

Micromagnetism

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October 30, 2003

1 The basic equations

The fundamental equations governing the evolution of the magnetization \mathbf{M} in a ferromagnetic material are the *Landau-Livshits-Gilbert equations*:

$$\frac{d\mathbf{M}}{dt} = -\gamma \mathbf{M} \times \mathbf{H}_{\text{eff}} - \frac{\gamma\alpha}{M_s} \mathbf{M} \times (\mathbf{M} \times \mathbf{H}_{\text{eff}}) . \quad (1)$$

Here, γ is the gyromagnetic constant with physical units $\text{m}^2\text{V}^{-1}\text{s}^{-2}$, whereas α is dimensionless and plays the role of a phenomenological damping parameter. The parameter M_s represents the saturation magnetization and has dimension Am^{-1} . The equations are valid in the domain $\Omega \subset \mathbb{R}^3$ occupied by the ferromagnetic material.

The effective magnetic field \mathbf{H}_{eff} is the first variation of the Gibbs free energy \mathcal{E} with respect to the magnetization. In the presence of an exciting magnetic field \mathbf{H}_e , it reads

$$\mathcal{E}(\mathbf{M}) = \int_{\Omega} \frac{\bar{A}}{M_s^2} |\mathbf{grad} \mathbf{M}|^2 + \Phi(\mathbf{M}/M_s) - 2\mu_0 \mathbf{H}_e \cdot \mathbf{M} \, d\mathbf{x} + \frac{1}{2}\mu_0 \int_{\mathbb{R}^3} |\mathbf{grad} \psi|^2 \, d\mathbf{x} . \quad (2)$$

The Gibbs free energy is composed of several parts:

- the exchange term modelling the coupling between neighboring elementary magnets.
- the anisotropy energy with

$$\Phi(\mathbf{u}) = Q(1 - (\mathbf{d} \cdot \mathbf{u})^2), \quad (3)$$

where the unit vector $\mathbf{d} \in \mathbb{R}^3$ denotes the “easy axis”.

- the stray field term $\int_{\mathbb{R}^3} |\mathbf{grad} \psi|^2 \, d\mathbf{x}$ representing the magnetic energy of the magnetic field due to the magnetization. Here, ψ is the scalar magnetic potential, which, besides decay conditions at ∞ , satisfies

$$\mu_0 \Delta \psi = \text{div} \mathbf{M} \quad \text{in } \mathbb{R}^3 . \quad (4)$$

Scaling the Landau-Livshits equation and the Gibbs free energy we arrive at the non-dimensional form for the scaled magnetization $\mathbf{m} = (m_1, m_2, m_3)$.

$$\frac{d\mathbf{m}}{dt} = -\mathbf{m} \times \mathbf{h}_{\text{eff}} - \alpha \mathbf{m} \times (\mathbf{m} \times \mathbf{h}_{\text{eff}}) , \quad (5)$$

where the effective magnetic field can be obtained as

$$\mathbf{h}_{\text{eff}} = -\frac{\partial \mathcal{E}(\mathbf{m})}{\partial \mathbf{m}} . \quad (6)$$

The scaled free energy reads

$$\mathcal{E}(\mathbf{m}) = \frac{1}{2} \int_{\Omega} \eta |\mathbf{grad} \mathbf{m}|^2 + Q(1 - (\mathbf{d} \cdot \mathbf{m})^2) - 2\mathbf{h}_e \cdot \mathbf{m} \, d\mathbf{x} + \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{m}^0|^2 \, d\mathbf{x} ,$$

where \mathbf{m}^0 is the **curl**-free component of the $\mathbf{L}^2(\mathbb{R}^3)$ -orthogonal *Helmholtz decomposition* of \mathbf{m} :

$$\mathbf{m}^0 = \mathbf{grad} \psi : \int_{\mathbb{R}^3} \mathbf{grad} \psi \cdot \mathbf{grad} \varphi \, d\mathbf{x} = \int_{\mathbb{R}^3} \mathbf{m} \cdot \mathbf{grad} \varphi \, d\mathbf{x} \quad \forall \varphi .$$

The equations have to be supplied with initial conditions and boundary conditions on $\partial\Omega$. It will turn out that

$$\frac{\partial m_i}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \quad i = 1, 2, 3 , \quad (7)$$

are the right natural boundary conditions. The symbol \mathbf{n} stands for the exterior normal vector on $\partial\Omega$.

2 Underlying balance laws

We form the inner product of (5) with \mathbf{m} and obtain

$$\frac{d\mathbf{m}}{dt} \cdot \mathbf{m} = 0 \quad \Rightarrow \quad \frac{d|\mathbf{m}|^2}{dt} = 2 \frac{d\mathbf{m}}{dt} \cdot \mathbf{m} = 0 .$$

Hence $|\mathbf{m}|$ is invariant in time. We will make the assumption that

$$|\mathbf{m}(\mathbf{x}, 0)| = 1 \quad \forall \mathbf{x} \in \Omega . \quad (8)$$

This will hold for any time and reflects the assumption of microscopic magnetic saturation. Multiplying (5) with \mathbf{h}_{eff} and using the identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c} \cdot (\mathbf{a} \cdot \mathbf{b}) \quad (9)$$

we get

$$\frac{d\mathbf{m}}{dt} \cdot \mathbf{h}_{\text{eff}} = -\alpha(\mathbf{m}(\mathbf{m} \cdot \mathbf{h}_{\text{eff}}) - \mathbf{h}_{\text{eff}} \cdot \mathbf{h}_{\text{eff}}) = -\alpha((\mathbf{m} \cdot \mathbf{h}_{\text{eff}})^2 - |\mathbf{h}_{\text{eff}}|^2) = \alpha|\mathbf{m} \times \mathbf{h}_{\text{eff}}|^2 .$$

This means that in the case of no excitation, that is, $\mathbf{H}_e = 0$,

$$\frac{d\mathcal{E}(\mathbf{m}(t))}{dt} = \left\langle \frac{\partial \mathcal{E}(\mathbf{m})}{\partial \mathbf{m}}, \frac{d\mathbf{m}(t)}{dt} \right\rangle = \int_{\Omega} -\mathbf{h}_{\text{eff}} \cdot \frac{d\mathbf{m}}{dt} \, d\mathbf{x} = -\alpha \int_{\Omega} |\mathbf{m} \times \mathbf{h}_{\text{eff}}|^2 \, d\mathbf{x} .$$

It reveals that we are faced with a dissipative process and that α controls the strength of dissipation. In a sense, only the second term on the right hand side of (5) has a dissipative nature and forces the system towards an equilibrium state.

3 Reformulation

We can take the cross product of (5) with \mathbf{m} and get, using (9) and $|\mathbf{m}| = 1$,

$$\begin{aligned}\mathbf{m} \times \frac{d\mathbf{m}}{dt} &= -\mathbf{m} \times (\mathbf{m} \times \mathbf{h}_{\text{eff}}) + \alpha(\mathbf{m}(\mathbf{m} \cdot \mathbf{h}_{\text{eff}}) - \mathbf{h}_{\text{eff}}) \times \mathbf{m} \\ &= -\mathbf{m} \times (\mathbf{m} \times \mathbf{h}_{\text{eff}}) + \alpha(\mathbf{m} \times \mathbf{h}_{\text{eff}}) .\end{aligned}$$

This can be used to replace the term $\mathbf{m} \times (\mathbf{m} \times \mathbf{h}_{\text{eff}})$ in (5), which leads to the *Gilbert form* of the equations

$$\frac{d\mathbf{m}}{dt} - \alpha \mathbf{m} \times \frac{d\mathbf{m}}{dt} = -(1 + \alpha^2) \mathbf{m} \times \mathbf{h}_{\text{eff}} .$$

4 Timestepping

Given suitable initial data $\mathbf{m}(\mathbf{x}, 0)$ that satisfy (8), we aim to solve (5). We opt for a very simple single step method, related to the implicit trapezoidal rule (method of Heun). We assume a fixed timestep $\tau > 0$ and want to approximate the functions $\mathbf{m}(t)$ at instances $t_n = n\tau$, $n \in \mathbb{N}$. To that end we replace

$$\frac{d\mathbf{m}}{dt}(t_{n+\frac{1}{2}}) \approx \delta \mathbf{m}^{n+\frac{1}{2}} := \frac{\mathbf{m}^{n+1} - \mathbf{m}^n}{\tau} \quad , \quad \mathbf{m}(t_{n+\frac{1}{2}}) \approx \bar{\mathbf{m}}^{n+\frac{1}{2}} := \frac{\mathbf{m}^{n+1} + \mathbf{m}^n}{2} ,$$

where, for the sake of brevity, $\mathbf{m}^n \approx \mathbf{m}(t_n)$. It leads to the timestepping scheme

$$\delta \mathbf{m}^{n+\frac{1}{2}} = -\bar{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{h}_{\text{eff}}^{n+\frac{1}{2}} - \alpha \bar{\mathbf{m}}^{n+\frac{1}{2}} \times (\bar{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{h}_{\text{eff}}^{n+\frac{1}{2}}) . \quad (10)$$

Here the appropriate definition of the effective magnetic field is

$$\mathbf{h}_{\text{eff}}^{n+\frac{1}{2}} := \eta \Delta \bar{\mathbf{m}}^{n+\frac{1}{2}} + Q \mathbf{d} (\mathbf{d} \cdot \bar{\mathbf{m}}^{n+\frac{1}{2}}) - \mathbf{grad} \bar{\psi}^{n+\frac{1}{2}} + \mathbf{h}_e(t_{n+\frac{1}{2}}) , \quad (11)$$

with

$$-\Delta \bar{\psi}^{n+\frac{1}{2}} = -\text{div} \bar{\mathbf{m}}^{n+\frac{1}{2}} \quad \text{in } \mathbb{R}^3 .$$

In the case $\mathbf{h}_e = 0$, this implies

$$\begin{aligned}\mathcal{E}(\mathbf{m}^{n+1}) - \mathcal{E}(\mathbf{m}^n) &= \frac{1}{2} \int_{\Omega} \eta |\mathbf{grad} \mathbf{m}^{n+1}|^2 - |\mathbf{grad} \mathbf{m}^n|^2 + Q(1 - (\mathbf{d} \cdot \mathbf{m}^{n+1})^2) \\ &\quad - Q(1 - (\mathbf{d} \cdot \mathbf{m}^n)^2) \, d\mathbf{x} + \frac{1}{2} \int_{\mathbb{R}^3} |(\mathbf{m}^{n+1})^0|^2 - |(\mathbf{m}^n)^0|^2 \, d\mathbf{x} \\ &= \int_{\Omega} \eta \mathbf{grad}(\mathbf{m}^{n+1} - \mathbf{m}^n) \cdot \mathbf{grad} \bar{\mathbf{m}}^{n+\frac{1}{2}} \\ &\quad - Q(\mathbf{d} \cdot (\mathbf{m}^{n+1} - \mathbf{m}^n)(\mathbf{d} \cdot \bar{\mathbf{m}}^{n+\frac{1}{2}}) \, d\mathbf{x} + \int_{\mathbb{R}^3} (\mathbf{m}^{n+1} - \mathbf{m}^n) \cdot (\bar{\mathbf{m}}^{n+\frac{1}{2}})^0 \, d\mathbf{x} \\ &= -\tau \int_{\Omega} \delta \mathbf{m}^{n+\frac{1}{2}} \cdot \mathbf{h}_{\text{eff}}^{n+\frac{1}{2}} \, d\mathbf{x} = -\tau \alpha \int_{\Omega} |\bar{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{h}_{\text{eff}}^{n+\frac{1}{2}}|^2 \, d\mathbf{x} ,\end{aligned} \quad (12)$$

where the $\mathbf{L}^2(\mathbb{R}^3)$ -orthogonality of the Helmholtz decomposition has been exploited. This is the discrete analogue of the energy balance law.

