^{n\times n}$ the transformation $x \mapsto Rx = y$ results in $u \to \tilde{u} = u \circ R^{-1}$ and transforms the operator $A^{i,j}\partial_i\partial_j$ as well, i.e. we evaluate the partial derivatives to obtain that

$$\begin{aligned}
A^{i,j}\partial_i\partial_j u(x) &= A^{i,j}\partial_i\partial_j\tilde{u}(Rx) \\
&= \sum_{i,j,k=1}^{n} A_{i,j}\partial_i R_{k,j}\partial_k\tilde{u}\Big|_{Rx} \\
&= \sum_{i,j,k,l=1}^{n} A_{i,j}R_{l,i}\partial_l R_{k,j}\partial_k\tilde{u}\Big|_{Rx} \\
&= (RAR^\top)^{k,l}\partial_k\partial_l\tilde{u}(y) = \tilde{A}^{k,l}\partial_k\partial_l\tilde{u}(y).
\end{aligned}$$

Since the matrix A is symmetric and positive definite, we know from linear algebra that it can be diagonalized with an orthogonal matrix and has strictly positive eigenvalues $(\lambda_1, ..., \lambda_n)$. Let R be this orthogonal matrix. Let J be the diagonal matrix that results from the inverted square roots of the eigenvalues, i.e. $J = diag(\lambda_1^{-\frac{1}{2}}, ..., \lambda_n^{-\frac{1}{2}})$. If we transform the matrix A with $Q = JR$, then $\tilde{A}$ becomes the identity matrix and the transformed problem becomes the Poisson equation, i.e. $\Delta\tilde{u} = \tilde{f}$ for $\tilde{f} = f \circ Q^{-1}$ and $\tilde{u} = u \circ Q^{-1}$, where we redefined $\tilde{u}$.

The next step is to relate the norms of a function v and respectively $\tilde{v} = v \circ Q^{-1}$ to each other. We define the domain $\tilde{D} = Q(D)$. Since R is an orthogonal matrix and the eigenvalues of A satisfy that $\lambda \leq \lambda_1, ..., \lambda_n \leq \Lambda$, which form the other matrix J we obtain that for all $\xi \in \mathbb{R}^n$, it holds that

$$\Lambda^{-1}\|\xi\|^2_{\mathbb{R}^n} \leq \|Q\xi\|^2_{\mathbb{R}^n} \leq \lambda^{-1}||\xi\|^2_{\mathbb{R}^n}. \tag{8.20}$$

For $v \in C^k(\overline{D})$ and some integer $k \geq 0$, we examine the semi-norm of $\tilde{v}$. We compute the partial derivative of $\tilde{v}$ as above and obtain together with Inequality (8.20) that for $j \in \{1, ..., n\}$ and $y \in \tilde{D}$ it holds that

$$|\partial_i \tilde{v}(y)| = |\sum_{i=1}^{n} \partial_i v(x) Q_{i,j}| \leq \|Q \nabla v(x)\|_{\mathbb{R}^n} \leq \lambda^{-\frac{1}{2}} n \sup_{j=1,...,n} |\partial_j v(x)|,$$

where $x = Q^{-1}y$ and $\nabla = (\partial_1, ..., \partial_n)^\top$ is the usual gradient on subdomains of Euclidean space. This gives the estimate

$$|\tilde{v}|_{1,\tilde{D}} \leq \lambda^{-\frac{1}{2}} n |v|_{1,D}.$$

For higher order derivatives the above argument can be iterated in order to obtain for a multi-index $\boldsymbol{\beta} \in \mathbb{N}_0^n$ satisfying $|\boldsymbol{\beta}| = k$ that

$$|\partial_{\boldsymbol{\beta}} \tilde{v}(y)| \leq \lambda^{-\frac{k}{2}} n^k \sup_{|\boldsymbol{\alpha}|=k} |\partial_{\boldsymbol{\alpha}} v(x)|,$$

where $x = Q^{-1}y$. We obtain the estimate

$$|\tilde{v}|_{k,\tilde{D}} \leq \lambda^{-\frac{k}{2}} n^k |v|_{k,D}.$$

The whole argument also works similarly the other way around, where we bound the semi-norm of v with the one of $\tilde{v}$, i.e.

$$|v|_{k,D} \leq \Lambda^{\frac{k}{2}} n^k |\tilde{v}|_{k,\tilde{D}}.$$

Together we obtain that

$$\Lambda^{-\frac{k}{2}} n^{-k} |v|_{k,D} \leq |\tilde{v}|_{k,\tilde{D}} \leq \lambda^{-\frac{k}{2}} n^k |v|_{k,D}. \tag{8.21}$$

Note that the argument also applies similarly to the Hölder semi-norms $|\cdot|_{\gamma,k,D}$, i.e. for $v \in C^{k,\gamma}(\overline{D})$ it holds that

$$\lambda^{\frac{\gamma}{2}} \Lambda^{-\frac{k}{2}} n^{-k} |v|_{\gamma,k,D} \leq |\tilde{v}|_{\gamma,k,\tilde{D}} \leq \Lambda^{\frac{\gamma}{2}} \lambda^{-\frac{k}{2}} n^k |v|_{\gamma,k,D}. \tag{8.22}$$

Note that with Inequality (8.20) we obtain that for $x \in D$ and $\tilde{x} = Qx$ it holds that

$$\Lambda^{-\frac{1}{2}} d_x \leq \tilde{d}_{\tilde{x}} \leq \lambda^{-\frac{1}{2}} d_x, \tag{8.23}$$

where $\tilde{d}_{\tilde{x}} = dist(\tilde{x}, \partial\tilde{D})$. This inequality enables us to relate the semi-norms $|\cdot|^{(j)}_{\gamma,0,D}$ for some positive integer j and $[\cdot]^*_{\gamma,k,D}$ to each other with the help of Inequalities (8.20), (8.22) and (8.23), i.e.

$$\begin{aligned} |\tilde{v}|^{(j)}_{\gamma,0,\tilde{D}} &= \sup_{\tilde{x}\in\tilde{D}} \tilde{d}^j_{\tilde{x}} |\tilde{v}(\tilde{x})| + \sup_{\tilde{x},\tilde{y}\in\tilde{D},\tilde{x}\neq\tilde{y}} \tilde{d}^{j+\gamma}_{\tilde{x},\tilde{y}} \frac{|\tilde{v}(\tilde{x}) - \tilde{v}(\tilde{y})|}{\|\tilde{x} - \tilde{y}\|^\gamma_{\mathbb{R}^n}} \\ &\leq \lambda^{-\frac{j}{2}} [v]^{(j)}_{0,D} + \Lambda^{\frac{\gamma}{2}} \lambda^{-\frac{j+\gamma}{2}} [v]^{(j)}_{\gamma,0,D} \leq (\lambda^{-\frac{j}{2}} + \Lambda^{\frac{\gamma}{2}} \lambda^{-\frac{j+\gamma}{2}}) |v|^{(j)}_{\gamma,0,D} \end{aligned} \tag{8.24}$$

and

$$\Lambda^{-k-\frac{\gamma}{2}} \lambda^{\frac{\gamma}{2}} n^{-k} [v]^*_{\gamma,k,D} \leq [\tilde{v}]^*_{\gamma,k,\tilde{D}} \leq \lambda^{-k-\frac{\gamma}{2}} \Lambda^{\frac{\gamma}{2}} n^k [v]^*_{\gamma,k,D}. \tag{8.25}$$

Theorem 4.8 in [14] states that there exists a constant $C > 0$ that depends only on n and γ such that the following estimate holds for the function $\tilde{u}$,

$$[\tilde{u}]^*_{\gamma,2,\tilde{D}} \leq C \left(\|\tilde{u}\|_{C^0(\overline{\tilde{D}})} + |\tilde{f}|^{(2)}_{\gamma,0,\tilde{D}} \right). \tag{8.26}$$

We combine this estimate with the Estimates (8.24) and (8.25) to obtain that

$$\begin{aligned}
[u]^*_{\gamma,2,D} &\leq \Lambda^{2+\frac{\gamma}{2}} n^2 \lambda^{-\frac{\gamma}{2}} [\tilde{u}]^*_{\gamma,2,\tilde{D}} \\
&\leq C\Lambda^{2+\frac{\gamma}{2}} n^2 \lambda^{-\frac{\gamma}{2}} \left(\|\tilde{u}\|_{C^0(\overline{\tilde{D}})} + |\tilde{f}|^{(2)}_{\gamma,0,\tilde{D}} \right) \\
&\leq C\Lambda^{2+\frac{\gamma}{2}} n^2 \lambda^{-\frac{\gamma}{2}} \left(\|u\|_{C^0(\overline{D})} + (\lambda^{-1} + \Lambda^{\frac{\gamma}{2}} \lambda^{-1-\frac{\gamma}{2}}) |f|^{(2)}_{\gamma,0,D} \right).
\end{aligned}$$

□

The Schauder theory interprets the functions $(a_{i,j} : i,j = 1,...,n)$ as small perturbations from a frozen at a point $x_0 \in D$. This point x_0 will be varied after an upper bound of a difference quotient of the solution is established independently of this point x_0. The next theorem illustrates this point of view in its proof. It is Theorem 6.2 in [14]. Again we review its proof to emphasize the dependencies of the respective constants on the coefficients of the differential operator.

Theorem 8.9. *Let $u \in C^{2,\gamma}(\overline{D})$ and $f \in C^{0,\gamma}(\overline{D})$ for some $\gamma \in (0,1)$ satisfy $Lu = f$. Let $D' \subset\subset D$ be a closed subset such that $dist(D', \partial D) = d$ for some $d > 0$. If the coefficients of L are all Hölder continuous, i.e. $(a_{i,j} : i,j = 1,...,n), (b_i : i = 1,...,n), \{c\} \subset C^{0,\gamma}(\overline{D})$, then the following estimate holds:*

$$d^{2+\gamma} |u|_{\gamma,2,D'} \leq \mathcal{K} \left(\|u\|_{C^0(\overline{D})} + \|f\|_{C^{0,\gamma}(\overline{D})} \right).$$

The constant $\mathcal{K}$ depends implicitly on γ and r through a constant K, i.e.

$$\mathcal{K} = K \left(\left(\frac{(1 + \|\Lambda\|_{C^0(\overline{D})})^{2+\gamma}}{\min_{x \in \overline{D}} \lambda(x)^{1+\gamma}} \right)^{\frac{9}{1-\gamma}} (1 + a + b + c)^{\frac{9}{1-\gamma}} \right),$$

where

$$a = \sum_{i,j=1}^{n} |a_{i,j}|_{\gamma,0,D}, \qquad b = \sum_{i=1}^{n} \|b_i\|_{C^{0,\gamma}(\overline{D})} \qquad \text{and} \qquad c = \|c\|_{C^{0,\gamma}(\overline{D})}.$$

Proof. We fix two interior points $x_0, y_0 \in D$. We use x_0 and the equation $Lu = f$ to define the function F, i.e.

$$a^{i,j}(x_0) \partial_i \partial_j u = (a^{i,j}(x_0) - a^{i,j}) \partial_i \partial_j u - b^i \partial_i u - cu + f = F. \tag{8.27}$$

For $\mu \leq \frac{1}{2}$ we set $B_{\mu d_{x_0}} = B_{\mu d_{x_0}}(x_0) \subset D$. We interpret Equation (8.27) as an equation in $B_{\mu d_{x_0}}$ and apply the estimate of the previous lemma to obtain in the case that $y_0 \in B_{\frac{\mu d_{x_0}}{2}}(x_0)$ and $\boldsymbol{\alpha} \in \mathbb{N}_0^n$ with $|\boldsymbol{\alpha}| = 2$ it holds that

$$\begin{aligned}
&\left(\frac{\mu d_{x_0,y_0}}{2} \right)^{2+\gamma} \frac{|\partial_{\boldsymbol{\alpha}} u(x_0) - \partial_{\boldsymbol{\alpha}} u(y_0)|}{\|x_0 - y_0\|^{\gamma}_{\mathbb{R}^n}} \\
&\quad \leq C\Lambda(x_0)^{2+\frac{\gamma}{2}} n^2 \lambda(x_0)^{-\frac{\gamma}{2}} \left(\|u\|_{C^0(\overline{B_{\mu d_{x_0}}})} + (\lambda(x_0)^{-1} + \Lambda(x_0)^{\frac{\gamma}{2}} \lambda(x_0)^{-1-\frac{\gamma}{2}}) |F|^{(2)}_{\gamma,0,B_{\mu d_{x_0}}} \right).
\end{aligned}$$

In the case that $\|x_0 - y_0\|_{\mathbb{R}^n} \geq \frac{\mu d_{x_0}}{2}$ with the same multi-index $\boldsymbol{\alpha}$ it holds that

$$d_{x_0,y_0}^{2+\gamma} \frac{|\partial_{\boldsymbol{\alpha}} u(x_0) - \partial_{\boldsymbol{\alpha}} u(y_0)|}{\|x_0 - y_0\|_{\mathbb{R}^n}^{\gamma}} \leq \frac{2}{\mu^{\gamma}} (d_{x_0}^2 |\partial_{\boldsymbol{\alpha}} u(x_0)| + d_{y_0}^2 |\partial_{\boldsymbol{\alpha}} u(y_0)|) \leq \frac{4}{\mu^{\gamma}} [u]_{2,D}^*.$$

The combination of these two estimates yields that for all $x_0, y_0 \in D^\circ$ it holds that

$$\begin{aligned} & d_{x_0,y_0}^{2+\gamma} \frac{|\partial_{\boldsymbol{\alpha}} u(x_0) - \partial_{\boldsymbol{\alpha}} u(y_0)|}{\|x_0 - y_0\|_{\mathbb{R}^n}^{\gamma}} \\ & \quad \leq C \frac{2^{2+\gamma}}{\mu^{2+\gamma}} \frac{\Lambda(x_0)^{2+\frac{\gamma}{2}} n^2}{\lambda(x_0)^{\frac{\gamma}{2}}} \left(\|u\|_{C^0(\overline{D})} + \frac{\lambda(x_0)^{\frac{\gamma}{2}} + \Lambda(x_0)^{\frac{\gamma}{2}}}{\lambda(x_0)^{1+\frac{\gamma}{2}}} |F|_{\gamma,0,B_{\mu d_{x_0}}}^{(2)} \right) + \frac{4}{\mu^{\gamma}} [u]_{2,D}^*. \end{aligned} \tag{8.28}$$

To ease the notation, we set $B = B_{\mu d_{x_0}}$. The next step is to estimate $|F|_{\gamma,0,B}^{(2)}$. For this reason we will establish the below estimate, which will be Inequality (8.29), for functions $g \in C^{0,\gamma}(\overline{D})$ first. For $y \in B$ it holds that $d_y = dist(y, \partial D) > (1-\mu) d_{x_0}$. We use this property to bound $|g|_{\gamma,0,D}^{(2)} = [g]_{0,D}^{(2)} + [g]_{\gamma,0,D}^{(2)}$ from below. We simply apply the definition and obtain that

$$[g]_{0,D}^{(2)} = \sup_{x \in D} d_x^2 |g(x)| \geq (1-\mu)^2 d_{x_0}^2 \sup_{x \in B} |g(x)| = (1-\mu)^2 d_{x_0}^2 \|g\|_{C^0(\overline{B})}$$

and

$$\begin{aligned} [g]_{\gamma,0,D}^{(2)} &= \sup_{x,y \in D, x \neq y} d_{x,y}^{2+\gamma} \frac{|g(x) - g(y)|}{\|x-y\|_{\mathbb{R}^n}^{\gamma}} \\ &\geq (1-\mu)^{2+\gamma} d_{x_0}^{2+\gamma} \sup_{x,y \in B, x \neq y} \frac{|g(x) - g(y)|}{\|x-y\|_{\mathbb{R}^n}^{\gamma}} = (1-\mu)^{2+\gamma} d_{x_0}^{2+\gamma} |g|_{\gamma,0,B}. \end{aligned}$$

In the following chain of inequalities we apply first the fact that for $x \in B$ it holds that $dist(x, \partial B) \leq \mu d_{x_0}$, then we use the two estimates, which we just established, and at last that $\mu \leq \frac{1}{2}$ to obtain that

$$\begin{aligned} |g|_{\gamma,0,B}^{(2)} &\leq \mu^2 d_{x_0}^2 \|g\|_{C^0(\overline{B})} + \mu^{2+\gamma} d_{x_0}^{2+\gamma} |g|_{\gamma,0,B} \\ &\leq \frac{\mu^2}{(1-\mu)^2} [g]_{0,D}^{(2)} + \frac{\mu^{2+\gamma}}{(1-\mu)^{2+\gamma}} [g]_{\gamma,0,D}^{(2)} \\ &\leq 4\mu^2 [g]_{0,D}^{(2)} + 8\mu^{2+\gamma} [g]_{\gamma,0,D}^{(2)}. \end{aligned} \tag{8.29}$$

Now we begin to bound $|F|_{\gamma,0,B}^{(2)}$ and obtain with the triangle inequality that we can analyze the components of F individually:

$$|F|_{\gamma,0,B}^{(2)} \leq \sum_{i,j=1}^{n} |(a_{i,j}(x_0) - a^{i,j}) \partial_i \partial_j u|_{\gamma,0,B}^{(2)} + \sum_{i=1}^{n} |b_i \partial_i u|_{\gamma,0,B}^{(2)} + |cu|_{\gamma,0,B}^{(2)} + |f|_{\gamma,0,B}^{(2)}.$$

We will indeed estimate these components individually and start with those that involve $(a_{i,j} : i,j = 1, ..., n)$. We fix $i, j \in \{1, ..., n\}$ and apply Inequality (8.5) and Inequality (8.29) to obtain that

$$|(a_{i,j}(x_0) - a_{i,j}) \partial_i \partial_j u|_{\gamma,0,B}^{(2)} \leq |a_{i,j}(x_0) - a_{i,j}|_{\gamma,0,B}^{(0)} |\partial_i \partial_j u|_{\gamma,0,B}^{(2)}$$

$$\leq |a_{i,j}(x_0) - a_{i,j}|^{(0)}_{\gamma,0,B}(4\mu^2[\partial_i\partial_j u]^{(2)}_{0,D} + 8\mu^{2+\gamma}[\partial_i\partial_j u]^{(2)}_{\gamma,0,D})$$
$$\leq |a_{i,j}(x_0) - a_{i,j}|^{(0)}_{\gamma,0,B}(4\mu^2[u]^*_{2,D} + 8\mu^{2+\gamma}[u]^*_{\gamma,2,D}).$$

Furthermore we apply that for $x \in B$ it holds that $d_x > (1-\mu)d_{x_0} \geq \frac{d_{x_0}}{2}$, because $\mu \leq \frac{1}{2}$, and that $d_{x,y}(B) = \min\{dist(x, \partial B), dist(y, \partial B)\} < \mu d_{x_0}$ to obtain that

$$\begin{aligned}
|a_{i,j}(x_0) - a_{i,j}|^{(0)}_{\gamma,0,B} &= \sup_{x\in B} |a_{i,j}(x_0) - a_{i,j}(x)| + \sup_{x,y\in B} d_{x,y}(B)^\gamma \frac{|a_{i,j}(x) - a_{i,j}(y)|}{\|x-y\|^\gamma_{\mathbb{R}^n}} \\
&\leq \sup_{x\in B} \frac{|a_{i,j}(x_0) - a_{i,j}(x)|}{\|x_0 - x\|^\gamma_{\mathbb{R}^n}} (\mu d_{x_0})^\gamma + \sup_{x,y\in B} (\mu d_{x_0})^\gamma \frac{|a_{i,j}(x) - a_{i,j}(y)|}{\|x-y\|^\gamma_{\mathbb{R}^n}} \\
&\leq \mu^\gamma 2^\gamma [a_{i,j}]^*_{\gamma,0,D} + \mu^\gamma \sup_{x,y\in B} 2^\gamma d^\gamma_{x,y} \frac{|a_{i,j}(x) - a_{i,j}(y)|}{\|x-y\|^\gamma_{\mathbb{R}^n}} \leq 4\mu^\gamma [a_{i,j}]^*_{\gamma,0,D}.
\end{aligned}$$

We combine the last two computations and conclude using Inequality (8.7) from Lemma 8.7 with $\varepsilon = \mu^\gamma$ that

$$\begin{aligned}
\sum_{i,j=1}^n |(a_{i,j}(x_0) - a_{i,j})\partial_i\partial_j u|^{(2)}_{\gamma,0,B} &\leq 4\mu^\gamma \sum_{i,j=1}^n [a_{i,j}]^*_{\gamma,0,D}(4\mu^2[u]^*_{2,D} + 8\mu^{2+\gamma}[u]^*_{\gamma,2,D}) \\
&\leq 32\mu^{2+\gamma} \sum_{i,j=1}^n [a_{i,j}]^*_{\gamma,0,D} \left(\frac{64^2 2^{\frac{2}{\gamma}}}{\mu^2} \|u\|_{C^0(\overline{D})} + \mu^\gamma [u]^*_{\gamma,2,D} \right).
\end{aligned} \tag{8.30}$$

For the components that involve $(b_i : i = 1, ..., n)$ we fix $i \in \{1, ..., n\}$ and apply Inequality (8.5) and Inequality (8.29) to obtain that

$$|b_i\partial_i u|^{(2)}_{\gamma,0,B} \leq 8\mu^2 |b_i\partial_i u|^{(2)}_{\gamma,0,D} \leq 8\mu^2 |b_i|^{(1)}_{\gamma,0,D} |\partial_i u|^{(1)}_{\gamma,0,D} \leq 8\mu^2 |b_i|^{(1)}_{\gamma,0,D}([u]^*_{1,D} + [u]^*_{\gamma,1,D}).$$

In the next step we apply Inequality (8.6) and Inequality (8.9) from Lemma 8.7 with $\varepsilon = 2^{-1}\mu^{2\gamma}$ in each case and obtain a constant K dependent on γ such that

$$\begin{aligned}
\sum_{i=1}^n |b_i\partial_i u|^{(2)}_{\gamma,0,B} &\leq 8\mu^2 \sum_{i=1}^n |b_i|^{(1)}_{\gamma,0,D} \left[\left(\frac{1024}{\mu^{\frac{2\gamma}{\gamma+1}}} + \frac{K}{\mu^{\frac{8\gamma}{1-\gamma}}} \right) \|u\|_{C^0(\overline{D})} + \mu^{2\gamma}[u]^*_{\gamma,2,D} \right] \\
&\leq 8\mu^2 \sum_{i=1}^n |b_i|^{(1)}_{\gamma,0,D} \left(\frac{1024 + K}{\mu^{\frac{8\gamma}{1-\gamma}}} \|u\|_{C^0(\overline{D})} + \mu^{2\gamma}[u]^*_{\gamma,2,D} \right).
\end{aligned} \tag{8.31}$$

The component that involves the function c is estimated in a similar way. We apply Inequality (8.29) and Inequality (8.5) to obtain that

$$|cu|^{(2)}_{\gamma,0,B} \leq 8\mu^2 |cu|^{(2)}_{\gamma,0,D} \leq 8\mu^2 |c|^{(2)}_{\gamma,0,D}(\|u\|_{C^0(\overline{D})} + [u]^*_{\gamma,0,D}).$$

In the next step we apply Inequality (8.8) from Lemma 8.7 with $\varepsilon = \mu^{2\gamma}$ and obtain a constant K dependent on γ such that

$$|cu|^{(2)}_{\gamma,0,B} \leq 8\mu^2 |c|^{(2)}_{\gamma,0,D} \left(\frac{K}{\mu^{\frac{2\gamma}{1-\gamma}}} \|u\|_{C^0(\overline{D})} + \mu^{2\gamma}[u]^*_{\gamma,2,D} \right). \tag{8.32}$$

Furthermore Inequality (8.29) yields that

$$|f|^{(2)}_{\gamma,0,B} \leq 8\mu^2 |f|^{(2)}_{\gamma,0,D}. \tag{8.33}$$

Finally we apply Inequality (8.7) from Lemma 8.7 with $\varepsilon = \mu^{2\gamma}$ to $[u]^*_{2,D}$ and obtain that

$$[u]^*_{2,D} \leq \frac{64^2 2^{\frac{2}{\gamma}}}{\mu^4} \|u\|_{C^0(\overline{D})} + \mu^{2\gamma} [u]^*_{\gamma,2,D}. \tag{8.34}$$

We combine the contributions to $|F|^{(2)}_{\gamma,0,B}$ stated in Inequalities (8.30), (8.31), (8.32) and (8.33) and obtain that for a constant K, which only dependent on γ, it holds that

$$\begin{aligned}
|F|^{(2)}_{\gamma,0,B} &\leq K\mu^{2+\gamma} \sum_{i,j=1}^{n} [a_{i,j}]^*_{\gamma,0,D} \left(\frac{K}{\mu^2} \|u\|_{C^0(\overline{D})} + \mu^{\gamma} [u]^*_{\gamma,2,D} \right) \\
&\quad + K\mu^2 \sum_{i=1}^{n} |b_i|^{(1)}_{\gamma,0,D} \left(\frac{K}{\mu^{\frac{8\gamma}{1-\gamma}}} \|u\|_{C^0(\overline{D})} + \mu^{2\gamma} [u]^*_{\gamma,2,D} \right) \\
&\quad + K\mu^2 |c|^{(2)}_{\gamma,0,D} \left(\frac{K}{\mu^{\frac{2\gamma}{1-\gamma}}} \|u\|_{C^0(\overline{D})} + \mu^{2\gamma} [u]^*_{\gamma,2,D} \right) + K\mu^2 |f|^{(2)}_{\gamma,0,D} \\
&= \mathcal{K}_1 \mu^{2+2\gamma} [u]^*_{\gamma,2,D} + \mathcal{K}_2 \left(\|u\|_{C^0(\overline{D})} + |f|^{(2)}_{\gamma,0,D} \right),
\end{aligned}$$

where $\mathcal{K}_1$ and $\mathcal{K}_2$ are constants that depend on the coefficients of the operator L and γ. Note that only $\mathcal{K}_2$ depends additionally on μ. We insert the last estimate into Inequality (8.28) and obtain with Inequality (8.34) that

$$\begin{aligned}
d^{2+\gamma}_{x_0,y_0} \frac{|\partial_\alpha u(x_0) - \partial_\alpha u(y_0)|}{\|x_0 - y_0\|^{\gamma}_{\mathbb{R}^n}} &\leq C \frac{2^{2+\gamma}}{\mu^{2+\gamma}} \frac{\Lambda(x_0)^{2+\frac{\gamma}{2}} n^2}{\lambda(x_0)^{\frac{\gamma}{2}}} \left(\|u\|_{C^0(\overline{D})} + \frac{\lambda(x_0)^{\frac{\gamma}{2}} + \Lambda(x_0)^{\frac{\gamma}{2}}}{\lambda(x_0)^{1+\frac{\gamma}{2}}} |F|^{(2)}_{\gamma,0,B)} \right) \\
&\quad + \frac{4}{\mu^{\gamma}} [u]^*_{2,D} \\
&\leq \tilde{\mathcal{K}}_1 \mu^{\gamma} [u]^*_{\gamma,2,D} + \tilde{\mathcal{K}}_2 \left(\|u\|_{C^0(\overline{D})} + |f|^{(2)}_{\gamma,0,D} \right),
\end{aligned}$$

where the constants $\tilde{\mathcal{K}}_1$ and $\tilde{\mathcal{K}}_2$ will be discussed later in the proof. We emphasize that $\tilde{\mathcal{K}}_1$ is independent of μ. We take the supremum over $x_0, y_0 \in D$ and obtain that

$$[u]^*_{\gamma,2,D} \leq \tilde{\mathcal{K}}_1 \mu^{\gamma} [u]^*_{\gamma,2,D} + \tilde{\mathcal{K}}_2 \left(\|u\|_{C^0(\overline{D})} + |f|^{(2)}_{\gamma,0,D} \right).$$

