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# Numerical Pricing of American Options for general Bivariate Lévy Models 

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#### Abstract

In this work, we present a numerical pricing scheme for American options with two underlying assets for an exponential bivariate Lévy model. We consider a specific model structure that is constructed by a linear transformation of a Lévy process with independent components through a matrix-vector multiplication. We derive characteristics of the model such as the martingale property for each of its components and we examine how the jumps are correlated. Furthermore, we derive a system of inequalities that is solved by the American option price without the use of complicated Lévy copulas resulting in highly complicated and not tensorable stiffness matrices for the finite element discretisation. Well-posedness is proven for a variational formulation of the American option partial integrodifferential inequality (PIDI) and the continuous Galerkin finite element method is used for the discretisation of the pricing inequality. As a consequence of the linear transformation in the bivariate Lévy model we may completely tensorise the finite element matrices. This is a favourable feature for the implementation of the pricing scheme. Thereafter, we examine the semi-smooth Newton algorithm which solves the linear complementary problems (LCPs) arising from the continuous Galerkin finite element discretisation. We discuss the uniqueness of solutions to LCPs and analyse the convergence of the semi-smooth Newton algorithm. This is supported by numerical experiments that show the convergence of the semismooth Newton algorithm for different elliptic and parabolic variational problems with closed form solutions. Moreover, we test the convergence for the American basket put option using the overkill method as no closed form solution is available and observe a convergence rate of order $\mathcal{O}\left(\tilde{N}^{-1}\right)$ in $L^{2}$ where $\tilde{N}$ are the number of inner spatial grid points. The smooth pasting condition for the American put option with single underlying is also discussed and these conditions are applied to the numerical convergence tests of the American basket put option. In this case we observe no convergence. Additional to numerical convergence experiments, we analyse the speed of the semi-smooth Newton algorithm and compare results to the projective successive over-relaxation (PSOR) algorithm. We find that the semi-smooth Newton algorithm is faster than the PSOR algorithm using the MATLAB $®$ implementations. Finally we present two numerical examples of American options with two underlying assets, i.e. the American basket put option and the best-of American put option and depict their value surface, free boundary and the argument Greeks; theta, delta and gamma.


## Information

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To my parents
who always stood behind all of my decisions

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## Introduction

The first notion of option-like contracts was in the story of the mathematician and philosopher Thales of Miletus in the book of Aristotle [2, Book I (Chapter XI)] who during the winter hired all olive-presses in Chios and Miletus for the next harvesting session at a low cost since no one bid against him. The next olive harvest was so abundant and many farmers demanded olive-presses which Thales leased at a very high rate. Since that time options have become a very common type of financial derivative and during the last decades many kinds of options have been introduced, e.g. European, American, Asian, Bermuda, Israeli (options of game type), et cetera. Options are widely traded in many exchanges and in the over-the-counter (OTC) market. In 1973, exchange traded options became standardized contracts due to the opening of the Chicago Board Options Exchange (CBOE). In the year 2005, the volume of option contracts traded on the CBOE alone was approximately half a billion with a total notional value of more than $\$ 12$ trillion U.S. Dollar, see CBOE [8]. In simple terms, an option gives the buyer the right, but not the obligation, to perform a previously defined financial transaction within a set period of time. In this work we concentrate on one specific kind of option; American style options. These options give the holder the right to execute a financial transaction from the time of purchase until the expiration date, or maturity $T>0$.

Since the 1970's options have been priced using financial models. Black and Scholes 4 and Merton 37 developed a model in 1973, known as the BlackScholes model, which opened a new era of financial modelling. Within this model, closed-form solutions for the price of European options can be derived, see Black and Scholes [4]. In general, however, closed-form solutions are rarely available. In the past decades, extensions to the Black-Scholes model and more general Lévy models have been proposed to better mimic the characteristics of financial assets, e.g. Carr, et al. [7, Cox and Ross [11, Heston [23], Kou [32] and Merton [38]. For these complex models, closed-form solutions for general American options have, to our knowledge, not been established. For this reason, we analyse a numerical technique to approximate the price of American options in general bivariate Lévy models. We establish a pricing scheme based on the continuous Galerkin finite element method and propose a bivariate Lévy model to circumnavigate the use of elaborate Lévy copulas which result in highly complex stiffness matrices which do not tensorise. This is achieved by a matrix-vector multiplication in the exponent of the stock price process

$$
\left.\begin{array}{l}
S=\left(S_{t}\right)_{t \geq 0}=\left(\left(S_{t}^{1}\right)_{t \geq 0},\left(S_{t}^{2}\right)_{t \geq 0}\right)^{\top}, \\
\qquad\binom{S_{t}^{1}}{S_{t}^{2}}:=\binom{\exp \left(\left(r+w_{1}\right) t+\sum_{j=1}^{2} \boldsymbol{\Sigma}_{1 j} X_{t}^{j}\right)}{\exp \left(\left(r+w_{2}\right) t+\sum_{j=1}^{2} \boldsymbol{\Sigma}_{2 j} X_{t}^{j}\right.}
\end{array}\right), ~ l, ~ l
$$

where $r \in \mathbb{R}_{+}, \boldsymbol{\Sigma} \in \mathbb{R}^{2}, X=\left(X_{t}\right)_{t \geq 0}=\left(\left(X_{t}^{1}\right)_{t \geq 0},\left(X_{t}^{2}\right)_{t \geq 0}\right)^{\top}$ is a Lévy process with independent components and $w_{1}, w_{2}$ are constants subject to the martingale condition of $S^{1}$ and $S^{2}$, respectively. However, this comes at a cost of the dependence structure of the resulting asset price processes. We stress the fact that the dependence structure of the model proposed here is limited and should not be adopted without statistically significant arguments. Another result of the model above is that we have to deal with time-dependent obstacle $\tilde{g}(t, x)$ for the partial integro-differential inequality (PIDI) of the American option price. Though this has already been dealt with in the existing theory on variational inequalities. We use continuous Galerkin finite elements for the discretisation of the American option pricing problem with products of piecewise linear hat functions as basis for the finite element space $V_{N}$. That is,

$$
V_{N}=\operatorname{span}\left\{b_{l}\left(x_{1}\right) b_{k}\left(x_{2}\right): 1 \leq l, k \leq N\right\},
$$

where $b_{l}\left(x_{j}\right):=\max \left(0,1-h^{-1}\left|x_{j}-x_{l_{j}}\right|\right),\left(x_{l_{1}}, x_{l_{2}}\right)^{\top}$ with $1 \leq l_{1}, l_{2} \leq N \in \mathbb{N}$ are the inner nodes of the partition of the bounded domain $G \subset \mathbb{R}^{2}$ and $h>0$ is the spatial mesh size. The dimension of $V_{N}$ is $\tilde{N}=N^{2}$. Moreover, we employ Lagrangian multipliers to avoid the inequality resulting from the American PIDI in the discrete pricing scheme, see Hager, et al. [20, Section 3]. To solve this discrete pricing scheme numerically we utilize the primal dual active set strategy which is equivalent to a semi-smooth Newton algorithm, see Hintermüller, et al. [25, Chapter 2]. We analyse the convergence of the semi-smooth Newton method to solve the following linear complementary problems,

$$
\begin{aligned}
& \text { Find } \underline{x}, \underline{\lambda} \in \mathbb{R}^{n} \text { such that, } \\
& \mathbf{B} \underline{x}-\underline{\lambda}=\underline{b} \\
& C(\underline{x}, \underline{\lambda}):=\underline{\lambda}-\max (\underline{0}, \underline{\lambda}-\omega(\underline{x}-\underline{c}))=0
\end{aligned}
$$