We remark that timestepping can also be cast in Gilbert-form

$$\delta \mathbf{m}^{n+\frac{1}{2}} - \alpha \overline{\mathbf{m}}^{n+\frac{1}{2}} \times \delta \mathbf{m}^{n+\frac{1}{2}} = -(1 + \alpha^2) \overline{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{h}_{\text{eff}}^{n+\frac{1}{2}}. \quad (13)$$

5 Spatial discretization

Due to the rapid decay of the field $\overline{\psi}^{n+\frac{1}{2}}$ away from Ω it is possible to restrict the computations to a bounded domain D , whose boundary is sufficiently far away from Ω . This computational domain D will be equipped with a triangulation that resolves the boundary of Ω .

In order to apply a finite element method we have to cast (10) and (11) into weak form. We set out from

$$\begin{aligned} \delta \mathbf{m}^{n+\frac{1}{2}} - \alpha \overline{\mathbf{m}}^{n+\frac{1}{2}} \times \delta \mathbf{m}^{n+\frac{1}{2}} = & - (1 + \alpha^2) \overline{\mathbf{m}}^{n+\frac{1}{2}} \times \left(\eta \Delta \overline{\mathbf{m}}^{n+\frac{1}{2}} + Q \mathbf{d} (\mathbf{d} \cdot \overline{\mathbf{m}}^{n+\frac{1}{2}}) \right. \\ & \left. - \mathbf{grad} \overline{\psi}^{n+\frac{1}{2}} + \mathbf{h}_e(t_{n+\frac{1}{2}}) \right) \quad \text{in } \Omega, \end{aligned} \quad (14)$$

$$\begin{aligned} \frac{\partial m_i}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega, \quad i = 1, 2, 3, \\ -\Delta \overline{\psi}^{n+\frac{1}{2}} = -\text{div } \overline{\mathbf{m}}^{n+\frac{1}{2}} \quad \text{in } D, \\ \overline{\psi}^{n+\frac{1}{2}} = 0 \quad \text{on } \partial D. \end{aligned} \quad (15)$$

We seek \mathbf{m}^n in the space $(H^1(\Omega))^3$, whereas $\overline{\psi}^{n+\frac{1}{2}}$ is taken from $H_0^1(D)$. We multiply (14) by a test vector \mathbf{v} and use the Green identity and the boundary conditions for \mathbf{m} on $\partial\Omega$, to get the weak form. For the term containing the laplacian we have:

$$\int_{\Omega} (\overline{\mathbf{m}}^{n+\frac{1}{2}} \times \Delta \overline{\mathbf{m}}^{n+\frac{1}{2}}) \cdot \mathbf{v} \, d\mathbf{x} = \int_{\partial\Omega} (\mathbf{v} \times \overline{\mathbf{m}}^{n+\frac{1}{2}}) \cdot \underbrace{\mathbf{grad} \overline{\mathbf{m}}^{n+\frac{1}{2}}}_{0 \text{ on } \partial\Omega} \cdot \mathbf{n} \, dS + \int_{\Omega} \mathbf{grad} (\overline{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{v}) \cdot \mathbf{grad} \overline{\mathbf{m}}^{n+\frac{1}{2}} \, d\mathbf{x}$$

Thus, the weak form of (14) and (15) is:

$$\begin{aligned} & \frac{1}{1 + \alpha^2} \int_{\Omega} (\delta \mathbf{m}^{n+\frac{1}{2}} - \alpha \overline{\mathbf{m}}^{n+\frac{1}{2}} \times \delta \mathbf{m}^{n+\frac{1}{2}}) \cdot \mathbf{v} \, d\mathbf{x} = \\ & = \int_{\Omega} \eta \mathbf{grad} (\overline{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{v}) \cdot \mathbf{grad} \overline{\mathbf{m}}^{n+\frac{1}{2}} - Q (\overline{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{d}) (\mathbf{d} \cdot \overline{\mathbf{m}}^{n+\frac{1}{2}}) \mathbf{v} + \\ & \quad + (\overline{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{grad} \psi) \cdot \mathbf{v} - (\overline{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{h}_e) \cdot \mathbf{v} \, d\mathbf{x} \end{aligned} \quad (16)$$

for all $\mathbf{v} \in (H^1(\Omega))^3$, and for all $\varphi \in H_0^1(D)$

$$\int_D \mathbf{grad} \psi \cdot \mathbf{grad} \varphi \, d\mathbf{x} = \int_D \overline{\mathbf{m}}^{n+\frac{1}{2}} \cdot \mathbf{grad} \varphi \, d\mathbf{x}. \quad (17)$$

Then, the function spaces are replaced with finite element spaces. The simplest choice are continuous piecewise linear finite elements on D for each component of \mathbf{m}^n and the same type of finite elements (with zero boundary values) on D for ψ .

We cannot solve (16) and (17) for $\overline{\mathbf{m}}^{n+\frac{1}{2}}$ explicitly. Thus we replace $\overline{\mathbf{m}}^{n+\frac{1}{2}}$ in (17) by the old magnetisation \mathbf{m}^n , which leads us to a explicit scheme.

$$\int_D \mathbf{grad} \psi \cdot \mathbf{grad} \varphi \, d\mathbf{x} = \int_D \mathbf{m}^n \cdot \mathbf{grad} \varphi \, d\mathbf{x} . \quad (18)$$

From a known magnetisation \mathbf{m}^n we can calculate ψ with (17) and then use (16) to obtain $\overline{\mathbf{m}}^{n+\frac{1}{2}}$.

All integrations will be carried out using local quadrature formulas whose nodes coincide with the vertices of the elements.

6 Conservation of the norm of \mathbf{m}

In this section we show that we need to use a reduced integration for the first term on the right hand side of (16) in order that the norm of \mathbf{m} is conserved everywhere.

In (16) we chose the test function

$$\mathbf{v}(\mathbf{x}) = \begin{cases} \overline{\mathbf{m}}^{n+\frac{1}{2}}(\mathbf{x}_0) & \text{for } \mathbf{x} = \mathbf{x}_0 \\ 0 & \text{else} \end{cases}$$

Many terms in (16) vanish and we obtain

$$\frac{1}{1 + \alpha^2} \int_{\Omega} (\delta \mathbf{m}^{n+\frac{1}{2}}) \cdot \mathbf{v} \, d\mathbf{x} = \int_{\Omega} \eta \, \mathbf{grad}(\overline{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{v}) \cdot \mathbf{grad} \overline{\mathbf{m}}^{n+\frac{1}{2}} \, d\mathbf{x} \quad (19)$$

Calculating the scalar product on the left side leads to

$$\frac{1}{1 + \alpha^2} \int_{\Omega} (\delta \mathbf{m}^{n+\frac{1}{2}}) \cdot \mathbf{v} \, d\mathbf{x} \sim (|\mathbf{m}^{n+1}(\mathbf{x}_0)|^2 - |\mathbf{m}^n(\mathbf{x}_0)|^2) \quad (20)$$

If the norm of \mathbf{m} is conserved at any point \mathbf{x}_0 in space, $(|\mathbf{m}^{n+1}(\mathbf{x}_0)|^2 - |\mathbf{m}^n(\mathbf{x}_0)|^2)$ is zero and the right hand side of (19) should vanish. But this is not the case in general. Thus we introduce what is called a reduced integration. We replace $\mathbf{grad}(\overline{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{v})$ by $\mathbf{grad}(I(\overline{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{v}))$ where I denotes the linear interpolation with values of $(\overline{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{v})$ taken in the corners of an elementary lattice triangle.

Let's consider the triangle, where \mathbf{x}_0 lies on one corner of that triangle. Our special test function \mathbf{v} is zero in two corners and in \mathbf{x}_0 we have

$$\overline{\mathbf{m}}^{n+\frac{1}{2}}(\mathbf{x}_0) \times \mathbf{v}(\mathbf{x}_0) = \overline{\mathbf{m}}^{n+\frac{1}{2}}(\mathbf{x}_0) \times \overline{\mathbf{m}}^{n+\frac{1}{2}}(\mathbf{x}_0) = 0$$

Therefore the right hand side of (19) vanish by using this reduced integration and the norm of \mathbf{m} is conserved in \mathbf{x}_0 .

We can use this reasoning for any \mathbf{x}_0 in Ω , so that \mathbf{m} is conserved anywhere in Ω .

7 Model-problem

We consider a translation invariant setting, in which none of the quantities depends on the z -coordinate. This means that the problem is reduced to two dimensions. However, all vectors still retain their three components. Only if both the excitation \mathbf{h}_e and the initial magnetization \mathbf{m}^0 have vanishing z -component this will hold for all other fields. We are not going to assume this in the sequel.

The model problem is posed on the square $D =]-2; 2[^2$, and $\Omega =]-\frac{1}{2}, \frac{1}{2}[^2$. The excitation is provided by a spatially constant magnetic field $\mathbf{h}_e = \mathbf{h}_e(t)$ that may vary with time, however. As initial magnetization we choose $\mathbf{m} = (1, 0, 0)^T$.

We use the same regular uniform triangular mesh on Ω and D . The mesh width is h .

There are $v_D = 2/h - 1$ lattice points per length of D , and $n_D = (v_D)^2$ nodes total.

For Ω there are $v_\Omega = 1/h + 1$ lattice points per length of Ω , and $n_\Omega = (v_\Omega)^2$ total.

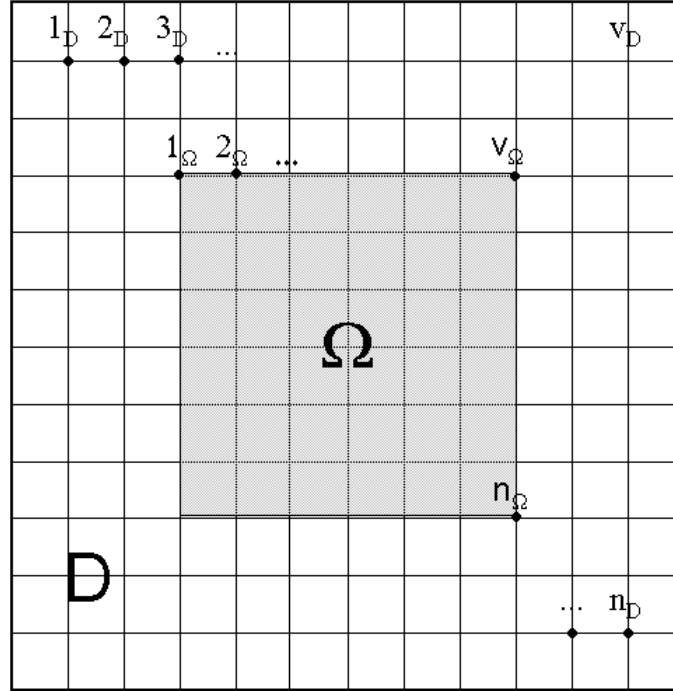


Figure 1: Triangular mesh over D and Ω

8 FEM

As function space to approximate the components of \mathbf{m} on Ω we use the continuous piecewise linear finite elements with basis functions Λ_p , where p is indexing the location of the basis function. The numbering starts from the upper left corner of Ω (see fig. 1). In this basis the components of \mathbf{m} can be written as:

$m_l(\mathbf{x}) = \sum_p^{n_\Omega} m_{p,l} \Lambda_p(\mathbf{x})$ where $m_{p,l}$ are the expansion coefficients of the function m_l in the basis Λ_p . n_Ω denotes the total number of lattice points in Ω .