If we choose the parameter μ small enough such that μ satisfies that $\tilde{\mathcal{K}}_1 \mu^{\gamma} \leq \frac{1}{2}$, then we obtain that

$$[u]^*_{\gamma,2,D} \leq 2\, \tilde{\mathcal{K}}_2 \left(\|u\|_{C^0(\overline{D})} + |f|^{(2)}_{\gamma,0,D} \right). \tag{8.35}$$

It remains to investigate how the constant $\tilde{\mathcal{K}}_2$ depends on the coefficients of the operator L and on γ. We observe that

$$\mathcal{K}_1 = K \left(\sum_{i,j=1}^{n} [a_{i,j}]^*_{\gamma,0,D} + \sum_{i=1}^{n} |b_i|^{(1)}_{\gamma,0,D} + |c|^{(2)}_{\gamma,0,D} \right)$$

and that

$$\mathcal{K}_2 = K\left(K\sum_{i,j=1}^{n}[a_{i,j}]^*_{\gamma,0,D} + \frac{K}{\mu^{\frac{8\gamma}{1-\gamma}}}\sum_{i=1}^{n}|b_i|^{(1)}_{\gamma,0,D} + \frac{K}{\mu^{\frac{2\gamma}{1-\gamma}}}|c|^{(2)}_{\gamma,0,D} + 1\right),$$

where we applied that $\mu \leq \frac{1}{2}$. For a new constant K we obtain that

$$\tilde{\mathcal{K}}_1 = K\left(1 + \frac{\|\Lambda\|^{2+\frac{\gamma}{2}}_{C^0(\overline{D})}\|\lambda\|^{\frac{\gamma}{2}}_{C^0(\overline{D})} + \|\Lambda\|^{2+\gamma}_{C^0(\overline{D})}}{\min_{x\in\overline{D}}\lambda(x)^{1+\gamma}}\right)(1+\mathcal{K}_1)$$

and that

$$\tilde{\mathcal{K}}_2 = K\frac{\|\Lambda\|^{2+\frac{\gamma}{2}}_{C^0(\overline{D})}}{\min_{x\in\overline{D}}\lambda(x)^{\frac{\gamma}{2}}}\left(1 + \frac{\|\lambda\|^{\frac{\gamma}{2}}_{C^0(\overline{D})} + \|\Lambda\|^{\frac{\gamma}{2}}_{C^0(D)}}{\min_{x\in\overline{D}}\lambda(x)^{1+\frac{\gamma}{2}}}\mathcal{K}_2\right) + \frac{K}{\mu^{4+\gamma}}.$$

If we set $\mu = (2\tilde{\mathcal{K}}_1)^{-\frac{1}{\gamma}}$ and insert μ into $\mathcal{K}_2$, we obtain that

$$\mathcal{K}_2 \leq K\left(\left(\frac{(1+\|\Lambda\|_{C^0(\overline{D})})^{2+\gamma}}{\min_{x\in D}\lambda(x)^{1+\gamma}}\right)^{\frac{8}{1-\gamma}}(1+a+b+c)^{\frac{9}{1-\gamma}}\right),$$

where we tacitly applied that $1+\frac{8}{1-\gamma} \leq \frac{9}{1-\gamma}$. To obtain the exponents $\frac{8}{1-\gamma}$ and $\frac{9}{1-\gamma}$ in the last inequality, it was important that $\tilde{\mathcal{K}}_1 \geq 1$. We insert the last estimate as the redefined constant $\mathcal{K}_2$ into $\tilde{\mathcal{K}}_2$ and obtain for a new constant K that

$$\tilde{\mathcal{K}}_2 = K\left(\left(\frac{(1+\|\Lambda\|_{C^0(\overline{D})})^{2+\gamma}}{\min_{x\in\overline{D}}\lambda(x)^{1+\gamma}}\right)^{\frac{9}{1-\gamma}}(1+a+b+c)^{\frac{9}{1-\gamma}}\right) = \mathcal{K}.$$

Here we could leave the exponent $\frac{9}{1-\gamma}$, because $4+\gamma < \frac{9}{1-\gamma}$. Again it was important that here $\mathcal{K}_2 \geq 1$.
Note that we were able to bound the appearing semi-norms $|\cdot|^{(\tau)}_{\gamma,0,D}$ for $\tau \geq 0$ of a function $f \in C^{0,\gamma}(\overline{D})$ in the following way:

$$\begin{aligned}|f|^{(\tau)}_{\gamma,0,D} &\leq \sup_{x\in D} r^\tau|f(x)| + \sup_{x,y\in D, x\neq y} r^{\tau+\gamma}\frac{|f(x)-f(y)|}{\|x-y\|^\gamma_{\mathbb{R}^n}}\\ &\leq (r^\tau + r^{\tau+\gamma})\|f\|_{C^{0,\gamma}(\overline{D})}.\end{aligned}$$

The factors consisting of combinations of r were tacitly included in the respective constant K. A similar relation holds for the case $[a_{i,j}]^*_{\gamma,2,D}$ for $i,j \in \{1,...,n\}$. Also for the inner domain D' it holds that

$$d^{2+\gamma}|u|_{\gamma,2,D'} \leq [u]^*_{\gamma,2,D}. \qquad \square$$

The main goal of this section is to achieve higher order regularity of functions u satisfying $Lu = f$ depending on the regularity of the coefficients of the operator L and on the right hand side f. This is achieved in the next theorem. It is Theorem 6.17 in [14]. Again we review its proof to distinguish the dependence of the respective constants on the coefficients of the differential operator L.

Theorem 8.10. *Let $u \in C^2(\overline{D})$ and $f \in C^{k,\gamma}(\overline{D})$ for some $\gamma \in (0,1)$ and $k \in \mathbb{N}_0$ satisfy $Lu = f$. Let $D' \subset\subset D$ be a closed subset such that $dist(D', \partial D) = d$ for some $d > 0$. If the coefficients of L are in $C^{k,\gamma}(\overline{D})$, then $u \in C^{k+2,\gamma}(D')$ and for $k \geq 1$ the following estimate holds:*

$$|u|_{\gamma,k+2,D'} \leq \left(1 + \frac{\mathcal{K}}{d^{2+\gamma}}\left(1 + \frac{\mathcal{K}}{d^{2+\gamma}}\right)\right)^k (1 + a + b + c)^{2k} \left(\|u\|_{C^0(\overline{D})} + \|f\|_{C^{k,\gamma}(\overline{D})}\right),$$

where $\mathcal{K}$ is the constant in the statement of Theorem 8.9 and

$$a = \sum_{i,j=1}^{n} \|a_{i,j}\|_{C^{k,\gamma}(\overline{D})}, \qquad b = \sum_{i=1}^{n} \|b_i\|_{C^{k,\gamma}(\overline{D})} \qquad \text{and} \qquad c = \|c\|_{C^{k,\gamma}(\overline{D})}.$$

Note that for $k = 0$ an estimate for $|u|_{\gamma,2,D'}$ is already given in Theorem 8.9.

Proof of Theorem 8.10. For $k = 0$ Lemma 6.16 in [14] implies that $u \in C^{2,\gamma}(D')$. Therefore, we assume in the following that $k \geq 1$. We will prove the claim with the help of Theorem 8.9. For a function g on $\mathbb{R}^n$, a real number $h > 0$ and a unit coordinate vector e_l for $l \in \{1, ..., n\}$ the difference quotient at $x \in \mathbb{R}^n$ in the x_l-direction is defined as

$$\Delta_l^h g(x) = \frac{g(x + he_l) - g(x)}{h}.$$

The idea behind the following argument is to apply the Schauder interior estimate in Theorem 8.9 to the problem $L(\Delta_l^h u) = F^h$ and eventually obtain the desired regularity of u. We expand $L(\Delta_l^h u)$ and add terms that sum to zero to meet $Lu = f$ repeatedly to obtain that

$$\begin{aligned} L(\Delta_l^h u) = \frac{1}{h}(L\overline{u} - Lu) &= \frac{1}{h}(a^{i,j}\partial_i\partial_j\overline{u} + b^i\partial_i\overline{u} + c\overline{u} - f) \\ &= \frac{1}{h}(a^{i,j}\partial_i\partial_j\overline{u} + b^i\partial_i\overline{u} + c\overline{u} - \overline{f}) + \Delta_l^h f \\ &= -(\Delta_l^h a^{i,j})\partial_i\partial_j\overline{u} - (\Delta_l^h b^i)\partial_i\overline{u} - (\Delta_l^h c)\overline{u} + \Delta_l^h f = F^h, \end{aligned}$$

where we introduced for a function g the notation $\overline{g}(x) = g(x+he_l)$. We fix two subdomains B', B of D that satisfy $D' \subset\subset B' \subset\subset B \subset\subset D$ and $dist(D', \partial B') = dist(B', \partial B) = dist(B, \partial D) = \frac{d}{3}$. Lemma 6.16 in [14] implies that $u \in C^{2,\gamma}(\overline{B})$. Since we assume that $f \in C^{1,\gamma}(\overline{D})$ we can manipulate $\Delta_l^h f$ according to the fundamental theorem of calculus to obtain that for $x \in B'$ and $h < \frac{d}{3}$ it holds that

$$\Delta_l^h f(x) = \frac{1}{h}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(x + the_l)\mathrm{d}t = \frac{1}{h}\int_0^1 Df\Big|_{x+the_l} he_l \mathrm{d}t = \int_0^1 \partial_l f(x + the_l)\mathrm{d}t.$$

This equality also implies that for every $0 < h < \frac{d}{3}$ it holds that $\Delta_l^h f \in C^{0,\gamma}(\overline{B'})$ and in particular it provides bounds of the $C^{0,\gamma}(\overline{B'})$-norm of $\Delta_l^h f$ independently of h, i.e. $\|\Delta_l^h f\|_{C^0(\overline{B'})} \leq \|\partial_l f\|_{C^0(\overline{B})}$ and $|\Delta_l^h f|_{\gamma,0,B'} \leq |\partial_l f|_{\gamma,0,B}$. Note that the same estimates also apply to the coefficients of the operator L, that are by assumption in $C^{1,\gamma}(\overline{D})$ as well and they apply to the function u and its first derivative, since $u \in C^{2,\gamma}(\overline{B})$ due to Lemma 6.16 in [14] as mentioned earlier. To be able to apply the Schauder interior estimate in Theorem 8.9 we have to discuss the Hölder-norm of the function F^h. Inequality (4.7) in [14] with $\alpha, \beta = \gamma$ says that for functions $g_1, g_2 \in C^{0,\gamma}(\overline{D})$ it holds that

$$\|g_1 g_2\|_{C^{0,\gamma}(\overline{D})} \leq \|g_1\|_{C^{0,\gamma}(\overline{D})}\|g_2\|_{C^{0,\gamma}(\overline{D})}. \tag{8.36}$$

This enables us to obtain the following bound independently of h

$$\|F^h\|_{C^{0,\gamma}(\overline{B'})} \leq \|\partial_l f\|_{C^{0,\gamma}(\overline{B})} + \sum_{i,j=1}^{n} \|\partial_l a_{i,j}\|_{C^{0,\gamma}(\overline{B})} \|\partial_i \partial_j u\|_{C^{0,\gamma}(\overline{B})}$$
$$+ \sum_{i=1}^{n} \|\partial_l b_i\|_{C^{0,\gamma}(\overline{B})} \|\partial_i u\|_{C^{0,\gamma}(\overline{B})} + \|\partial_l c\|_{C^{0,\gamma}(\overline{B})} \|u\|_{C^{0,\gamma}(\overline{B})}, \tag{8.37}$$

which implies that $F^h \in C^{0,\gamma}(\overline{B'})$. Now we can apply the interior estimate in Theorem 8.9 to the problem $L(\Delta_l^h u) = F^h$ with the nested domains $D' \subset\subset B'$ and obtain for $i, j \in \{1, ..., n\}$ that

$$d^{2+\gamma} |\partial_i \partial_j \Delta_l^h u|_{\gamma,0,D'} \leq \mathcal{K}(\|\Delta_l^h u\|_{C^0(\overline{B'})} + \|F^h\|_{C^{0,\gamma}(\overline{B'})}), \tag{8.38}$$

where we tacitly included the factor $3^{2+\gamma}$ into the constant $\mathcal{K}$. Note that $\mathcal{K}$ is the constant, that is stated in Theorem 8.9, which depends on the coefficients of the operator L. As pointed out above the right hand side is bounded independently of h. Since the left hand side is a Hölder semi-norm it follows that the set of functions $(\partial_i \partial_j \Delta_l^h u : h < \frac{d}{3})$ is equicontinuous in D' for $i, j \in \{1, ..., n\}$. The Arzelà–Ascoli theorem, which is Theorem 4.44 in [12], implies that a subsequence of $(\partial_i \partial_j \Delta_l^h u : h < \frac{d}{3})$ converges to a continuous function. Since this sequence is a differential quotient, this is sufficient to conclude that $\partial_i \partial_j \Delta_l^h u$ converges uniformly on compact subsets of D' to $\partial_i \partial_j \partial_l u$ as $h \to 0$ and $\partial_i \partial_j \partial_l u$ is continuous. The fact that the convergence is only on compact subsets of D' can be safely disregarded in the following, since D' is compact itself. Since the Hölder semi-norm of $\partial_i \partial_j \Delta_l^h u$ is bounded independently of h for all $i, j \in \{1, ..., n\}$ and $l \in \{1, ..., n\}$ was chosen arbitrarily, it follows that $u \in C^{3,\gamma}(D')$. To finish the proof of the case $k = 1$ we have to compute the bound of the Hölder semi-norm of third order partial derivatives of u. We combine Inequality (8.38) and Equation (6.3) to obtain that

$$d^{2+\gamma} |\partial_i \partial_j \partial_l u|_{\gamma,0,D'} = \lim_{h \to 0} d^{2+\gamma} |\partial_i \partial_j \Delta_l^h u|_{\gamma,0,D'} \leq \lim_{h \to 0} \mathcal{K}(\|\Delta_l^h u\|_{C^0(\overline{B'})} + \|F^h\|_{C^{0,\gamma}(\overline{B'})})$$
$$\leq \lim_{h \to 0} \mathcal{K}\tilde{K}(\|u\|_{C^0(\overline{B'})} + |u|_{\gamma,2,B'} + \|F^h\|_{C^{0,\gamma}(\overline{B'})}),$$

where the constant $\tilde{K}$ is due to the equivalence of Hölder norms given in Equation (6.3). We continue the estimation and apply Inequality (8.37) and bound the appearing norms of first and second order partial derivatives of u with Equation (6.3) and the Schauder interior estimate in Theorem 8.9 applied to $Lu = f$ with the nested sets $B \subset\subset D$. We obtain that

$$|u|_{\gamma,3,D'} \leq \frac{\mathcal{K}}{d^{2+\gamma}} \left(1 + \frac{\mathcal{K}}{d^{2+\gamma}}\right) \left(1 + \sum_{i,j=1}^{n} \|a_{i,j}\|_{C^{1,\gamma}(\overline{D})} + \sum_{i=1}^{n} \|b_i\|_{C^{1,\gamma}(\overline{D})} + \|c\|_{C^{1,\gamma}(\overline{D})}\right)$$
$$\cdot \left(\|u\|_{C^0(\overline{D})} + \|f\|_{C^{1,\gamma}(\overline{D})}\right), \tag{8.39}$$

where we tacitly included the constant $\tilde{K}$ that depends on the domain D into $\mathcal{K}$.

For the general case $k > 1$ we proceed by induction with respect to the order of differentiation. We assume that there are subdomains $D' = D_k \subset\subset D_{k-1} \subset\subset ... \subset\subset D_1 \subset\subset D_0 = D$ such that $dist(D_l, \partial D_{l-1}) = \frac{d}{k}$ for all $l \in \{1, ..., k\}$. We apply the proof of the case $k = 1$ with the nested sets $D_1 \subset\subset D$ and obtain that

$$|u|_{\gamma,3,D_1} \leq \frac{\mathcal{K}}{d^{2+\gamma}} \left(1 + \frac{\mathcal{K}}{d^{2+\gamma}}\right) (1 + a + b + c) \left(\|u\|_{C^0(\overline{D})} + \|f\|_{C^{1,\gamma}(\overline{D})}\right),$$

where we tacitly included the factor $k^{2+\gamma}$ into $\mathcal{K}$ and we used the following notation:

$$a = \sum_{i,j=1}^{n} \|a_{i,j}\|_{C^{k,\gamma}(\overline{D})}, \qquad b = \sum_{i=1}^{n} \|b_i\|_{C^{k,\gamma}(\overline{D})} \qquad \text{and} \qquad c = \|c\|_{C^{k,\gamma}(\overline{D})}.$$

We want to prove that for all $l \in \{1, ..., k\}$ it holds that $u \in C^{l+2,\gamma}(\overline{D_l})$ and it holds that

$$|u|_{\gamma,l+2,D_l} \leq \left(1 + \frac{\mathcal{K}}{d^{2+\gamma}}\left(1 + \frac{\mathcal{K}}{d^{2+\gamma}}\right)\right)^l (1 + a + b + c)^{2l} \left(\|u\|_{C^0(\overline{D})} + \|f\|_{C^{l,\gamma}(\overline{D})}\right). \quad (8.40)$$

This is already established for $l = 1$. Now, we assume this claim for $l \in \{1, ..., k-1\}$ and want to prove that this implies the claim for $l + 1$, i.e. that $u \in C^{l+3,\gamma}(\overline{D_{l+1}})$ and that Inequality (8.40) holds for $l + 1$.
This means that $u \in C^{l+2,\gamma}(\overline{D_l})$ and f and the coefficients of the operator L are in $C^{k,\gamma}(\overline{D})$. For a multi-index $\boldsymbol{\beta} \in \mathbb{N}_0^n$ such that $|\boldsymbol{\beta}| = l$, we differentiate the equation $Lu = f$ for l times and obtain that for $\tilde{u} = \partial_{\boldsymbol{\beta}} u$ it holds that

$$\begin{aligned} L\tilde{u} = L(\partial_{\boldsymbol{\beta}} u) = \partial_{\boldsymbol{\beta}} f &- \sum_{0 \neq \boldsymbol{\alpha} \leq \boldsymbol{\beta}} \binom{\boldsymbol{\beta}}{\boldsymbol{\alpha}} \partial_{\boldsymbol{\alpha}} a^{i,j} \partial_{\boldsymbol{\beta}-\boldsymbol{\alpha}} \partial_i \partial_j u \\ &- \sum_{0 \neq \boldsymbol{\alpha} \leq \boldsymbol{\beta}} \binom{\boldsymbol{\beta}}{\boldsymbol{\alpha}} \partial_{\boldsymbol{\alpha}} b^i \partial_{\boldsymbol{\beta}-\boldsymbol{\alpha}} \partial_i u \\ &- \sum_{0 \neq \boldsymbol{\alpha} \leq \boldsymbol{\beta}} \binom{\boldsymbol{\beta}}{\boldsymbol{\alpha}} \partial_{\boldsymbol{\alpha}} c \partial_{\boldsymbol{\beta}-\boldsymbol{\alpha}} u = \tilde{f}_{\boldsymbol{\beta}}. \end{aligned}$$

The idea of the induction argument is to apply the proof of the case $k = 1$. From the above equation we see that the right hand side $\tilde{f}_{\boldsymbol{\beta}} \in C^{1,\gamma}(\overline{D_l})$ and then we can apply the proof of the case $k = 1$ with the nested sets $D_{l+1} \subset\subset D_l$ to obtain that $\partial_{\boldsymbol{\beta}} u \in C^{3,\gamma}(\overline{D_{l+1}})$ and that the following estimate holds:

$$|\partial_{\boldsymbol{\beta}} u|_{\gamma,3,D_{l+1}} \leq \tilde{\mathcal{K}} \left(\|\partial_{\boldsymbol{\beta}} u\|_{C^0(\overline{D_l})} + \|\tilde{f}_{\boldsymbol{\beta}}\|_{C^{1,\gamma}(\overline{D_l})}\right),$$

where $\tilde{\mathcal{K}}$ is the constant from the proof of the case $k = 1$, i.e.

$$\tilde{\mathcal{K}} = \frac{\mathcal{K}}{d^{2+\gamma}}\left(1 + \frac{\mathcal{K}}{d^{2+\gamma}}\right)(1 + a + b + c).$$

Since $\boldsymbol{\beta}$ is an arbitrary multi-index referring to a partial derivative of order l, it follows that $u \in C^{l+3,\gamma}(\overline{D_{l+1}})$. We now seek to develop an estimate of the $C^{l+3,\gamma}(\overline{D_{l+1}})$ semi-norm of u in terms of $\|u\|_{C^0(D)}$ and norms of the coefficients of the operator L and the function f. We apply Equation (6.2) and tacitly include the appearing constant into $\tilde{\mathcal{K}}$ and obtain that

$$|u|_{\gamma,l+3,D_{l+1}} \leq \tilde{\mathcal{K}} \left(\|u\|_{C^0(\overline{D_l})} + |u|_{l,D_l} + \max_{|\boldsymbol{\beta}|=l} \|\tilde{f}_{\boldsymbol{\beta}}\|_{C^{1,\gamma}(\overline{D_l})}\right). \quad (8.41)$$

To estimate $\max_{|\boldsymbol{\beta}|=l} \|\tilde{f}_{\boldsymbol{\beta}}\|_{C^{1,\gamma}(\overline{D_l})}$, we bound $\|\tilde{f}_{\tilde{\boldsymbol{\beta}}}\|_{C^{1,\gamma}(\overline{D_l})}$ for an arbitrary multi-index $\tilde{\boldsymbol{\beta}}$ that satisfies that $|\tilde{\boldsymbol{\beta}}| = l$ and try to find a bound, which does not depend on $\tilde{\boldsymbol{\beta}}$. Therefore we analyze the norm of $\tilde{f}_{\tilde{\boldsymbol{\beta}}}$. We apply that the product of two Hölder functions can be

bounded with the product of the individual Hölder norms, that is Inequality (6.4), to obtain the following estimate for the Hölder norm of $\tilde{f}_{\tilde{\boldsymbol{\beta}}}$:

$$\begin{aligned}\|\tilde{f}_{\tilde{\boldsymbol{\beta}}}\|_{C^{1,\gamma}(\overline{D_l})} \leq \|f\|_{C^{l+1,\gamma}(\overline{D_l})} + K \sum_{0\neq\boldsymbol{\alpha}\leq\tilde{\boldsymbol{\beta}}} \binom{\tilde{\boldsymbol{\beta}}}{\boldsymbol{\alpha}} \Bigg(&\sum_{i,j=1}^{n} \|a_{i,j}\|_{C^{|\boldsymbol{\alpha}|+1,\gamma}(\overline{D_l})} \|u\|_{C^{|\tilde{\boldsymbol{\beta}}-\boldsymbol{\alpha}|+3,\gamma}(\overline{D_l})} \\ &+ \sum_{i=1}^{n} \|b_i\|_{C^{|\boldsymbol{\alpha}|+1,\gamma}(\overline{D_l})} \|u\|_{C^{|\tilde{\boldsymbol{\beta}}-\boldsymbol{\alpha}|+2,\gamma}(\overline{D_l})} \\ &+ \|c\|_{C^{|\boldsymbol{\alpha}|+1,\gamma}(\overline{D_l})} \|u\|_{C^{|\tilde{\boldsymbol{\beta}}-\boldsymbol{\alpha}|+1,\gamma}(\overline{D_l})} \Bigg),\end{aligned}$$

where the constant K is due to the estimate in Inequality (6.4). In the above sum for the multi-indices $\boldsymbol{\alpha}$ and $\tilde{\boldsymbol{\beta}}$ it holds that $0 < |\boldsymbol{\alpha}| \leq l$ and $|\tilde{\boldsymbol{\beta}} - \boldsymbol{\alpha}| \leq l-1$, since $\boldsymbol{\alpha} \neq 0$. Therefore, we can bound the components of the above sum with Equation (6.3) in this way:

$$\begin{aligned}\|a_{i,j}\|_{C^{|\boldsymbol{\alpha}|+1,\gamma}(\overline{D_l})} \|u\|_{C^{|\tilde{\boldsymbol{\beta}}-\boldsymbol{\alpha}|+3,\gamma}(\overline{D_l})} &\leq Ca(\|u\|_{C^0(\overline{D_l})} + |u|_{\gamma,l+2,D_l}), \\ \|b_i\|_{C^{|\boldsymbol{\alpha}|+1,\gamma}(\overline{D_l})} \|u\|_{C^{|\tilde{\boldsymbol{\beta}}-\boldsymbol{\alpha}|+2,\gamma}(\overline{D_l})} &\leq Cb(\|u\|_{C^0(\overline{D_l})} + |u|_{\gamma,l+2,D_l}), \\ \|c\|_{C^{|\boldsymbol{\alpha}|+1,\gamma}(\overline{D_l})} \|u\|_{C^{|\tilde{\boldsymbol{\beta}}-\boldsymbol{\alpha}|+1,\gamma}(\overline{D_l})} &\leq Cc(\|u\|_{C^0(\overline{D_l})} + |u|_{\gamma,l+2,D_l}).\end{aligned}$$