where $\mathbf{B} \in \mathbb{R}^{n \times n}, \underline{b}, \underline{c} \in \mathbb{R}^{n}, \omega>0$ is a penalty constant and $C(x, \lambda)$ is called the complementary function where the maximum is taken componentwise. For the general discrete American option pricing scheme we prove the existence of a unique solution and local convergence of the solution. Moreover, we infer conditions on the Lévy process $X$ for the global convergence of the unique solutions. Additionally, we conduct numerical experiments of one-dimensional elliptic and parabolic variational inequalities with a non-degenerate linear operator of order two and in presence of a closed-form solution. We conclude that in these cases the convergence rates of the semi-smooth Newton method are optimal under some regularity conditions on the obstacle and the linear operator. For the parabolic problem we distinguish between two cases. First, we analyse the case where the obstacle $\tilde{g}(x) \in V_{N}$, i.e. the non-continuously differentiable point of the obstacle $\tilde{g}(x)$ is part of the spatial grid. Second, we consider the case where $\tilde{g}(x) \notin V_{N}$. In the latter case the convergence rate in $H^{1}$ drastically decreases whereas the convergence rates in the $L^{\infty}$ - and $L^{2}$ error estimators remain identical. As we do not have a closed-form solution
for the American option problem, we exploit the overkill method to measure the convergence of the pricing scheme. For the American basket option with time-dependent obstacle $\tilde{g}(t, x) \notin V_{N}$ we procure a convergence rate of almost $\mathcal{O}\left(\tilde{N}^{-1}\right)$ in $L^{2}$ where $\tilde{N} \in \mathbb{N}$ is the number of inner nodes. This convergence rate is in line with the numerical experiments in one dimension taking into account the 'curse of dimension.' However the convergence rates in $L^{\infty}$ and $H^{1}$ are of order $\frac{1}{2}$ in terms of $\tilde{N}$. Further, we analyse the recent findings of Lamberton and Mikou 34 on the smooth pasting property of the plain vanilla American put option and compute the convergence of the American basket put option under the violation of the the smooth pasting property as given in Lamberton and Mikou [34, Theorem 4.2]. We find that the finite element solution does not converge under these circumstances. Thereafter, we compare the speed of the semi-smooth Newton algorithm to the well-known projective successive overrelaxation (PSOR) algorithm and find that the MATLAB® implementation of the semi-smooth Newton algorithm is much faster than the PSOR algorithm for the same accuracy.

This paper is structured as follows. In Chapter 1, we lay out the preliminaries that are needed for the analysis of American option pricing in a bivariate Lévy model. We give an introduction to multidimensional Lévy processes and discuss important properties such as the Lévy-Itô decomposition and the LévyKhinchin representation. Moreover, we present the specific bivariate Lévy model for the asset price processes in Section 1.3.1 and give two concrete examples of the underlying Lévy processes used for the asset pricing model, namely the Kou and the CGMY process. Thereafter we shortly discuss Lévy copulas and semigroups of Lévy processes. In Chapter 2 we derive the continuous Galerkin finite element pricing scheme. First, we formally derive the system of inequalities for general American options with two underlying assets. Thereafter, we formulate the variational inequalities for the weak solution of the American option price and discuss the well-posedness of the variational problem. In the last section of Chapter 2 we deduce the fully discrete pricing scheme using continuous, linear finite elements in space and finite difference in time and reformulate the problem using Lagrangian multipliers. In Chapter 3 we delineate the semi-smooth Newton algorithm and cover some general characteristics of the convergence and the uniqueness of solutions to linear complementary problems. In Section 3.3, we give some a priori error estimates for elliptic and parabolic variational inequalities. In addition, we numerically test the convergence of the semi-smooth Newton algorithm for different elliptic and parabolic variational problems in one and two dimensions in Section 3.4. The last section of Chapter 3is devoted to a speed analysis of the semi-smooth Newton algorithm. We compare it to the well-known PSOR algorithm. In the last chapter (Chapter 4), we depict two numerical examples. First, we consider the American basket put option and second, the best-of American put option. We display the value as well as the free boundary of these options and show their argument Greeks, i.e. delta, gamma and theta.

## Chapter 1

## Preliminaries

In this chapter, we discuss some preliminaries about financial modelling with Lévy processes. First, we lay out the concept of Lévy processes itself. Thereafter we present the model for introducing correlation among stock price processes driven by independent Lévy processes applying a matrix-vector multiplication. Further, we discuss the martingale property of each of the stock price processes and calculate the necessary drift term. Moreover, we consider two different Lévy processes within the model, the CGMY process in Carr, et al. 7] and the process proposed in Kou [32. Finally, we cover some properties of Lévy copulas and make a connection between Lévy processes and Markov processes through semigroups. This is necessary for the derivation of the partial integrodifferential equation (PIDE) for American options in Chapter 2. Let us first introduce some notation.

### 1.1 Notation

In this section, we introduce some notation that we use throughout this paper. The extended real line is denoted by $\overline{\mathbb{R}}:=\mathbb{R} \cup\{+\infty\}=(-\infty, \infty]$. We write the Euclidean norm in $\mathbb{R}^{d}$ with $d \in \mathbb{N}$ as $\|\underline{x}\|_{l^{2}}^{2}=\sum_{i=1}^{d}\left|x_{i}\right|^{2}$ and the supremum norm as $\|\underline{x}\|_{l^{\infty}}=\sup _{i}\left|x_{i}\right|$ for $\underline{x}=\left(x_{1}, \ldots, x_{d}\right)^{\top} \in \mathbb{R}^{d}$. We denote the inner product on $\mathbb{R}^{2}$ by $\langle\underline{x}, \underline{y}\rangle=\sum_{i=1}^{d} x_{i} y_{i}$ for $\underline{x}, \underline{y} \in \mathbb{R}^{d}$. For $\phi, \psi \in L^{2}(G)$ the inner product on $L^{2}(\bar{G})$ is denoted by $(\phi, \psi)=\int_{G} \phi(\underline{x}) \psi(\underline{x}) d \underline{x}$ where $G \subset$ $\mathbb{R}^{d}$ is bounded in $\mathbb{R}^{d}$. Moreover, we denote the support of a function $\psi(\underline{x})$ as $\operatorname{supp} \psi$. If $\mathcal{V}$ is a Hilbert space with dual $\mathcal{V}^{*}$ then we denote the pairing of $\mathcal{V}$ and $\mathcal{V}^{*}$ with $(\cdot, \cdot)_{\mathcal{V}^{*}, \mathcal{V}}$. Furthermore, we write for $\underline{x} \in \mathbb{R}^{d}$ that $\underline{x}_{\mathcal{I}}=$ $\left(x_{i}\right)_{i \in \mathcal{I}}$ for $\mathcal{I} \subset\{1, \ldots, d\}$ and $\underline{0}=(0, \ldots, 0)^{\top}$ is a vector of zeros. Matrices are bold and $\mathbf{M}_{(\mathcal{I}, \mathcal{J})}=\left(\mathbf{M}_{i j}\right)_{i \in \mathcal{I}, j \in \mathcal{J}}$ with $\mathcal{I}, \mathcal{J} \subset\{1, \ldots, d\}$ and $\mathbf{M} \in \mathbb{R}^{d \times d}$. $\mathbf{M}_{i}=\left(\mathbf{M}_{i 1}, \ldots, \mathbf{M}_{i d}\right)$ denotes the $i^{\text {th }}$ row of the matrix $\mathbf{M} \in \mathbb{R}^{d \times d}$ and $|\mathbf{M}|$ denotes the determinant of matrix M. The Kronecker product to two matrices is define in Definition C.1. Also we use the convention,

$$
\underline{z}=\underline{x}+\underline{y}_{\mathcal{I}} \text { then } \underline{z}_{i}= \begin{cases}x_{i}+y_{i}, & \text { if } i \in \mathcal{I}, \\ x_{i}, & \text { if } i \notin \mathcal{I},\end{cases}
$$