In a similar way ψ is approximated on D : $\psi(\mathbf{x}) = \sum_k^{n_D} \psi_k Z_k(\mathbf{x})$ where the basis function Z_k are identical to Λ_p , but with a different numbering. As ψ is a scalar function we only need one basis function per lattice point.

Λ_p (resp. Z_p) is 1 at p and zero on the whole boundary of the six triangles shown below (o is v_D for Z_p and v_Ω for Λ_p).

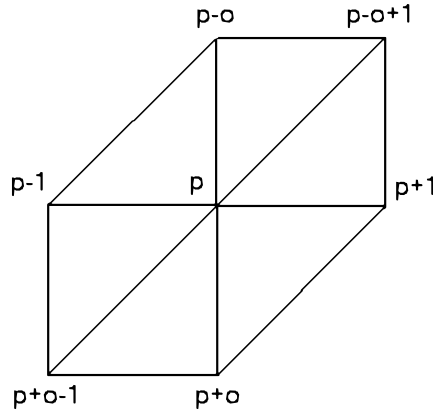


Figure 2: Supp of Λ_p

We will also need the derivatives of these basis functions, which vanish for the z -component:

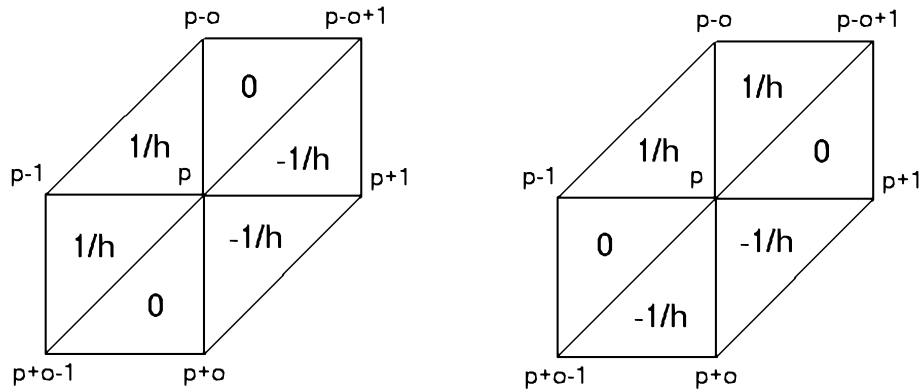


Figure 3: Derivatives of basis functions with respect to x (left side) and y (right side)

An integral over one triangle is calculated by the quadrature formula:

$$\int_{\Delta} f dx \approx \frac{\text{area}(\Delta)}{3} \sum_{i=1}^3 f(a_i) \quad (21)$$

whereas a_i , $i = \{1, 2, 3\}$ are the corners of the triangle Δ .

9 Discrete forms

We will denote $m_{i,l}$ as the expansion coefficients of the l th component of \mathbf{m}^{n+1} , $\tilde{m}_{i,l}$ for the l th component of \mathbf{m}^n , and $\bar{m}_{i,l}$ for the l th component of $\bar{\mathbf{m}}^{n+\frac{1}{2}} := \frac{\mathbf{m}^{n+1} + \mathbf{m}^n}{2}$.

9.1 Discrete form of potential problem (18)

Equation (18) leads to a simple linear system of equations for the coefficients ψ_k ,

$$\begin{aligned} \int_D \mathbf{grad} \psi \cdot \mathbf{grad} Z_k dx &= \sum_i^{n_D} \psi_i \int_D \mathbf{grad} Z_i \mathbf{grad} Z_k dx \\ &= \sum_i^{n_D} M_{ki} \psi_i, \quad k \in \{1, 2, \dots, n_D\} \\ M_{ki} &= \int_D (\partial_x Z_i)(\partial_x Z_k) + (\partial_y Z_i)(\partial_y Z_k) dx \end{aligned} \quad (22)$$

A is a $n_D \times n_D$ constant block matrix with 5 non zero diagonals and block sizes $v_d \times v_d$.

$$M = \begin{pmatrix} C & -I & & & \\ -I & C & -I & & \\ & -I & C & -I & \\ & & \dots & \dots & \dots \\ & & & -I & C & -I \\ & & & & -I & C \end{pmatrix}$$

$$C = \begin{pmatrix} 4 & -1 & & & \\ -1 & 4 & -1 & & \\ & -1 & 4 & -1 & \\ & & \dots & \dots & \dots \\ & & & -1 & 4 & -1 \\ & & & & -1 & 4 \end{pmatrix}$$

This block structure is typical for a matrix of a f.e.m problem. We will deal with similar matrices later.

The right hand side of (18) is

$$\int_D \mathbf{m}^n \cdot \mathbf{grad} Z_k \, d\mathbf{x} = \int_D \sum_i^{n_\Omega} \sum_{l=1}^2 \tilde{m}_{i,l} \Lambda_i \partial_l Z_k \, d\mathbf{x} = \sum_i^{n_\Omega} \tilde{m}_{i,1} B_{ik,1} + \tilde{m}_{i,2} B_{ik,2}$$

with

$$B_{ik,l} = \int_\Omega (\partial_l Z_k) \Lambda_i \, d\mathbf{x}$$

where we replaced $\overline{\mathbf{m}}^{n+\frac{1}{2}}$ in (18) by \mathbf{m}^n to get an explicit scheme.

B_l is of dimension $n_\Omega \times n_D$. Taking the Neumann boundary conditions on Ω into account the matrix B_l takes the following block structure:

$$B_l = \frac{h}{6} \begin{pmatrix} \dots & E & \dots & U & & & & \\ & L & \dots & D & \dots & U & & \\ & \dots & \dots & L & \dots & D & \dots & U \\ & & & & \ddots & & & \\ & & & & & & & \\ & & & & & & & \\ & & & \dots & \dots & L & \dots & D & \dots & U \\ & & & & & \dots & L & \dots & F & \dots \end{pmatrix} \quad (23)$$

The first element of block B is at position $v_D(\frac{v_D-v_\Omega}{2} - 1) + \frac{v_D-v_\Omega}{2} + 1$ of the matrix. The blocks E,D,D,...,D,F are not arranged diagonally in the matrix but are shifted horizontally. This shift of $v_D - v_\Omega$ positions is due to the fact, that the area D is bigger than Ω and that we therefore have a different numbering of the basis functions Z_k and Λ_p .

For $l=1$ the blocks are:

$$\begin{aligned} E &= \begin{pmatrix} -1 & 1 & & & & & & \\ -1 & 0 & 1 & & & & & \\ & -1 & 0 & 1 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & & -1 & 0 & 1 & \\ & & & & & -1 & 1 & \end{pmatrix} & D &= \begin{pmatrix} -2 & 2 & & & & & & \\ -2 & 0 & 2 & & & & & \\ & -2 & 0 & 2 & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & & -2 & 0 & 2 & \\ & & & & & -2 & 2 & \end{pmatrix} \\ L &= \begin{pmatrix} -1 & 1 & & & & & & \\ 0 & -1 & 1 & & & & & \\ & & -1 & 1 & & & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & -1 & 1 & & \\ & & & & & 0 & 0 & \end{pmatrix} & U &= \begin{pmatrix} 0 & & & & & & & \\ -1 & 1 & & & & & & \\ & -1 & 1 & & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & -1 & 1 & & & \\ & & & & -1 & 1 & & \\ & & & & & -1 & 1 & \end{pmatrix} \\ F &= E \end{aligned}$$

For $l=2$ the blocks are:

$$\begin{aligned}
E &= \begin{pmatrix} -1 & 0 & & & & \\ -1 & -2 & & & & \\ & -1 & -2 & & & \\ & & \ddots & \ddots & & \\ & & & -1 & -2 & \\ & & & & -1 & -1 \end{pmatrix} & F &= \begin{pmatrix} 1 & & & & & \\ 1 & 2 & & & & \\ & 1 & 2 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 2 & \\ & & & & 1 & 1 \end{pmatrix} \\
D &= \begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & 1 & & & \\ & -1 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix} \\
L &= \begin{pmatrix} -1 & -1 & & & & \\ 0 & -2 & -1 & & & \\ & & -2 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -2 & -1 & \\ & & & & -1 & \end{pmatrix} & U &= \begin{pmatrix} 1 & & & & & \\ 1 & 2 & & & & \\ & 1 & 2 & & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 2 & \\ & & & & 1 & 1 \end{pmatrix}
\end{aligned}$$

These matrices B_x and B_y will be needed also for the fifth term of equation 16.

At each iteration step the following linear system for the coefficients ψ_i , $i \in \{1, 2, \dots, n_D\}$ has to be solved:

$$M\psi^c = B_x^T m_x^c + B_y^T m_y^c \quad (24)$$

By ψ^c (resp. m_i^c) the vector of the expansion coefficients of the function ψ (resp. m_i) is meant.

9.2 Discrete form of (16)

Most matrices (numbered from A^1 to A^6) don't depend on $\overline{\mathbf{m}}^{n+\frac{1}{2}}$. Some of them are even diagonal, so that they can be easily stored in a vector.

Some matrices are not the same for each component of \mathbf{m} . For example we will have A_l^3 , $l = \{1, 2, 3\}$ three different matrices for the third term in (16).

Let's start with the first term on the left hand side (lth component) of (16).

Term 1: $\int_{\Omega} (\delta \mathbf{m}^{n+\frac{1}{2}})_l \Lambda_p \, d\mathbf{x}$

$$\begin{aligned}
\int_{\Omega} (\delta \mathbf{m}^{n+\frac{1}{2}})_l \Lambda_p \, d\mathbf{x} &= \int_{\Omega} \left(\frac{\mathbf{m}^{n+1} - \mathbf{m}^n}{\tau} \right)_l \cdot \Lambda_p \, d\mathbf{x} = \frac{1}{\tau} \int_{\Omega} \sum_i^{n_{\Omega}} (m_{i,l} - \widetilde{m}_{i,l}) \Lambda_i \Lambda_p \, d\mathbf{x} \\
&= \frac{1}{\tau} \sum_i^{n_{\Omega}} A_{pi}^1 (m_{i,l} - \widetilde{m}_{i,l}), \quad l = \{1, 2, 3\}, \quad p \in \{1, 2, \dots, n_{\Omega}\}
\end{aligned}$$

$$A_{pi}^1 = \int_{\Omega} \Lambda_i \Lambda_p \, d\mathbf{x} \quad (25)$$

Using the quadrature formula and taking account of the Neumann boundary conditions leads to a diagonal matrix (can be stored as a vector):

$$A^1 = \frac{h^2}{6} \begin{pmatrix} B & & & & & \\ & D & & & & \\ & & D & & & \\ & & & D & & \\ & & & & \ddots & \\ & & & & & D \\ & & & & & & C \end{pmatrix}$$

where the blocks A,B and C are

$$B = \begin{pmatrix} 1 & & & & & \\ & 3 & & & & \\ & & 3 & & & \\ & & & \ddots & & \\ & & & & 3 & \\ & & & & & 2 \end{pmatrix} \quad D = \begin{pmatrix} 3 & & & & & \\ & 6 & & & & \\ & & 6 & & & \\ & & & \ddots & & \\ & & & & 6 & \\ & & & & & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 2 & & & & & \\ & 3 & & & & \\ & & 3 & & & \\ & & & \ddots & & \\ & & & & 3 & \\ & & & & & 1 \end{pmatrix}$$