The constant C is due to the equivalence of Hölder norms, which is Equation (6.3). We insert these three estimates in the previous one and obtain that

$$\|\tilde{f}_{\tilde{\boldsymbol{\beta}}}\|_{C^{1,\gamma}(\overline{D_l})} \leq \|f\|_{C^{l+1,\gamma}(\overline{D_l})} + KN(a+b+c)(\|u\|_{C^0(\overline{D_l})} + |u|_{\gamma,l+2,D_l}),$$

where $N = \sum_{0\neq\boldsymbol{\alpha}\leq\tilde{\boldsymbol{\beta}}} \binom{\tilde{\boldsymbol{\beta}}}{\boldsymbol{\alpha}}$ and the constant C is tacitly included into K. Since there are only finitely many multi-indices $\boldsymbol{\beta} \in \mathbb{N}_0^n$ that satisfy that $|\boldsymbol{\beta}| = l$, N can be bounded independently of $\tilde{\boldsymbol{\beta}}$. We simply redefine $N = \max_{|\boldsymbol{\beta}|=l} \sum_{0\neq\boldsymbol{\alpha}\leq\boldsymbol{\beta}} \binom{\boldsymbol{\beta}}{\boldsymbol{\alpha}}$. Hence, the previous estimate is also valid for $\max_{|\boldsymbol{\beta}|=l} \|\tilde{f}_{\boldsymbol{\beta}}\|_{C^{1,\gamma}(\overline{D_l})}$. We insert this estimate into Inequality (8.41) and obtain with another application of the equivalent Hölder norms in Equation (6.3) that

$$\begin{aligned}|u|_{\gamma,l+3,D_{l+1}} &\leq \tilde{\mathcal{K}} \left(\|u\|_{C^0(\overline{D_l})} + |u|_{l,D_l} + (a+b+c)(\|u\|_{C^0(\overline{D_l})} + |u|_{l+2,D_l}) + \|f\|_{C^{l+1,\gamma}(\overline{D_l})} \right) \\ &\leq \tilde{\mathcal{K}}(1+a+b+c) \left(\|u\|_{C^0(\overline{D_l})} + |u|_{\gamma,l+2,D_l} + \|f\|_{C^{l+1,\gamma}(\overline{D_l})} \right),\end{aligned}$$

where we tacitly included the constants K and N into $\tilde{\mathcal{K}}$ in the above computation assuming $KN \geq 1$. Now we apply the induction hypothesis and insert the estimate in Inequality (8.40) for l into the above inequality and obtain with a few manipulations that

$$\begin{aligned}|u|_{\gamma,l+3,D_{l+1}} &\leq \tilde{\mathcal{K}}(1+a+b+c) \left(\|u\|_{C^0(\overline{D_l})} + \tilde{\mathcal{K}}_l(\|u\|_{C^0(\overline{D})} + \|f\|_{C^{l,\gamma}(\overline{D})}) + \|f\|_{C^{l+1,\gamma}(\overline{D_l})} \right) \\ &\leq \tilde{\mathcal{K}}(1+a+b+c)\tilde{\mathcal{K}}_l(\|u\|_{C^0(\overline{D})} + \|f\|_{C^{l+1,\gamma}(\overline{D})}),\end{aligned}$$

where we introduced the notation $\tilde{\mathcal{K}}_l = (1 + \mathcal{K}d^{-2-\gamma}(1+\mathcal{K}d^{-2-\gamma}))^l(1+a+b+c)^{2l}$. We observe that

$$\tilde{\mathcal{K}}(1+a+b+c)\tilde{\mathcal{K}}_l \leq \left(1 + \frac{\mathcal{K}}{d^{2+\gamma}}\left(1 + \frac{\mathcal{K}}{d^{2+\gamma}}\right)\right)^{l+1} (1+a+b+c)^{2l+2},$$

which finishes the induction argument and the proof of the theorem, because we can take $l = k-1$ and conclude the desired result for $l = k$. □

8.1.2. The Schauder interior estimates for weak solutions

Another interesting case is when the operator L is given in divergence form, i.e.

$$L = \partial_i(a^{i,j}\partial_j + b^i) + c^i\partial_i + e.$$

We are interested in the problem

$$Lu = \partial_i(a^{i,j}\partial_j u + b^i u) + c^i\partial_i u + eu = g + \partial_i f^i. \tag{8.42}$$

We will then consider the variational formulation, i.e. u satisfies that

$$\int_D (a^{i,j}\partial_j + b^i)u\partial_i v - c^i\partial_i uv - euv \,\mathrm{d}x = \int_D -gv + f^i\partial_i v \,\mathrm{d}x, \tag{8.43}$$

for all $v \in C_0^1(D)$ and $g \in L^p(D)$ and $(f_i : i = 1, ..., n) \subset C^{0,\gamma}(\overline{D})$ for some $p \in [1, \infty)$ and $\gamma \in (0, 1)$. In this case we will say that $u \in W^{1,2}(D)$ satisfies Equation (8.42) in the sense of distributions. This setup is also discussed in Gilbarg and Trudinger [14] in Chapter 8. For the subsequent discussion we have to introduce another semi-norm. For $g \in L^p(D)$ and $\tau \in \mathbb{R}$ we define

$$|g|^{(\tau)}_{L^p(D)} = \|gd^\tau_{(\cdot)}\|_{L^p(D)}$$

for $p \in [1, \infty)$. The first step is to find an analogue of Lemma 8.8.

Lemma 8.11. *Let $A \in \mathbb{R}^{n\times n}$ be a constant symmetric matrix such that for two constants $\lambda, \Lambda > 0$ and all $\xi \in \mathbb{R}^n$ it holds that*

$$\lambda\|\xi\|^2_{\mathbb{R}^n} \le A^{i,j}\xi_i\xi_j \le \Lambda\|\xi\|^2_{\mathbb{R}^n}.$$

If $u \in C^{1,\gamma}(\overline{D})$, $g \in L^p(D)$ and $(f_i : i = 1, ..., n) \subset C^{0,\gamma}(\overline{D})$ for $\gamma = 1 - \frac{n}{p}$ satisfy $A^{i,j}\partial_i\partial_j u = g + \partial_i f^i$ in the sense of distributions, then for a constant C it holds that

$$[u]^*_{\gamma,1,D} \le C\frac{\Lambda^{1+\frac{\gamma}{2}}}{\lambda^{\frac{\gamma}{2}}}\left(\|u\|_{C^0(\overline{D})} + \frac{|g|^{(1+\gamma)}_{L^p(D)}}{\lambda^{\frac{1+\gamma}{2}}} + \frac{\lambda^{\frac{\gamma}{2}} + \Lambda^{\frac{\gamma}{2}}}{\lambda^{1+\frac{\gamma}{2}}}\sum_{i=1}^n |f_i|^{(1)}_{\gamma,0,D}\right).$$

Proof. The proof of this lemma is very similar to the proof of Lemma 8.8. We apply the same change of coordinates, i.e. $Q \in \mathbb{R}^{n\times n}$, as in the proof of Lemma 8.8 to be able to transfer results about solutions to the Poisson equation to our elliptic equation here. Note that $\partial_i f^i$ transforms under Q as:

$$\int_D f^i(x)\partial_i v(x) \,\mathrm{d}x = \int_{\tilde{D}} \hat{f}^i(y)\partial_j\tilde{v}(y)Q^{j,i}|\det(Q^{-1})| \,\mathrm{d}y,$$

where $\hat{f}_i = f_i \circ Q^{-1}$ and $\tilde{v} = v \circ Q^{-1}$. We observe that $\tilde{u}$ satisfies $\Delta\tilde{u} = \tilde{g} + \partial_i\tilde{f}^i$ in the sense of distributions, where $\tilde{u} = u \circ Q^{-1}$, $\tilde{g} = g \circ Q^{-1}$ and $\tilde{f}_i = (f_j \circ Q^{-1})Q^{j,i}$.
In Gilbarg and Trudinger [14], the same proof for Theorem 4.8 in [14] can be applied to Estimate (4.45) in [14]. Together with the remark about Estimate (4.45) in [14] at the very end of Chapter 4 in [14], we obtain that for a constant C it holds that

$$[\tilde{u}]^*_{\gamma,1,\tilde{D}} \le C\left(\|\tilde{u}\|_{C^0(\overline{\tilde{D}})} + |\tilde{g}|^{(1+\gamma)}_{L^p(\tilde{D})} + \sum_{i=1}^n |\tilde{f}_i|^{(1)}_{\gamma,0,\tilde{D}}\right).$$

We apply a similar argument as in the proof of Lemma 8.8 with Inequalities (8.23), (8.24) and (8.25) and obtain that

$$
\begin{aligned}
[u]^*_{\gamma,1,D} \leq \frac{\Lambda^{1+\frac{\gamma}{2}}}{\lambda^{\frac{\gamma}{2}}} n[\tilde{u}]^*_{\gamma,1,\tilde{D}} &\leq \frac{\Lambda^{1+\frac{\gamma}{2}}}{\lambda^{\frac{\gamma}{2}}} C \left(\|\tilde{u}\|_{C^0(\overline{\tilde{D}})} + |\tilde{g}|^{(1+\gamma)}_{L^p(\tilde{D})} + \sum_{i=1}^n |\tilde{f}_i|^{(1)}_{\gamma,0,\tilde{D}} \right) \\
&\leq \frac{\Lambda^{1+\frac{\gamma}{2}}}{\lambda^{\frac{\gamma}{2}}} C \left(\|u\|_{C^0(\overline{D})} + \frac{|g|^{(1+\gamma)}_{L^p(D)}}{\lambda^{\frac{1+\gamma}{2}}} + \frac{\lambda^{\frac{\gamma}{2}} + \Lambda^{\frac{\gamma}{2}}}{\lambda^{1+\frac{\gamma}{2}}} \sum_{i=1}^n |f_i|^{(1)}_{\gamma,0,D} \right),
\end{aligned}
$$

where we tacitly used that Q can be seen as a bounded bilinear form on $\mathbb{R}^n \times \mathbb{R}^n$ with norm smaller or equal to $\lambda^{-\frac{1}{2}}$ due to Inequality (8.20), which appears as an additional factor in the coefficient of $\sum_{i=1}^n |f_i|^{(1)}_{\gamma,0,D}$. □

The following result corresponds to Theorem 8.32 in [14] and states an estimate for the Hölder semi-norm of the solution. We provide the proof in order to track the dependence of the constants on the coefficients of L in the resulting bound.

Theorem 8.12. *Let $u \in C^{1,\gamma}(\overline{D})$, $g \in L^p(D)$ and $(f_i : i = 1, ..., n) \subset C^{0,\gamma}(\overline{D})$ for some $\gamma \in (0,1)$ such that $p = \frac{n}{1-\gamma}$ satisfy the variational problem in Equation (8.43). Let $D' \subset\subset D$ be a closed subset such that dist$(D', \partial D) = d$ for some $d > 0$.*
If the coefficients of L are Hölder continuous, i.e. $(a_{i,j} : i, j = 1, ..., n), (b_i : i = 1, ..., n) \subset C^{0,\gamma}(D)$ and $(c_i : i = 1, ..., n), \{e\} \subset L^p(D)$, then the following estimate holds:

$$
d^{1+\gamma} |u|_{\gamma,1,D'} \leq \mathcal{K} \left(\|u\|_{C^0(\overline{D})} + \|g\|_{L^p(D)} + \sum_{i=1}^n \|f_i\|_{C^{0,\gamma}(\overline{D})} \right).
$$

The constant $\mathcal{K}$ depends implicitly on r and γ through a constant K, i.e.

$$
\mathcal{K} = K \left(\frac{(1 + \|\Lambda\|_{C^0(\overline{D})})^{2+\gamma}}{\min_{x \in \overline{D}} \lambda(x)^{1+\gamma}} \right)^{\frac{2}{1-\gamma}} (1 + a + b + c + e)^{\frac{2}{1-\gamma}},
$$

where

$$
a = \sum_{i,j=1}^n \|a_{i,j}\|_{C^{0,\gamma}(\overline{D})}, \quad b = \sum_{i=1}^n \|b_i\|_{C^{0,\gamma}(\overline{D})}, \quad c = \sum_{i=1}^n \|c_i\|_{L^p(D)} \quad \text{and} \quad e = \|e\|_{L^p(D)}.
$$

Proof. The proof is similar to the proof of Theorem 8.9. We will therefore often refer to the development of the proof of this theorem, when arguments can be applied to our setup here. However we will not repeat the whole argument, but rather state the respective inequality in the respective step.

We fix again two interior points $x_0, y_0 \in D$ and rewrite the equation $Lu = g + \partial_i f^i$, i.e.

$$
a^{i,j}(x_0)\partial_i\partial_j u = \partial_i[(a^{i,j}(x_0) - a^{i,j})\partial_j u - b^i u] - c^i \partial_i u - eu + g + \partial_i f^i, = G + \partial_i F^i
$$

where

$$
G = -c^i \partial_i u - eu + g \qquad \text{and} \qquad F^i = (a^{i,j}(x_0) - a^{i,j})\partial_j u - b^i u + f^i
$$

for $i = 1, ..., n$. We interpret this equation in $B = B_{\mu d_{x_0}}(x_0) \subset D$ for some $\mu \leq \frac{1}{2}$ and apply the estimate in Lemma 8.11. With a similar argument as in the proof of Theorem 8.9 we obtain that for all $x_0, y_0 \in D$ and $i \in \{1, ..., n\}$ it holds that

$$d_{x_0,y_0}^{1+\gamma} \frac{|\partial_i u(x_0) - \partial_i u(y_0)|}{\|x_0 - y_0\|_{\mathbb{R}^n}^{\gamma}}$$
$$\leq C \frac{4}{\mu^{1+\gamma}} \frac{\Lambda(x_0)^{1+\frac{\gamma}{2}}}{\lambda(x_0)^{\frac{\gamma}{2}}} \left(\|u\|_{C^0(\overline{D})} + \frac{|G|_{L^p(B)}^{(1+\gamma)}}{\lambda(x_0)^{\frac{1+\gamma}{2}}} + \frac{\lambda(x_0)^{\frac{\gamma}{2}} + \Lambda(x_0)^{\frac{\gamma}{2}}}{\lambda(x_0)^{1+\frac{\gamma}{2}}} \sum_{i=1}^n |F_i|_{\gamma,0,B}^{(1)} \right) + \frac{4}{\mu^{\gamma}} [u]_{1,D}^*.$$

The next step is to estimate $|G|_{L^p(B)}^{(1+\gamma)}$ and $\sum_{i=1}^n |F_i|_{\gamma,0,B}^{(1)}$. In the proof of Theorem 8.9 we relied on Inequality (8.29) for functions $h \in C^{0,\gamma}(\overline{D})$. With the same argument we can show that for $h \in C^{0,\gamma}(\overline{D})$ it holds that

$$|h|_{\gamma,0,B}^{(1)} \leq 2\mu [h]_{0,D}^{(1)} + 4\mu^{1+\gamma} [h]_{\gamma,0,D}^{(1)}. \tag{8.44}$$

First we consider the contributions in $\sum_{i=1}^n |F_i|_{\gamma,0,B}^{(1)}$, i.e.

$$\sum_{i=1}^n |F_i|_{\gamma,0,B}^{(1)} \leq \sum_{i,j=1}^n |(a_{i,j}(x_0) - a_{i,j})\partial_j u|_{\gamma,0,B}^{(1)} + \sum_{i=1}^n |b_i u|_{\gamma,0,B}^{(1)} + \sum_{i=1}^n |f_i|_{\gamma,0,B}^{(1)}.$$

With the same argument that we applied to obtain Inequality (8.30) that relied on Inequality (8.29) and Inequality (8.7) from Lemma 8.7, we obtain in our case with Inequality (8.44) and Inequality (8.10) from Lemma 8.7 that for a constant K, which is independent of μ, it holds that

$$\sum_{i,j=1}^n |(a_{i,j}(x_0) - a_{i,j})\partial_j u|_{\gamma,0,B}^{(1)} \leq K\mu^{1+\gamma} \sum_{i,j=1}^n [a_{i,j}]_{\gamma,0,D}^* \left(\frac{K}{\mu} \|u\|_{C^0(\overline{D})} + \mu^{\gamma} [u]_{\gamma,1,D}^* \right).$$

Similarly to the development of Inequality (8.32), we obtain with Inequality (8.44) and Inequality (8.11) from Lemma 8.7 that for a constant independent of μ, say also K, it holds that

$$\sum_{i=1}^n |b_i u|_{\gamma,0,B}^{(1)} \leq K\mu \sum_{i=1}^n |b_i|_{\gamma,0,D}^{(1)} \left((K\ \mu^{-\frac{2}{1-\gamma}} + 1) \|u\|_{C^0(\overline{D})} + \mu^{2\gamma} [u]_{\gamma,1,D}^* \right).$$

Inequality (8.44) also implies that

$$\sum_{i=1}^n |f_i|_{\gamma,0,B}^{(1)} \leq 4\mu \sum_{i=1}^n |f_i|_{\gamma,0,D}^{(1)}.$$

Now we estimate the contributions in $|G|_{L^p(B)}^{(1+\gamma)}$, i.e.

$$|G|_{L^p(B)}^{(1+\gamma)} \leq |c^i \partial_i u|_{L^p(B)}^{(1+\gamma)} + |eu|_{L^p(B)}^{(1+\gamma)} + |g|_{L^p(B)}^{(1+\gamma)}.$$

With a similar argument as we used to proof Inequality (8.29) in the proof of Theorem 8.9 we obtain that

$$|h|_{L^p(B)}^{(1+\gamma)} \leq 4\mu^{1+\gamma} |h|_{L^p(D)}^{(1+\gamma)}. \tag{8.45}$$

With Inequality (8.45) and Inequality (8.10) from Lemma 8.7 we obtain that

$$\begin{aligned}|c^i\partial_i u|^{(1+\gamma)}_{L^p(B)} \leq 4\mu^{1+\gamma}|c^i\partial_i u|^{(1+\gamma)}_{L^p(D)} &\leq 4\mu^{1+\gamma}\sum_{i=1}^n |c_i|^{(\gamma)}_{L^p(D)}[u]^*_{1,D} \\ &\leq 4\mu^{1+\gamma}\sum_{i=1}^n |c_i|^{(\gamma)}_{L^p(D)}\left(\frac{2\ 4^{\frac{1}{\gamma}}}{\mu}\|u\|_{C^0(\overline{D})} + \mu^\gamma[u]^*_{\gamma,1,D}\right).\end{aligned}$$

For the other two terms we obtain with Inequality (8.45) that

$$|eu|^{(1+\gamma)}_{L^p(B)} \leq 4\mu^{1+\gamma}|e|^{(1+\gamma)}_{L^p(B)}\|u\|_{C^0(\overline{D})} \qquad \text{and} \qquad |g|^{(1+\gamma)}_{L^p(B)} \leq 4\mu^{1+\gamma}|g|^{(1+\gamma)}_{L^p(D)}.$$

Similar as in the proof of Theorem 8.9 we obtain with the derived estimates for $|G|^{(1+\gamma)}_{L^p(B)}$ and $\sum_{i=1}^n |F_i|^{(1)}_{\gamma,0,B}$ that

$$[u]^*_{\gamma,1,D} \leq \mathcal{K}_1\mu^\gamma[u]^*_{\gamma,1,D} + \mathcal{K}_2\left(\|u\|_{C^0(\overline{D})} + \|g\|_{L^p(D)} + \sum_{i=1}^n \|f_i\|_{C^{0,\gamma}(\overline{D})}\right).$$

When we achieve that $\mathcal{K}_1\mu^\gamma \leq \frac{1}{2}$ we obtain that

$$[u]^*_{\gamma,1,D} \leq 2\ \mathcal{K}_2\left(\|u\|_{C^0(\overline{D})} + \|g\|_{L^p(D)} + \sum_{i=1}^n \|f_i\|_{C^{0,\gamma}(\overline{D})}\right).$$

The constants $\mathcal{K}_1$ and $\mathcal{K}_2$ are given by

$$\mathcal{K}_1 = K\left(\frac{(1+\|\Lambda\|_{C^0(\overline{D})})^{2+\gamma}}{\min_{x\in\overline{D}}\lambda(x)^{1+\gamma}}\right)(1+a+b+c)$$

and

$$\mathcal{K}_2 = K\left(\frac{(1+\|\Lambda\|_{C^0(\overline{D})})^{2+\gamma}}{\min_{x\in\overline{D}}\lambda(x)^{1+\gamma}}\right)^{\frac{2}{1-\gamma}}(1+a+b+c+e)^{\frac{2}{1-\gamma}}. \qquad \square$$

8.2. Regularity of solutions of elliptic partial differential equations on the sphere

In this section we return to the elliptic problem in Equation (8.1) and the respective weak form in Equation (8.2). The respective elliptic operator was induced by a coefficient $A \in C^0(S^2)$ such that $\min_{x\in S^2} A(x) > 0$. We recall the problem in Equation (8.2). In the beginning of Chapter 8, we already proved that there exists a unique $u \in H^1(S^2)/\mathbb{R}$ such that

$$b(u,v) = \int_{S^2} A\nabla_{S^2}u\cdot\nabla_{S^2}v\mathrm{d}\sigma = \int_{S^2} fv\mathrm{d}\sigma = \ell_f(v)$$

for all $v \in H^1(S^2)/\mathbb{R}$, where $f \in L^2(S^2,\mathbb{R})$ satisfies that $\int_{S^2} f\mathrm{d}\sigma = 0$. In this section we impose higher regularity on the coefficient A and on the right hand side f and aim to prove which regularity the solution u will have. Also we are interested in estimates of the solution in terms of the right hand side and the coefficient A. In particular we emphasize the dependence on the coefficient A, which is not always explicitly analyzed in the literature. This is, where the analysis of the Schauder interior estimates from Section 8.1 will be applied.

8.2.1. L^p estimates of solutions of elliptic partial differential equations on the sphere

We start with the situation, that the coefficient A is Hölder continuous. The solution to the problem in Equation (8.2) is in $H^1(S^2)/\mathbb{R}$ and therefore an equivalence class of functions in $H^1(S^2)$, we will consider the representative u of the solution that satisfies that $\int_{S^2} u \mathrm{d}\sigma = 0$. Charrier, Scheichl and Teckentrup have analyzed such situation in bounded domains in Euclidean space in [5]. In particular Proposition 3.1 in [5] yields the membership of the weak solution u in a Sobolev space with more than one weak derivative, which leads to stronger integrability of the first weak derivative in terms of its L^p-norm. Their argument is essentially the development of the proof of Theorem 9.1.16 by Hackbusch in [15].

Proposition 8.13. *Let $u \in H^1(S^2)$ be the representative of the solution in $H^1(S^2)/\mathbb{R}$ to the problem in Equation (8.2) that satisfies that $\int_{S^2} u \ \mathrm{d}\sigma = 0$.*
If $A \in C^{0,t}(S^2)$ for some $t \in (0,1)$, then for all $\gamma \in (0,t)$ it holds that $u \in W^{1,p}(S^2)$, where $p = \frac{2}{1-\gamma}$. Furthermore, there exists a constant K independently of A, f and the solution u such that u satisfies the estimate:

$$\|u\|_{W^{1,p}(S^2)} \leq K \frac{(1 + \|A\|_{C^{0,t}(S^2)})^2}{(\min_{x \in S^2} A(x))^2} \|f\|_{L^2(S^2,\mathbb{R})}.$$

Let $A_1, A_2 \in C^{0,t}(S^2) \cap \{\tilde{A} : \min_{x \in S^2} \tilde{A}(x) > 0\}$ be two coefficients that induce bilinear forms. For the respective solutions u_1 with respect to A_1 and u_2 with respect to A_2 with the same right hand side $f \in L^2(S^2, \mathbb{R}) \cap \{\tilde{f} : \int_{S^2} \tilde{f} \mathrm{d}\sigma = 0\}$ that satisfy for $k = 1, 2$ that $\int_{S^2} u_k \ \mathrm{d}\sigma = 0$ it holds that

$$\|u_1 - u_2\|_{W^{1,p}(S^2)} \leq K \frac{\|A_1\|_{C^{0,t}(S^2)}}{(\min\limits_{x \in S^2} A_1(x))^2} \frac{(1 + \|A_2\|_{C^{0,t}(S^2)})^2}{(\min\limits_{x \in S^2} A_2(x))^2} \|f\|_{L^2(S^2,\mathbb{R})} \|A_1 - A_2\|_{C^{0,t}(S^2)},$$

where K is a constant that is independent of A_1, A_2, f and the solutions u_1, u_2. Moreover the solution u depends continuously on the coefficient A, i.e. the mapping $A \mapsto u$ is continuous from $C^{0,t}(S^2) \cap \{\tilde{A} : \min_{x \in S^2} \tilde{A}(x) > 0\}$ to $W^{1,p}(S^2)$.