where $\underline{x}, \underline{y}, \underline{z} \in \mathbb{R}^{d}$ and $\mathcal{I} \subset\{1, \ldots, d\}$. If $\mathcal{I}=\{i\} \subset\{1, \ldots, d\}$ then we simply write $\underline{z}=\underline{x}+y_{i}$. Moreover, for a vector $\underline{x} \in \mathbb{R}^{d^{2}}$ we use $x_{j}=x_{j_{1}(d-1)+j_{2}}=: x_{j_{1}, j_{2}}$ for $j=j_{1}(d-1)+j_{2}$ with $1 \leq j_{1}, j_{2} \leq d$. We denote the number of elements in a set $\mathcal{I}$ by $\# \mathcal{I}$. If we have a set $A \subset \mathbb{R}^{d}$ then we denote $A^{c}=\mathbb{R}^{d} \backslash A \subset \mathbb{R}^{d}$ its complement in $\mathbb{R}^{d}$. Orthotopes are defined as $(\underline{a}, \underline{b}]=\left(a_{1}, b_{1}\right] \times \cdots \times\left(a_{d}, b_{d}\right] \subset \overline{\mathbb{R}}^{d}$. Finally, proofs and definitions are ended by $\square$ and $\perp$, respectively.

### 1.2 Introduction to Lévy Processes

In this section, we discuss some properties of general Lévy processes. We mostly follow the work of Sato [46] and Applebaum [1]. In the course of this paper, we work on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual condition in Delbaen and Schachermayer [12, Chapter 7] with filtration $\mathbb{F}=\left\{\mathcal{F}_{t}: t \geq 0\right\}$. Moreover, a stochastic process (SP) $L=\left(L_{t}\right)_{t \geq 0}$ is called adapted if $L_{t}$ is $\mathcal{F}_{t^{-}}$ measurable, i.e. $\left\{L_{t} \leq \underline{x}\right\} \in \mathcal{F}_{t}$ for all $t \geq 0$ and $\underline{x} \in \mathbb{R}^{d}$. A specific kind of stochastic process is a Lévy process (LP).

Definition 1.1 (Lévy Process). An adapted, càdlàg $\mathbb{R}^{d}$-valued SP $L=\left(L_{t}\right)_{t \geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ as above with $d \geq 1$ is a Lévy Process if it possesses the following properties:
(L1) $L_{0}=0 \quad$ a.s.
(L2) Independent increments: For any sequence $t_{0}<\cdots<t_{n}$ the $\mathbb{R}^{d}$-valued random variables (rv) $L_{t_{0}}, L_{t_{1}}-L_{t_{0}}, \ldots, L_{t_{n}}-L_{t_{n-1}}$ are independent.
(L3) Stationary increments: the law of $L_{t+h}-L_{t}$ does not depend on $t$, i.e. $\mathcal{L}\left(L_{t+h}-L_{t}\right)=\mathcal{L}\left(L_{h}\right)$ where $\mathcal{L}\left(L_{t}\right)$ is the law of $L_{t}$ for $t \geq 0$.
(L4) Stochastic continuity: $\forall \epsilon>0, \lim _{h \rightarrow 0} \mathbb{P}\left(\left|L_{t+h}-L_{t}\right| \geq \epsilon\right)=0$.

This definition can also be found in Applebaum [1, Section 1.3]. The following two theorems are extremely useful tools for calculations with Lévy processes.

Theorem 1.2 (Lévy-Itô Decomposition). Let $L=\left(L_{t}\right)_{t \geq 0}$ be a d-dimensional $L P$. Then there is a drift $\underline{\gamma} \in \mathbb{R}^{d}$, an $\mathbb{R}^{d}$-valued Brownian motion (BM) $W=\left(W_{t}\right)_{t \geq 0}$ with covariance matrix $\mathbf{Q}$ with $\mathbf{Q}=\mathbf{Q}^{\frac{1}{2}}\left(\mathbf{Q}^{\frac{1}{2}}\right)^{\top}$ and an independent Poisson random measure, also called jump measure, $J_{L}$ on $[0, T] \times \mathbb{R}^{d}$ such that,

$$
\begin{aligned}
d L_{t} & =\underline{\gamma} d t+\mathbf{Q}^{\frac{1}{2}} d W_{t}+\int_{\|\underline{z}\|_{l^{2}} \geq 1} J_{L}(d t, d \underline{z})+\lim _{\epsilon \downarrow 0} \int_{\epsilon \leq\|\underline{z}\|_{l^{2}} \leq 1} \underline{z}\left(J_{L}(d t, d \underline{z})-\nu(d \underline{z}) d t\right) \\
& =\underline{\gamma} d t+\mathbf{Q}^{\frac{1}{2}} d W_{t}+\Delta L_{t} \mathbb{1}_{\left[\left\|\Delta L_{t}\right\|_{l^{2} \geq 1}\right]}+\lim _{\epsilon \downarrow 0} \int_{\epsilon \leq\|\underline{z}\|_{l^{2}} \leq 1} \underline{z} \tilde{J}_{L}(d t, d \underline{z}),
\end{aligned}
$$

where $\tilde{J}_{L}(d t, d \underline{z}):=J_{L}(d t, d \underline{z})-\nu(d \underline{z}) d t$ is the compensated jump measure and the measure $\nu(d \underline{z})$ is called the Lévy measure defined in Sato [46, Definition 8.2].

Proof. The proof to this theorem is given in Applebaum [1, Theorem 2.4.11].
The drift $\underline{\gamma}$, covariance matrix $\mathbf{Q}$ and Lévy measure $\nu(d \underline{z})$ are called the characteristic triplet of the Lévy process, denoted as ( $\underline{\gamma}, \mathbf{Q}, \nu)$. The Lévy measure satisfies $\int_{\mathbb{R}^{d}} 1 \wedge\|\underline{z}\|_{l^{2}}^{2} \nu(d \underline{z})<\infty$.
Theorem 1.3 (Lévy-Khinchin Representation). Let $L=\left(L_{t}\right)_{t \geq 0}$ be a ddimensional $L P$ with characteristic triplet $(\underline{\gamma}, \mathbf{Q}, \nu)$. Then for $t \geq 0$ and $\underline{\xi} \in \mathbb{R}^{d}$,

$$
\begin{aligned}
& \mathrm{E}\left[e^{i\left\langle\underline{\xi}, L_{t}\right\rangle}\right]=e^{t \psi(\underline{\xi})}, \text { where } \\
& \psi(\underline{\xi})=i\langle\underline{\gamma}, \underline{\xi}\rangle-\frac{1}{2}\langle\underline{\xi}, \mathbf{Q} \underline{\xi}\rangle+\int_{\mathbb{R}^{d} \backslash\{0\}}\left(e^{i\langle\underline{z}, \underline{\xi}\rangle}-1-i\langle\underline{z}, \underline{\xi}\rangle \mathbb{1}_{\left[\|\underline{z}\|_{l^{2}} \leq 1\right]}\right) \nu(d \underline{z}) .
\end{aligned}
$$

Proof. The proof to this theorem is given in Applebaum [1, Theorem 1.3.3].
Another characteristic of LPs is finite and infinite activity. As in some of the proofs throughout this paper we distinguish between finite and infinite activity LPs let us give the definition here.
Definition 1.4. Let $L=\left(L_{t}\right)_{t \geq 0}$ be a $d$-dimensional LP with characteristic triplet $(\underline{\gamma}, \mathbf{Q}, \nu)$. If $\nu\left(\mathbb{R}^{d}\right)<\infty$, then we say that $L$ is of finite activity. On the other hand, if $\nu\left(\mathbb{R}^{d}\right)=\infty$ the process $L$ is said to be of infinite activity. $\perp$

Another characteristic of LPs is finite variation. The definition of finite variation can be found in Sato [46, Section 4.21].