Term 2: $\int_{\Omega} (\overline{\mathbf{m}}^{n+\frac{1}{2}} \times \delta \mathbf{m}^{n+\frac{1}{2}})_l \Lambda_p \, d\mathbf{x}$

$$\begin{aligned} \int_{\Omega} (\overline{\mathbf{m}}^{n+\frac{1}{2}} \times \delta \mathbf{m}^{n+\frac{1}{2}})_l \Lambda_p \, d\mathbf{x} &= \\ &= \frac{1}{2\tau} \int_{\Omega} (\mathbf{m}^{n+1} + \mathbf{m}^n)_{l+1} (\mathbf{m}^{n+1} - \mathbf{m}^n)_{l+2} - (\mathbf{m}^{n+1} + \mathbf{m}^n)_{l+2} (\mathbf{m}^{n+1} - \mathbf{m}^n)_{l+1} \Lambda_p \, d\mathbf{x} = \\ &= \frac{1}{2\tau} \int_{\Omega} (2\mathbf{m}_{l+1}^n \mathbf{m}_{l+2}^{n+1} - 2\mathbf{m}_{l+2}^n \mathbf{m}_{l+1}^{n+1}) \Lambda_p \, d\mathbf{x} = \\ &= \frac{1}{\tau} \int_{\Omega} \sum_i^{n_{\Omega}} \sum_j^{n_{\Omega}} (\tilde{m}_{j,l+1} m_{i,l+2} - \tilde{m}_{j,l+2} m_{i,l+1}) \Lambda_i \Lambda_j \Lambda_p \, d\mathbf{x} = \\ &= \frac{1}{\tau} (A_{l+1}^2 m_{l+2} - A_{l+2}^2 m_{l+1}) \quad l = \{1, 2, 3\}, \quad p \in \{1, 2, \dots, n_{\Omega}\}. \end{aligned}$$

$$A_{pi,l}^2 = \sum_j^{n_\Omega} \tilde{m}_{j,l} \int_{\Omega} \Lambda_i \Lambda_j \Lambda_p d\mathbf{x} = (A^1 \cdot \text{diag}(\tilde{m}_l))_{pi,l} \quad (26)$$

The second matrix A^2 is similar to the first matrix A^1 except that each component j of the diagonal of A^1 is multiplied by the j th component of \tilde{m}_l .

Term 3: $\int_{\Omega} \eta \mathbf{grad}(\bar{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{v}) \cdot \mathbf{grad} \bar{\mathbf{m}}^{n+\frac{1}{2}} d\mathbf{x}$

For this term we use a reduced integration. We write $I(\bar{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{v})$ for the linear interpolation over the three corners of the corresponding triangle.

$$\int_{\Omega} \eta \mathbf{grad}(I(\bar{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{v})) \cdot \mathbf{grad} \bar{\mathbf{m}}^{n+\frac{1}{2}} d\mathbf{x} = \eta \int_{\Omega} \mathbf{grad} \begin{pmatrix} I(\bar{m}_2 v_3 - \bar{m}_3 v_2) \\ I(\bar{m}_3 v_1 - \bar{m}_1 v_3) \\ I(\bar{m}_1 v_2 - \bar{m}_2 v_1) \end{pmatrix} \mathbf{grad} \begin{pmatrix} \bar{m}_1 \\ \bar{m}_2 \\ \bar{m}_3 \end{pmatrix} d\mathbf{x}$$

l th component:

We replace $\mathbf{grad}(I(\bar{m}_l \Lambda_p))$ with $\bar{m}_{p,l} \mathbf{grad} \Lambda_p$ where $\bar{m}_{p,l}$ is the value l th component of \bar{m}_l at lattice point p .

We get

$$\begin{aligned} & \eta \sum_{i=1}^{n_\Omega} \int_{\Omega} (\bar{m}_{p,l+2} \bar{m}_{i,l+1} - \bar{m}_{p,l+1} \bar{m}_{i,l+2}) ((\partial_x \Lambda_p)(\partial_x \Lambda_i) + (\partial_y \Lambda_p)(\partial_y \Lambda_i)) d\mathbf{x} = \\ & = \eta \sum_{i=1}^{n_\Omega} (A_{pi,l+2}^3 \bar{m}_{i,l+1} - A_{pi,l+1}^3 \bar{m}_{i,l+2}) d\mathbf{x}, \quad l = \{1, 2, 3\}, \quad p \in \{1, 2, \dots, n_\Omega\} \end{aligned}$$

with

$$A_{pi,l}^3 = \bar{m}_{p,l} \int_{\Omega} ((\partial_x \Lambda_p)(\partial_x \Lambda_i) + (\partial_y \Lambda_p)(\partial_y \Lambda_i)) d\mathbf{x} = \bar{m}_{p,l} A_{Npi} \quad (27)$$

A_N is the stiffness matrix for the laplace operator with Neumann boundary conditions. It has the block form

$$A_N = \begin{pmatrix} E & C & & & & \\ C & D & C & & & \\ & C & D & C & & \\ & & \ddots & \ddots & \ddots & \\ & & & & C & D & C \\ & & & & & C & F \end{pmatrix} \quad (28)$$

with the following blocks

$$\begin{aligned}
D &= \begin{pmatrix} 2 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & & \ddots & & \\ & & & -1 & 4 & -1 \\ & & & & -1 & 2 \end{pmatrix} & C &= \begin{pmatrix} -\frac{1}{2} & & & & & \\ & -1 & & & & \\ & & -1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \\ & & & & & & -\frac{1}{2} \end{pmatrix} \\
E &= \begin{pmatrix} 1 & -\frac{1}{2} & & & & \\ -\frac{1}{2} & 2 & -\frac{1}{2} & & & \\ & -\frac{1}{2} & 2 & -\frac{1}{2} & & \\ & & & \ddots & & \\ & & & -\frac{1}{2} & 2 & -\frac{1}{2} \\ & & & & -\frac{1}{2} & 1 \end{pmatrix} & F &= E
\end{aligned}$$

Term 4: $\int_{\Omega} Q(\bar{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{d})(\mathbf{d} \cdot \bar{\mathbf{m}}^{n+\frac{1}{2}})_l \Lambda_p d\mathbf{x}$

$$\begin{aligned}
\int_{\Omega} Q(\bar{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{d})(\mathbf{d} \cdot \bar{\mathbf{m}}^{n+\frac{1}{2}})_l \Lambda_p d\mathbf{x} &= \\
&= \int_{\Omega} Q\left(\sum_{k=1}^3 d_k \bar{m}_k^{n+\frac{1}{2}}\right)(\bar{m}_{l+1}^{n+\frac{1}{2}} d_{l+2} - \bar{m}_{l+2}^{n+\frac{1}{2}} d_{l+1}) \Lambda_p d\mathbf{x} = \\
&= \sum_i^{n_{\Omega}} (A_{pi,l+2}^4 \bar{m}_{i,l+1} - A_{pi,l+1}^4 \bar{m}_{i,l+2}), \quad l = \{1, 2, 3\}, \quad p \in \{1, 2, \dots, n_{\Omega}\} \\
A_{pi,l}^4 &= Q d_l \sum_j^{n_{\Omega}} \left(\sum_{k=1}^3 d_k \bar{m}_{j,k} \right) \int_{\Omega} \Lambda_i \Lambda_j \Lambda_p d\mathbf{x} = Q d_l \cdot (A^1 \cdot \text{diag}(\sum_{k=1}^3 d_k \bar{m}_{i,k}))_{pi} \quad (29)
\end{aligned}$$

The matrix A^4 is diagonal and corresponds to the matrix A^1 , where each component j of the diagonal is multiplied by $\sum_{k=1}^3 d_k \bar{m}_{j,k}$.

Term 5: $\int_{\Omega} (\bar{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{grad} \psi)_l \Lambda_p d\mathbf{x}$

$$\begin{aligned}
\int_{\Omega} (\bar{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{grad} \psi)_l \Lambda_p d\mathbf{x} &= \int_{\Omega} (\bar{m}_{l+1}^{n+\frac{1}{2}} \partial_{l+2} \psi - \bar{m}_{l+2}^{n+\frac{1}{2}} \partial_{l+1} \psi) \Lambda_p d\mathbf{x} = \\
&= \int_{\Omega} \sum_i^{n_{\Omega}} \sum_j^{n_D} (\bar{m}_{i,l+1} \psi_j \partial_{l+2} Z_j - \bar{m}_{i,l+2} \psi_j \partial_{l+1} Z_j) \Lambda_i \Lambda_p d\mathbf{x} \\
&= \sum_i^{n_{\Omega}} (A_{pi,l+2}^5 \bar{m}_{i,l+1} - A_{pi,l+1}^5 \bar{m}_{i,l+2}), \quad l = \{1, 2, 3\}, \quad p \in \{1, 2, \dots, n_{\Omega}\}.
\end{aligned}$$

$$A_{pi,l}^5 = \sum_j^{n_D} \psi_j \int_{\Omega} (\partial_l Z_j) \Lambda_i \Lambda_p \, d\mathbf{x} = \delta_{pi} \sum_j^{n_D} b_{pj,l} \psi_j \quad (30)$$

$$A_{pi,z}^5 = 0$$

$$B_{pj,l} = \delta_{pi} \int_{\Omega} (\partial_l Z_j) \Lambda_i \Lambda_p \, d\mathbf{x} \quad (31)$$

The matrices B_l are given in (23).

Term 6

$$\begin{aligned} \int_{\Omega} (\bar{\mathbf{m}}^{n+\frac{1}{2}} \times \mathbf{h}_e)_l \Lambda_p \, d\mathbf{x} &= \int_{\Omega} (\bar{m}_{l+1}^{n+\frac{1}{2}} h_{el+2} - \bar{m}_{l+2}^{n+\frac{1}{2}} h_{el+1}) \Lambda_p \, d\mathbf{x} = \\ &= \sum_i^{n_{\Omega}} (A_{pi,l+2}^6 \bar{m}_{i,l+1} - A_{pi,l+1}^6 \bar{m}_{i,l+2}), \quad l = \{1, 2, 3\}, \quad p \in \{1, 2, \dots, n_{\Omega}\}. \end{aligned}$$

$$A_{pi,l}^6 = \int_{\Omega} h_{e,l} \Lambda_i \Lambda_p \, d\mathbf{x} = (A^1)_{pi} \cdot h_{el}(x_p) \delta_{pi} \quad (32)$$

This matrix is also diagonal. $h_{el}(x_p)$ is the value of the l th component of the external field at the lattice point p .

10 Test of the matrices

All the matrices of each term of (16) and (18) have been tested one by one by comparing the f.e.m solution with the analytical calculated integral. We took 3 different mesh widths to see the diminution of the error of order h^2 ($h_0 = 1/24$). For testing we chose arbitrary magnetic fields and test functions on Ω :

$$\begin{aligned}
\mathbf{m}_1 &= \begin{pmatrix} m_{1x} \\ m_{1y} \\ m_{1z} \end{pmatrix} = \begin{pmatrix} x \cdot y^3 + 1 \\ x^2 \cdot y + y^2 + x^2 - 1 \\ x \cdot y + x^2 \cdot y + y^3 \end{pmatrix} \\
\mathbf{m}_2 &= \begin{pmatrix} m_{2x} \\ m_{2y} \\ m_{2z} \end{pmatrix} = \begin{pmatrix} x^2 \cdot y^3 - x \cdot y^2 + 1 \\ -x^2 + y \\ x^2 \cdot y^4 - x \cdot y^2 \end{pmatrix} \\
\mathbf{v} &= \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} x + x^4 \cdot y - x \cdot y^4 + \exp(x) \\ x \cdot y + \exp(y) \\ x + y^2 - \exp(x) \cdot y \end{pmatrix}
\end{aligned}$$

The following tables show the relative error between the f.e.m. result and the exact value for each term.