Proof. We will consider the variational problem in Equation (8.2) with respect to this A. Since the mentioned result of Charrier, Scheichl and Teckentrup in [5] or respectively of Hackbusch in [15] is for bounded domains in Euclidean space, we will pull the bilinear form b in Equation (8.2) back to the chart domains. We will in general follow the proof of Lemma A.4 in [5], which is part of the proof of the mentioned result Proposition 3.1 in [5]. We remind of our usual atlas $(U_i, \eta_i : i = 1, ..., 6)$ and partition of unity Ψ. Due to the definition of the Sobolev norms on S^2 we are interested in the behavior of $(u\Psi_i)_{\eta_i}$ for all $i \in \{1, ..., 6\}$. Since $u\Psi_i$ is compactly supported, the statement of Lemma A.4 in [5] is sufficient for our purposes. We fix $i \in \{1, ..., 6\}$ and obtain with the Leibniz rule that $u\Psi_i$ satisfies that

$$-\nabla_{S^2} \cdot (A \nabla_{S^2} u \Psi_i) = f\Psi_i - A\nabla_{S^2} u \cdot \nabla_{S^2} \Psi_i - \nabla_{S^2} \cdot (Au \nabla_{S^2} \Psi_i) = F_i.$$

Since Ψ_i is compactly supported in U_i, we obtain the variational formulation for $u\Psi_i$, i.e.

$$\int_{U_i} A \nabla_{S^2}(u\Psi_i) \cdot \nabla_{S^2} v \ d\sigma = \int_{S^2} (f\Psi_i - A\nabla_{S^2} u \cdot \nabla_{S^2} \Psi_i) v + Au \nabla_{S^2} \Psi_i \cdot \nabla_{S^2} v \mathrm{d}\sigma$$

for all $v \in H^1(S^2)/\mathbb{R}$. We pull the problem back to the chart domain and obtain that

$$\int_{\eta_i^{-1}(U_i)} A_{\eta_i} \left(\partial_\theta (u\Psi_i)_{\eta_i} \partial_\theta v + \frac{1}{\sin^2(\theta)} \partial_\varphi (u\Psi_i)_{\eta_i} \partial_\varphi v \right) \sin(\theta) \mathrm{d}x = \ell_{(F_i)_{\eta_i} \sin(\theta)}(v), \tag{8.46}$$

for all $v \in W_0^{1,2}(\eta_i^{-1}(U_i))$, where we tacitly used Proposition 6.5. The functional $\ell_{(F_i)_{\eta_i} \sin(\theta)}$ is given by

$$\begin{aligned} \ell_{(F_i)_{\eta_i} \sin(\theta)}(v) = \int_{\eta_i^{-1}(U_i)} & (f\Psi_i)_{\eta_i} \sin(\theta) v - A_{\eta_i} \sin(\theta) \partial_\theta u_{\eta_i} \partial_\theta (\Psi_i)_{\eta_i} v - \frac{A_{\eta_i}}{\sin(\theta)} \partial_\varphi u_{\eta_i} \partial_\varphi (\Psi_i)_{\eta_i} v \\ & + (Au)_{\eta_i} \sin(\theta) \partial_\theta (\Psi_i)_{\eta_i} \partial_\theta v + \frac{(Au)_{\eta_i}}{\sin(\theta)} \partial_\varphi (\Psi_i)_{\eta_i} \partial_\varphi v \mathrm{d}x. \end{aligned}$$

for all $v \in W_0^{1,2}(\eta_i^{-1}(U_i))$. It holds formally that

$$\begin{aligned} (F_i)_{\eta_i} \sin(\theta) = (f\Psi_i)_{\eta_i} \sin(\theta) - A_{\eta_i} \sin(\theta) \partial_\theta u_{\eta_i} \partial_\theta (\Psi_i)_{\eta_i} - \frac{A_{\eta_i}}{\sin(\theta)} \partial_\varphi u_{\eta_i} \partial_\varphi (\Psi_i)_{\eta_i} \\ - \partial_\theta \Big((Au)_{\eta_i} \sin(\theta) \partial_\theta (\Psi_i)_{\eta_i} \Big) - \partial_\varphi \Big(\frac{(Au)_{\eta_i}}{\sin(\theta)} \partial_\varphi (\Psi_i)_{\eta_i} \Big). \end{aligned}$$

We will abuse notation for the rest of the proof of this proposition and write the dual pairing of $\ell_{(F_i)_{\eta_i} \sin(\theta)}$ and $v \in W^{-1,2}(\eta_i^{-1}(U_i))$ as

$$\ell_{(F_i)_{\eta_i} \sin(\theta)}(v) = \int_{\eta_i^{-1}(U_i)} (F_i)_{\eta_i} \sin(\theta) v \mathrm{d}x.$$

This notation is more convenient, because we want to exploit that the components of $(F_i)_{\eta_i} \sin(\theta)$ are compactly supported and change the domain of integration, which is $\eta_i^{-1}(U_i)$ at the moment. The subtlety here is that we cannot give a meaning to $(F_i)_{\eta_i}$. It would be a product of a function and a distribution, i.e. since A_{η_i} is only Hölder continuous

$$\partial_\theta \Big((Au)_{\eta_i} \sin(\theta) \partial_\theta (\Psi_i)_{\eta_i} \Big)$$

has to be understood as a distribution and the product of the latter with $\frac{1}{\sin(\theta)}$ has no clear meaning. In the following we will disregard the notation $\ell_{(F_i)_{\eta_i} \sin(\theta)}$ and use $(F_i)_{\eta_i} \sin(\theta)$ instead, also when we mean the functional. Clearly $(F_i)_{\eta_i} \sin(\theta) \in W^{-1,2}(\eta_i^{-1}(U_i))$. Moreover, we claim that $(F_i)_{\eta_i} \sin(\theta) \in W^{\gamma-1,2}(\eta_i^{-1}(U_i))$. This will be verified at a later point in the proof.

Let D be a subdomain of $\eta_i^{-1}(U_i)$ with smooth boundary such that $supp((\Psi_i)_{\eta_i}) \subset\subset D \subset\subset \eta_i^{-1}(U_i)$. Since $\eta_i(\overline{D})$ is relatively closed in S^2, Lemma 6.2 implies that there exists a partition of unity $\hat{\Psi}$ subordinate to the open cover $(U_j : j = 1, ..., 6)$ such that on $\overline{D}$ it holds that $(\hat{\Psi}_i)_{\eta_i} = 1$. We wish to extend A_{η_i} to all of $\mathbb{R}^2$, i.e. we choose $\chi \in C^\infty(\mathbb{R}^2, [0,1])$ such that $\chi = 0$ on $supp((\Psi_i)_{\eta_i})$ and $\chi = 1$ on the complement of D and define

$$\overline{A}_{\eta_i} = \begin{cases} (A_{\eta_i}(x)(1-\chi(x)) + \min_{y \in \overline{D}} A_{\eta_i}(y)\chi(x) & \text{if } x \in D \\ \min_{y \in \overline{D}} A_{\eta_i}(y)\chi & \text{else} \end{cases}.$$

The pulled back problem in Equation (8.46) implies that

$$\int_{\mathbb{R}^2} \overline{A}_{\eta_i} \left(\partial_\theta (u\Psi_i)_{\eta_i} \partial_\theta v + \frac{1}{\sin^2(\theta)} \partial_\varphi (u\Psi_i)_{\eta_i} \partial_\varphi v \right) \sin(\theta) \mathrm{d}x = \int_{\mathbb{R}^2} (F_i)_{\eta_i} \sin(\theta) v \mathrm{d}x, \tag{8.47}$$

for all $v \in C_0^\infty(\mathbb{R}^2)$, where we tacitly used the fact that $(F_i)_{\eta_i}\sin(\theta)$ and $(u\Psi_i)_{\eta_i}$ are compactly supported in $\eta_i^{-1}(U_i)$ and can therefore be extended with zero in the rest of $\mathbb{R}^2$. Now we are able to apply Lemma 3.2 in [5] with the matrix $diag(\sin(\theta)A_{\eta_i}, \frac{1}{\sin(\theta)}A_{\eta_i})$ and obtain that $(u\Psi_i)_{\eta_i} \in W^{1+\gamma,2}(\mathbb{R}^2)$. Furthermore for a constant K, which is independent of $(u\Psi_i)_{\eta_i}$, $\overline{A}_{\eta_i}$ and $(F_i)_{\eta_i}\sin(\theta)$, it holds that

$$\begin{aligned}\|(u\Psi_i)_{\eta_i}\|_{W^{1+\gamma,2}(\mathbb{R}^2)} \leq K\frac{1}{\min_{x\in\mathbb{R}^2}\overline{A}_{\eta_i}(x)}&\left(\|\overline{A}_{\eta_i}\|_{C^{0,t}(\mathbb{R}^2)}\|(u\Psi_i)_{\eta_i}\|_{W^{1,2}(\mathbb{R}^2)}\right.\\&\left.+\|(F_i)_{\eta_i}\sin(\theta)\|_{W^{\gamma-1,2}(\mathbb{R}^2)}\right)+\|(u\Psi_i)_{\eta_i}\|_{W^{1,2}(\mathbb{R}^2)}.\end{aligned}$$

Since $\overline{A}_{\eta_i}$ is constant outside of D and χ is smooth, there exists a constant independently of A_{η_i} such that

$$\begin{aligned}\|\overline{A}_{\eta_i}\|_{C^{0,t}(\mathbb{R}^2)} \leq K\|A_{\eta_i}\|_{C^{0,t}(\overline{D})} &= K\|(A\hat{\Psi}_i)_{\eta_i}\|_{C^{0,t}(\overline{D})}\\&\leq K\|(A\hat{\Psi}_i)_{\eta_i}\|_{C^{0,t}(\overline{\eta_i^{-1}(U_i)})} \leq K\|A\|_{C^{0,t}(S^2)},\end{aligned}$$

where we applied that $(\hat{\Psi}_i)_{\eta_i} = 1$ on $\overline{D}$. Since $(u\Psi_i)_{\eta_i}$ and $(F_i)_{\eta_i}\sin(\theta)$ are compactly supported in D, we obtain with the last two estimates that

$$\begin{aligned}\|(u\Psi_i)_{\eta_i}\|_{W^{1+\gamma,2}(D)} \leq K\frac{1}{\min_{x\in\overline{D}}A_{\eta_i}(x)}&\Big(\|A_{\eta_i}\|_{C^{0,t}(\overline{D})}\|(u\Psi_i)_{\eta_i}\|_{W^{1,2}(D)}\\&+\|(F_i)_{\eta_i}\sin(\theta)\|_{W^{\gamma-1,2}(D)}\Big)+\|(u\Psi_i)_{\eta_i}\|_{W^{1,2}(D)}.\end{aligned} \tag{8.48}$$

Note that $\partial_\theta, \partial_\varphi$ are linear and continuous mappings from $W^{\gamma,2}(D)$ to $W^{\gamma-1,2}(D)$. We apply our knowledge about $\partial_\theta, \partial_\varphi$ as operators and then we use $\hat{\Psi}_i$ to meet the definition of the Sobolev and Hölder norms on S^2, i.e. we obtain that

$$\begin{aligned}\|(F_i)_{\eta_i}\sin(\theta)\|_{W^{\gamma-1,2}(D)} &\leq K\Big(\|(f\Psi_i)_{\eta_i}\|_{W^{\gamma-1,2}(D)}+\|(A\hat{\Psi}_i)_{\eta_i}\|_{C^0(\overline{D})}\|(u\hat{\Psi}_i)_{\eta_i}\|_{W^{\gamma,2}(D)}\\&\qquad+\|(A\hat{\Psi}_i)_{\eta_i}\|_{C^{0,t}(\overline{D})}\|(u\hat{\Psi}_i)_{\eta_i}\|_{W^{1,2}(D)}\Big)\\&\leq K\Big(\|(f\Psi_i)_{\eta_i}\|_{W^{\gamma-1,2}(\eta_i^{-1}(U_i))}\\&\qquad+2\|(A\hat{\Psi}_i)_{\eta_i}\|_{C^{0,t}(\overline{\eta_i^{-1}(U_i)})}\|(u\hat{\Psi}_i)_{\eta_i}\|_{W^{1,2}(\eta_i^{-1}(U_i))}\Big)\\&\leq K\left(\|f\|_{L^2(S^2,\mathbb{R})}+2\|A\|_{C^{0,t}(S^2)}\|u\|_{W^{1,2}(S^2)}\right),\end{aligned} \tag{8.49}$$

where we included the contributions of Ψ_i into the constant K. Also we tacitly applied the continuous embedding $L^2(D,\mathbb{R}) \subset W^{\gamma-1,2}(D)$, which is due to Theorem 4.6.1.(c) in [25]. Since $(F_i)_{\eta_i}\sin(\theta)$ is compactly supported in D, we showed that $(F_i)_{\eta_i}\sin(\theta) \in W^{\gamma-1,2}(\eta_i^{-1}(U_i))$. With Proposition 6.5, Proposition 8.3 and Inequality (8.3) we obtain that for a constant K it holds that

$$\|u\|_{W^{1,2}(S^2)} \leq K\|u\|_{H^1(S^2)/\mathbb{R}} \leq \frac{K}{\sqrt{2}}\frac{\|f\|_{L^2(S^2,\mathbb{R})}}{\min_{x\in S^2}A(x)}.$$

Theorem 4.6.1.(c) in [25] implies that $W^{1+\gamma,2}(D) \subset W^{1,p}(D)$ for $p = \frac{2}{1-\gamma}$. Therefore, for a constant K it holds that

$$\|(u\Psi_i)_{\eta_i}\|_{W^{1,p}(\eta_i^{-1}(U_i))} = \|(u\Psi_i)_{\eta_i}\|_{W^{1,p}(D)} \leq K\|(u\Psi_i)_{\eta_i}\|_{W^{1+\gamma,2}(D)},$$

where we also applied that $(u\Psi_i)_{\eta_i}$ is compactly supported in D. We combine the last four inequalities and consider the definition of the Hölder norms on S^2 to obtain that for a new constant K' it holds that

$$\|(u\Psi_i)_{\eta_i}\|_{W^{1,p}(\eta_i^{-1}(U_i))} \leq K\|(u\Psi_i)_{\eta_i}\|_{W^{1+\gamma,2}(\eta_i^{-1}(U_i))} \leq K'\frac{(1+\|A\|_{C^{0,t}(S^2)})^2}{(\min_{x\in S^2} A(x))^2}\|f\|_{L^2(S^2,\mathbb{R})}. \tag{8.50}$$

This argument applies to all other $j \in \{1,...,6\}\backslash\{i\}$. Hence, the estimate in Inequality (8.50) holds for all $i \in \{1,...,6\}$. Since $(u\Psi_i)_{\eta_i}$ is compactly supported in $\eta_i^{-1}(U_i)$ for all $i \in \{1,...,6\}$, it holds that $(u\Psi_i)_{\eta_i} \in W_0^{1,p}(\eta_i^{-1}(U_i))$. Therefore, we conclude that $u \in W^{1,p}(S^2)$ and the first claim of the proposition is proven.

For the second claim, we look at two coefficients $A_1, A_2 \in C^{0,t}(S^2)\cap\{\tilde{A} : \min_{x\in S^2}\tilde{A}(x) > 0\}$ with respective bilinear forms b_1 and b_2. Let $u_1, u_2 \in H^1(S^2)$ be the respective solutions to the problem in Equation (8.2) with respect to b_1 and b_2 and the same right hand side f that satisfy that $\int_{S^2} u_k \mathrm{d}\sigma = 0$ for $k = 1, 2$. We are interested in the problem that $u_1 - u_2$ satisfies. We obtain that weakly

$$\begin{aligned}
-\nabla_{S^2}\cdot(A_1\nabla_{S^2}(u_1-u_2)) &= f + \nabla_{S^2}\cdot(A_1\nabla_{S^2}u_2)\\
&= f - f - \nabla_{S^2}\cdot(A_2\nabla_{S^2}u_2) + \nabla_{S^2}\cdot(A_1\nabla_{S^2}u_2)\\
&= \nabla_{S^2}\cdot((A_1-A_2)\nabla_{S^2}u_2) = F.
\end{aligned}$$

We fix $i \in \{1,...,6\}$ and observe that $(u_1-u_2)\Psi_i$ satisfies that

$$\begin{aligned}
-\nabla_{S^2}\cdot(A_1\nabla_{S^2}((u_1-u_2)\Psi_i)) &= F\Psi_i - A_1\nabla_{S^2}(u_1-u_2)\cdot\nabla_{S^2}\Psi_i\\
&\quad - \nabla_{S^2}\cdot(A_1(u_1-u_2)\nabla_{S^2}\Psi_i)\\
&= (\nabla_{S^2}\cdot((A_1-A_2)\nabla_{S^2}u_2))\Psi_i\\
&\quad - A_1\nabla_{S^2}(u_1-u_2)\cdot\nabla_{S^2}\Psi_i\\
&\quad - \nabla_{S^2}\cdot(A_1(u_1-u_2)\nabla_{S^2}\Psi_i) = F_i^{1,2}.
\end{aligned}$$

Since $(F\Psi_i)_{\eta_i}\sin(\theta) \in W^{\gamma-1,2}(D)$, we obtain that $(F_i^{1,2})_{\eta_i}\sin(\theta) \in W^{\gamma-1,2}(D)$. Therefore the argument in the proof of the first claim of this proposition is applicable. Especially with Inequality (8.48) we conclude that $(u_1\Psi_i)_{\eta_i}$, $(u_2\Psi_i)_{\eta_i}$, $((u_1-u_2)\Psi_i)_{\eta_i} \in W^{1+\gamma,2}(D)$. Also Inequality (8.48) implies an estimate for the $W^{1+\gamma,2}(D)$-norm of $((u_1-u_2)\Psi_i)_{\eta_i}$, i.e.

$$\begin{aligned}
\|((u_1-u_2)\Psi_i)_{\eta_i}\|_{W^{1+\gamma,2}} \leq K\frac{1}{\min_{x\in\overline{D}}(A_1)_{\eta_i}(x)}&\Big(\|(A_1)_{\eta_i}\|_{C^{0,t}}\|((u_1-u_2)\Psi_i)_{\eta_i}\|_{W^{1,2}}\\
&+\|(F_i^{1,2})_{\eta_i}\sin(\theta)\|_{W^{\gamma-1,2}}\Big) + \|((u_1-u_2)\Psi_i)_{\eta_i}\|_{W^{1,2}},
\end{aligned} \tag{8.51}$$

where we excluded the domain D, respectively $\overline{D}$, in the above inequality for notational convenience. We remind that D was chosen in the proof of the first claim and satisfies that $supp(\Psi_i)_{\eta_i} \subset\subset D \subset\subset \eta_i^{-1}(U_i)$. As in the proof of the first claim we have to estimate the norm of the right hand side $(F_i^{1,2})_{\eta_i}\sin(\theta)$. With Inequality (8.49) we obtain that

$$\begin{aligned}
\|(F_i^{1,2})_{\eta_i}\sin(\theta)\|_{W^{\gamma-1,2}(D)} \leq K\Big(&\|(F\Psi_i)_{\eta_i}\sin(\theta)\|_{W^{\gamma-1,2}(\eta_i^{-1}(U_i))}\\
&+2\|(A_1\hat{\Psi}_i)_{\eta_i}\|_{C^{0,t}(\overline{\eta_i^{-1}(U_i)})}\|((u_1-u_2)\hat{\Psi}_i)_{\eta_i}\|_{W^{1,2}(\eta_i^{-1}(U_i))}\Big),
\end{aligned}$$

where $\hat{\Psi}$ is the partition of unity subordinate to the open cover $(U_j : j = 1, ..., 6)$, that was chosen in the proof of the first claim. We remind that on D it holds that $(\hat{\Psi}_i)_{\eta_i} = 1$. Since $\partial_\theta, \partial_\varphi$ are continuous linear operators from $W^{\gamma-1,2}(D)$ to $W^{\gamma,2}(D)$ and due to the fact that $(F\Psi_i)_{\eta_i} \sin(\theta)$ is compactly supported in D, we can estimate the $W^{\gamma-1,2}(\eta_i^{-1}(U_i))$-norm of $(F\Psi_i)_{\eta_i} \sin(\theta)$, i.e.

$$\begin{aligned}
\|(F\Psi_i)_{\eta_i} \sin(\theta)\|_{W^{\gamma-1,2}(\eta_i^{-1}(U_i))} &= \|(F\Psi_i)_{\eta_i} \sin(\theta)\|_{W^{\gamma-1,2}(D)} \\
&= \|((\nabla_{S^2} \cdot ((A_1 - A_2)\nabla_{S^2} u_2))\Psi_i)_{\eta_i} \sin(\theta)\|_{W^{\gamma-1,2}(D)} \\
&\leq K(\|(A_1 - A_2)_{\eta_i} \partial_\theta (u_2)_{\eta_i}\|_{W^{\gamma,2}(D)} + \|(A_1 - A_2)_{\eta_i} \partial_\varphi (u_2)_{\eta_i}\|_{W^{\gamma,2}(D)}) \\
&\leq K' \|(A_1 - A_2)_{\eta_i}\|_{C^{0,t}(\overline{D})} (\|\partial_\theta (u_2)_{\eta_i}\|_{W^{\gamma,2}(D)} + \|\partial_\varphi (u_2)_{\eta_i}\|_{W^{\gamma,2}(D)}) \\
&\leq K' \|((A_1 - A_2)\hat{\Psi}_i)_{\eta_i}\|_{C^{0,t}(\overline{\eta_i^{-1}(U_i)})} \|(u_2 \hat{\Psi}_i)_{\eta_i}\|_{W^{1+\gamma,2}(\eta_i^{-1}(U_i))} \\
&\leq K'' \|A_1 - A_2\|_{C^{0,t}(S^2)} \frac{(1 + \|A_2\|_{C^{0,t}(S^2)})^2}{(\min_{x \in S^2} A_2(x))^2} \|f\|_{L^2(S^2,\mathbb{R})},
\end{aligned}$$

where we tacitly included the contributions of Ψ_i into the constant K and applied Inequality (8.50) in the last step. Note that the estimate in Inequality (8.50) was applicable, since its development also applies with $\hat{\Psi}_i$ instead of Ψ_i. We insert the last two estimates and the estimate in Proposition 8.6, which is Inequality (8.4), into Inequality (8.51) to obtain that

$$\begin{aligned}
&\|((u_1 - u_2)\Psi_i)_{\eta_i}\|_{W^{1+\gamma,2}(\eta_i^{-1}(U_i))} \\
&\leq K' \left(\frac{\|A_1\|_{C^{0,t}(S^2)}}{\min\limits_{x \in S^2} A_1(x)} \|u_1 - u_2\|_{W^{1,2}(S^2)} + \frac{\|(F\Psi_i)_{\eta_i} \sin(\theta)\|_{W^{\gamma-1,2}(\eta_i^{-1}(U_i))}}{\min\limits_{x \in S^2} A_1(x)} \right) \\
&\leq K \left(\frac{\|A_1\|_{C^{0,t}(S^2)}}{\min\limits_{x \in S^2} A_1(x)} \|u_1 - u_2\|_{H^1(S^2)/\mathbb{R}} + \frac{(1 + \|A_2\|_{C^{0,t}(S^2)})^2}{(\min\limits_{x \in S^2} A_2(x))^2} \frac{\|f\|_{L^2(S^2,\mathbb{R})}}{\min\limits_{x \in S^2} A_1(x)} \|A_1 - A_2\|_{C^{0,t}(S^2)} \right) \\
&\leq K'' \frac{\|A_1\|_{C^{0,t}(S^2)}}{(\min\limits_{x \in S^2} A_1(x))^2} \frac{(1 + \|A_2\|_{C^{0,t}(S^2)})^2}{(\min\limits_{x \in S^2} A_2(x))^2} \|f\|_{L^2(S^2,\mathbb{R})} \|A_1 - A_2\|_{C^{0,t}(S^2)}.
\end{aligned}$$

We remind that Theorem 4.6.1.(c) in [25] implies that $W^{1+\gamma,2}(\eta_i^{-1}(U_i)) \subset W^{1,p}(\eta_i^{-1}(U_i))$ for $p = \frac{2}{1-\gamma}$. This implies that there exists a constant K independently of u_1, u_2 and Ψ_i such that

$$\|((u_1 - u_2)\Psi_i)_{\eta_i}\|_{W^{1,p}(\eta_i^{-1}(U_i))} \leq K \|((u_1 - u_2)\Psi_i)_{\eta_i}\|_{W^{1+\gamma,2}(\eta_i^{-1}(U_i))}.$$

We combine the last two inequality and since the whole argument applies for all $i \in \{1, ..., 6\}$ we obtain the second claim of the proposition, i.e. there exists a constant K independently of A_1, A_2, f, u_1 and u_2 such that

$$\|u_1 - u_2\|_{W^{1,p}(S^2)} \leq K \frac{\|A_1\|_{C^{0,t}(S^2)}}{(\min\limits_{x \in S^2} A_1(x))^2} \frac{(1 + \|A_2\|_{C^{0,t}(S^2)})^2}{(\min\limits_{x \in S^2} A_2(x))^2} \|f\|_{L^2(S^2,\mathbb{R})} \|A_1 - A_2\|_{C^{0,t}(S^2)}. \tag{8.52}$$

For the proof of the third claim let $(A_n : n \in \mathbb{N}), \{A\} \subset C^{0,t}(S^2) \cap \{\tilde{A} : \min_{x \in S^2} \tilde{A}(x) > 0\}$ be such that $A_n \to A$ in the $C^{0,t}(S^2)$-norm as $n \to \infty$, i.e. $\|A_n - A\|_{C^{0,t}(S^2)} = \varepsilon_n \to 0$ as

$n \to \infty$. For all $n \in \mathbb{N}$ let $u, u_n \in W^{1,p}(S^2)$ be the solutions with respect to the bilinear forms induced by A and A_n to the problem in Equation (8.2) that satisfy that $\int_{S^2} u \, \mathrm{d}\sigma = 0$ and $\int_{S^2} u_n \, \mathrm{d}\sigma = 0$. We already established in the proof of Proposition 8.6 that there exists $N \in \mathbb{N}$ such that for integers $n \geq N$ it holds that

$$\frac{1}{\min\limits_{x \in S^2} A_n(x)} \leq \frac{2}{\min\limits_{x \in S^2} A(x)}.$$

There exists an integer, say also N, such that $\|A_n - A\|_{C^{0,t}(S^2)} \leq \|A\|_{C^{0,t}(S^2)}$ for every integer $n \geq N$. With the triangle inequality we obtain that for every $n \geq N$ it holds that

$$\|A_n\|_{C^{0,t}(S^2)} \leq \|A_n - A\|_{C^{0,t}(S^2)} + \|A\|_{C^{0,t}(S^2)} \leq 2\|A\|_{C^{0,t}(S^2)}.$$

Therefore, $(\min_{x \in S^2} A_n(x))^{-2}$ and $\|A_n\|_{C^{0,t}(S^2)}$ can be bounded independently of $n \in \mathbb{N}$ and we conclude with Inequality (8.52) that $u_n \to u$ in the $W^{1,p}(S^2)$-norm as $n \to \infty$. Therefore, the mapping $A \mapsto u$ from $C^{0,t}(S^2) \cap \{\tilde{A} : \min_{x \in S^2} \tilde{A}(x) > 0\}$ to $W^{1,p}(S^2)$ is continuous. □

Corollary 8.14. *Let t, γ, p, u, A, u_1, u_2, A_1, A_2 and f be as in Proposition 8.13. There exist continuous representatives $\tilde{u}, \tilde{u}_1, \tilde{u}_2 \in C^{0,\gamma}(S^2)$ of u and of u_1, u_2 respectively such that the two estimates in Proposition 8.13 hold for the $C^{0,\gamma}(S^2)$-norm of $\tilde{u}$ and $\tilde{u}_1 - \tilde{u}_2$ respectively with an unchanged right hand side of the estimates.*
Moreover, $\tilde{u}$ depends continuously on the coefficient A, i.e. the mapping $A \mapsto \tilde{u}$ is continuous from $C^{0,t}(S^2) \cap \{\tilde{A} : \min_{x \in S^2} \tilde{A}(x) > 0\}$ to $C^{0,\gamma}(S^2)$.