Definition 1.5. Let $L=\left(L_{t}\right)_{t \geq 0}$ be a $d$-dimensional LP with characteristic triplet $(\underline{\gamma}, \mathbf{Q}, \nu)$ and $\Omega_{1} \subset \Omega$ with $\mathbb{P}\left(\Omega_{1}\right)=1$. Then $L$ is a finite variation process $\overline{\text { if }}$ and only if,

$$
\sup _{\Pi_{N} \in \Pi_{T}} \sum_{i=1}^{N}\left\|L_{t_{i}}(\omega)-L_{t_{i-1}}(\omega)\right\|_{l^{2}}<\infty \quad \forall \omega \in \Omega_{1} \text { and } T>0
$$

where the supremum is taken over all partitions,

$$
\Pi_{T}:=\left\{\Pi_{N}=\left(0=t_{0}<\cdots<t_{N}=T\right): N \in \mathbb{N} \text { and } T>0\right\}
$$

of the interval $[0, T]$.
Proposition 1.6. Let $L=\left(L_{t}\right)_{t \geq 0}$ be a d-dimensional LP with characteristic triplet $(\gamma, \mathbf{Q}, \nu)$. Then $L$ is of finite variation if and only if $\mathbf{Q}=\mathbf{0}$ and $\int_{\|\underline{z}\|_{l^{2}} \leq 1}\|\underline{z}\|_{l^{2}} \nu(d \underline{z})<\infty$.
Proof. The proof may be found in Applebaum [1, Theorem 2.4.25].
One of the direct consequences of Proposition 1.6 is stated in the following corollary.
Corollary 1.7. Let $L=\left(L_{t}\right)_{t \geq 0}$ be a d-dimensional LP of finite variation with characteristic triplet $(\underline{\gamma}, \mathbf{0}, \nu)$. Then we can write,

$$
L_{t}=\int_{0}^{t} \underline{\gamma}_{0} d s+\int_{0}^{t} \int_{\mathbb{R}^{d} \backslash\{0\}} \underline{z} J_{L}(d s, d \underline{z})
$$

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and,

$$
\psi(\underline{\xi})=i\left\langle\underline{\gamma}_{0}, \underline{\xi}\right\rangle+\int_{\mathbb{R}^{d} \backslash\{0\}}\left(e^{i\langle\underline{z}, \underline{\xi}\rangle}-1\right) \nu(d \underline{z}),
$$

where $\underline{\gamma}_{0}=\underline{\gamma}-\int_{\|\underline{z}\|_{l^{2}} \leq 1} \underline{z} \nu(d \underline{z})$.
Proof. Using Theorem 1.2 and Proposition 1.6 we get,

$$
\begin{aligned}
d L_{t} & =\underline{\gamma} d t+\mathbf{Q}^{\frac{1}{2}} d W_{t}+\int_{\|\underline{z}\|_{l^{2}} \geq 1} J_{L}(d t, d \underline{z})+\lim _{\epsilon \downarrow 0} \int_{\epsilon \leq\|\underline{z}\|_{l^{2}} \leq 1} \underline{z}\left(J_{L}(d t, d \underline{z})-\nu(d \underline{z}) d t\right) \\
& =\underline{\gamma} d t+\int_{\mathbb{R}^{d} \backslash\{0\}} \underline{z} J_{L}(d t, d \underline{z})-\int_{\|\underline{z}\|_{l^{2}} \leq 1} \underline{z} \nu(d \underline{z}) d t \\
& =\underline{\gamma}_{0} d t+\int_{\mathbb{R}^{d} \backslash\{0\}} \underline{z} J_{L}(d t, d \underline{z}) .
\end{aligned}
$$

Furthermore, Theorem 1.3 and Proposition 1.6 yield,

$$
\begin{aligned}
\psi(\underline{\xi}) & =i\langle\underline{\gamma}, \underline{\xi}\rangle-\frac{1}{2}\langle\underline{\xi}, \mathbf{Q} \underline{\xi}\rangle+\int_{\mathbb{R}^{d} \backslash\{0\}}\left(e^{i\langle\underline{z}, \underline{\xi}\rangle}-1-i\langle\underline{z}, \underline{\xi}\rangle \mathbb{1}_{\left[\|\underline{z}\|_{l^{2}} \leq 1\right]}\right) \nu(d \underline{z}) \\
& =i\langle\underline{\gamma}, \underline{\xi}\rangle+\int_{\mathbb{R}^{d} \backslash\{0\}}\left(e^{i\langle\underline{z}, \underline{\xi}\rangle}-1\right) \nu(d \underline{z})-\int_{\mathbb{R}^{d} \backslash\{0\}} i\langle\underline{z}, \underline{\xi}\rangle \mathbb{1}_{\left[\|\underline{z}\|_{l^{2}} \leq 1\right]} \nu(d \underline{z}) \\
& =i\left\langle\underline{\gamma}_{0}, \underline{\xi}\right\rangle+\int_{\mathbb{R}^{d} \backslash\{0\}}\left(e^{i\langle\underline{z}, \underline{\xi}\rangle}-1\right) \nu(d \underline{z}),
\end{aligned}
$$

by the definition of $\underline{\gamma}_{0}$.
To make sure that the LP has finite $p^{\text {th }}$ moments with $p \geq 1$ we use the following proposition.

Proposition 1.8. Let $L=\left(L_{t}\right)_{t \geq 0}$ be a d-dimensional LP with characteristic triplet $(\underline{\gamma}, \mathbf{Q}, \nu)$. Then $L$ has finite $p^{t h}$ moment, i.e $\mathrm{E}\left[\left|L_{t}\right|^{p}\right]<\infty$, if and only if $\int_{\|\underline{z}\|_{l^{2} \geq 1} \geq} \bar{\Pi} \underline{z} \|_{l^{2}}^{p} \nu(d \underline{z})<\infty$ for $p \geq 0$.
Proof. The proof can be found in Sato [46, Theorem 25.3].
A direct consequence of Proposition 1.8 for an LP with finite first moment is given in the next corollary.

Corollary 1.9. Let $L=\left(L_{t}\right)_{t \geq 0}$ be a d-dimensional LP with characteristic triplet $(\underline{\gamma}, \mathbf{Q}, \nu)$ and suppose $L$ has finite first moment. Then we can write $L$ as follows,

$$
L_{t}=\int_{0}^{t} \underline{\gamma}^{c} d s+\int_{0}^{t} \mathbf{Q}^{\frac{1}{2}} d W_{s}+\int_{0}^{t} \int_{\mathbb{R}^{d} \backslash\{0\}} \underline{z} \tilde{J}_{L}(d s, d \underline{z})
$$

and the Lévy exponent can also be represented differently,

$$
\psi(\underline{\xi})=i\left\langle\underline{\gamma}^{c}, \underline{\xi}\right\rangle-\frac{1}{2}\langle\underline{\xi}, \mathbf{Q} \underline{\xi}\rangle+\int_{\mathbb{R}^{d} \backslash\{0\}}\left(e^{i\langle\underline{z}, \underline{\xi}\rangle}-1-i\langle\underline{z}, \underline{\xi}\rangle\right) \nu(d \underline{z}),
$$

where $\underline{\gamma}^{c}=\underline{\gamma}+\int_{\|\underline{z}\|_{l^{2}} \geq 1} \underline{z} \nu(d \underline{z})$.
Proof. Idem to proof of Proposition 1.7. Employing Proposition 1.8 instead of Proposition 1.6 .

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Inter alia, it is crucial for option pricing to have finite exponential moment, i.e. $\mathrm{E}\left[e^{L_{t}}\right]<\infty$. We elaborate on this in Assumptions 1.14 . One important consequence of this is that finite exponential moment is equivalent to a integrability condition on the Lévy measure $\nu(d \underline{z})$. We formulate this in the next proposition.