Term 1: $\int_{\Omega} \mathbf{m}_1 \cdot \mathbf{v} d\mathbf{x}$

h	rel. error x	rel. error y	rel. error z
h_0	$1.0612 \cdot 10^{-4}$	$3.3282 \cdot 10^{-5}$	$1.3339 \cdot 10^{-2}$
$h_0/2$	$2.6971 \cdot 10^{-5}$	$7.5802 \cdot 10^{-6}$	$3.3332 \cdot 10^{-3}$
$h_0/4$	$7.1826 \cdot 10^{-6}$	$1.1707 \cdot 10^{-6}$	$8.3256 \cdot 10^{-4}$

Term 2: $\int_{\Omega} (\mathbf{m}_1 \times \mathbf{m}_2) \cdot \mathbf{v} d\mathbf{x}$

h	rel. error x	rel. error y	rel. error z
h_0	$1.3584 \cdot 10^{-2}$	$1.7462 \cdot 10^{-3}$	$1.1540 \cdot 10^{-2}$
$h_0/2$	$3.3947 \cdot 10^{-3}$	$4.3677 \cdot 10^{-4}$	$2.8831 \cdot 10^{-3}$
$h_0/4$	$8.4801 \cdot 10^{-4}$	$1.0933 \cdot 10^{-4}$	$7.2133 \cdot 10^{-4}$

Term 3: $\int_{\Omega} \mathbf{grad}(\mathbf{m}_1 \times \mathbf{v}) \cdot \mathbf{grad} \mathbf{m}_1 d\mathbf{x}$

h	rel. error x	rel. error y	rel. error z
h_0	$2.0760 \cdot 10^{-3}$	$1.5707 \cdot 10^{-3}$	$4.6545 \cdot 10^{-3}$
$h_0/2$	$5.2174 \cdot 10^{-4}$	$3.9231 \cdot 10^{-4}$	$1.1667 \cdot 10^{-3}$
$h_0/4$	$1.2955 \cdot 10^{-4}$	$9.7748 \cdot 10^{-5}$	$2.9456 \cdot 10^{-4}$

Term 4: $\int_{\Omega} (\mathbf{m}_1 \times \mathbf{d}) \cdot \mathbf{v}(\mathbf{d} \cdot \mathbf{m}_1) d\mathbf{x}$

h	rel. error x	rel. error y	rel. error z
h_0	$2.9558 \cdot 10^{-3}$	$1.5585 \cdot 10^{-3}$	$1.3111 \cdot 10^{-2}$
$h_0/2$	$7.4250 \cdot 10^{-4}$	$3.8977 \cdot 10^{-4}$	$3.2752 \cdot 10^{-3}$
$h_0/4$	$1.8578 \cdot 10^{-4}$	$9.7316 \cdot 10^{-5}$	$8.1723 \cdot 10^{-4}$

Term 5: $\int_{\Omega} (\mathbf{m}_1 \times \mathbf{grad} \psi) \cdot \mathbf{v} d\mathbf{x}$

h	rel. error x	rel. error y	rel. error z
h_0	$3.1348 \cdot 10^{-3}$	$1.1708 \cdot 10^{-3}$	$2.6638 \cdot 10^{-3}$
$h_0/2$	$7.8324 \cdot 10^{-4}$	$2.9172 \cdot 10^{-4}$	$6.6497 \cdot 10^{-4}$
$h_0/4$	$1.9555 \cdot 10^{-4}$	$7.2398 \cdot 10^{-5}$	$1.6568 \cdot 10^{-4}$

Term 6: $\int_{\Omega} (\mathbf{m}_1 \times \mathbf{h}_e) \cdot \mathbf{v} d\mathbf{x}$
Same form as term 2.

For the domain $D [-2, 2]^2$ we tested the discretization with the following functions, satisfying the Dirichelet boundary conditions:

$$\begin{aligned}\psi &= (x-2)(x+2)(x-2)(y+2)(x-3)(y^3-1) \\ t_1 &= \sin(2\pi x) \sin(\pi y) \\ t_2 &= x \sin(\pi x) \sin(\pi y) \\ t_3 &= y \sin(\pi x) \sin(3\pi y)\end{aligned}$$

The relative errors for the terms in (18) are given in the two following tables.

Term LHS: $\int_D \mathbf{grad} \psi \cdot \mathbf{grad} t_i d\mathbf{x}$

h	rel. error i=1	rel. error i=2	rel. error i=3
h_0	$4.8654 \cdot 10^{-3}$	$1.4299 \cdot 10^{-3}$	$1.0046 \cdot 10^{-2}$
$h_0/2$	$1.2144 \cdot 10^{-3}$	$3.5710 \cdot 10^{-4}$	$2.5020 \cdot 10^{-3}$
$h_0/4$	$3.0347 \cdot 10^{-4}$	$8.9251 \cdot 10^{-5}$	$6.2490 \cdot 10^{-4}$

Term RHS: $\int_D \mathbf{m}_1 \cdot \mathbf{grad} t_i d\mathbf{x}$

h	rel. error i=1	rel. error i=2	rel. error i=3
h_0	$1.0865 \cdot 10^{-1}$	$1.3798 \cdot 10^{-3}$	$1.3170 \cdot 10^{-2}$
$h_0/2$	$2.7288 \cdot 10^{-2}$	$3.4430 \cdot 10^{-4}$	$3.2730 \cdot 10^{-3}$
$h_0/4$	$6.8316 \cdot 10^{-3}$	$8.5699 \cdot 10^{-5}$	$8.1761 \cdot 10^{-4}$

The results show, that all the f.e.m matrices are correct.

11 Newton method

As the resulting system of equation for the expansion coefficients of the magnetization is not linear, we will use the Newton method to determine them.

In this section m_l denotes the vector of expansion coefficients of $(\mathbf{m}^{n+1}(\mathbf{x}))_l$, and \tilde{m}_l for the expansion coefficients of $(\mathbf{m}^n(\mathbf{x}))_l$. \bar{m}_l is $\frac{m_l + \tilde{m}_l}{2}$.

The left hand side of (16) expressed with the matrices calculated in section 9 leads to (lth component):

$$\begin{aligned} & \frac{1}{1 + \alpha^2} \frac{1}{\tau} (A^1(m_l - \tilde{m}_l) + \alpha(A_{l+2}^2 m_{l+1} - A_{l+1}^2 m_{l+2})) = \\ & = \frac{1}{1 + \alpha^2} \frac{1}{\tau} (A^1(m_l - \tilde{m}_l) + \alpha(A^1 \tilde{m}_{l+2} * m_{l+1} - A^1 \tilde{m}_{l+1} * m_{l+2})) = \\ & = \frac{1}{1 + \alpha^2} \frac{1}{\tau} A^1(m_l - \tilde{m}_l + \alpha(\tilde{m}_{l+2} * m_{l+1} - \tilde{m}_{l+1} * m_{l+2})) \end{aligned}$$

The star $*$ stands for component-wise multiplication of two vectors.
The right hand side of (16) is

$$\begin{aligned} & -\eta(A_{l+2}^3 \bar{m}_{l+1} - A_{l+1}^3 \bar{m}_{l+2}) + A_{l+1}^4 \bar{m}_{l+2} - A_{l+2}^4 \bar{m}_{l+1} \\ & + A_{l+2}^5 \bar{m}_{l+2} - A_{l+1}^5 \bar{m}_{l+1} + A_{l+1}^6 \bar{m}_{l+2} - A_{l+2}^6 \bar{m}_{l+1} = \\ & = -\eta(A_{l+2}^3 \bar{m}_{l+1} - A_{l+1}^3 \bar{m}_{l+2}) + (A_{l+1}^4 - A_{l+1}^5 + A_{l+1}^6) \bar{m}_{l+2} - (A_{l+2}^4 - A_{l+2}^5 + A_{l+2}^6) \bar{m}_{l+1} \end{aligned}$$

We define

$$A_l = A_l^4 - A_l^5 + A_l^6 = Q d_l \cdot (A^1 \cdot \text{diag}(\sum_{k=1}^3 d_k \bar{m}_{i,k})) - \text{diag}(b_l \cdot \psi^c) + A^1 \cdot \text{diag}(h_{el}(x_p))$$

where $\text{diag}(\text{vector})$ means the diagonal matrix with *vector* as diagonal. ψ^c is the vector of the expansion coefficients of ψ .

Moving all terms to the left side we arrive at

$$\begin{aligned} f_l(m) &= k A^1(m_l - \tilde{m}_l + \alpha(\tilde{m}_{l+2} m_{l+1} - \tilde{m}_{l+1} m_{l+2})) \\ &+ \eta(A_{l+2}^3 \bar{m}_{l+1} - A_{l+1}^3 \bar{m}_{l+2}) - A_{l+1}^4 \bar{m}_{l+2} + A_{l+2}^4 \bar{m}_{l+1} = 0 \end{aligned} \quad (33)$$

with $k = \frac{1}{1 + \alpha^2} \frac{1}{\tau}$.

The derivative of f has takes the form

$$df = \begin{pmatrix} D1 & U1 & U2 \\ L1 & D2 & U3 \\ L2 & L3 & D3 \end{pmatrix} \quad (34)$$

The sparsity pattern of each block corresponds to the matrix AN (28). The blocks are

$$D1 = kA^1 + \frac{Qd_1}{2} \cdot \text{diag}(A^1(d_3\bar{m}_2 - d_2\bar{m}_3)) \quad (35)$$

$$D2 = kA^1 + \frac{Qd_2}{2} \cdot \text{diag}(A^1(d_1\bar{m}_3 - d_3\bar{m}_1)) \quad (36)$$

$$D3 = kA^1 + \frac{Qd_3}{2} \cdot \text{diag}(A^1(d_2\bar{m}_1 - d_1\bar{m}_2)) \quad (37)$$

$$U1 = \text{diag}(\alpha k A^1 \tilde{m}_3 - \frac{\eta}{2} A_N \bar{m}_3 + \frac{1}{2} A_z + \frac{Qd_2}{2} A^1(d_3\bar{m}_2 - d_2\bar{m}_3)) + \frac{\eta}{2} \text{diag}(\bar{m}_3) A_N \quad (38)$$

$$U2 = -\text{diag}(\alpha k A^1 \tilde{m}_2 - \frac{\eta}{2} A_N \bar{m}_2 + \frac{1}{2} A_y - \frac{Qd_3}{2} A^1(d_3\bar{m}_2 - d_2\bar{m}_3)) - \frac{\eta}{2} \text{diag}(\bar{m}_2) A_N \quad (39)$$

$$U3 = \text{diag}(\alpha k A^1 \tilde{m}_1 - \frac{\eta}{2} A_N \bar{m}_1 + \frac{1}{2} A_x + \frac{Qd_3}{2} A^1(d_1\bar{m}_3 - d_3\bar{m}_1)) + \frac{\eta}{2} \text{diag}(\bar{m}_1) A_N \quad (40)$$

$$L1 = -\text{diag}(\alpha k A^1 \tilde{m}_3 - \frac{\eta}{2} A_N \bar{m}_3 + \frac{1}{2} A_z - \frac{Qd_1}{2} A^1(d_1\bar{m}_3 - d_3\bar{m}_1)) - \frac{\eta}{2} \text{diag}(\bar{m}_3) A_N \quad (41)$$

$$L2 = \text{diag}(\alpha k A^1 \tilde{m}_2 - \frac{\eta}{2} A_N \bar{m}_2 + \frac{1}{2} A_y + \frac{Qd_1}{2} A^1(d_2\bar{m}_1 - d_1\bar{m}_2)) + \frac{\eta}{2} \text{diag}(\bar{m}_2) A_N \quad (42)$$

$$L3 = -\text{diag}(\alpha k A^1 \tilde{m}_1 - \frac{\eta}{2} A_N \bar{m}_1 + \frac{1}{2} A_x - \frac{Qd_2}{2} A^1(d_2\bar{m}_1 - d_1\bar{m}_2)) - \frac{\eta}{2} \text{diag}(\bar{m}_1) A_N \quad (43)$$

(44)

The Newton method works as follows:

Initialisation:

Start with $m = \tilde{m}$

Calculate the ψ_i .