Proof. Due to Proposition 8.13, it holds that u, u_1, $u_2 \in W^{1,p}(S^2)$. The Sobolev embedding theorem on S^2, which is Theorem 6.10, implies that $W^{1,p}(S^2) \subset C^{0,\gamma}(S^2)$ with continuous embedding, i.e. there exists a constant K such that for all $v \in W^{1,p}(S^2)$ it holds that $\|\tilde{v}\|_{C^{0,\gamma}(S^2)} \leq K\|v\|_{W^{1,p}(S^2)}$, where the continuous representative of v is denoted by $\tilde{v}$. This implies the existence of $\tilde{u}, \tilde{u}_1, \tilde{u}_2 \in C^{0,\gamma}(S^2)$ and the two estimates.
The proof of the third claim is almost identical to the proof of third claim of Proposition 8.13. Let $(A_n : n \in \mathbb{N}) \subset C^{0,t}(S^2) \cap \{\tilde{A} : \min_{x \in S^2} \tilde{A}(x) > 0\}$ be such that $A_n \to A$ in the $C^{0,t}(S^2)$-norm as $n \to \infty$. For all $n \in \mathbb{N}$ let $\tilde{u}_n \in C^{0,\gamma}(S^2)$ be the continuous solution with respect to the bilinear forms induced by A_n to the problem in Equation (8.2) that satisfy $\int_{S^2} \tilde{u}_n \, \mathrm{d}\sigma = 0$. Since we already established in the proof of the third claim of Proposition 8.13 that $(\min_{x \in S^2} A_n(x))^{-2}$ and $\|A_n\|_{C^{0,t}(S^2)}$ can be bounded independently of $n \in \mathbb{N}$, the second claim of this corollary implies the claim, i.e. there exists a constant K independently of n, $\tilde{u}$, $\tilde{u}_n$, A, A_n and f such that

$$\|\tilde{u} - \tilde{u}_n\|_{C^{0,\gamma}(S^2)} \leq K \frac{\|A\|_{C^{0,t}(S^2)}}{(\min\limits_{x \in S^2} A(x))^2} \frac{(1 + \|A_n\|_{C^{0,t}(S^2)})^2}{(\min\limits_{x \in S^2} A_n(x))^2} \|f\|_{L^2(S^2,\mathbb{R})} \|A - A_n\|_{C^{0,t}(S^2)}. \quad \square$$

Note that in Proposition 8.13 we were generous with our right hand side $f \in L^2(S^2, \mathbb{R})$. The proof of the previous proposition relied on Lemma 3.2 in [5]. In Lemma 3.2 in [5], which is for certain elliptic problems in domains of Euclidean space, it was needed that $f \in W^{\gamma-1,2}$ to obtain that the solution will be in $W^{1+\gamma,2}$. If one would discuss Sobolev spaces on S^2 of fractional order, then the assumption on the right hand side f could be sharpened accordingly. We preferred to have have $f \in L^2(S^2, \mathbb{R})$ to avoid Sobolev spaces of fractional order on S^2 in this discussion. In the following subsection we aim at higher regularity of the solution to Equation (8.2). Then we will need at least that $f \in L^p(S^2, \mathbb{R})$ for $p > 2$. Since the results of the next subsection are our main aim, the discussion with $f \in L^2(S^2, \mathbb{R})$ was not too restrictive.

8.2.2. Schauder estimates on the sphere

In this section we develop Schauder estimates for elliptic equations of second order on the sphere. We return to the problem equation, which we introduced in the beginning of Chapter 8 as Equation (8.1). However, for the estimate of first order partial derivatives, we consider the problem:

$$-\nabla_{S^2} \cdot (A\nabla_{S^2} u) + cu = g + \nabla_{S^2} \cdot f. \tag{8.53}$$

The right hand side g and the coefficient c are at least in $L^p(S^2, \mathbb{R})$ for $p > 2$ and the coefficient A is a Hölder continuous, strictly positive function on S^2. The part of the right hand side f, that is in divergence form, is a Hölder continuous vector field on S^2. In our usual coordinates a vector field X on S^2 has a $\hat{\theta}$-component denoted by X_θ and a $\hat{\varphi}$-component that is denoted by X_φ. If $u \in H^1(S^2)$ satisfies that

$$\int_{S^2} A\nabla_{S^2} u \cdot \nabla_{S^2} v + cuv \, \mathrm{d}\sigma = \int_{S^2} gv - f \cdot \nabla_{S^2} v \, \mathrm{d}\sigma \tag{8.54}$$

for all $v \in H^1(S^2)$, we call u a weak solution of Equation (8.53). We want to develop estimates of the solution u on S^2 analogue to the Schauder interior estimates on domains of Euclidean space. Since Hölder continuity is a local property we analyze the solution u multiplied with a bumb function and pull this product back to the chart domains. On the chart domains the function will be compactly supported and the Schauder interior estimates can be applied.

Theorem 8.15. *For some $\gamma \in (0,1)$ let $u \in C^0(S^2) \cap W^{1,p}(S^2)$ be a weak solution of Equation (8.53) for $p = \frac{2}{1-\gamma}$. If $g, c \in L^p(S^2, \mathbb{R})$, $c \geq 0$ and $A, f_\theta, f_\varphi \in C^{0,\gamma}(S^2)$, then $u \in C^{1,\gamma}(S^2)$ and u satisfies the estimate that*

$$\|u\|_{C^{1,\gamma}(S^2)} \leq K \left(\frac{(1 + \|A\|_{C^{0,\gamma}(S^2)} + \|c\|_{L^p(S^2,\mathbb{R})})^4}{\min_{x \in S^2} A(x)^{1+\gamma}} \right)^{\frac{2}{1-\gamma}} \Big(\|u\|_{W^{1,p}(S^2)} + \|g\|_{L^p(S^2,\mathbb{R})} + \sum_{\beta \in \{\theta, \varphi\}} \|f_\beta\|_{C^{0,\gamma}(S^2)} \Big),$$

where K is a constant that is independent of the solution u and A, c, g and f.

Proof. Since Hölder continuity is a local property, we multiply the solution with a cut-off function and pull this product back to the chart domains, where we are able to apply the Schauder interior estimate from Section 8.1.2. We remind of our usual atlas $(U_i, \eta_i : i = 1, ..., 6)$ with partition of unity Ψ subordinate to the open cover $(U_i : i = 1, ..., 6)$. We fix $i \in \{1, ..., 6\}$ and set $D' = supp((\Psi_i)_{\eta_i})$ and let D be a subdomain of $\eta_i^{-1}(U_i)$ with smooth boundary such that $D' \subset\subset D \subset\subset \eta_i^{-1}(U_i)$. Since $\eta_i(\overline{D})$ is relatively closed in S^2, we can apply Lemma 6.2 and conclude that there exists a partition of unity $\hat{\Psi}$ subordinate to the open cover $(U_j : j = 1, ..., 6)$ such that $(\hat{\Psi}_i)_{\eta_i} = 1$ on $\overline{D}$. We observe that $u\Psi_i$ satisfies weakly that

$$-\nabla_{S^2} \cdot (A\nabla_{S^2}(u\Psi_i)) + cu\Psi_i = F_i,$$

where F_i is given by

$$F_i = g\Psi_i + (\nabla_{S^2} \cdot f)\Psi_i - A\nabla_{S^2} u \cdot \nabla_{S^2} \Psi_i - \nabla_{S^2} \cdot (Au\nabla_{S^2} \Psi_i)$$

$$= g\Psi_i + \nabla_{S^2} \cdot (f\Psi_i) - f \cdot \nabla_{S^2}\Psi_i - A\nabla_{S^2}u \cdot \nabla_{S^2}\Psi_i - \nabla_{S^2} \cdot (Au\nabla_{S^2}\Psi_i).$$

Since Ψ_i is compactly supported in U_i, $u\Psi_i$ satisfies that

$$\int_{U_i} A\nabla_{S^2}(u\Psi_i) \cdot \nabla_{S^2}v + c(u\Psi_i)v \,\mathrm{d}\sigma = \int_{U_i} (g\Psi_i)v - (f\Psi_i) \cdot \nabla_{S^2}v - (f \cdot \nabla_{S^2}\Psi_i)v \\ - A\nabla_{S^2}u \cdot \nabla_{S^2}\Psi_i + Au\nabla_{S^2}\Psi_i \cdot \nabla_{S^2}v \,\mathrm{d}\sigma$$

for all $v \in H^1(S^2)$. We pull the problem back to the chart domains and obtain that in our local coordinates $u\Psi_i$ satisfies that

$$\begin{aligned}\int_D A_{\eta_i}\left(\partial_\theta(u\Psi_i)_{\eta_i}\partial_\theta v + \frac{1}{\sin^2(\theta)}\partial_\varphi(u\Psi_i)_{\eta_i}\partial_\varphi v\right)\sin(\theta) + c_{\eta_i}\sin(\theta)(u\Psi_i)_{\eta_i}v \,\mathrm{d}x \\ = \int_D g_{\eta_i}\sin(\theta)v - ((f_\theta)_{\eta_i}\sin(\theta)\partial_\theta(\Psi_i)_{\eta_i} + (f_\varphi)_{\eta_i}\partial_\varphi(\Psi_i)_{\eta_i})v \\ - (f_\theta\Psi_i)_{\eta_i}\sin(\theta)\partial_\theta v - (f_\varphi\Psi_i)_{\eta_i}\partial_\varphi v \\ - A_{\eta_i}\left(\partial_\theta u_{\eta_i}\partial_\theta(\Psi_i)_{\eta_i} + \frac{1}{\sin^2(\theta)}\partial_\varphi u_{\eta_i}\partial_\varphi(\Psi_i)_{\eta_i}\right)\sin(\theta)v \\ + (Au)_{\eta_i}\sin(\theta)\partial_\theta(\Psi_i)_{\eta_i}\partial_\theta v + (Au)_{\eta_i}\frac{1}{\sin(\theta)}\partial_\varphi(\Psi_i)_{\eta_i}\partial_\varphi v \,\mathrm{d}x,\end{aligned}$$

for all $v \in W_0^{1,2}(D)$. We divide the right hand side into the part that is given in divergence form and the part that is not. We define:

$$\begin{aligned}\tilde{G} = -(g\Psi_i)_{\eta_i}\sin(\theta) + (f_\theta)_{\eta_i}\sin(\theta)\partial_\theta(\Psi_i)_{\eta_i} + (f_\varphi)_{\eta_i}\partial_\varphi(\Psi_i)_{\eta_i} \\ + A_{\eta_i}\left(\partial_\theta u_{\eta_i}\partial_\theta(\Psi_i)_{\eta_i} + \frac{1}{\sin^2(\theta)}\partial_\varphi u_{\eta_i}\partial_\varphi(\Psi_i)_{\eta_i}\right)\sin(\theta)\end{aligned}$$

and

$$\tilde{F} = \left(-(f_\theta\Psi_i)_{\eta_i}\sin(\theta) + (Au)_{\eta_i}\sin(\theta)\partial_\theta(\Psi_i)_{\eta_i},\ -(f_\varphi\Psi_i)_{\eta_i} + (Au)_{\eta_i}\partial_\varphi\frac{(\Psi_i)_{\eta_i}}{\sin(\theta)}\right)^\top.$$

Moreover we define the coefficient matrix of the above elliptic operator as

$$a = diag\left(A_{\eta_i}\sin(\theta), \frac{A_{\eta_i}}{\sin(\theta)}\right).$$

We obtain that $u\Psi_i$ is a weak solution of the following equation in our local coordinates:

$$\begin{aligned}\partial_l(a^{l,k}\partial_k(u\Psi_i)_{\eta_i}) - c_{\eta_i}\sin(\theta)(u\Psi_i)_{\eta_i} = \tilde{G} + \partial_l\tilde{F}^l \\ (u\Psi_i)_{\eta_i}\Big|_{\partial D} = 0\end{aligned} \tag{8.55}$$

We have to analyze the regularity of the coefficients of the operator and the right hand side. Since $\frac{1}{\sin(\theta)}$ is smooth on the closure of $\eta_i^{-1}(U_i)$, Inequality (6.4) implies that the components of a are in $C^{0,\gamma}(\overline{D})$. The Sobolev embedding theorem, which is Theorem 4.6.1.(e) in [25], implies that $(u\Psi_i)_{\eta_i} \in C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})$ and for a constant K it holds that

$$\|(u\Psi_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})} \leq K\|(u\Psi_i)_{\eta_i}\|_{W^{1,p}(\eta_i^{-1}(U_i))}. \tag{8.56}$$

Since $(f_\theta \Psi_i)_{\eta_i}, (f_\varphi \Psi_i)_{\eta_i}, (u\Psi_i)_{\eta_i} \in C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})$ and $A_{\eta_i} \in C^{0,\gamma}(\overline{D})$, Inequality (6.4) implies that the components of $\tilde{F}$ are in $C^{0,\gamma}(\overline{D})$, where we tacitly used that Ψ_i is smooth and compactly supported. Also $\tilde{G} \in L^p(\eta_i^{-1}(U_i), \mathbb{R})$. Therefore the theory in Gilbarg and Trudinger [14] becomes applicable.
Since the operator in Equation (8.55) is strictly elliptic in the sense of Section 8.1.1 and $-c \leq 0$, Theorem 8.34 in [14] together with the remark at the very end of Section 8.11 in [14] imply that the Dirichlet problem in Equation (8.55) has a unique weak solution in $C^{1,\gamma}(\overline{D})$. This implies that $(u\Psi_i)_{\eta_i} \in C^{1,\gamma}(\overline{\eta_i^{-1}(U_i)})$, because $(\Psi_i)_{\eta_i}$ is compactly supported in D. Since $i \in \{1, ..., 6\}$ was arbitrarily fixed, we conclude that $u \in C^{1,\gamma}(S^2)$.

For the proof of the second claim we fix the same $i \in \{1, ..., 6\}$ and consult the Schauder interior estimate for weak solutions in Section 8.1.2. Since $\tilde{G}, \tilde{F}$ and $(u\Psi_i)_{\eta_i}$ are compactly supported in D and $(\hat{\Psi}_i)_{\eta_i} = 1$ on $\overline{D}$, we can add $\hat{\Psi}_i$ as a factor to the right hand side and the solution. Now we can apply Theorem 8.12 with the nested sets $D' \subset\subset D$ and we obtain that

$$\begin{aligned}
|(u\Psi_i)_{\eta_i}|_{\gamma,1,D'} &\leq \mathcal{K} \Big(\|(u\Psi_i)_{\eta_i}\|_{C^0(\overline{D})} + \|\tilde{G}(\hat{\Psi}_i^2)_{\eta_i}\|_{L^p(D,\mathbb{R})} \\
&\qquad + \|\tilde{F}_\theta(\hat{\Psi}_i^2)_{\eta_i}\|_{C^{0,\gamma}(\overline{D})} + \|\tilde{F}_\varphi(\hat{\Psi}_i^2)_{\eta_i}\|_{C^{0,\gamma}(\overline{D})} \Big) \\
&\leq \mathcal{K}K \Big(\|(u\Psi_i)_{\eta_i}\|_{W^{1,p}(\eta_i^{-1}(U_i))} + \|\tilde{G}(\hat{\Psi}_i^2)_{\eta_i}\|_{L^p(D,\mathbb{R})} \\
&\qquad + \|\tilde{F}_\theta(\hat{\Psi}_i^2)_{\eta_i}\|_{C^{0,\gamma}(\overline{D})} + \|\tilde{F}_\varphi(\hat{\Psi}_i^2)_{\eta_i}\|_{C^{0,\gamma}(\overline{D})} \Big)
\end{aligned} \tag{8.57}$$

We analyze the right hand side of the above inequality and obtain that

$$\begin{aligned}
\|\tilde{G}(\hat{\Psi}_i^2)_{\eta_i}\|_{L^p(D,\mathbb{R})} &\leq K \Big(\|g_{\eta_i}\|_{L^p(\eta_i^{-1}(U_i),\mathbb{R})} + \sum_{\beta \in \{\theta,\varphi\}} \|(f_\beta \Psi_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})} \\
&\qquad + \|(A\hat{\Psi}_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})} \|(u\hat{\Psi}_i)_{\eta_i}\|_{W^{1,p}(\eta_i^{-1}(U_i))} \Big) \\
&\leq K \Big(\|g\|_{L^p(S^2,\mathbb{R})} + \sum_{\beta \in \{\theta,\varphi\}} \|f_\beta\|_{C^{0,\gamma}(S^2)} + \|A\|_{C^{0,\gamma}(S^2)} \|u\|_{W^{1,p}(S^2)} \Big).
\end{aligned} \tag{8.58}$$

Since a product of two Hölder functions results again a Hölder function, which is shown in Inequality (6.4), and due to Inequality (8.56) we obtain that

$$\begin{aligned}
\|\tilde{F}_\theta(\hat{\Psi}_i^2)_{\eta_i}\|_{C^{0,\gamma}(\overline{D})} &\leq K \Big(\|(f_\theta \Psi_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})} \\
&\qquad + \|(A\hat{\Psi}_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{\eta_i^{-1}(U_i)})} \|(u\hat{\Psi}_i)_{\eta_i}\|_{W^{1,p}(\eta_i^{-1}(U_i))} \Big) \\
&\leq K \left(\|f_\theta\|_{C^{0,\gamma}(S^2)} + \|A\|_{C^{0,\gamma}(S^2)} \|u\|_{W^{1,p}(S^2)} \right),
\end{aligned} \tag{8.59}$$

where the same estimate also holds for the $C^{0,\gamma}(\overline{D})$-norm of $\tilde{F}_\varphi(\hat{\Psi}_i^2)_{\eta_i}$. Note that the contributions from $\sin(\theta)$ and Ψ_i are included in the constant K. This causes no problem because $\sin(\theta)$ and Ψ_i are smooth and Ψ_i is in addition compactly supported.

The next step is to analyze the constant $\mathcal{K}$. Since $\frac{1}{\sin(\theta)}$ is a smooth function on $supp((\hat{\Psi}_i)_{\eta_i})$, for the coefficients in the matrix a and a constant K it holds that

$$\sum_{l,k=1}^{2} \|a_{l,k}\|_{C^{0,\gamma}(\overline{D})} \leq K \|(A\hat{\Psi}_i)_{\eta_i}\|_{C^{0,\gamma}(\overline{D})} \leq K \|A\|_{C^{0,\gamma}(S^2)}.$$

The functions Λ and λ from Section 8.1 that satisfy that $\lambda\|\xi\|^2_{\mathbb{R}^2} \leq \|a\xi\|^2_{\mathbb{R}^2} \leq \Lambda\|\xi\|^2_{\mathbb{R}^2}$ for all $\xi \in \mathbb{R}^2$ are bounded from above and below in this way:

$$\min_{(\theta,\varphi)\in\overline{D}} \sin(\theta) \min_{x\in\eta_i(\overline{D})} A(x) \leq \lambda \leq \Lambda \leq \max_{x\in\eta_i(\overline{D})} A(x) \max_{(\theta,\varphi)\in\overline{D}} \frac{1}{\sin(\theta)}. \tag{8.60}$$

The constant $\mathcal{K}$ can be bounded with Theorem 8.12 and the last two inequalities, i.e.

$$\mathcal{K} \leq K \left(\frac{(1+\|A\|_{C^0(S^2)})^{2+\gamma}}{\min_{x\in S^2} A(x)^{1+\gamma}} \right)^{\frac{2}{1-\gamma}} \left(1 + \|A\|_{C^{0,\gamma}(S^2)} + \|c\|_{L^p(S^2,\mathbb{R})}\right)^{\frac{2}{1-\gamma}},$$

where K is a constant independent of the coefficients A and c and the solution u. We combine the last inequality with the Inequalities (8.57), (8.58) and (8.59) and apply the fact that $(u\Psi_i)_{\eta_i}$ is equal to zero outside of D' to obtain that

$$\begin{aligned}
\|(u\Psi_i)_{\eta_i}\|_{C^{1,\gamma}(\overline{\eta_i^{-1}(U_i)})} &\leq K \left(\frac{(1+\|A\|_{C^0(S^2)})^{2+\gamma}}{\min_{x\in S^2} A(x)^{1+\gamma}} \right)^{\frac{2}{1-\gamma}} \left(1 + \|A\|_{C^{0,\gamma}(S^2)} + \|c\|_{L^p(S^2,\mathbb{R})}\right)^{\frac{2}{1-\gamma}+1} \\
&\quad \cdot \Big(\|u\|_{W^{1,p}(S^2)} + \|g\|_{L^p(S^2,\mathbb{R})} + \sum_{\beta\in\{\theta,\varphi\}} \|f_\beta\|_{C^{0,\gamma}(S^2)} \Big) \\
&\leq K \left(\frac{(1+\|A\|_{C^{0,\gamma}(S^2)} + \|c\|_{L^p(S^2,\mathbb{R})})^{4}}{\min_{x\in S^2} A(x)^{1+\gamma}} \right)^{\frac{2}{1-\gamma}} \\
&\quad \cdot \Big(\|u\|_{W^{1,p}(S^2)} + \|g\|_{L^p(S^2,\mathbb{R})} + \sum_{\beta\in\{\theta,\varphi\}} \|f_\beta\|_{C^{0,\gamma}(S^2)} \Big).
\end{aligned}$$

This argument applies to all other $j \in \{1, ..., 6\}\backslash\{i\}$. Hence the second claim of the theorem is also proven. □

A similar localization procedure on the chart domains can be applied when the right hand side and the coefficient satisfy higher order Hölder regularity. We will obtain a form of Schauder estimates for higher order Hölder regularity on the sphere. For estimates of higher order partial derivatives, we consider the problem:

$$-\nabla_{S^2} \cdot (A\nabla_{S^2} u) + cu = f, \tag{8.61}$$

where first order partial derivatives of the coefficient A and the right hand side f are at least Hölder continuous. If $u \in H^1(S^2)$ satisfies that

$$\int_{S^2} A\nabla_{S^2} u \cdot \nabla_{S^2} v + cuv \mathrm{d}\sigma = \int_{S^2} fv \mathrm{d}\sigma$$

for all $v \in H^1(S^2)$, then we call u a weak solution of Equation (8.61)

Theorem 8.16. *For some $\gamma \in (0,1)$ let $u \in C^{1,\gamma}(S^2)$ be a weak solution of Equation (8.61). If $f, c \in C^{\iota-1,\gamma}(S^2)$, $c \geq 0$ and $A \in C^{\iota,\gamma}(S^2)$ for some positive integer $\iota \geq 1$, then $u \in C^{\iota+1,\gamma}(S^2)$ and u satisfies the estimate that*

$$\|u\|_{C^{\iota+1,\gamma}(S^2)} \leq K \left(\frac{(1+\|A\|_{C^{\iota,\gamma}(S^2)} + \|c\|_{C^{\iota-1,\gamma}(S^2)})^{5}}{\min_{x\in S^2} A(x)^{1+\gamma}} \right)^{\frac{18\iota^2}{1-\gamma}} \left(\|u\|_{C^{1,\gamma}(S^2)} + \|f\|_{C^{\iota-1,\gamma}(S^2)}\right),$$

where K is a constant that is independent of the solution u and A, c and f.

Proof. The proof of this theorem will follow an iterative argument. For $\kappa \in \{1, ..., \iota\}$ we assume that we already proved that $u \in C^{\kappa,\gamma}(S^2)$ and we want to show that this implies that $u \in C^{\kappa+1,\gamma}(S^2)$ and we want to establish an estimate of the $C^{\kappa+1,\gamma}(S^2)$-norm of u in terms of the coefficients A and c, the right hand side f and the $C^{\kappa,\gamma}(S^2)$-norm of u.