Proposition 1.10. Let $L=\left(L_{t}\right)_{t \geq 0}$ be a d-dimensional LP with characteristic triplet $(\underline{\gamma}, \mathbf{Q}, \nu)$. Then $L$ has finite exponential moment, i.e. $\mathrm{E}\left[e^{L_{t}}\right]<\infty$ for all $t \geq 0$ if and only if $\int_{\|\underline{z}\|_{l^{2}}>1} e^{\underline{z}} \nu(d \underline{z})<\infty$.

Proof. The proof is given in Sato [46, Theorem 25.17].
Remark. In subsequent sections and chapters we work with specific Lévy processes $X$ which always have finite first moment. Therefore, in the remainder of this paper we refer to $\left(\underline{\gamma}^{c}, \mathbf{Q}, \nu\right)$ as the characteristic triplet where $\underline{\gamma}^{c}$ is called the center of the process given by $\underline{\gamma}^{c}=\underline{\gamma}+\int_{\|\underline{z}\|_{L_{2}} \geq 1} \underline{z} \nu(d \underline{z})$. We also remain using the notation for the Lévy exponent as in Corollary 1.9 if not explicitly stated otherwise.

### 1.3 Bivariate Lévy Models

This section introduces the bivariate Lévy model for dependence among stock price processes discussed in this paper. First, we give the general setup for any two-dimensional Lévy process $X$ with independent components on the complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ as above. Second, we give some properties of the dependence structure within the model setup and point out some limitations of dependent jumps. Further, let $\mathbb{Q}$ be an equivalent martingale measure with respect to $\mathbb{P}$. Under $\mathbb{Q}$ we state the martingale condition that has to be satisfied such that each discounted stock price process is a martingale. Third, we describe two specific Lévy processes which are employed for numerical examples throughout this paper as well as state the specific martingale condition for the model in terms of each of the two underlying Lévy processes. Finally, we review some of the theory of Lévy copulas which we need in subsequent chapters.

### 1.3.1 The General Bivariate Model and Dependence Properties

To introduce dependence in the bivariate Lévy model we use a matrix-vector multiplication of the form

$$
\begin{equation*}
Y_{t}:=\boldsymbol{\Sigma} X_{t} . \tag{1.1}
\end{equation*}
$$

Thus let the stock price process $S=\left(S_{t}\right)_{t \geq 0}=\left(\left(S_{t}^{1}\right)_{t \geq 0},\left(S_{t}^{2}\right)_{t \geq 0}\right)^{\top}$ be defined as follows

$$
\begin{equation*}
\binom{S_{t}^{1}}{S_{t}^{2}}:=\binom{\exp \left(\left(r+w_{1}\right) t+Y_{t}^{1}\right)}{\exp \left(\left(r+w_{2}\right) t+Y_{t}^{2}\right)}=\binom{\exp \left(\left(r+w_{1}\right) t+\sum_{j=1}^{2} \boldsymbol{\Sigma}_{1 j} X_{t}^{j}\right)}{\exp \left(\left(r+w_{2}\right) t+\sum_{j=1}^{2} \boldsymbol{\Sigma}_{2 j} X_{t}^{j}\right)}, \tag{1.2}
\end{equation*}
$$

where $r \in \mathbb{R}_{+}, X^{j}=\left(X_{t}^{j}\right)_{t \geq 0}$ for $j=1,2$ are independent Lévy processes on $\left(\Omega, \mathcal{F}, \mathcal{F}_{t}, \mathbb{Q}\right)$ with center zero, i.e. the Lévy triplet of $X=\left(X^{1}, X^{2}\right)^{\top}$ is
$\left(\underline{0}, \mathbf{Q}, \nu_{X}\right)$ where $\mathbf{Q}=\left(\begin{array}{cc}\sigma_{1}^{2} & 0 \\ 0 & \sigma_{2}^{2}\end{array}\right) \in \mathbb{R}^{2 \times 2}$ with $\sigma_{1}, \sigma_{2} \geq 0$ and $\nu_{X}(d \underline{z})$ is the twodimensional Lévy measure. The constants $w_{1}, w_{2}$ are to be determined. They are closely related to the martingale property as we will observe in Theorem 1.15. By the Lévy-Itô decomposition we have the following general expression for $X_{t}^{j}$,

$$
X_{t}^{j}=\sigma_{j} W_{t}^{j}+\int_{0}^{t} \int_{\mathbb{R} \backslash\{0\}} z_{j} \tilde{J}_{X^{j}}\left(d s, d z_{j}\right) \text { for } j=1,2
$$

where $\tilde{J}_{X^{j}}\left(d t, d z_{j}\right)=J_{X^{j}}\left(d t, d z_{j}\right)-\nu_{X^{j}}\left(d z_{j}\right) d t$ and $W=\left(W_{t}\right)_{t \geq 0}$ is a twodimensional BM. An equivalent model can be found in Eberlein and Madan [13.
Remark. For Lévy processes $X^{j}$ where $\int_{\left|z_{j}\right| \leq 1}\left|z_{j}\right| v_{X^{j}}\left(d z_{j}\right)<\infty$ for $j=1,2$ we take $\gamma_{0}^{j}$ to be zero and write

$$
\begin{equation*}
X_{t}^{j}=\sigma_{j} W_{t}^{j}+\int_{0}^{t} \int_{\mathbb{R} \backslash\{0\}} z_{j} J_{X^{j}}\left(d s, d z_{j}\right) \tag{1.3}
\end{equation*}
$$

To analyse the dependence created by the model in 1.2, let us state some characteristics of the two-dimensional process $Y=\left(Y_{t}\right)_{t \geq 0}$ in 1.1). Due to the result in Cont and Tankov [10, Theorem 4.1] for $n=d=2$ we know that $Y$ is a Lévy process. We derive its characteristic triplet in the next theorem.

Theorem 1.11. Let $X=\left(X_{t}\right)_{t \geq 0}$ be a two-dimensional Lévy process with independent components and characteristic triplet $\left(\underline{\gamma}_{X}^{c}, \mathbf{Q}_{X}, \nu_{X}\right)$ where $\mathbf{Q}_{X}=\left(\begin{array}{cc}\sigma_{1} & 0 \\ 0 & \sigma_{2}\end{array}\right)$ with $\sigma_{1}, \sigma_{2} \geq 0$ and $\underline{\gamma}_{X}^{c}=\underline{0}$. Further, let $\boldsymbol{\Sigma} \in \mathbb{R}^{2 \times 2}$ be a constant matrix then $Y=\left(Y_{t}\right)_{t \geq 0}:=\left(\boldsymbol{\Sigma} X_{t}\right)_{t \geq 0}$ is again a Lévy process with characteristic triplet $\left(\underline{\gamma}_{Y}^{c}, \mathbf{Q}_{Y}, \nu_{Y}\right)$ given by,

$$
\begin{align*}
\mathbf{Q}_{Y} & =\left(\begin{array}{cc}
\sigma_{1}^{2} \boldsymbol{\Sigma}_{11}^{2}+\sigma_{2}^{2} \boldsymbol{\Sigma}_{12}^{2} & \sigma_{1}^{2} \boldsymbol{\Sigma}_{11} \boldsymbol{\Sigma}_{21}+\sigma_{2}^{2} \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22} \\
\sigma_{1}^{2} \boldsymbol{\Sigma}_{11} \boldsymbol{\Sigma}_{21}+\sigma_{2}^{2} \boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22} & \sigma_{1}^{2} \boldsymbol{\Sigma}_{21}^{2}+\sigma_{2}^{2} \boldsymbol{\Sigma}_{22}^{2}
\end{array}\right),  \tag{1.4a}\\
\nu_{Y}(B) & =\nu_{X}\left(\left\{\underline{x} \in \mathbb{R}^{2}: \boldsymbol{\Sigma} \underline{x} \in B\right\}\right)  \tag{1.4b}\\
\underline{\gamma}_{Y}^{c} & =\underline{0} . \tag{1.4c}
\end{align*}
$$