Iteration:

Do while $\text{norm}(f(m)) > \text{tolerance}$

 Calculate Df

$m^{it+1} = m^{it} - (Df)^{-1} f(m^{it})$

 if it > itmax, break and error

12 Calculation of the energy

We want to calculate the energy at each iteration step.

$$\mathcal{E}(\mathbf{m}) = \frac{1}{2} \int_{\Omega} \eta |\mathbf{grad} \mathbf{m}|^2 + Q(1 - (\mathbf{d} \cdot \mathbf{m})^2) - 2\mathbf{h}_e \cdot \mathbf{m} \, d\mathbf{x} + \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{m}^0|^2 \, d\mathbf{x}$$

The first term can be calculated as follows

$$\begin{aligned} \int_{\Omega} |\mathbf{grad} \mathbf{m}|^2 \, d\mathbf{x} &= \sum_{k,l} \sum_{i,j} m_{i,l} m_{j,l} \left(\int_{\Omega} \partial_k \Lambda_i \partial_k \Lambda_j \, d\mathbf{x} \right) = \\ &= \sum_l \sum_{i,j} m_{i,l} m_{j,l} A_{Nij} \end{aligned}$$

where A_N is the stiffness matrix of the laplacian with Neumann boundary conditions. We defined it already in (28)

The middle integrand terms $f(m) := Q(1 - (\mathbf{d} \cdot \mathbf{m})^2) - 2\mathbf{h}_e \cdot \mathbf{m}$ are integrated by the following quadrature formula:

$$I_{\Omega} = \sum_{\square_k \in \Omega} \frac{h^2}{4} \sum_{i \in \square_k} f(m(x_i))$$

where $\square_k \in \Omega$ means all elementary lattice squares in Ω .

The last integrand is

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{m}^0|^2 \, d\mathbf{x} &= \frac{1}{2} \int_{\mathbb{R}^3} |\mathbf{grad} \psi|^2 \, d\mathbf{x} = \\ &= \frac{1}{2} \sum_{i,j} \psi_i^c \psi_j^c \int_{\mathbb{R}^3} \partial_x \Lambda_i \partial_x \Lambda_j + \partial_y \Lambda_i \partial_y \Lambda_j \, d\mathbf{x} = \frac{1}{2} \sum_{i,j} \psi_i^c \psi_j^c A_{ij} \end{aligned}$$

where A_{ij} is the stiffness matrix to the laplacian operator with Dirichelet boundary conditions.

13 Interpretation of the equation

A short interpretation of the equation 1 and the energy term 2 may help to understand the results.

The first term $\gamma \mathbf{M} \times \mathbf{H}_{\text{eff}}$ lets the spins rotate (precession) around the axis \mathbf{H}_{eff} . The second term $\frac{\gamma\alpha}{M_s} \mathbf{M} \times (\mathbf{M} \times \mathbf{H}_{\text{eff}})$ has a dissipative character, i.e the angle between \mathbf{H}_{eff} and \mathbf{M} decreases in time.

We can solve equation 1 for a single spin with an ODE solver in Matlab. \mathbf{M} depends only on the time and \mathbf{H}_{eff} is taken as a constant vector.

Figure 4 shows the evolution in time of a spin vector.

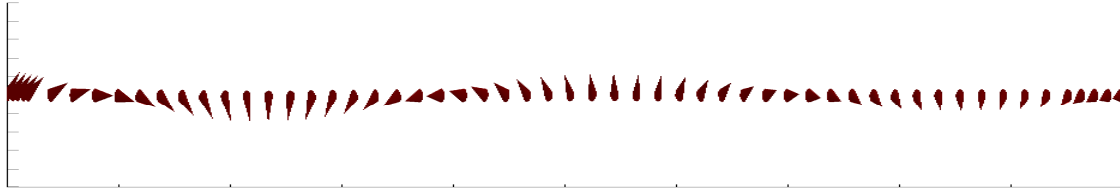


Figure 4: Evolution of m for a single spin

Let us shortly discuss the different energy terms, which configuration leads to a minimum:

Exchange term: $\int_{\Omega} \eta |\mathbf{grad} \mathbf{m}|^2 d\mathbf{x}$

↘ for aligned spins (ferromagnet)

Anisotropy term: $\int_{\Omega} (1 - (\mathbf{d} \cdot \mathbf{m})^2) d\mathbf{x}$

↘ for spins in direction of the easy axis

External magnetic field: $\int_{\Omega} (-2\mathbf{h}_e \cdot \mathbf{m}) d\mathbf{x}$

↘ for spins along the external magnetic field

Stray field term: $\int_{\mathbb{R}^3} |\mathbf{m}^0| d\mathbf{x}$

↘ for small variation of ψ

14 Results

14.1 Example

We took the following parameters as an example of a simulation:

Size of domain D : 4

Size of domain Ω : 1

Number of lattice points per length of Ω : $v_0=33$

Timestep: $dt=0.2$

Strength of anisotropy term: $Q=0.5$

Direction of easy axis: $d = 1/\sqrt{3}(111)^T$

Dissipation parameter: $\alpha=0.1$

Strength of coupling: $\eta=0.001$

External field (x-component): $h_e=0.1$

Figure 14.1 shows the evolution of the energy of the system. The total energy decreases monotonously and converges to a finite value. The curves below the total energy are the different contributions to the energy defined in (2). It is possible that a single energy contribution increases in time. This is due to an energy transition from another energy contribution.