We remind of our usual atlas $(U_i, \eta_i : i = 1, ..., 6)$ with partition of unity Ψ subordinate to the open cover $(U_i : i = 1, ..., 6)$. We fix $i \in \{1, ..., 6\}$ and set $D' = supp((\Psi_i)_{\eta_i})$ and let D be a subdomain of $\eta_i^{-1}(U_i)$ with smooth boundary such that $D' \subset\subset D \subset\subset \eta_i^{-1}(U_i)$. Since $\eta_i(\overline{D})$ is relatively closed in S^2, we can apply Lemma 6.2 and conclude that there exists a partition of unity $\hat{\Psi}$ subordinate to the open cover $(U_j : j = 1, ..., 6)$ such that $(\hat{\Psi}_i)_{\eta_i} = 1$ on $\overline{D}$. Since u is a weak solution of Equation (8.54) we observe that $u\Psi_i$ satisfies weakly that

$$-\nabla_{S^2} \cdot (A\nabla_{S^2}(u\Psi_i)) + c(u\Psi_i) = F_i,$$

where F_i is given by

$$F_i = f\Psi_i - A\nabla_{S^2} u \cdot \nabla_{S^2}\Psi_i - \nabla_{S^2} \cdot (Au\nabla_{S^2}\Psi_i).$$

From the proof of the previous theorem we recall that $u\Psi_i$ satisfies the Dirichlet problem in Equation (8.55) in our local coordinates in the sense of distributions:

$$\begin{aligned} \partial_l(a^{l,k}\partial_k(u\Psi_i)_{\eta_i}) - c_{\eta_i}\sin(\theta)(u\Psi_i)_{\eta_i} &= (F_i)_{\eta_i}\sin(\theta) \\ (u\Psi_i)_{\eta_i}\Big|_{\partial D} &= 0, \end{aligned} \tag{8.62}$$

where $a = diag(A_{\eta_i}\sin(\theta),\ A_{\eta_i}\sin^{-1}(\theta))$. Moreover $(u\Psi_i)_{\eta_i}$ is the unique solution of the above Dirichlet problem. This was due to Theorem 8.34 in [14]. Note that $(F_i)_{\eta_i} \in C^{0,\gamma}(\overline{D})$ and is compactly supported in D. In our case the coefficients of the differential operator are continuously differentiable, therefore we are not in the situation of an operator in divergence form as we were in the proof of the previous theorem. Therefore, we consider the following Dirichlet problem:

$$\begin{aligned} a^{l,k}\partial_l\partial_k\tilde{u} + b^l\partial_l\tilde{u} - c_{\eta_i}\sin(\theta)\tilde{u} &= (F_i)_{\eta_i}\sin(\theta) \\ \tilde{u}\Big|_{\partial D} &= 0. \end{aligned} \tag{8.63}$$

The coefficients of the operator in Equation (8.63) are given by:

$$a = diag\left(A_{\eta_i}\sin(\theta), \frac{A_{\eta_i}}{\sin(\theta)}\right)$$

and

$$b = (\partial_l a^{l,\theta}, \partial_l a^{l,\varphi})^\top = \left(\sin(\theta)\partial_\theta A_{\eta_i} + \cos(\theta)A_{\eta_i}, \partial_\varphi\frac{A_{\eta_i}}{\sin(\theta)}\right)^\top.$$

Every function $\tilde{u} \in C^2(\overline{D})$ that satisfies the Dirichlet problem in Equation (8.63) also satisfies the Dirichlet problem in Equation (8.62) weakly and is by uniqueness equal to $(u\Psi_i)_{\eta_i}$. The uniqueness was established in the proof of the previous theorem and is due to Theorem 8.34 in [14].
As in the previous proof we have to analyze the regularity of the coefficients and the right hand side. Since $\frac{1}{\sin(\theta)}$ is a smooth function on $\overline{D}$, we observe that the components of a

are in $C^{\kappa,\gamma}(\overline{D})$ and similarly the components of b are in $C^{\kappa-1,\gamma}(\overline{D})$. Since the operator in Equation (8.63) is strictly elliptic in the sense of Section 8.1.1 and $-c \leq 0$, Theorem 6.14 in [14] implies that there exists a unique solution $\tilde{u} \in C^{2,\gamma}(\overline{D})$ to the respective Dirichlet problem in Equation (8.63). The uniqueness implies that $\tilde{u} = (u\Psi_i)_{\eta_i} \in C^{2,\gamma}(\overline{D})$ and that $(u\Psi_i)_{\eta_i}$ satisfies Equation (8.63) in D.
We assumed that $u \in C^{\kappa,\gamma}(S^2)$, this implies that $(F_i)_{\eta_i} \in C^{\kappa-1,\gamma}(\overline{D})$. Therefore, we can apply Theorem 8.10 with the nested domains $D' \subset\subset D$ to the problem in Equation (8.63). It implies that $(u\Psi_i)_{\eta_i} \in C^{\kappa+1,\gamma}(D')$. Due to the fact that $supp((\Psi_i)_{\eta_i}) = D'$, it holds that $(u\Psi_i)_{\eta_i} \in C^{\kappa+1,\gamma}(\overline{\eta_i^{-1}(U_i)})$. Since $i \in \{1, ..., 6\}$ was arbitrarily fixed, it follows that $u \in C^{\kappa+1,\gamma}(S^2)$. Note that in the case that $\kappa = 1$, we made the assumption $u \in C^{1,\gamma}(S^2)$ in this theorem. Therefore, we have proven that $u \in C^{\iota+1,\gamma}(S^2)$.

Now we have to prove the estimate of the $C^{\kappa+1,\gamma}(S^2)$-norm of the solution u. Theorem 8.10 also yields an estimate for the $C^{\kappa+1,\gamma}(D')$-semi-norm of $(u\Psi_i)_{\eta_i}$, i.e.

$$|(u\Psi_i)_{\eta_i}|_{\gamma,\kappa+1,D'} \leq \mathcal{K}\left(\|(u\Psi_i)_{\eta_i}\|_{C^0(\overline{D})} + \|(F_i)_{\eta_i}\|_{C^{\kappa-1,\gamma}(\overline{D})}\right). \tag{8.64}$$

The Hölder norm of the right hand side $(F_i)_{\eta_i}$ can be bounded in terms of f, A and u, i.e. there exists a constant K independently of A, f and u such that

$$\begin{aligned}
\|(F_i)_{\eta_i}\|_{C^{\kappa-1,\gamma}(\overline{D})} &\leq K\left(\|f\|_{C^{\kappa-1,\gamma}(S^2)} + \|A_{\eta_i}\|_{C^{\kappa,\gamma}(\overline{D})}\|u_{\eta_i}\|_{C^{\kappa,\gamma}(\overline{D})}\right) \\
&\leq K\left(\|f\|_{C^{\kappa-1,\gamma}(S^2)} + \|(A\hat{\Psi}_i)_{\eta_i}\|_{C^{\kappa,\gamma}(\overline{\eta_i^{-1}(U_i)})}\|(u\hat{\Psi}_i)_{\eta_i}\|_{C^{\kappa,\gamma}(\overline{\eta_i^{-1}(U_i)})}\right) \\
&\leq K\left(\|f\|_{C^{\kappa-1,\gamma}(S^2)} + \|A\|_{C^{\kappa,\gamma}(S^2)}\|u\|_{C^{\kappa,\gamma}(S^2)}\right).
\end{aligned}$$

We insert the estimate in the previous inequality of the right hand side $(F_i)_{\eta_i}$ into Inequality (8.64) and obtain with Inequality (6.3) that

$$\|(u\Psi_i)_{\eta_i}\|_{C^{\kappa+1,\gamma}(\overline{\eta_i^{-1}(U_i)})} \leq (1+\mathcal{K})\left(\|u\|_{C^0(S^2)} + \|f\|_{C^{\kappa-1,\gamma}(S^2)} + \|A\|_{C^{\kappa,\gamma}(S^2)}\|u\|_{C^{\kappa,\gamma}(S^2)}\right),$$

where we tacitly included the constant K into $\mathcal{K}$ and applied the fact that $(u\Psi_i)_{\eta_i}$ is equal to zero outside of D'. Since the argument also applies to all other $j \in \{1, ..., 6\}\backslash\{i\}$, we obtain with the definition of Hölder norms on S^2 that

$$\|u\|_{C^{\kappa+1,\gamma}(S^2)} \leq (1+\mathcal{K})\left(\|u\|_{C^0(S^2)} + \|f\|_{C^{\kappa-1,\gamma}(S^2)} + \|A\|_{C^{\kappa,\gamma}(S^2)}\|u\|_{C^{\kappa,\gamma}(S^2)}\right). \tag{8.65}$$

The next step is to analyze the constant $\mathcal{K}$. We remind of the proof of Theorem 7.7, where we estimated the Hölder norm of the product of a Hölder and a smooth function on a bounded domain with the Faà di Bruno formula. We obtained that the Hölder norm of the product can be bounded by the Hölder norm of the Hölder function multiplied with a constant that depends on the smooth function. Since $\frac{1}{\sin(\theta)}$ and $\cos(\theta)$ are a smooth functions on the closure of $\eta_i^{-1}(U_i)$, for the coefficients a and b and a constant K it holds that

$$\sum_{l,k=1}^{2} \|a_{l,k}\|_{C^{\kappa,\gamma}(\overline{D})}, \quad \sum_{l=1}^{2} \|b_l\|_{C^{\kappa-1,\gamma}(\overline{D})} \leq K\|A\|_{C^{\kappa,\gamma}(S^2)}.$$

The constant $\mathcal{K}$ in Inequality (8.64) is bounded with Theorem 8.9 and Theorem 8.10. We additionally apply the last inequality and Inequality (8.60) to obtain that for a constant K, which is independent of A, c, f and u, it holds that

$$\mathcal{K} \leq K \left(\frac{(1+\|A\|_{C^0(S^2)})^{2+\gamma}}{\min_{x\in S^2} A(x)^{1+\gamma}} \right)^{\frac{18\kappa}{1-\gamma}} \left(1+\|A\|_{C^{\kappa,\gamma}(S^2)}+\|c\|_{C^{\kappa-1,\gamma}(S^2)}\right)^{\frac{18\kappa}{1-\gamma}+2\kappa},$$

Note that all factors in the above estimate of $\mathcal{K}$ are greater than 1. We insert the estimate of the constant $\mathcal{K}$ into Inequality (8.65) and obtain that

$$\begin{aligned}
\|u\|_{C^{\kappa+1,\gamma}(S^2)} &\leq K \left(\frac{(1+\|A\|_{C^0(S^2)})^{2+\gamma}}{\min_{x\in S^2} A(x)^{1+\gamma}} \right)^{\frac{18\kappa}{1-\gamma}} \left(1+\|A\|_{C^{\kappa,\gamma}(S^2)}+\|c\|_{C^{\kappa-1,\gamma}(S^2)}\right)^{\frac{18\kappa}{1-\gamma}+2\kappa} \\
&\quad \cdot \left(\|u\|_{C^0(S^2)}+\|f\|_{C^{\kappa-1,\gamma}(S^2)}+\|A\|_{C^{\kappa,\gamma}(S^2)}\|u\|_{C^{\kappa,\gamma}(S^2)}\right) \\
&\leq K' \left(\frac{(1+\|A\|_{C^0(S^2)})^{2+\gamma}}{\min_{x\in S^2} A(x)^{1+\gamma}} \right)^{\frac{18\kappa}{1-\gamma}} \left(1+\|A\|_{C^{\kappa,\gamma}(S^2)}+\|c\|_{C^{\kappa-1,\gamma}(S^2)}\right)^{\frac{18\kappa}{1-\gamma}+2\kappa+1} \\
&\quad \cdot \left(\|u\|_{C^{\kappa,\gamma}(S^2)}+\|f\|_{C^{\kappa-1,\gamma}(S^2)}\right) \\
&\leq K' \left(\frac{(1+\|A\|_{C^{\kappa,\gamma}(S^2)}+\|c\|_{C^{\kappa-1,\gamma}(S^2)})^{5}}{\min_{x\in S^2} A(x)^{1+\gamma}} \right)^{\frac{18\kappa}{1-\gamma}} \left(\|u\|_{C^{\kappa,\gamma}(S^2)}+\|f\|_{C^{\kappa-1,\gamma}(S^2)}\right).
\end{aligned}$$

Therefore, we have obtained an iterative formula for the bound of the $C^{\iota+1,\gamma}(S^2)$-norm of u. We expand this recursion and conclude that

$$\|u\|_{C^{\iota+1,\gamma}(S^2)} \leq K \left(\frac{(1+\|A\|_{C^{\iota,\gamma}(S^2)}+\|c\|_{C^{\iota-1,\gamma}(S^2)})^{5}}{\min_{x\in S^2} A(x)^{1+\gamma}} \right)^{\frac{18\iota^2}{1-\gamma}} \left(\|u\|_{C^{1,\gamma}(S^2)}+\|f\|_{C^{\iota-1,\gamma}(S^2)}\right).$$

□

8.3. Random elliptic partial differential equations on the sphere

In this section we want to further discuss the problem in Equation (8.1) and in Equation (8.2). In particular we want to take the function A to be a 2-weakly isotropic log-normal spherical random field such that the angular power spectrum of the respective continuous 2-weakly isotropic Gaussian spherical random field T satisfies that $(C_l l^{1+2\iota+\delta} : l \geq 0)$ is summable for some $\delta \in (0,2]$ and an integer $\iota \geq 0$.
We fix $\gamma \in (0, \frac{\delta}{2})$ for this section. Theorem 7.7 implies that there exists an indistinginguishable modification A^* of A such that $A^* \subset C^{\iota,\gamma}(S^2)$. This means that there exists a measurable set of full probability Ω^* such that $A^* \mathbb{1}_{\Omega^*} = A\mathbb{1}_{\Omega^*}$ On the compliment of Ω^* we set $A = 1$ and can therefore disregard the indistinguishable modification A^* in the following. Also Theorem 7.7 implies that $A \in L^p_P(\Omega, C^{\iota,\gamma}(S^2))$ and that $A^L = \exp(T^L)$ converges to A in the $L^p_P(\Omega, C^{j,\gamma}(S^2))$-norm as $L \to \infty$ for all $j \in \{0,1,...,\iota\}$ and all $p \in (0,\infty)$, i.e. for all $p \in (0,\infty)$ there exists a constant K_p independently of L and $(C_l : l \geq 0)$ such that

$$\|A - A^L\|_{L^p_P(\Omega, C^{j,\gamma}(S^2))} = E\left[\|A-A^L\|^p_{C^{j,\gamma}(S^2)}\right]^{\frac{1}{p}} \leq K_p \left(\sum_{l>L} C_l l^{1+2j+\delta} \right)^{\frac{1}{2}}, \tag{8.66}$$

for all $j \in \{0, 1, ..., \iota\}$. The Hölder regularity of A is therefore dependent on ι and γ.

Now we return to the problem in Equation (8.2). Since T is continuous and therefore bounded on S^2 it holds that $A = \exp(T)$ is continuous and $\min_{x \in S^2} A(x)$ is strictly positive. Therefore, due to the discussion at the beginning of this chapter, for all $\omega \in \Omega$ the problem to find $u(\omega) \in H^1(S^2)/\mathbb{R}$ such that

$$b_\omega(u(\omega), v) = \int_{S^2} A(\omega) \nabla_{S^2} u(\omega) \cdot \nabla_{S^2} v \, \mathrm{d}\sigma = \int_{S^2} f v \, \mathrm{d}\sigma = \ell_f(v) \tag{8.67}$$

for all $v \in H^1(S^2)/\mathbb{R}$, admits a unique solution $u(\omega)$ such that Inequality (8.3) holds, i.e.

$$\|u(\omega)\|_{H^1(S^2)/\mathbb{R}} \leq \frac{1}{\sqrt{2}} \frac{\|f\|_{L^2(S^2,\mathbb{R})}}{\min_{x \in S^2} A(\omega, x)}, \tag{8.68}$$

where $\ell_f = (v \mapsto \int_{S^2} f v \mathrm{d}\sigma)$ for $f \in L^2(S^2, \mathbb{R})$ such that $\int_{S^2} f \mathrm{d}\sigma = 0$ as in the beginning of this chapter. We remind that at the beginning of Chapter 6, we established that the mapping $\omega \mapsto T(\omega)$ is $\mathcal{A} - \mathcal{B}(C^0(S^2))$ measurable. Therefore also $\omega \mapsto A(\omega) = \exp(T(\omega))$ is. Proposition 8.6 implies that the mapping $A \mapsto u$ is continuous from $C^0(S^2)$ to $H^1(S^2)/\mathbb{R}$. Hence it is also $\mathcal{B}(C^0(S^2)) - \mathcal{B}(H^1(S^2)/\mathbb{R})$ measurable, where $\mathcal{B}(H^1(S^2)/\mathbb{R})$ is the Borel σ-algebra of $H^1(S^2)/\mathbb{R}$. The mapping $\omega \mapsto u(\omega)$ from Ω to $H^1(S^2)/\mathbb{R}$ can be seen as a composition of these mappings and is consequently $\mathcal{A} - \mathcal{B}(H^1(S^2)/\mathbb{R})$ measurable. Remark 7.6 on Proposition 7.5 implies that $(\min_{x \in S^2} A(x))^{-1} \in L^p_P(\Omega, \mathbb{R})$ for all $p \in (0, \infty)$. Hence, for all $p \in (0, \infty)$ we obtain the estimate

$$\|u\|_{L^p_P(\Omega, H^1(S^2)/\mathbb{R})} = E\left[\|u\|^p_{H^1(S^2)/\mathbb{R}}\right]^{\frac{1}{p}} \leq E\left[\left(\frac{1}{\min_{x \in S^2} A(x)}\right)^p\right]^{\frac{1}{p}} \frac{\|f\|_{L^2(S^2,\mathbb{R})}}{\sqrt{2}} < \infty. \tag{8.69}$$

We conclude that $u \in L^p_P(\Omega, H^1(S^2)/\mathbb{R})$ for all $p \in (0, \infty)$.

8.3.1. Basic properties and approximation

For all $L \in \mathbb{N}_0$ let b^L be the bilinear form that results from $A^L = \exp(T^L)$, where T^L is the truncated expansion of T. Since T^L is also continuous, A^L is continuous and $\min_{x \in S^2} A^L(x)$ is strictly positive.
By the same argument as before we obtain that for all $\omega \in \Omega$ and all $L \in \mathbb{N}_0$ we can find a unique $u^L(\omega) \in H^1(S^2)/\mathbb{R}$ such that

$$b^L_\omega(u^L(\omega), v) = l_f(v), \tag{8.70}$$

for all $v \in H^1(S^2)/\mathbb{R}$. As in the discussion about the solution u to the problem in Equation (8.67), for all $L \in \mathbb{N}_0$ it holds that u^L is $\mathcal{A} - \mathcal{B}(H^1(S^2)/\mathbb{R})$ measurable and it holds that

$$\|u^L\|_{H^1(S^2)/\mathbb{R}} \leq \frac{1}{\sqrt{2}} \frac{\|f\|_{L^2(S^2,\mathbb{R})}}{\min_{x \in S^2} A^L(x)}$$

Due to Remark 7.6 on Proposition 7.5, for all $p \in (0, \infty)$ there exists a constant K_p, which is independent of L, such that

$$\|u^L\|_{L^p_P(\Omega, H^1(S^2)/\mathbb{R})} = E\left[\|u^L\|^p_{H^1(S^2)/\mathbb{R}}\right]^{\frac{1}{p}} \leq K_p \|f\|_{L^2(S^2,\mathbb{R})}.$$

Therefore $u^L \in L^p_P(\Omega, H^1(S^2)/\mathbb{R})$ for all $L \in \mathbb{N}_0$ and all $p \in (0,\infty)$ and the norm can be bounded independently of L. The following proposition discusses the convergence of u^L to u in the $L^p_P(\Omega, H^1(S^2)/\mathbb{R})$-norm as $L \to \infty$ for all $p \in (0,\infty)$.

Proposition 8.17. *For $L \in \mathbb{N}_0$ and $p \in (0,\infty)$ let $u, u^L \in L^p_P(\Omega, H^1(S^2)/\mathbb{R})$ be the solutions to Equation* (8.67) *and Equation* (8.70)*, then for all $\delta \geq \varepsilon > 0$ there exists a constant $K_{p,\varepsilon}$ independently of L such that*

$$\|u - u^L\|_{L^p_P(\Omega,H^1(S^2)/\mathbb{R})} \leq K_{p,\varepsilon}\|f\|_{L^2(S^2,\mathbb{R})}\left(\sum_{l>L} C_l l^{1+\varepsilon}\right)^{\frac{1}{2}}.$$

We indicate ε in the constant in the above proposition to emphasize that the constant may become very large for borderline values of ε.

Proof. Proposition 8.6 implies an estimate of the $H^1(S^2)/\mathbb{R}$-norm of $u - u^L$, i.e.

$$\|u - u^L\|_{H^1(S^2)/\mathbb{R}} \leq \frac{1}{\sqrt{2}}\frac{\|f\|_{L^2(S^2,\mathbb{R})}}{(\min_{x\in S^2} A(x))(\min_{x\in S^2} A^L(x))}\|A - A^L\|_{C^0(S^2)}.$$

Remark 7.6 implies that the $L^{p'}_P(\Omega,\mathbb{R})$-norm of $(\min_{x\in S^2} A^L(x))^{-1}$ can be bounded independently of L for all $p' \in (0,\infty)$. With a twofold application of the Cauchy–Schwarz inequality we obtain with Remark 7.6 that there exists a constant K independently of L such that

$$\begin{aligned} E\left[\|u - u^L\|^p_{H^1(S^2)/\mathbb{R}}\right]^{\frac{1}{p}} &\leq \frac{1}{\sqrt{2}} E\left[\left(\frac{\|f\|_{L^2(S^2,\mathbb{R})}}{(\min_{x\in S^2} A(x))(\min_{x\in S^2} A^L(x))}\right)^p \|A - A^L\|^p_{C^0(S^2)}\right]^{\frac{1}{p}} \\ &\leq \frac{K}{\sqrt{2}}\|f\|_{L^2(S^2,\mathbb{R})} E\left[\|A - A^L\|^{2p}_{C^0(S^2)}\right]^{\frac{1}{2p}}. \end{aligned}$$

The second factor in the above inequality is treated with Theorem 7.7 for $\gamma = \frac{\varepsilon}{2}$ and $\delta = \varepsilon$. We conclude that there exists a constant $K_{p,\varepsilon}$ independently of L such that

$$E\left[\|u - u^L\|^p_{H^1(S^2)/\mathbb{R}}\right]^{\frac{1}{p}} \leq K_{p,\varepsilon}\|f\|_{L^2(S^2,\mathbb{R})}\left(\sum_{l>L} C_l l^{1+\varepsilon}\right)^{\frac{1}{2}}. \qquad \square$$

Note that in the preceding discussion in this section, we could have also taken an arbitrary $\ell \in (H^1(S^2)/\mathbb{R})^*$.

We remind that by our assumptions at the beginning of this section the 2-weakly isotropic log-normal spherical random field A results from a continuous 2-weakly isotropic Gaussian spherical random field, whose angular power spectrum satisfies that $(C_l l^{1+2\iota+\delta} : l \geq 0)$ is summable for some $\delta \in (0,2]$ and an integer $\iota \geq 0$. In the case that $\iota = 0$, it was a sufficient condition that $\delta > 0$ such that there exists a continuous 2-weakly isotropic lognormal spherical random field, which is also Hölder continuous. It seems that we cannot lower the assumptions on the angular power spectrum in order to only obtain P-a.s. the membership of the realizations of A in $C^0(S^2)$ or even in $L^\infty(S^2)$, because if we set $\delta = 0$ we only obtain that A is P-a.s. in $L^2(S^2,\mathbb{R})$ by Lemma 3.3 or more precisely in $L^{2k}_{P\otimes \mathrm{d}\sigma}(\Omega \times S^2, \mathbb{R})$ for all $k \in \mathbb{N}$, where the norm depends on k.

But we are able to exploit this Hölder continuity of A and obtain higher regularity of the solution u of Equation (8.67) with Proposition 8.13. In the following we want to fix a particular representative of the solution u. For all $\omega \in \Omega$ we will always consider the representative $u(\omega) \in H^1(S^2)$ in $[u(\omega)] \in H^1(S^2)/\mathbb{R}$ that satisfies that $\int_{S^2} u(\omega)\mathrm{d}\sigma = 0$. We used the parentheses $[\cdot]$ to distinguish between the equivalence class and the representatives. Due to Lemma 8.3 $[v] \in H^1(S^2)/\mathbb{R}$ is mapped continuously to the representative $v \in [v]$ that satisfies that $\int_{S^2} v\mathrm{d}\sigma = 0$. Therefore, we observe that the mapping $\omega \mapsto u(\omega)$ is $\mathcal{A} - \mathcal{B}(H^1(S^2))$ measurable.

Proposition 8.18. *Let u be the representative of the unique weak solution of Equation* (8.67) *that satisfies that $\int_{S^2} u\mathrm{d}\sigma = 0$ with right hand side $f \in L^2(S^2, \mathbb{R})$ such that $\int_{S^2} f\mathrm{d}\sigma = 0$ and let u^L be the respective representative of the unique weak solution of Equation* (8.70) *for $L \in \mathbb{N}_0$ with the same right hand side.*
The mapping $\omega \mapsto u(\omega)$ from Ω to $W^{1,q}(S^2)$ is $\mathcal{A} - \mathcal{B}(W^{1,q}(S^2))$ measurable. For $q = \frac{2}{1-\gamma}$ it holds that $u \in L^p_P(\Omega, W^{1,q}(S^2))$ for all $p \in (0, \infty)$ and there exists a constant K_p independently of f and u such that

$$\|u\|_{L^p_P(\Omega, W^{1,q}(S^2))} \leq K_p \|f\|_{L^2(S^2,\mathbb{R})}.$$

For all $L \in \mathbb{N}_0$, u^L is also in $L^p_P(\Omega, W^{1,q}(S^2))$ for all $p \in (0, \infty)$ and there exists a constant K_p independently of L, f and u^L such that

$$\|u^L\|_{L^p_P(\Omega, W^{1,q}(S^2))} \leq K_p \|f\|_{L^2(S^2,\mathbb{R})}.$$

Moreover for all $p \in (0, \infty)$ there exists a constant K_p independently of L, f, u and u^L such that

$$\|u - u^L\|_{L^p_P(\Omega, W^{1,q}(S^2))} \leq K_p \|f\|_{L^2(S^2,\mathbb{R})} \left(\sum_{l>L} C_l l^{1+\delta} \right)^{\frac{1}{2}}.$$

Proof. We apply Theorem 7.7 with $t = \frac{2\gamma+\delta}{4}$ and obtain that on a measurable set of full probability, say also Ω^* as in the beginning of Section 8.3, $A\mathbb{1}_{\Omega^*} \subset C^{0,t}(S^2)$. As before we set $A = 1$ on the compliment of Ω^*. Also Theorem 7.7 implies that $A, A^L \in L^{p'}_P(\Omega, C^{0,t}(S^2))$ for all $p' \in (0, \infty)$ and all $L \in \mathbb{N}_0$. Moreover the $L^{p'}_P(\Omega, C^{0,t}(S^2))$-norm of A^L can be bounded independently of L.
Now we apply Proposition 8.13 that $u \subset W^{1,q}(S^2)$. It also implies that the mapping $A \mapsto u$ from $C^{0,t}(S^2) \cap \{\tilde{A} : \min_{x \in S^2} \tilde{A}(x) > 0\}$ to $W^{1,q}(S^2)$ is continuous. Therefore it is also $\mathcal{B}(C^{0,t}(S^2)) - \mathcal{B}(W^{1,q}(S^2))$ measurable. Remark 6.11 implies that $\mathcal{B}(C^{0,t}(S^2)) = \mathcal{B}(C^0(S^2))$. We can interpret the mapping $\omega \mapsto u(\omega)$ as a composition of measurable mappings and conclude that $\omega \mapsto u(\omega)$ is $\mathcal{A} - \mathcal{B}(W^{1,q}(S^2))$ measurable.