Proof. The drift of the process $X$ is given by $\underline{\gamma}_{X}=-\int_{\|\underline{z}\|_{l^{2} \geq 1} \underline{z}} \nu_{X}(d \underline{z})<\infty$. By Cont and Tankov [10, Theorem 4.1] for $n=d=2$ we have to following expression for the drift of $Y$,

$$
\begin{aligned}
& \gamma_{Y}=\boldsymbol{\Sigma} \underline{\gamma}_{X}+\int_{\mathbb{R}^{2} \backslash\{\underline{0}\}} \frac{y}{y}\left(\mathbb{1}_{\left[\|\underline{y}\|_{l^{2}} \leq 1\right]}-\mathbb{1}_{\left[\underline{y} \in S_{\boldsymbol{\Sigma}}\right]}\right) \nu_{Y}(d \underline{y}) \\
& =-\int_{A_{\boldsymbol{\Sigma}}} \underline{y} \nu_{Y}(d \underline{y})+\int_{\mathbb{R}^{2} \backslash\{\underline{0}\}} \underline{y}\left(\mathbb{1}_{\left[\|\underline{y}\|_{l^{2}} \leq 1\right]}-\mathbb{1}_{\left[\underline{y} \in S_{\boldsymbol{\Sigma}}\right]}\right) \nu_{Y}(d \underline{y}) \\
& =\int_{\mathbb{R}^{2} \backslash\{\underline{\underline{\}}}\}} \frac{y}{}\left(\mathbb{1}_{\left[\|\underline{y}\|_{\left.l^{2} \leq 1\right]}\right]}-\mathbb{1}_{\left[\underline{y} \in S_{\boldsymbol{\Sigma}} \cup A_{\boldsymbol{\Sigma}}\right]}\right) \nu_{Y}(d \underline{y}),
\end{aligned}
$$

where $A_{\boldsymbol{\Sigma}}=\left\{\boldsymbol{\Sigma} \underline{x}:\|\underline{x}\|_{l^{2}}>1\right\}$ and $S_{\boldsymbol{\Sigma}}=\left\{\boldsymbol{\Sigma} \underline{x}:\|\underline{x}\|_{l^{2}} \leq 1\right\}$ is the transformation of the unit sphere in $\mathbb{R}^{2}$ under $\boldsymbol{\Sigma}$. Since $\boldsymbol{\Sigma}$ is positive definite we have that $S_{\boldsymbol{\Sigma}} \cup A_{\boldsymbol{\Sigma}}=\mathbb{R}^{2}$ and $\underline{\gamma}_{Y}=-\int_{\|\underline{y}\|_{l^{2}} \geq 1} \underline{y} \nu_{Y}(d \underline{y})<\infty$. Hence $Y$ is also a centered

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LP with finite first moment and we get the following expression for the Lévy exponent of $Y$,

$$
\psi_{Y}(\underline{\xi})=-\frac{1}{2}\left\langle\underline{\xi}, \mathbf{Q}_{Y} \underline{\xi}\right\rangle+\int_{\mathbb{R}^{2} \backslash\{\underline{0}\}}\left(e^{i\langle\underline{y}, \underline{\xi}\rangle}-1-i\langle\underline{y}, \underline{\xi}\rangle\right) \nu_{Y}(d \underline{y}) .
$$

This gives us the characteristic triplet $\left(\underline{\gamma}_{Y}^{c}, \mathbf{Q}_{Y}, \nu_{Y}\right)$ stated in Theorem 1.11.
In the Brownian motion case we can deduce the dependence structure from 1.4a). From Glasserman [18, Section 3.1.2], we know that the linear transformation of a two-dimensional BM is again a BM. Hence $Y$ is a two-dimensional BM with covariance matrix $\mathbf{Q}_{Y}$ and the matrix $\mathbf{Q}_{Y}$ fully determines the dependence structure of $Y$, as $Y_{t}-Y_{s} \sim N\left(\underline{0}, \mathbf{Q}_{Y}^{\frac{1}{2}}(t-s)\right)$ for $t \geq s \geq 0$. However, in presence of jumps this is not the case. To characterise the dependence of the jumps of $Y$ through the expression of its Lévy measure $\nu_{Y}(d \underline{y})$ in terms of the Lévy measure $\nu_{X}(d \underline{z})$ we state the next proposition.

Proposition 1.12. Let $X=\left(X_{t}\right)_{\geq 0}$ and $Y=\left(Y_{t}\right)_{\geq 0}$ be two-dimensional LPs defined as in Theorem 1.11 with corresponding Lévy measures $\nu_{X}(\underline{z})$ and $\nu_{Y}(d \underline{y})$ and constant matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{2 \times 2}$ with $\boldsymbol{\Sigma}_{i j}>0$ for $i, j=1,2$. Further, let $B=[\underline{a}, \underline{b}]=\left[a_{1}, b_{1}\right] \times\left[a_{2}, b_{2}\right] \subset \mathbb{R}^{2}$ be a compact subset of $\mathbb{R}^{2}$, then we have the expression $\nu_{Y}(B)=\nu_{X^{1}}\left(\tilde{B}_{1}\right)+\nu_{X^{2}}\left(\tilde{B}_{2}\right)$ where,

$$
\begin{equation*}
\tilde{B}_{j}=\left[\frac{a_{1}}{\boldsymbol{\Sigma}_{1 j}}, \frac{b_{1}}{\boldsymbol{\Sigma}_{1 j}}\right] \cap\left[\frac{a_{2}}{\boldsymbol{\Sigma}_{2 j}}, \frac{b_{2}}{\boldsymbol{\Sigma}_{2 j}}\right] . \tag{1.5}
\end{equation*}
$$

Proof. Since the components of $X$ are independent, we know that the set $A:=\left\{\underline{z} \in \mathbb{R}^{2}: z_{1} z_{2} \neq 0\right\}$ has measure zero under $\nu_{X}$, i.e. $\nu_{X}(A)=0$, by Cont and Tankov [10, Proposition 5.3]. Using this and Theorem 1.11 we have,

$$
\begin{aligned}
\nu_{Y}(B) & =\nu_{X}\left(\left\{\underline{z} \in \mathbb{R}^{2}: \boldsymbol{\Sigma} \underline{z} \in B\right\}\right) \\
& =\nu_{X}\left(\left\{\underline{z} \in \mathbb{R}^{2}: \boldsymbol{\Sigma}_{i 1} z_{1}+\boldsymbol{\Sigma}_{i 2} z_{2} \in\left[a_{i}, b_{i}\right] \text { for } i=1,2\right\}\right) \\
& =\nu_{X}(\underbrace{\left\{\underline{z} \in \mathbb{R}^{2}: \boldsymbol{\Sigma}_{i 1} z_{1}+\boldsymbol{\Sigma}_{i 2} z_{2} \in\left[a_{i}, b_{i}\right] \text { for } i=1,2\right\} \cap A^{c}}_{(*)})
\end{aligned}
$$

Since the set in $(*)$ is a subset of $A^{c}=\left\{\underline{z} \in \mathbb{R}^{2} \mid z_{1} z_{2}=0\right\}$ we know that at least one of the two components of $\underline{z}$ must be zero. Therefore, the sum $\boldsymbol{\Sigma}_{i 1} z_{1}+\boldsymbol{\Sigma}_{i 2} z_{2}$ is always equal to one of its summands. Define the sets $\tilde{B}_{j}$ for $j=1,2$ as,

$$
\tilde{B}_{j}:=\left\{z_{j} \in \mathbb{R}: \boldsymbol{\Sigma}_{i j} z_{j} \in\left[a_{i}, b_{i}\right] \text { for } i=1,2\right\}=\left[\frac{a_{1}}{\boldsymbol{\Sigma}_{1 j}}, \frac{b_{1}}{\boldsymbol{\Sigma}_{1 j}}\right] \cap\left[\frac{a_{2}}{\boldsymbol{\Sigma}_{2 j}}, \frac{b_{2}}{\boldsymbol{\Sigma}_{2 j}}\right] .
$$