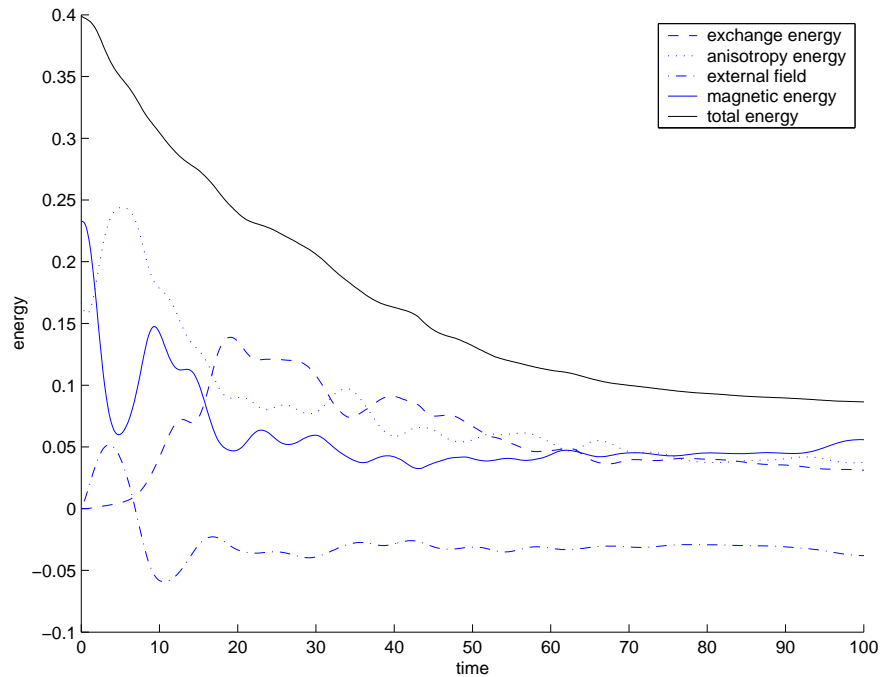
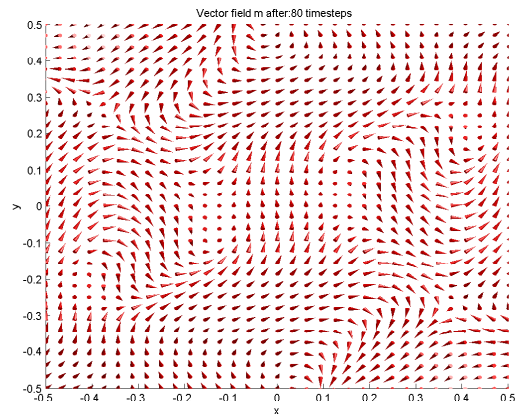
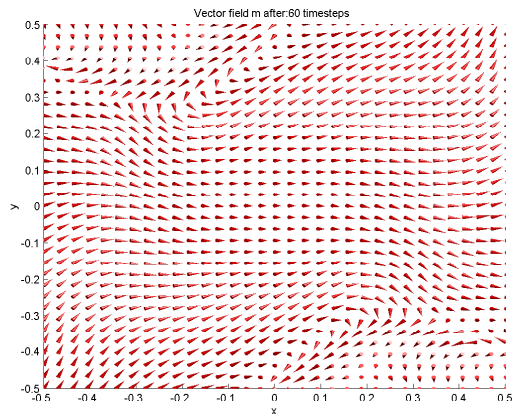
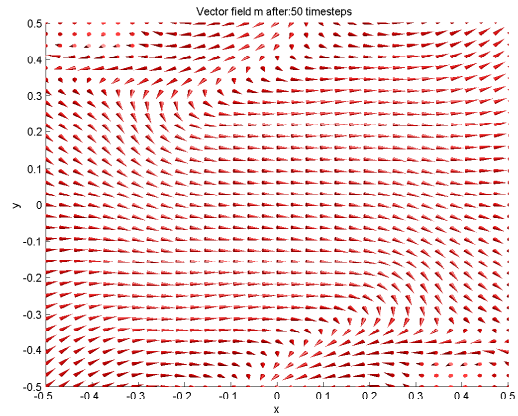
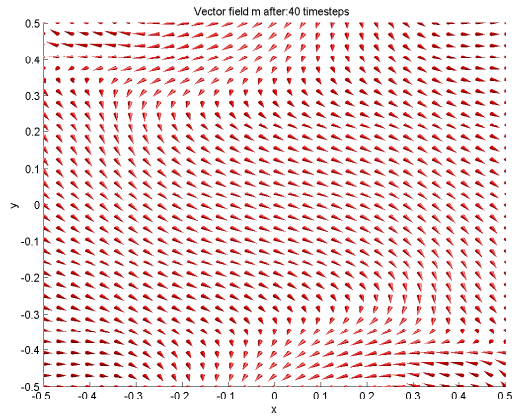
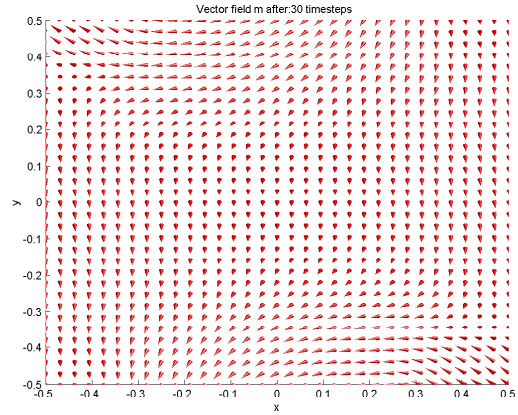
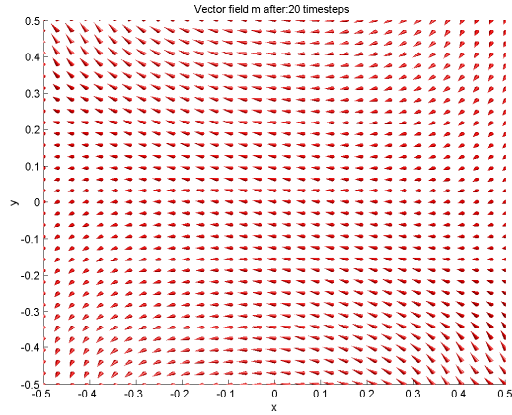
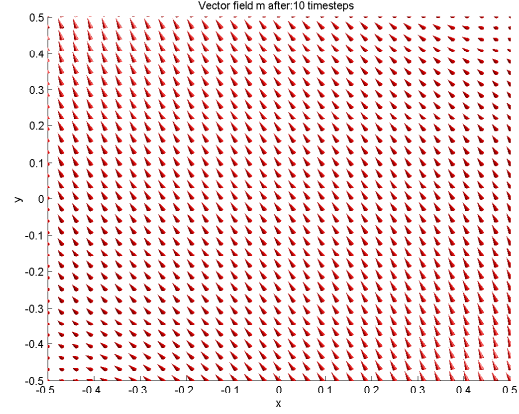
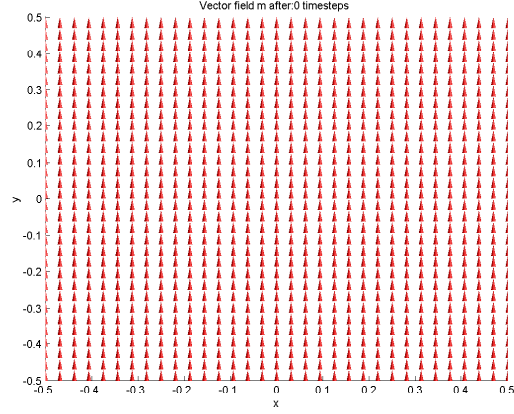
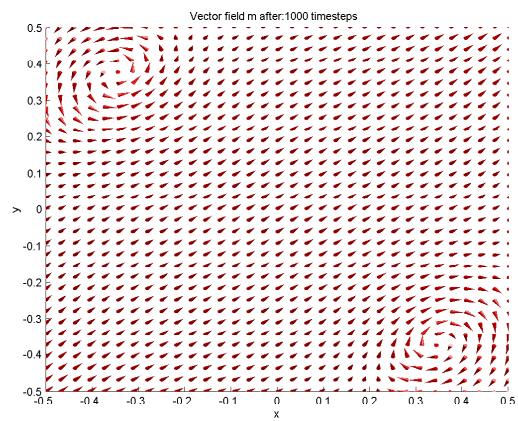
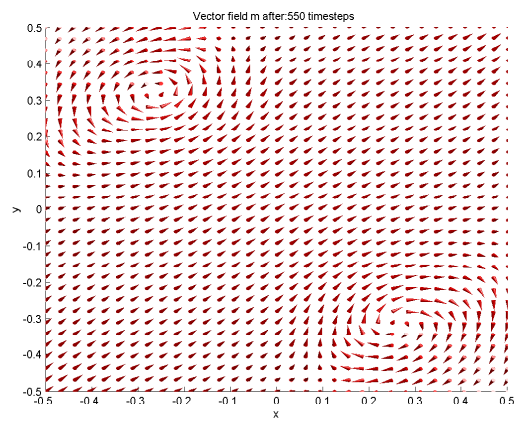
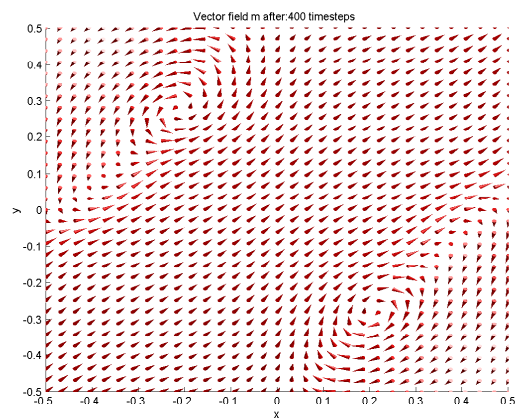
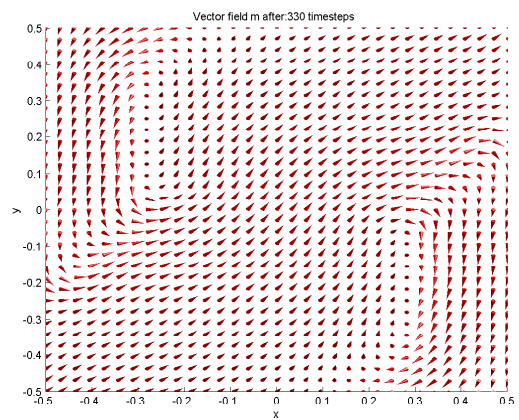
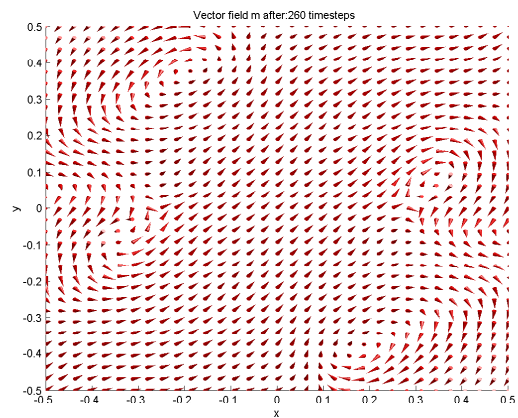
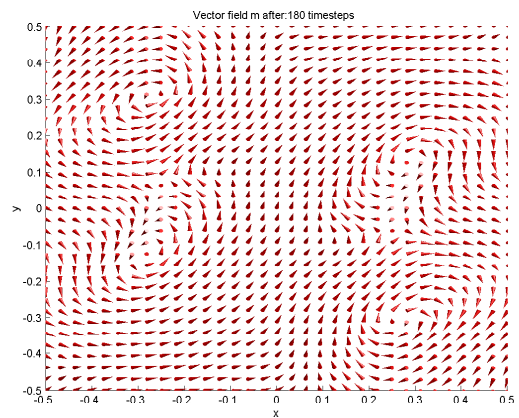
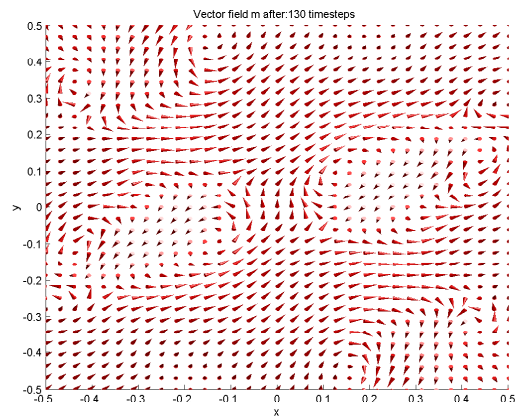
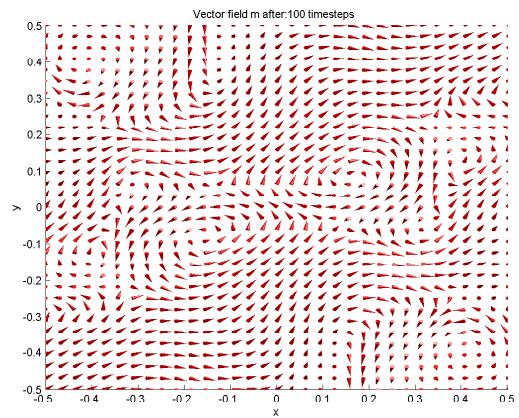


Figure 5: Different energy contributions

14.2 Screenshots at different timesteps

The following screenshots show the evolution of \mathbf{m} of our example. We observe the formation of vortices, which finally disappear at the upper left and the lower right corners. In the final state, the elementary magnets tend to point in the same direction.





14.3 Dependencies

In this section we study the dependency of the results on the meshwidth, on the timestep and on the size of the outer domain D .

In figure 14.3 the total energies of four simulations with different meshwidths are plotted. The finest grid consists of 97×97 points and the smallest of 25×25 points. Up to the physical time $t \approx 38$ the difference between the energies is smaller than 1%. For $t > 38$ the curves separate. All the simulations converge to the same final state, whereas the simulation with the finest grid takes longest.

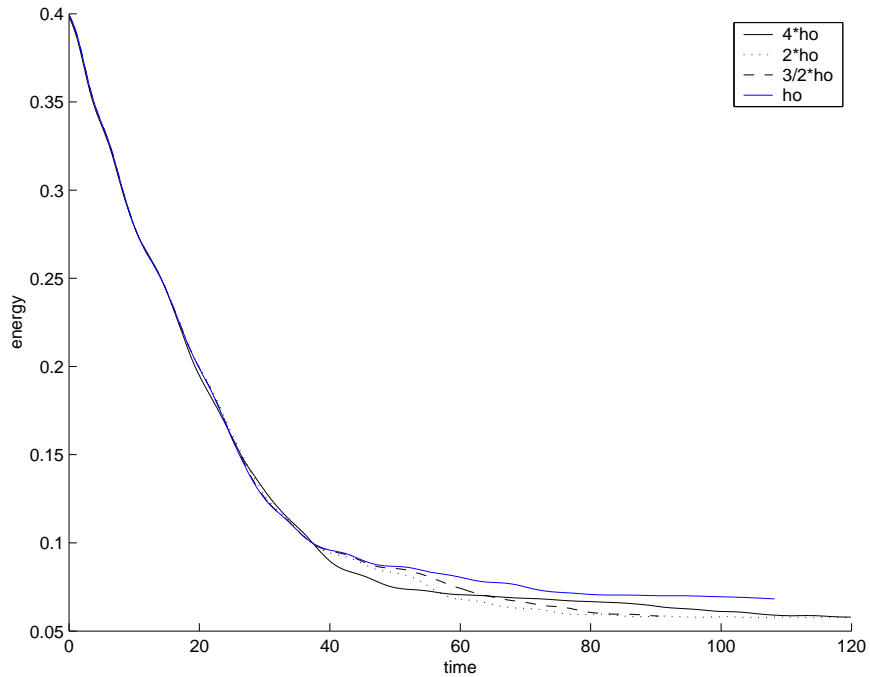


Figure 6: Total energies of four simulations with different meshwidths

Figure 14.3 shows the energies of the same example using three different timesteps. Qualitatively we get the same result for the evolution of the magnetization, but there are differences in the speed of the evolution. The example with the smallest timestep seems to evolve a little bit faster than the examples with bigger timesteps.

Figure 14.3 compares the energies of two simulations with different size of D . From the rather big difference in the results we can conclude, that we should chose the size of the domain D bigger than only four times of the size of Ω . This means, that we have to take much more meshpoints into account for the domain D , which increases the computing time considerably. A solution would be to use an adaptive grid in D , which gets less fine for points far away from Ω .