Proposition 8.13 also implies that u satisfies the estimate that

$$\|u\|_{W^{1,q}(S^2)} \leq K \frac{(1 + \|A\|_{C^{0,t}(S^2)})^2}{(\min_{x \in S^2} A(x))^2} \|f\|_{L^2(S^2,\mathbb{R})},$$

where the constant K is independent of A, f and the solution u. Remark 7.6 implies that $(\min_{x \in S^2} A(x))^{-1}$ is in $L^{p'}_P(\Omega, \mathbb{R})$ for all $p' \in (0, \infty)$. Therefore a twofold application of the Cauchy–Schwarz inequality implies the second claim.

Remark 7.6 also implies that the $L_P^{p'}(\Omega,\mathbb{R})$-norm of $(\min_{x\in S^2} A^L(x))^{-1}$ can be bounded independently of $L\in\mathbb{N}_0$ for all $p'\in(0,\infty)$. We apply Proposition 8.13 to u^L and obtain the third claim in the same way.

For the proof of the fourth claim we consult the second estimate in Proposition 8.13, i.e. there exists a constant K independently of u, u^L, A, A^L and the right hand side f such that

$$\|u-u^L\|_{W^{1,q}(S^2)} \leq K\frac{\|A\|_{C^{0,t}(S^2)}}{(\min\limits_{x\in S^2} A(x))^2}\frac{(1+\|A^L\|_{C^{0,t}(S^2)})^2}{(\min\limits_{x\in S^2} A^L(x))^2}\|f\|_{L^2(S^2,\mathbb{R})}\|A-A^L\|_{C^{0,t}(S^2)}.$$

We recall that Theorem 7.7 and Remark 7.6 imply that $\|A\|_{C^{0,t}(S^2)}$, $(\min_{x\in S^2} A(x))^{-2}\in L_P^{p'}(\Omega,\mathbb{R})$ for all $p'\in(0,\infty)$. Moreover Theorem 7.7 and Remark 7.6 also imply that $\|A^L\|_{C^{0,t}(S^2)}$, $(\min_{x\in S^2} A^L(x))^{-2}\in L_P^{p'}(\Omega,\mathbb{R})$ and their $L_P^{p'}(\Omega,\mathbb{R})$-norm can be bounded independently of L for all $p'\in(0,\infty)$. Now a fourfold application of the Cauchy–Schwarz inequality implies that for all $p\in(0,\infty)$ there exist constants K_p, K_p' independently of f such that

$$\begin{aligned}E\left[\|u-u^L\|^p_{W^{1,q}(S^2)}\right]^{\frac{1}{p}} &\leq K_p\|f\|_{L^2(S^2,\mathbb{R})}E\left[\|A-A^L\|^{2p}_{C^{0,t}(S^2)}\right]^{\frac{1}{2p}}\\ &\leq K_p'\|f\|_{L^2(S^2,\mathbb{R})}\left(\sum_{l>L} C_l l^{1+\delta}\right)^{\frac{1}{2}},\end{aligned}$$

where we applied Theorem 7.7 another time to obtain the estimate for the $L_P^{2p}(\Omega, C^{0,t}(S^2))$-norm of $A-A^L$. □

In the following proposition we prove the existence of continuous solutions. Note that the measurability will be implied by the fact that the solution depends continuously on the coefficient A, which is measurable.

Proposition 8.19. *There exists a unique, continuous weak solution $\hat{u}\subset C^{0,\gamma}(S^2)$ to the problem in Equation* (8.67) *that satisfies that $\int_{S^2}\hat{u}\mathrm{d}\sigma=0$ with right hand side $f\in L^2(S^2,\mathbb{R})$ such that $\int_{S^2} f\mathrm{d}\sigma=0$ and for all $L\in\mathbb{N}_0$ there exists a unique, continuous weak solution $\hat{u}^L\subset C^{0,\gamma}(S^2)$ to the problem in Equation* (8.70) *that satisfies that $\int_{S^2}\hat{u}^L\mathrm{d}\sigma=0$ with the same right hand side $f\in L^2(S^2,\mathbb{R})$ such that $\int_{S^2} f\mathrm{d}\sigma=0$.*
The mappings $\omega\mapsto\hat{u}(\omega)$ and $\omega\mapsto\hat{u}^L(\omega)$ from Ω to $C^{0,\gamma}(S^2)$ are $\mathcal{A}-\mathcal{B}(C^0(S^2))$ measurable for all $L\in\mathbb{N}_0$.

Proof. As in the proof of Proposition 8.18, we apply Theorem 7.7 with $t=\frac{2\gamma+\delta}{4}$ and obtain that on a measurable set of full probability, say also Ω^* as in the beginning of Section 8.3, $A\mathbb{1}_{\Omega^*}\subset C^{0,t}(S^2)$. As before we set $A=1$ on the compliment of Ω^*. Due to Corollary 8.14, for all $\omega\in\Omega$ there exists $\hat{u}(\omega)$ such that $\hat{u}(\omega)$ is a solution of the problem in Equation (8.67). We interpret the mapping $\omega\mapsto\hat{u}(\omega)$ as a composition of the mappings $\omega\mapsto A(\omega)$ and $A\mapsto\hat{u}$. The first of the two mappings is $\mathcal{A}-\mathcal{B}(C^0(S^2))$ measurable. Corollary 8.14 implies that the mapping $A\mapsto\hat{u}$ is continuous from $C^{0,t}(S^2)\cap\{\tilde{A}:\min_{x\in S^2}\tilde{A}(x)>0\}$ to $C^{0,\gamma}(S^2)$. Since Remark 6.11 implies that $\mathcal{B}(C^{0,t}(S^2))=\mathcal{B}(C^{0,\gamma}(S^2))=\mathcal{B}(C^0(S^2))$, we conclude that the mapping $\omega\mapsto\hat{u}(\omega)$ is a composition of measurable mappings and is therefore $\mathcal{A}-\mathcal{B}(C^0(S^2))$ measurable.
The uniqueness of the solution was already established pathwise in $H^1(S^2)\cap\{\tilde{u}:\int_{S^2}\tilde{u}\mathrm{d}\sigma=0\}$. If we assume that for some $\omega\in\Omega$ there exist two continuous weak solution in $H^1(S^2)\cap$

$\{\tilde{u} : \int_{S^2} \tilde{u} \mathrm{d}\sigma = 0\}$, then they have to agree on a dense subset of S^2. Since they are continuous, they agree everywhere in S^2.
The proof of the second claim about the existence and uniqueness of a continuous weak solution $\hat{u}^L$ to the problem in Equation (8.70) for all $L \in \mathbb{N}_0$ is completely analogous. $\square$

Remark 8.20. *Let $\tilde{u}$, $(\tilde{u}^L : L \in \mathbb{N}_0)$ be as in the previous proposition. Proposition 8.18 implies with the Sobolev embedding theorem on S^2, which is Theorem 6.10, that $\tilde{u}, \tilde{u}^L \in L^p_P(\Omega, C^{0,\gamma}(S^2))$ for all $p \in (0,\infty)$ and all $L \in \mathbb{N}_0$. It also implies that for all $p \in (0,\infty)$ there exists a constant K_p independently of L and f such that*

$$\|\tilde{u}^L\|_{L^p_P(\Omega, C^{0,\gamma}(S^2))} \leq K_p \|f\|_{L^2(S^2,\mathbb{R})}.$$

Moreover for all $p \in (0,\infty)$ there exists a constant K_p independently of L , f, $\tilde{u}$ and $\tilde{u}^L$ such that

$$\|\tilde{u} - \tilde{u}^L\|_{L^p_P(\Omega, C^{0,\gamma}(S^2))} \leq K_p \|f\|_{L^2(S^2,\mathbb{R})} \left(\sum_{l>L} C_l l^{1+\delta} \right)^{\frac{1}{2}}.$$

Proposition 8.18 of course also applies to the continuous weak solution $\tilde{u}$ that exists due to Propsition 8.19, i.e. $\tilde{u} \in L^p_P(\Omega, W^{1,q}(S^2))$ for all $p \in (0,\infty)$ and the respective estimates in Proposition 8.18 hold, where $q = \frac{2}{1-\gamma}$ as in Proposition 8.18. The analogous statement applies to $\tilde{u}^L$ for all $L \in \mathbb{N}_0$, where $(\tilde{u}^L : L \in \mathbb{N}_0)$ is as in Propsition 8.19.
In the following we will always consider the continuous solutions to the problems in Equation (8.67) and in Equation (8.70). We will denote them with u instead of $\tilde{u}$ and u^L instead of $\tilde{u}^L$ for all $L \in \mathbb{N}_0$ respectively.

8.3.2. Higher order regularity of solutions

The Schauder regularity theory from the previous section can be applied to obtain higher regularity of the solution of the random partial differential equation. We are interested in the solution of the following problem: to find u such that

$$-\nabla_{S^2} \cdot (A \nabla_{S^2} u) = f \qquad \text{with} \qquad \int_{S^2} u \, \mathrm{d}\sigma = 0, \tag{8.71}$$

where $\int_{S^2} f \mathrm{d}\sigma = 0$. We have solved this in the variational formulation, i.e. Equation (8.67), and obtained a weak solution with realizations in $C^{0,\gamma}(S^2) \cap W^{1,q}(S^2)$ for $q = \frac{2}{1-\gamma}$. The first step in this subsection will be to impose higher regularity on the right hand side f to obtain with the Schauder estimates that first order partial derivatives of the realizations of the solution are Hölder continuous. In the second step, higher regularity of the coefficient A and on the right hand side f will lead to higher regularity of the solution in terms of Hölder continuity of its higher order partial derivatives. The next theorem is the first part of the precise version of Theorem 1.4 from the introduction.

Theorem 8.21. *Let u be the unique, continuous weak solution of the problem in Equation (8.67) that satisfies that $\int_{S^2} u \mathrm{d}\sigma = 0$ with right hand side $f \in L^q(S^2, \mathbb{R})$ such that $\int_{S^2} f \mathrm{d}\sigma = 0$ for $q = \frac{2}{1-\gamma}$ and for all $L \in \mathbb{N}_0$ let u^L be the respective unique, continuous weak solution of the problem in Equation (8.70) with the same right hand side that satisfies that $\int_{S^2} u^L \mathrm{d}\sigma = 0$.*

We obtain that $u \in L^p_P(\Omega, C^{1,\gamma}(S^2))$ for all $p \in (0,\infty)$ and there exists a constant K_p independently of f and u such that

$$\|u\|_{L^p_P(\Omega,C^{1,\gamma}(S^2))} \leq K_p \|f\|_{L^q(S^2,\mathbb{R})}.$$

For all $L \in \mathbb{N}_0$, u^L is also in $L^p_P(\Omega, C^{1,\gamma}(S^2))$ for all $p \in (0,\infty)$ and there exists a constant K_p independently of L, f and u^L such that

$$\|u^L\|_{L^p_P(\Omega,C^{1,\gamma}(S^2))} \leq K_p \|f\|_{L^q(S^2,\mathbb{R})}.$$

Proof. Note that $A \subset C^{0,\gamma}(S^2)$. Proposition 8.18 implies that $u \subset W^{1,q}(S^2)$. Therefore, $u \subset C^0(S^2) \cap W^{1,q}(S^2)$ and we can apply Theorem 8.15 and obtain that $u \subset C^{1,\gamma}(S^2)$. Moreover it implies that there exists a constant K, which is independent of A, f and u, such that

$$\|u\|_{C^{1,\gamma}(S^2)} \leq K \left(\frac{(1+\|A\|_{C^{0,\gamma}(S^2)})^4}{(\min_{x\in S^2} A(x))^{1+\gamma}} \right)^{\frac{2}{1-\gamma}} \left(\|u\|_{W^{1,q}(S^2)} + \|f\|_{L^q(S^2,\mathbb{R})} \right),$$

where we tacitly applied that for $\alpha > 0$ it holds that $\min_{x\in I} \exp(x)^\alpha = (\min_{x\in I} \exp(x))^\alpha$ for $I \subset \mathbb{R}$. Remark 7.6 implies that $(\min_{x\in S^2} A(x))^{-1} \in L^{p'}_P(\Omega,\mathbb{R})$ for all $p' \in (0,\infty)$. This also holds for $\|A\|_{C^{0,\gamma}(S^2)}$ due to Theorem 7.7. Proposition 8.18 implies this property for $\|u\|_{W^{1,q}(S^2)}$ and that for all $p' \in (0,\infty)$ there exists a constant $K_{p'}$ independently of f such that

$$E\left[\|u\|^{p'}_{W^{1,q}(S^2)} \right]^{\frac{1}{p'}} \leq K_{p'} \|f\|_{L^2(S^2,\mathbb{R})}.$$

Since $L^q(S^2,\mathbb{R}) \subset L^2(S^2,\mathbb{R})$ with continuous embedding, a threefold application of the Cauchy–Schwarz inequality implies the first claim.

The proof of the second claim is very similar. For all $L \in \mathbb{N}_0$ we can establish the respective estimate of the $C^{1,\gamma}(S^2)$-norm of u^L in terms of A^L and the $W^{1,q}(S^2)$-norm of u^L. We note that the $L^{p'}_P(\Omega,\mathbb{R})$-norm of $(\min_{x\in S^2} A^L(x))^{-1}$, $\|A^L\|_{C^{0,\gamma}(S^2)}$ and $\|u^L\|_{W^{1,q}(S^2)}$ can be bounded independently of $L \in \mathbb{N}_0$ for all $p' \in (0,\infty)$ due to Remark 7.6, Theorem 7.7 and Proposition 8.18. Also Proposition 8.18 implies that that for all $p' \in (0,\infty)$ there exists a constant $K_{p'}$ independently of L and f such that

$$E\left[\|u^L\|^{p'}_{W^{1,q}(S^2)} \right]^{\frac{1}{p'}} \leq K_{p'} \|f\|_{L^2(S^2,\mathbb{R})}. \qquad \square$$

Corollary 8.22. *Let $q = \frac{2}{1-\gamma}$, u, u^L and f be as in the previous theorem. For all $p \in (0,\infty)$ there exists a constant K_p independently of L, f, u and u^L such that*

$$\|u - u^L\|_{L^p_P(\Omega,C^{1,\gamma}(S^2))} \leq K_p \|f\|_{L^q(S^2,\mathbb{R})} \left(\sum_{l>L} C_l l^{1+\delta} \right)^{\frac{1}{2}}.$$

Proof. We give a brief proof of this corollary. We observe that $u - u^L$ satisfies weakly that

$$\begin{aligned} -\nabla_{S^2} \cdot (A \nabla_{S^2}(u - u^L)) &= f + \nabla_{S^2} \cdot (A \nabla_{S^2} u^L) \\ &= f - f - \nabla_{S^2} \cdot (A^L \nabla_{S^2} u^L) + \nabla_{S^2} \cdot (A \nabla_{S^2} u^L) \end{aligned}$$

$$= \nabla_{S^2} \cdot ((A - A^L)\nabla_{S^2} u^L).$$

Since $A, A^L \subset C^{0,\gamma}(S^2)$ and $u^L \subset C^{1,\gamma}(S^2)$, it holds that the components of the vector field $(A - A^L)\nabla_{S^2} u^L$ are in $C^{0,\gamma}(S^2)$. Therefore we can apply the Schauder estimate in Theorem 8.15 and obtain that

$$\|u - u^L\|_{C^{1,\gamma}(S^2)} \leq K \left(\frac{(1 + \|A\|_{C^{0,\gamma}(S^2)})^4}{(\min_{x \in S^2} A(x))^{1+\gamma}} \right)^{\frac{2}{1-\gamma}} \Big(\|u - u^L\|_{W^{1,q}(S^2)} + \sum_{\beta \in \{\theta, \varphi\}} \|(A - A^L)(\nabla_{S^2} u^L)_\beta\|_{C^{0,\gamma}(S^2)} \Big). \tag{8.72}$$

We observe that for $\beta = \theta, \varphi$ and a constant K, which is independent of A, A^L and u^L, it holds that

$$\|(A - A^L)(\nabla_{S^2} u^L)_\beta\|_{C^{0,\gamma}(S^2)} \leq K \|A - A^L\|_{C^{0,\gamma}(S^2)} \|u^L\|_{C^{1,\gamma}(S^2)}.$$

Proposition 8.18 implies that for all $p' \in (0, \infty)$ there exists a constant $K_{p'}$ independently of L and f such that

$$E\left[\|u - u^L\|^{p'}_{W^{1,q}(S^2)}\right]^{\frac{1}{p'}} \leq K_{p'} \|f\|_{L^2(S^2,\mathbb{R})} \left(\sum_{l > L} C_l l^{1+\delta} \right)^{\frac{1}{2}}.$$

The previous theorem implies that the $L^{p'}_P(\Omega, C^{1,\gamma}(S^2))$-norm of u^L can be bounded independently of L for all $p' \in (0, \infty)$. Note that Theorem 7.7 implies that for all $p' \in (0, \infty)$ there exists a constant $K_{p'}$ independently of L such that

$$E\left[\|A - A^L\|^{p'}_{C^{0,\gamma}(S^2)}\right]^{\frac{1}{p'}} \leq K_{p'} \left(\sum_{l > L} C_l l^{1+\delta} \right)^{\frac{1}{2}}.$$

For $p \in (0, \infty)$ we consider the $L^p_P(\Omega, \mathbb{R})$-norm of Inequality (8.72). With these estimates the claim of the corollary follows with a fourfold application of the Cauchy–Schwarz inequality in the same way as in the proof of the previous theorem. □

We remind that at the beginning of Section 8.3 we considered a particular 2-weakly isotropic lognormal spherical random field $A = \exp(T)$ that resulted from a continuous 2-weakly isotropic Gaussian spherical random field T, whose angular power spectrum satisfies that $(C_l l^{1+2\iota+\delta} : l \geq 0)$ is summable for an integer $\iota \geq 0$ and some $\delta \in (0, 2]$. We fixed $\gamma \in (0, \frac{\delta}{2})$ and established that after a modification on a measurable set of zero probability it holds that $A \subset C^{\iota,\gamma}(S^2)$. This was an implication of Theorem 7.7. In other words, the summability of the angular power spectrum of T in terms of ι and δ gives sufficient conditions for higher order Hölder continuity of A. Up to this point, all results are true for $\iota \geq 0$ and we obtained that with a suitable right hand side the first order partial derivatives of the solution u are P-a.s. Hölder continuous. Now we want to exploit the higher order Hölder regularity of the coefficient A and focus on the case $\iota \geq 1$. We observe that higher order Hölder regularity of the coefficient of the operator $-\nabla_{S^2} \cdot (A\nabla_{S^2})$ and of the right hand side transfers to the solution through the Schauder estimates. This was the content of Section 8.2.2. We want to show that the summability of the angular power spectrum of T in terms of ι and

δ also gives sufficient conditions for P-a.s. higher order Hölder continuity of the solution u. The parameter δ influence through $\gamma \in (0, \frac{\delta}{2})$, that we already fixed at the beginning of Section 8.3. The next theorem is the second part of the precise version of Theorem 1.4 from the introduction.

Theorem 8.23. *Let u be the unique, continuous weak solution of the problem in Equation* (8.67) *that satisfies that $\int_{S^2} u \mathrm{d}\sigma = 0$ with right hand side $f \in C^{m,\alpha}(S^2)$ that satisfies that $\int_{S^2} f \mathrm{d}\sigma = 0$ for an integer $m \geq 0$ and $\alpha \in (0,1)$. For all $L \in \mathbb{N}_0$ let u^L be the respective unique, continuous weak solution of the problem in Equation* (8.70) *with the same right hand side that satisfies that $\int_{S^2} u^L \mathrm{d}\sigma = 0$.*
If $\iota \geq 1$, then we obtain that $u \in L^p_P(\Omega, C^{k,\beta}(S^2))$ with $k = \min\{\iota + 1, m + 2\}$ and $\beta = \min\{\gamma, \alpha\}$ for all $p \in (0, \infty)$ and that there exists a constant K_p independently of f and u such that

$$\|u\|_{L^p_P(\Omega, C^{k,\beta}(S^2))} \leq K_p \|f\|_{C^{k-2,\beta}(S^2)}.$$

Moreover if $\iota \geq 1$, then for all $L \in \mathbb{N}_0$, u^L is also in $L^p_P(\Omega, C^{k,\beta}(S^2))$ for all $p \in (0, \infty)$ and there exists a constant K_p independently of L and f and u^L such that

$$\|u^L\|_{L^p_P(\Omega, C^{k,\beta}(S^2))} \leq K_p \|f\|_{C^{k-2,\beta}(S^2)}.$$

Proof. The proof is similar to the proof of the previous theorem. Theorem 8.21 implies that $u \subset C^{1,\gamma}(S^2)$. Note that $A \subset C^{k-1,\beta}(S^2)$ and $f \in C^{k-2,\beta}(S^2)$, because $C^{\iota,\gamma}(S^2) \subset C^{k-1,\beta}(S^2)$ and $C^{m,\alpha}(S^2) \subset C^{k-2,\beta}(S^2)$ both with continuous embeddings. Therefore, we can apply Theorem 8.16 and obtain that $u \subset C^{k,\beta}(S^2)$. Moreover it implies that there exists a constant K independently of A, f and u such that

$$\|u\|_{C^{k,\beta}(S^2)} \leq K \left(\frac{(1 + \|A\|_{C^{\iota,\gamma}(S^2)})^5}{(\min_{x \in S^2} A(x))^{1+\gamma}} \right)^{\frac{18\iota^2}{1-\gamma}} \left(\|u\|_{C^{1,\gamma}(S^2)} + \|f\|_{C^{k-2,\beta}(S^2)} \right).$$

Remark 7.6 implies that $(\min_{x \in S^2} A(x))^{-1} \in L^{p'}_P(\Omega, \mathbb{R})$ for all $p' \in (0, \infty)$, which also holds for $\|u\|_{C^{1,\gamma}(S^2)}$ due to Theorem 8.21 as well as for $\|A\|_{C^{\iota,\gamma}(S^2)}$ due to Theorem 7.7. Moreover Theorem 8.21 implies that for all $p' \in (0, \infty)$ there exists a constant $K_{p'}$ independently of f such that

$$E\left[\|u\|^{p'}_{C^{1,\gamma}(S^2)} \right]^{\frac{1}{p'}} \leq K_{p'} \|f\|_{L^q(S^2, \mathbb{R})},$$

where $q = \frac{2}{1-\gamma}$. Since $C^{k-2,\beta}(S^2) \subset L^q(S^2, \mathbb{R})$ with continuous embedding, a threefold application of the Cauchy–Schwarz inequality implies the first claim.

The proof of the second claim is very similar. For all $L \in \mathbb{N}_0$ we can establish the respective estimate of the $C^{k,\beta}(S^2)$-norm of u^L in terms of A^L and the $C^{1,\gamma}(S^2)$-norm of u^L. We note that the $L^{p'}_P(\Omega, \mathbb{R})$-norm of $(\min_{x \in S^2} A^L(x))^{-1}$, $\|A^L\|_{C^{0,\gamma}(S^2)}$ and $\|u^L\|_{W^{1,q}(S^2)}$ can be bounded independently of $L \in \mathbb{N}$ for all $p' \in (0, \infty)$ due to Remark 7.6, Theorem 7.7 and Theorem 8.21. Also Theorem 8.21 implies that for all $p' \in (0, \infty)$ there exists a constant $K_{p'}$ independently of L and f such that

$$E\left[\|u^L\|^{p'}_{C^{1,\gamma}(S^2)} \right]^{\frac{1}{p'}} \leq K_{p'} \|f\|_{L^q(S^2, \mathbb{R})},$$

where $q = \frac{2}{1-\gamma}$. The second claim is then obtained in the same way as the first claim, which we already proved. □

Corollary 8.24. *Let $\iota \geq 1$, $k = \min\{\iota + 1, m + 2\}$, $\beta = \min\{\gamma, \alpha\}$, u, u^L and f be as in the previous theorem. For all $p \in (0, \infty)$ there exists a constant K_p independently of L, f, u and u^L such that*

$$\|u - u^L\|_{L_P^p(\Omega, C^{k,\beta}(S^2))} \leq K_p \|f\|_{C^{k-2,\beta}(S^2)} \left(\sum_{l>L} C_l l^{1+2(k-1)+\delta} \right)^{\frac{1}{2}}.$$

Proof. We outline the proof of this corollary briefly. We observe that $u - u^L$ satisfies weakly that

$$-\nabla_{S^2} \cdot (A \nabla_{S^2}(u - u^L)) = \nabla_{S^2} \cdot ((A - A^L) \nabla_{S^2} u^L).$$

Since u, $u^L \subset C^{k,\beta}(S^2)$ and A, $A^L \subset C^{k-1,\beta}(S^2)$, we observe that the realizations of the right hand side, i.e. $\nabla_{S^2} \cdot ((A - A^L) \nabla_{S^2} u^L)$, are in $C^{k-2,\beta}(S^2)$. Moreover we observe that there exists a constant K independently of A, A^L and u^L such that

$$\|\nabla_{S^2} \cdot ((A - A^L) \nabla_{S^2} u^L)\|_{C^{k-2,\gamma}(S^2)} \leq K \|A - A^L\|_{C^{k-1,\gamma}(S^2)} \|u^L\|_{C^{k,\gamma}(S^2)}.$$

Therefore we can apply Theorem 8.16 and obtain that exists a constant K independently of A, A^L, u, u^L and f such that

$$\begin{aligned} &\|u - u^L\|_{C^{k,\beta}(S^2)} \\ &\quad \leq K \left(\frac{(1 + \|A\|_{C^{\iota,\gamma}(S^2)})^5}{(\min_{x \in S^2} A(x))^{1+\gamma}} \right)^{\frac{18\iota^2}{1-\gamma}} \left(\|u - u^L\|_{C^{1,\gamma}(S^2)} + \|A - A^L\|_{C^{k-1,\beta}(S^2)} \|u^L\|_{C^{k,\gamma}(S^2)} \right). \end{aligned}$$

The previous theorem implies that the $L_P^{p'}(\Omega, C^{k,\beta}(S^2))$-norm of u^L can be bounded independently of L for all $p' \in (0, \infty)$. For $p \in (0, \infty)$ we consider the $L_P^p(\Omega, \mathbb{R})$-norm the above inequality. The claim of this corollary follows with a fourfold application of the Cauchy–Schwarz inequality and Corollary 8.22 and Theorem 7.7. □

In the case that $\iota \geq 1$ Theorem 8.23 implies that the solution $u(\omega)$ to Equation (8.67) is twice continuously differentiable for all $\omega \in \Omega^*$, where Ω^* is suitable measurable set of full probability. For all test functions $v \in C^1(S^2)$ we can partially integrate in Equation (8.67) with Lemma 8.1 to obtain that

$$-\int_{S^2} \nabla_{S^2} \cdot (A(\omega) \nabla_{S^2} u(\omega)) v \, \mathrm{d}\sigma = \int_{S^2} f v \, \mathrm{d}\sigma$$

for all $\omega \in \Omega^*$ and all $v \in C^1(S^2)$. One could localize this equation as we did in Section 8.2 and argue with the de Bois Reymond lemma on the chart domains that $u(\omega)$ satisfies Equation (8.71) classically for all $\omega \in \Omega^*$.