Then we have,

$$
\begin{aligned}
\nu_{Y}(B) & =\nu_{X}\left(\left[\tilde{B}_{1} \times\{0\} \cup\{0\} \times \tilde{B}_{2}\right] \cap A^{c}\right) \\
& =\nu_{X}\left(\left[\tilde{B}_{1} \times \tilde{B}_{2}\right] \cap A^{c}\right) \\
& =\nu_{X}\left(\tilde{B}_{1} \times \tilde{B}_{2}\right)=\nu_{X^{1}}\left(\tilde{B}_{1}\right)+\nu_{X^{2}}\left(\tilde{B}_{2}\right),
\end{aligned}
$$

since $\nu_{X}(\tilde{A})=\nu_{X}\left(\tilde{A} \cap A^{c}\right)$ for all set $\tilde{A} \subset \mathbb{R}^{2}$. The last equality follows from Cont and Tankov [10, Proposition 5.3].

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Remark. Proposition 1.12 also holds if the matrix $\boldsymbol{\Sigma}$ has negative entries, i.e. $\Sigma_{i j}<0$ for $i, j=1,2$. However, the consequence of this would be that we have to reverse the upper and lower bounds of the intervals in 1.5).
Finally, let us rewrite the expression for the characteristic function of LP $Y$ in terms of $\psi_{X^{j}}\left(\xi_{j}\right)$ the Lévy exponent of $X^{j}$ for $j=1,2$.

Proposition 1.13. Let $X=\left(X_{t}\right)_{\geq 0}$ and $Y=\left(Y_{t}\right)_{\geq 0}$ be two dimensional LPs defined as in Theorem 1.11 and $\psi_{X^{j}}(\xi)$ be the Lévy exponents of $X^{j}$ for $j=1,2$. Then we have,

$$
\begin{aligned}
\mathrm{E}\left[e^{i\left\langle\underline{\xi}, Y_{t}\right\rangle}\right] & =e^{t \psi_{Y}(\underline{\xi})}, \text { with } \\
\psi_{Y}(\underline{\xi}) & =\psi_{X^{1}}\left(\boldsymbol{\Sigma}_{11} \xi_{1}+\boldsymbol{\Sigma}_{21} \xi_{2}\right)+\psi_{X^{2}}\left(\boldsymbol{\Sigma}_{12} \xi_{1}+\boldsymbol{\Sigma}_{22} \xi_{2}\right)
\end{aligned}
$$

Proof. By Theorem 1.11 we know that $Y$ is a Lévy process and by the LévyKhinchin representation (Theorem 1.3 ) we have,

$$
\begin{aligned}
e^{t \psi_{Y}(\underline{\xi})} & =\mathrm{E}\left[e^{i\left\langle\xi, Y_{t}\right\rangle}\right] \\
& =\mathrm{E}\left[e^{i\left\{\xi_{1}\left(\boldsymbol{\Sigma}_{11} X_{t}^{1}+\boldsymbol{\Sigma}_{12} X_{t}^{2}\right)+\xi_{2}\left(\boldsymbol{\Sigma}_{21} X_{t}^{1}+\boldsymbol{\Sigma}_{22} X_{t}^{2}\right)\right\}}\right] \\
& =\mathrm{E}\left[e^{i\left\{\left(\boldsymbol{\Sigma}_{11} \xi_{1}+\boldsymbol{\Sigma}_{21} \xi_{2}\right) X_{t}^{1}+\left(\boldsymbol{\Sigma}_{12} \xi_{1}+\boldsymbol{\Sigma}_{22} \xi_{2}\right) X_{t}^{2}\right\}}\right] \\
& =\mathrm{E}\left[e^{i\left(\boldsymbol{\Sigma}_{11} \xi_{1}+\boldsymbol{\Sigma}_{21} \xi_{2}\right) X_{t}^{1}}\right] \mathrm{E}\left[e^{i\left(\boldsymbol{\Sigma}_{12} \xi_{1}+\boldsymbol{\Sigma}_{22} \xi_{2}\right) X_{t}^{2}}\right] \\
& \left.=e^{t\left[\psi_{X^{1}}\left(\boldsymbol{\Sigma}_{11} \xi_{1}+\boldsymbol{\Sigma}_{21} \xi_{2}\right)+\psi_{X^{2}}\left(\boldsymbol{\Sigma}_{12} \xi_{1}+\boldsymbol{\Sigma}_{22} \xi_{2}\right)\right.}\right]
\end{aligned}
$$

The properties in Proposition 1.12 and 1.13 show that the jumps of the components of $Y$ are only co-linearly dependent. Therefore, a note of caution is appropriate. Modelling of the dependence structure of two assets with the model in (1.2) is a severe simplification and one should be careful assuming such a dependence structure for financial assets. Further research is needed to assess the limitations of the dependence structure created by the model in (1.2). One has to examine the geometric interpretation of the set $\tilde{B}_{1} \times \tilde{B}_{2}$ in Proposition 1.12 and the exact consequences of the simple expression of the Lévy exponent $\psi_{Y}(\underline{\xi})$ in Proposition 1.13 However in the next chapter, we perceive the features of the model in (1.2) to be a great advantage for the efficiency of the numerical implementation for the option pricing scheme, as it is not necessary to use complicated Lévy copulas that result in large stiffness matrices which cannot be tensorised.

### 1.3.2 Market Model Assumptions and the Martingale Property

For the subsequent chapters we need some assumptions on the Lévy measure of $X$ to ensure uniqueness and existence of the solution to the option price problem. We list these assumptions here.

Market Model Assumptions 1.14. Let $X$ be a two-dimensional LP with characteristic triplet $\left(\underline{0}, \mathbf{Q}, \nu_{X}\right)$ such that $\nu_{X}(d \underline{z})=k_{X}(\underline{z}) d \underline{z}$. Further let $\nu_{X^{j}}\left(d z_{j}\right)=$ $k_{X^{j}}\left(z_{j}\right) d z_{j}$ for $j=1,2$ be the marginal Lévy measures of $X$.

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(A1) There are constants $\zeta_{-}^{j}>0, \zeta_{+}^{j}>1$ and $C^{j}>0$ such that for $j=1,2$,

$$
k_{X^{j}}\left(z_{j}\right) \leq \begin{cases}C^{j} e^{-\zeta_{-}^{j}\left|z_{j}\right|}, & \text { if } z_{j}<-1, \\ C^{j} e^{-\zeta_{+}^{j}\left|z_{j}\right|}, & \text { if } z_{j}>1\end{cases}
$$

(A2) There exist constants $\alpha_{j} \in(0,2)$ and $C_{+}^{j}>0$ such that for $j=1,2$,

$$
k_{X^{j}}\left(z_{j}\right) \leq \frac{C_{+}^{j}}{\left|z_{j}\right|^{1+\alpha_{j}}} \quad \text { where } 0<\left|z_{j}\right|<1
$$

(A3) If $\sigma_{j}=0$ for $j=1,2$ there additionally exists a constant $C_{-}^{j}>0$ such that,

$$
\frac{1}{2}\left(k_{X^{j}}\left(z_{j}\right)+k_{X^{j}}\left(-z_{j}\right)\right) \geq \frac{C_{-}^{j}}{\left|z_{j}\right|^{1+\alpha_{j}}} \quad \text { where } 0<\left|z_{j}\right|<1
$$

Models satisfying these assumptions are admissible market models. Note also that all these assumptions are strictly on the marginal Lévy measures.