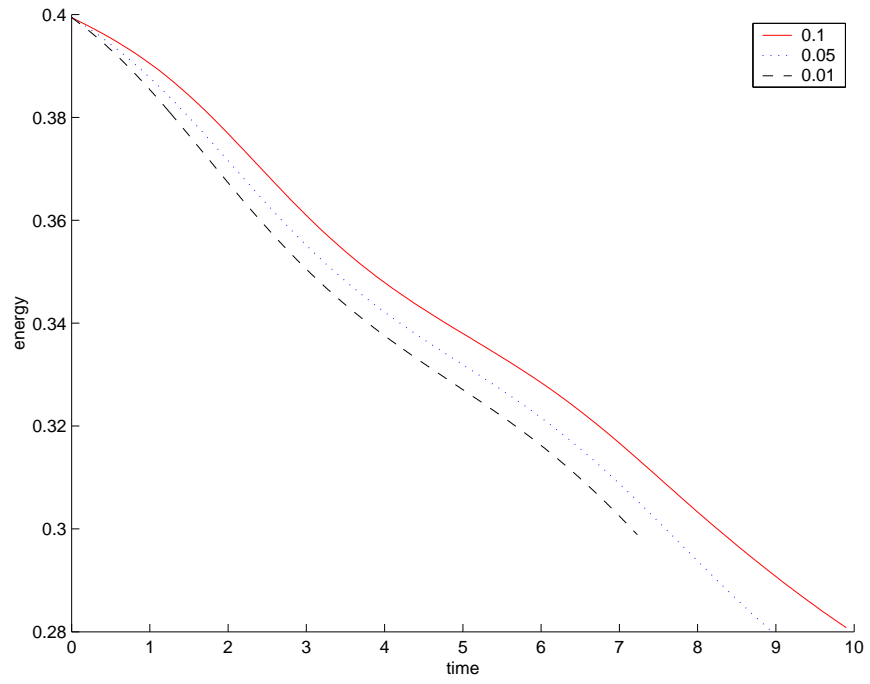


Figure 7: Total energies of four simulations with different timesteps

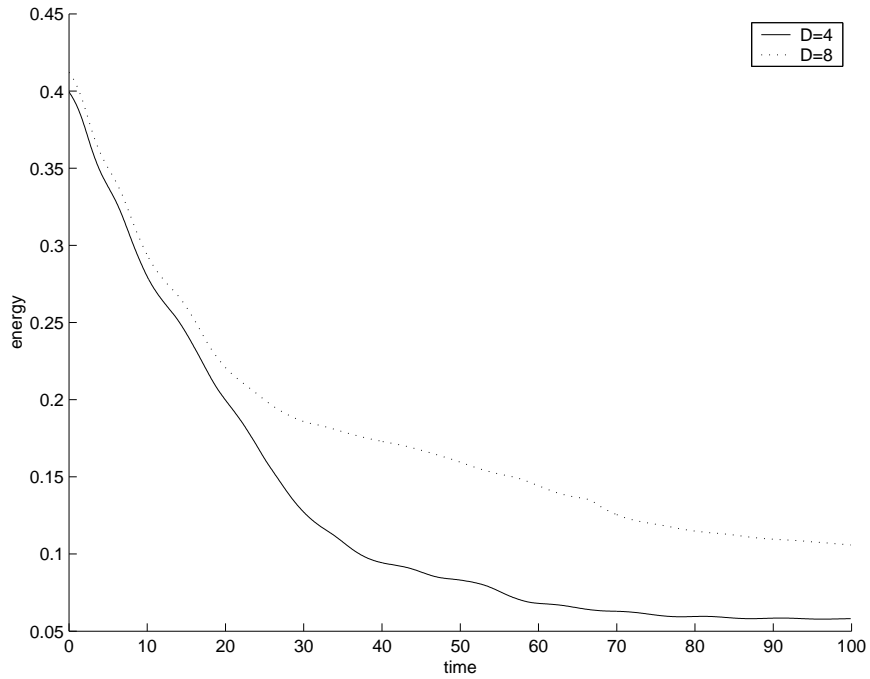


Figure 8: Total energies of two simulations with different size of the domain D

14.4 Convergence of the Newton algorithm

The convergence of the Newton-algorithm is fast, as figure 14.4 shows. The maximum number of steps is 3 to reach an error smaller than 10^{-12} .

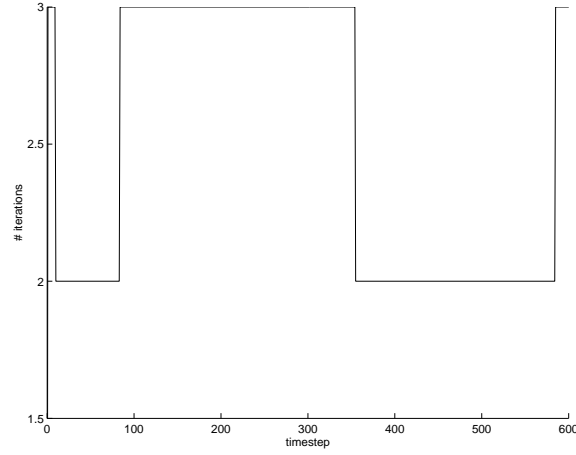


Figure 9: Number of iterations of the Newton algorithm used at each timestep

14.5 Energy conservation

When we set $\alpha = 0$ the energy remains constant (see figure 10). Even though the total energy remains constant, transitions between the different energy contributions can be observed.

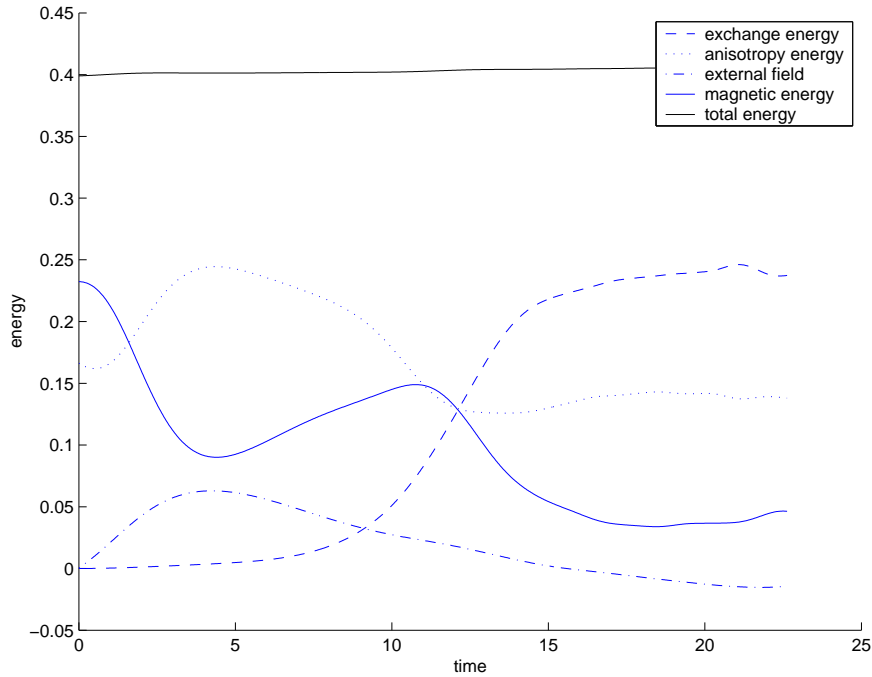


Figure 10: Energy conservation

A Matlab codes

In this section we present all the programs to run the simulation and to display the fields.

sim1.m (sim%.m)

A simulation file like `sim1.m` contains the initialization of the simulation parameters and of the initial magnetization. All the simulation data will be stored in a directory, given by the variable `simpath`. After the initialization the main routine `runsimulation.m` is called.

runsimulation.m

This is the main program to run a simulation. First the directory with name stored in the variable `simpath` is created, then the simulation parameters are printed out by the routine `dispparams.m`. The routines `init.m`, `calcpsi.m` and `energy.m` are called to initialize the matrices and to calculate the initial field ψ and the energy.

In the main loop of this routine, the Newton algorithm `newton.m` is called to perform a time step. The new magnetization is stored in the global variable `ms`. After each time step the values of all different energy contributions are calculated and stored in the file `En.bin` in the simulation directory. The scalar field ψ is stored in `psi.bin` and all components of the magnetization in `m.bin`.

contsim.m

Arguments:(`simpath`) Continues a interrupted simulation

dispparams.m

Arguments:(`simpath`)

Displays the parameters of the simulation in the directory `simpath`.

init.m

Initialization of the matrices A1, AN, M, Bx, By. Uses the function `basisd.m`.

basisd.m

Arguments:(`i`)

Converts the basis index `i` of the basis function Λ_i to the corresponding index `j` of the basis function Z_j , which is located at the same position. `j` is returned.

calcpsi.m

Arguments:(`m`)

Returns the scalar field ψ calculated from the magnetization `m`.

energy.m

Arguments:(`m,psi`)

Calculates all the energy contributions of the system and returns them in a vector [E1 E2 E3 E4], where E1 is the energy from the exchange term, E2 is the anisotropy energy, E3 is the contribution from the external magnetic field and E4 from the stray field term.

newton.m

Arguments:(`m`)

This routine contains the implementation of the Newton algorithm. It finds the root of the function `f.m`, which corresponds to the new magnetization. The number of iterations

is returned.

f.m

Arguments: (*m*)

This is the function $f(m)$ used in the Newton algorithm. The m which solves $f(m)=0$ corresponds to the new magnetization.

calcdm

Calculates 'dm' for the Newton step.

plotenergy.m

Arguments: (*simpath*)

Plots all energy contributions stored in *En.bin* in the directory *simpath*. The total energy is calculated and plotted too.

showm.m

Arguments: (*m*)

Plots the magnetization m as a coneplot. The size of the cones is set in the variable *conesize*.

showpsi.m

Arguments: (*psi*)

Plots the scalar field *psi*.

dispm.m

Arguments: (*simpath*, *step*)

Plots the magnetization stored in *m.bin* in the directory *simpath* at time step *step*. This routine uses *showm.m*.

disppsi.m

Arguments: (*simpath*, *step*)

Plots the scalar field ψ stored in *psi.bin* in the directory *simpath* at time step *step*. This routine uses *showpsi.m*.

makemoviem.m

Arguments: (*simpath*, *endstep*, *step*)

Creates movie of dynamical evolution of the magnetisation m of the simulation data in the path *simpath*. Every *step* time step a coneplot of m is appended to the film, up to time step *endstep*. The data of the film is stored in the file *Filmm.mat* in the directory *simpath*.

makemoviepsi.m

Arguments: (*simpath*, *endstep*, *step*)

Creates movie of dynamical evolution of the scalar field ψ of the simulation data in the path *simpath*. Every *step* time step a picture of ψ is appended to the film, up to time step *endstep*. The data of the film is stored in the file *Filmpsi.mat* in the directory *simpath*.

playmoviem.m

Arguments:(*simpath*)

Displays the film *Filmm.mat* stored in the directory *simpath*.

playmoviepsi.m

Arguments:(*simpath*)

Displays the film *Filmpsi.mat* stored in the directory *simpath*.

References

- [1] Zeidler Hackbusch, Schwarz. *Teubner-Taschenbuch der Mathematik*, volume 1. B.G. Teubner Stuttgart, Leipzig, 1996.