9. Conclusions

After we introduced 2-weakly spherical random fields, we developed an expansion with respect to the spherical harmonics of 2-weakly isotropic spherical random fields, where we made use of the representation theory of $SO(3)$.
The combination of 2-weakly isotropy and Gaussian distribution gave us an expansion with respect to the real spherical harmonics, which only depends on the angular power spectrum of the spherical random field. Therefore, 2-weakly isotropic Gaussian spherical random fields are characterized with their angular power spectrum. The angular power spectrum induces a symmetric nuclear operator Q, which is interpreted as a covariance operator of a Gaussian measure on $L^2(S^2, \mathbb{R})$. The discussion of the Q-Wiener process with respect to this covariance operator Q as an additive noise term in the heat equation showed that for the heat equation this noise is a very suitable choice, since the problem can be reduced to a decoupled system of stochastic ordinary differential equations.
In Chapter 5 we proved sufficient conditions on the angular power spectrum of a 2-weakly isotropic Gaussian spherical random field such that there exists a Hölder continuous modification. Since the modification is again a 2-weakly isotropic spherical random field with the same expansion, this result can be interpreted as an existence result for continuous 2-weakly isotropic Gaussian spherical random fields. In Chapter 6, we saw that this is part of a deeper principal. We could generalize these conditions such that a continuous 2-weakly isotropic Gaussian spherical random field T has P-a.s. Hölder continuous higher order partial derivatives, where the order and the Hölder coefficient depend on the subscribed condition on T. Moreover we obtained L^p bounds in the stochastic sense of the respective Hölder norms of T as well as convergence of the truncated expansion of T to T in the $L^p_P(\Omega, C^{\iota,\gamma}(S^2))$-norm, where the convergence is controlled with the angular power spectrum. For algebraically bounded angular power spectra, one would obtain a convergence rate, which is independent of p provided that p is finite, but potentially unbounded.
It is also noteworthy that the regularity results do not provide P-a.s. Lipschitz continuity for a continuous 2-weakly isotropic Gaussian spherical random field T or its partial derivatives. Our results readily imply differentiability of one order higher with Hölder continuous partial derivatives, i.e. assuming that for an integer $\iota \geq 0$ we are interested in the needed decay of the angular power spectrum of T such that P-a.s. realizations of T are in $C^{\iota,1}(S^2)$, then we have to demand the decay that implies P-a.s. the membership in $C^{\iota+1,\varepsilon}(S^2)$ for an arbitrarily small $\varepsilon > 0$ and employ the embedding $C^{\iota+1,\varepsilon}(S^2) \subset C^{\iota,1}(S^2)$. An explanation for this gap can be seen in the fact that we can provide a sufficient condition on the decay of the angular power spectrum to obtain that P-a.s. realizations of T are in $W^{\iota,q}(S^2)$ for all $q \in [1, \infty)$. Since the respective $W^{\iota,q}(S^2)$-norms are not uniformly bounded with respect to q, it seems that it is not possible to conclude the membership in $W^{\iota,\infty}(S^2)$ with the given decay of the angular power spectrum. The reason behind the unboundedness with respect to q of this norm is that higher order moments of the normal distribution are not bounded independently of the order.
Since the regularity can be transferred to log-normally distributed 2-weakly isotropic spherical random fields, we were able to discuss random elliptic partial differential equations on

the sphere, where we considered the elliptic operator $-\nabla_{S^2} \cdot (A \nabla_{S^2})$ for a 2-weakly isotropic log-normal spherical random field A. The Schauder theory turned out to be very suitable to discuss the regularity of the solution. We developed that with a sufficiently smooth right hand side the decay of the angular power spectrum of the continuous 2-weakly Gaussian spherical random field T that defined the log-normal coefficient $A = \exp(T)$ in the elliptic operator precisely implies the path regularity of the random solution with one order more of differentiability than the coefficient, i.e. if the decay of the angular power spectrum implies that P-a.s. the realizations of A are in $C^{\iota,\gamma}(S^2)$ for an integer $\iota \geq 0$ then P-a.s. the realization of the solution u are in $C^{\iota+1,\gamma}(S^2)$. For all integers $L \geq 0$ this analysis also applies to solutions u^L with respect to the coefficient A^L that results from the truncated expansion of the respective continuous 2-weakly isotropic Gaussian spherical random field. Since we provided a detailed convergence analysis of u^L converging to u for $L \to \infty$ in Section 8.3.1 and Section 8.3.2 and bounds for u^L independently of L, the way for further numerical analysis in terms of finite element and Monte Carlo methods is smoothed.
The regularity theory in Section 8.2 could be carried out similarly on closed compact manifolds. Also the discussion of random elliptic partial differential equation similarly applies on closed compact manifolds, provided that the log-normal coefficient is given and satisfies the conditions to form an elliptic operator and obeys the same regularity.

Another future aspect might be to investigate questions like, under which conditions a spherical random field or a 2-weakly isotropic spherical random field lies P-a.s. in some function space over the sphere. In this project we found conditions such that a continuous 2-weakly isotropic Gaussian spherical random field has P-a.s. Hölder continuous higher order partial derivatives and that it lies in the space of square integrable functions, the latter came without much work from the definition of 2-weakly isotropy. To investigate the membership of a 2-weakly isotropic spherical random field in other function spaces, it might be interesting to use a different definition, which does not rely on pointwise evaluation of the spherical random field, i.e. for all $x \in S^2$, $T(x)$ satisfies the 2-weakly isotropic property in Definition 3.2. Because not in all function spaces the pointwise evaluation of functions is well-defined.
If we have a similar setup as in this project, i.e. $(\Omega, \mathcal{A}, P)$ denotes a probability space and B is some function space over the sphere, we could take as definition: a spherical random field T, which takes values in some function space B and is in $L^2_P(\Omega, B)$ is 2-weakly isotropic if it satisfies that for all $\eta \in B^*$ and for all $g \in SO(3)$

$$E[{}_{B^*}\langle \eta, T \rangle_B] = E[{}_{B^*}\langle \eta, D(g)T \rangle_B], \tag{9.1}$$

and for all $\eta, \psi \in B^*$ and for all $g \in SO(3)$

$$E[{}_{B^*}\langle \eta, T \rangle_B \; {}_{B^*}\langle \psi, T \rangle_B] = E[{}_{B^*}\langle \eta, D(g)T \rangle_B \; {}_{B^*}\langle \psi, D(g)T \rangle_B], \tag{9.2}$$

where B^* denotes the dual space of B, ${}_{B^*}\langle ., . \rangle_B$ denotes the dual pairing of the function space B and $D(g)$ is the left regular representation, which was introduced in Chapter 2. Since we demanded that $T \in L^2_P(\Omega, B)$ the conditions in Equation (9.1) and Equation (9.2) are well-defined. The definition is motivated because in some function spaces the pointwise evaluation, which is commonly denoted by δ_x for some $x \in S^2$, is continuous and therefore an element of B^*. In these cases this new definition is a bit more restrictive than the definition in this project for 2-weakly isotropy, because for $\eta = \delta_x$ and $\psi = \delta_y$ it reduces to the definition in this project.

Of course this suggestion as the definition for 2-weakly isotropy raises some non-trivial question for example the existence of such a spherical random field depending on the function space B or if there are also expansions in some basis, which could be helpful for simulation purposes. The proof of the latter, the expansion of a 2-weakly isotropic spherical random field, relied on the property of 2-weakly isotropy being defined pointwise. So similar results for this new definition could involve quite some effort. But this definition puts the discussion more in the framework of function spaces and if one could overcome some difficulties other questions could become more accessible since these function spaces often also provide useful technology.

A. MATLAB code for the 2-weakly isotropic Gaussian spherical random field

This is an implementation of the real spherical harmonics:

```
function Y=Y_lm_full(l,N)
%this function evaluates the real spherical harmonics for fixed l and
%m=-l,...,l. N is connected to the resolution.


theta = linspace(0,pi,N);
phi = linspace(0,2*pi,2*N)';
Y = zeros(2*N,N,2*l+1);

%evaluate the normed associated legendre polynomials
L = sqrt(1/(2*pi))*legendre(l,cos(theta),'norm');

for m=-l:l
    if (m>0)
        Y(:,:,l+m+1) = sqrt(2)*cos(phi*m)*L(m+1,:);
    elseif (m<0)
        Y(:,:,l+m+1) = sqrt(2)*sin(phi*m)*L(-m+1,:);
    else
        Y(:,:,l+m+1) = ones(2*N,1)*L(1,:);
    end
end

end
```

This is an implementation of the truncated 2-weakly isotropic Gaussian random field:

```
function [x,y,z,T]=RandomField(alpha,N,L)
%this function realizes a 2-weakly isotropic spherical random field
%alpha cotrolls the decay of the angular power C_l=(1+l)^(-alpha)
%L is the truncation
%N is the space discretization
%the function 'Y_lm_full' is needed

T = zeros(2*N,N);
theta = linspace(0,pi,N);
phi = linspace(0,2*pi,2*N)';

T = randn(1,1)*1/2*sqrt(1/pi)*T;

%simulate T up to order L
C = (1:L+1).^(-alpha);
for l=0:L
```

```
    beta=randn(1,2*l+1);
    Y = Y_lm_full(l,N);
    for m=-l:l
        T = T + sqrt(C(l+1))*beta(1,l+m+1)*Y(:,:,l+m+1);
    end
end

%set the coordinates
x = (cos(phi)*sin(theta));
y = (sin(phi)*sin(theta));
z = (ones(2*N,1)*cos(theta));

%plot the truncated T
surf(x,y,z,T)
shading flat
colorbar

end
```

B. MATLAB code for the stochastic heat equation with 2-weakly isotropic Q-Wiener noise

This function computes the coefficients of the expansion in the real spherical harmonics for a function f in $L^2(S^2, \mathbb{R})$:

```
function A=coeff(f,l,N)
%this function computes the coefficients of a real L^2(S^2) function for
%fixed l and space discretization N
%the function 'Y_lm_full' is needed.


A = zeros(2*l+1,1);
theta = linspace(0,pi,N);
%phi = linspace(0,2*pi,2*N)';


Y=Y_lm_full(l,N);
    for m=-l:l
    %using trapezoidal rule
        I = f.*Y(:,:,l+m+1).*(ones(2*N,1)*sin(theta));

        A(l+1+m) = sum(sum(I))*(2*pi^2)/(2*N^2);
    end
end
```

This is an implementation of the truncated solution of the stochastic heat equation with 2-weakly isotropic Q-Wiener noise Equation (1.1):

```
function [V,x,y,z,W]=st_heat_eq
% this is an implementation of the stochastic heat equation on the sphere,
% as initial condition we take the for L=20 truncated expansion of an
% indicator function f.
% M is time discretization
% N is the space discretization
% T is the time horizon
%
% to run this function, the functions 'Y_lm_full' and 'coeff' are needed.
% W is the solution with respect to the coordinates 'x,y,z' in the time
% horizon [0,T]
% V is a video generated with the single plots of W

clear all
close all
alpha = 3;
T = 3;
M = 100;
```

```
N = 200;
L = 50;
h = T/M;

theta = linspace(0,pi,N);
phi = linspace(0,2*pi,2*N)';

%allocate memory
W = zeros(2*N,N,M+1);


%set the initial condition
f = zeros(2*N,N);
f(N/2:3*N/2,N/4:3*N/4) = 1;

%set the angular power spectrum
C = (1:L+1).^(-alpha);


for l=0:L
    A = coeff(f,l,N);
    beta = randn(M+1,2*l+1);
    Y = Y_lm_full(l,N);
    for m=-l:l
        for j=0:M
            W(:,:,j+1) = W(:,:,j+1) + ...
               Y(:,:,l+m+1)*(A(l+m+1)*exp(-l*(l+1)*h*j) +...
            sqrt(C(l+1))*sqrt(h)*exp(-l*(l+1)*(h*j - ...
               h*(0:j)))*beta(1:(j+1),l+m+1));
        end
    end
end

%define the coordinates
x = (cos(phi)*sin(theta));
y = (sin(phi)*sin(theta));
z = (ones(2*N,1)*cos(theta));

%create the video
for j=1:(M+1)
    surf(x,y,z,W(:,:,j));
    shading flat
    V(j)= getframe;
end

end
```

C. Interpolation theory

In the following we will summarize the needed material from interpolation theory. We will very briefly define the main objects and state the needed results. We will follow the exposition of Tartar in [24] and will cite some specific results from the book of Triebel [25]. This short exposition however requires some prior knowledge of the reader.

Let $(E_0, \|\cdot\|_{E_0}), (E_1, \|\cdot\|_{E_1})$ be two normed vector spaces which are both continuously imbedded in another topological vector space $\mathcal{E}$. Elements in E_0 will be denoted by a_0 and elements in E_1 will be denoted by a_1. We obtain the spaces

$$\begin{aligned} &E_0 \cap E_1 \quad \text{equipped with the norm} \quad \|a\|_{E_0 \cap E_1} = \max\{\|a\|_{E_0}, \|a\|_{E_1}\}, \\ &E_0 + E_1 \quad \text{equipped with the norm} \quad \|a\|_{E_0+E_1} = \inf_{a=a_0+a_1} \{\|a_0\|_{E_0} + \|a_1\|_{E_1}\}. \end{aligned}$$

E_0, E_1 is called an interpolation couple. We will follow the so called K-method and define for $a \in E_0 + E_1$ and $t > 0$ the so called K-functional

$$K(t, a, E_0, E_1) = \inf_{a=a_0+a_1} \{\|a_0\|_{E_0} + t\|a_1\|_{E_1}\}.$$

The dependency on the spaces in the K-functional will be disregarded, whenever they are clear. We define the interpolation spaces of E_0 and E_1.

Definition C.1. *For $\theta \in (0,1)$ and $p \in [1, \infty)$ we define the space*

$$(E_0, E_1)_{\theta,p} = \left\{ a \in E_0 + E_1 : \int_0^\infty t^{-\theta p} K(t,a)^p \frac{\mathrm{d}t}{t} < \infty \right\}$$

and for $p = \infty$ we define

$$(E_0, E_1)_{\theta,\infty} = \left\{ a \in E_0 + E_1 : \sup_{t>0} t^{-\theta} K(t,a) < \infty \right\}.$$

For $a \in (E_0, E_1)_{\theta,p}$ the respective norms are given by

$$\|a\|_{(E_0,E_1)_{\theta,p}} = \left(\int_0^\infty t^{-\theta p} K(t,a)^p \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}} \qquad \text{and} \qquad \|a\|_{(E_0,E_1)_{\theta,\infty}} = \sup_{t>0} t^{-\theta} K(t,a).$$

The interpolation spaces and the K-functional have many interesting properties, we however focus on what is necessary.

Lemma C.2. *For $\theta \in (0,1)$ and $1 \le p \le q \le \infty$ it holds that $(E_0, E_1)_{\theta,p} \subset (E_0, E_1)_{\theta,q}$ with continuous embedding.*

Proof. This is Lemma 22.2 in [24]. □

If we consider a second interpolation couple F_0, F_1 we can interpolate linear bounded operators.

Lemma C.3. *Let A be a linear mapping that maps from E_0+E_1 to F_0+F_1. If A maps E_0 into F_0 such that $\|Ax_0\|_{F_0} \leq M_0\|x_0\|_{E_0}$ for all $x_0 \in E_0$ and if A maps E_1 into F_1 such that $\|Ax_1\|_{F_1} \leq M_1\|x_1\|_{E_1}$ for all $x_1 \in E_1$, then A is a linear bounded mapping from $(E_0, E_1)_{\theta,p}$ to $(F_0, F_1)_{\theta,p}$ for all $\theta \in (0,1)$ and $p \in [1,\infty]$. Moreover we obtain a bound on the operator norm of A, i.e. for all $x \in (E_0, E_1)_{\theta,p}$ it holds that*

$$\|Ax\|_{(F_0,F_1)_{\theta,p}} \leq M_0^{1-\theta} M_1^{\theta} \|x\|_{(E_0,E_1)_{\theta,p}}.$$

Proof. This is Lemma 22.3 in [24]. □

With this general setup we are interested in the interpolation spaces that result from the sequence space $\ell^2_{k,\delta}(\mathbb{N})$ for integers $k \geq 0$ and $\delta \in (0,2]$ and from $L^p_P(\Omega, C^{\iota,\gamma}(S^2))$ for integers $\iota \geq 0$, $p \in (0,\infty)$ and $\gamma \in (0,1)$, where $(\Omega, \mathcal{A}, P)$ is a probability space. Note that these spaces were introduced in Chapter 6.

Lemma C.4. *For an integer $k \geq 0$ and $\delta \in (0,2]$ it holds with equivalent norms that*

$$\left(\ell^2_{k,\delta}(\mathbb{N}), \ell^2_{k+2,\delta}(\mathbb{N})\right)_{\frac{1}{2},2} = \ell^2_{k+1,\delta}(\mathbb{N}).$$

Proof. Since the sequence spaces can be interpreted as a L^p-space with the counting measure, we can cite a result from Triebel [25] for L^p-spaces with weights. In the notation of [25] we have that $\ell^2_{k,\delta}(\mathbb{N}) = L_{2,w_0^2}(\mathbb{N})$ and $\ell^2_{k+2,\delta}(\mathbb{N}) = L_{2,w_1^2}(\mathbb{N})$, where $w_0(x) = x^{(1+2k+\delta)/2}$ and $w_1(x) = x^{(1+2(k+2)+\delta)/2}$. Theorem 1.18.5 in [25] implies that with equivalent norms it holds that

$$\left(\ell^2_{k,\delta}(\mathbb{N}), \ell^2_{k+2,\delta}(\mathbb{N})\right)_{\frac{1}{2},2} = \left(L_{2,w_0^2}(\mathbb{N}), L_{2,w_1^2}(\mathbb{N})\right)_{\frac{1}{2},2} = L_{2,w^2}(\mathbb{N}) = \ell^2_{k+1,\delta}(\mathbb{N}),$$

where $w^2(x) = w_0(x)w_1(x) = x^{1+2(k+1)+\delta}$. □

Lemma C.5. *For an integer $\iota \geq 0$, $p \in [2,\infty)$ and $\gamma \in (0,1)$ it holds with continuous embedding that*

$$\left(L^p_P(\Omega, C^{\iota,\gamma}(S^2)), L^p_P(\Omega, C^{\iota+2,\gamma}(S^2))\right)_{\frac{1}{2},2} \subset L^p_P(\Omega, C^{\iota+1,\gamma}(S^2)).$$

Proof. In this proof we also cite some tools from Triebel [25]. Since $C^{\iota+2,\gamma}(S^2) \subset C^{\iota,\gamma}(S^2)$ these two Hölder spaces are an interpolation couple. With Lemma C.2 we obtain that with continuous embedding it holds that

$$\left(L^p_P(\Omega, C^{\iota,\gamma}(S^2)), L^p_P(\Omega, C^{\iota+2,\gamma}(S^2))\right)_{\frac{1}{2},2} \subset \left(L^p_P(\Omega, C^{\iota,\gamma}(S^2)), L^p_P(\Omega, C^{\iota+2,\gamma}(S^2))\right)_{\frac{1}{2},p}.$$

Theorem 1.18.4 in [25] is applicable and implies that with equivalent norms it holds that

$$\left(L^p_P(\Omega, C^{\iota,\gamma}(S^2)), L^p_P(\Omega, C^{\iota+2,\gamma}(S^2))\right)_{\frac{1}{2},p} = L^p_P\left(\Omega, \left(C^{\iota,\gamma}(S^2), C^{\iota+2,\gamma}(S^2)\right)_{\frac{1}{2},p}\right).$$

If $B \subset \mathbb{R}^2$ is a bounded domain with smooth boundary Theorem 4.5.1 in [25] implies that with equivalent norms

$$(C^{\iota,\gamma}(\overline{B}), C^{\iota+2,\gamma}(\overline{B}))_{\frac{1}{2},\infty} = C^{\iota+1,\gamma}(\overline{B}).$$

In particular, there exists a constant K such that for all $\tilde{f} \in (C^{\iota,\gamma}(\overline{B}), C^{\iota+2,\gamma}(\overline{B}))_{\frac{1}{2},\infty}$ it holds that

$$\|\tilde{f}\|_{C^{\iota+1,\gamma}(\overline{B})} \leq K\|\tilde{f}\|_{(C^{\iota,\gamma}(\overline{B}),C^{\iota+2,\gamma}(\overline{B}))_{\frac{1}{2},\infty}}. \tag{C.1}$$

In Section 6.1 we defined the Hölder spaces on S^2 with respect to an atlas. Let $(V_j, \beta_j : j \in \mathcal{J})$ be a finite, smooth atlas of S^2 such that the boundary of $\beta_j^{-1}(V_j)$ is smooth for all $j \in \mathcal{J}$ and let Ψ be a partition of unity subordinate to the open cover $(V_j : j \in \mathcal{J})$ of S^2. We arbitrarily fix $f \in (C^{\iota,\gamma}(S^2), C^{\iota+2,\gamma}(S^2))_{\frac{1}{2},\infty}$ and $j \in \mathcal{J}$ and apply the mentioned interpolation result of Hölder spaces on subdomains of Euclidean space, in particular Inequality (C.1), to obtain that

$$\begin{aligned}
&\|f\|_{(C^{\iota,\gamma}(S^2),C^{\iota+2,\gamma}(S^2))_{\frac{1}{2},\infty}} \\
&\quad= \sup_{t>0} t^{-\frac{1}{2}} \left(\inf_{f=f_1+f_2} \left\{ \|f_1\|_{C^{\iota,\gamma}(S^2)} + t\|f_2\|_{C^{\iota+2,\gamma}(S^2)} \right\} \right) \\
&\quad\geq \sup_{t>0} t^{-\frac{1}{2}} \left(\inf_{f=f_1+f_2} \left\{ \|(f_1\Psi_j)_{\beta_j}\|_{C^{\iota,\gamma}(\overline{\beta_j^{-1}(V_j)})} + t\|(f_2\Psi_j)_{\beta_j}\|_{C^{\iota+2,\gamma}(\overline{\beta_j^{-1}(V_j)})} \right\} \right) \\
&\quad\geq \sup_{t>0} t^{-\frac{1}{2}} \left(\inf_{(f\Psi_j)_{\beta_j}=f_{1,j}+f_{2,j}} \left\{ \|f_{1,j}\|_{C^{\iota,\gamma}(\overline{\beta_j^{-1}(V_j)})} + t\|f_{2,j}\|_{C^{\iota+2,\gamma}(\overline{\beta_j^{-1}(V_j)})} \right\} \right) \\
&\quad= \|(f\Psi_j)_{\beta_j}\|_{(C^{\iota,\gamma}(\overline{\beta_j^{-1}(V_j)}),C^{\iota+2,\gamma}(\overline{\beta_j^{-1}(V_j)}))_{\frac{1}{2},\infty}} \\
&\quad\geq K^{-1}\|(f\Psi_j)_{\beta_j}\|_{C^{\iota+1,\gamma}(\overline{\beta_j^{-1}(V_j)})},
\end{aligned}$$

where the infimum is taken over the respective functions such that $f_1 \in C^{\iota,\gamma}(S^2)$, $f_2 \in C^{\iota+2,\gamma}(S^2)$, $f_{1,j} \in C^{\iota,\gamma}(\overline{\beta_j^{-1}(V_j)})$ and $f_{2,j} \in C^{\iota+2,\gamma}(\overline{\beta_j^{-1}(V_j)})$ is maintained. This argument can be repeated for all $j \in \mathcal{J}$. Since the atlas is finite, we conclude that $f \in C^{\iota+1,\gamma}(S^2)$ and that there exists a constant K such that for all $f \in (C^{\iota,\gamma}(S^2), C^{\iota+2,\gamma}(S^2))_{\frac{1}{2},\infty}$ it holds that

$$\|f\|_{C^{\iota+1,\gamma}(S^2)} \leq K\|f\|_{(C^{\iota,\gamma}(S^2),C^{\iota+2,\gamma}(S^2))_{\frac{1}{2},\infty}}.$$

Hence, it holds with continuous embedding that

$$\left(C^{\iota,\gamma}(S^2), C^{\iota+2,\gamma}(S^2)\right)_{\frac{1}{2},\infty} \subset C^{\iota+1,\gamma}(S^2).$$

We apply Lemma C.2 and obtain that

$$L^p_P\left(\Omega, \left(C^{\iota,\gamma}(S^2), C^{\iota+2,\gamma}(S^2)\right)_{\frac{1}{2},p}\right) \subset L^p_P(\Omega, C^{\iota+1,\gamma}(S^2)),$$

which implies the claim together with the first two inequalities. □

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