The martingale condition for the model in (1.2) is stated in the next theorem.
Theorem 1.15. The processes $e^{-r t} S^{i}$ are martingales if and only if for $i=1,2$,

$$
\begin{equation*}
w_{i}:=-\sum_{j=1}^{2} \frac{\sigma_{j}^{2} \boldsymbol{\Sigma}_{i j}^{2}}{2}+\int_{\mathbb{R}}\left(e^{\boldsymbol{\Sigma}_{i j} z_{j}}-1-\boldsymbol{\Sigma}_{i j} z_{j}\right) \nu_{X^{j}}\left(d z_{j}\right) . \tag{1.6}
\end{equation*}
$$

Remark. Theorem 1.15 is given for general Lévy processes $X$. In the case the processes $X^{j}$ for $j=1,2$ have finite variation jump parts, as in 1.3), we may state a different martingale condition which we will employ in Section 1.3 .4 for the Kou process. This condition is,

$$
w_{i}=-\sum_{j=1}^{2} \frac{\sigma_{j}^{2} \boldsymbol{\Sigma}_{i j}^{2}}{2}+\int_{\mathbb{R}}\left(e^{\boldsymbol{\Sigma}_{i j} z_{j}}-1\right) \nu_{X^{j}}\left(d z_{j}\right)
$$

for $i=1,2$.
Proof. To show that $e^{-r t} S^{i}$ are martingales for $i=1,2$, we must have,

$$
\mathrm{E}\left[e^{-r t} S_{t}^{i} \mid \mathcal{F}_{s}\right]=\mathrm{E}\left[e^{w_{i} t+\sum_{j=1}^{2} \boldsymbol{\Sigma}_{i j} X_{t}^{j}} \mid \mathcal{F}_{s}\right]=e^{w_{i} s+\sum_{j=1}^{2} \boldsymbol{\Sigma}_{i j} X_{s}^{j}}=e^{-r s} S_{s}^{i}
$$

or equivalently, since $X_{s}^{j}$ are $\mathcal{F}_{s}$-measurable and $X^{j}$ are independent for $j=1,2$,

$$
\begin{equation*}
\mathrm{E}\left[e^{w_{i}(t-s)+\sum_{j=1}^{2} \boldsymbol{\Sigma}_{i j}\left(X_{t}^{j}-X_{s}^{j}\right)} \mid \mathcal{F}_{s}\right]=e^{w_{i}(t-s)} \prod_{j=1}^{2} \mathrm{E}\left[e^{\boldsymbol{\Sigma}_{i j} X_{t-s}^{j}}\right]=1 \tag{1.7}
\end{equation*}
$$

By Sato [46, Theorem 25.17(iii)] we have for $\xi=-i \boldsymbol{\Sigma}_{i j}$ and $j=1,2$,

$$
\begin{equation*}
\mathrm{E}\left[e^{\boldsymbol{\Sigma}_{i j} X_{t-s}^{j}}\right]=\mathrm{E}\left[e^{i \xi X_{t-s}^{j}}\right]=e^{\psi_{j}(\xi)(t-s)}, \tag{1.8}
\end{equation*}
$$

| Values of $\alpha$ | Properties |
| :--- | :--- |
| $\alpha<0$ | Finite variation and finite activity |
| $0 \leq \alpha<1$ | Finite variation and infinite activity |
| $1 \leq \alpha<2$ | Infinite variation and infinite activity |

Table 1.1: Properties for the CGMY process for different values of the parameter $\alpha$. For details see Carr, et al. [7, Theorem 2].
with,

$$
\begin{align*}
\psi_{j}(\xi) & =-\frac{\sigma_{j}^{2} \xi^{2}}{2}+\int_{\mathbb{R}}\left(e^{i \xi z_{j}}-1-i \xi z_{j}\right) \nu_{X^{j}}\left(d z_{j}\right)  \tag{1.9a}\\
& =\frac{\sigma_{j}^{2} \boldsymbol{\Sigma}_{i j}^{2}}{2}+\int_{\mathbb{R}}\left(e^{\boldsymbol{\Sigma}_{i j} z_{j}}-1-\boldsymbol{\Sigma}_{i j} z_{j}\right) \nu_{X^{j}}\left(d z_{j}\right) . \tag{1.9b}
\end{align*}
$$

By using the expression of $\psi_{j}(\xi)$ in 1.9 b and plugging (1.8) into 1.7) proves the theorem.

### 1.3.3 CGMY Process

The CGMY process is a pure jump process presented by Carr, Geman, Madan and Yor in Carr, et al. 77. For this reason, the covariance matrix $\mathbf{Q}$ of the Lévy triplet is zero. The CGMY Lévy process belongs to the class of tempered stable LPs, since it has an additional exponential term in its Lévy measure with respect to stable Lévy processes. The definition and further details about stable LPs can be found in Sato [46, Chapter 3] or Applebaum [1, Example 1.3.14]. The Lévy measure of a one-dimensional CGMY process reads as follows,

$$
\begin{equation*}
\nu^{\mathrm{CGMY}}(d z)=\left(c \frac{e^{-\beta_{+}|z|}}{|z|^{1+\alpha}} \mathbb{1}_{[z>0]}+c \frac{e^{-\beta_{-}|z|}}{|z|^{1+\alpha}} \mathbb{1}_{[z<0]}\right) d z, \tag{1.10}
\end{equation*}
$$

where $c>0, \beta_{+} \geq 1, \beta_{-} \geq 0$ and $\alpha \in(0,2)$. We need $\beta_{+} \geq 1$ in order to have finite exponential moment. In Table 1.1, we list the properties of the CGMY process for different values of $\alpha$. The proof for this can be found in Carr, et al. [7, Theorem 2]. Furthermore, the CGMY process satisfies the conditions in Assumptions 1.14. see Appendix A. For the martingale property to hold in the case of a CGMY process we need to compute the integral in 1.6 for which the expression strongly depends on the value of $\alpha$. The result is stated in the next proposition.

Proposition 1.16. Let $S=\left(S_{t}\right)_{t \geq 0}$ be defined as in (1.2), where $X$ is a twodimensional CGMY process where the Lévy measure $\nu_{X^{j}}^{C G M Y}$ of each independent component is defined as in 1.10 then for $i=1,2, e^{-r t} S^{i}$ is a martingale if and only if $\left|\boldsymbol{\Sigma}_{i j}\right|<\beta_{+}^{j}$ for $i, j=1,2$ and $w_{i}:=-\sum_{j=1}^{2} w_{\alpha_{j}}^{j}$, where for $j=1,2$,
$\alpha_{j} \in(0,1) \cup(1,2)$ implies,

$$
\begin{array}{r}
w_{\alpha_{j}}^{j}=c_{j} \Gamma\left(-\alpha_{j}\right)\left[\left(\beta_{+}^{j}\right)^{\alpha_{j}}\left\{\left(\frac{\beta_{+}^{j}-\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{\alpha_{j}}+\frac{\alpha_{j} \boldsymbol{\Sigma}_{i j}-\beta_{+}^{j}}{\beta_{+}^{j}}\right\}\right. \\
\left.+\left(\beta_{-}^{j}\right)^{\alpha_{j}}\left\{\left(\frac{\beta_{-}^{j}+\boldsymbol{\Sigma}_{i j}}{\beta_{-}^{j}}\right)^{\alpha_{j}}-\frac{\alpha_{j} \boldsymbol{\Sigma}_{i j}+\beta_{-}^{j}}{\beta_{-}^{j}}\right\}\right]
\end{array}
$$

and for $\alpha_{j}=0$ or $\alpha_{j}=1$,

$$
w_{0}^{j}=c_{j}\left\{\ln \left(\frac{\beta_{+}^{j}}{\beta_{+}^{j}-\boldsymbol{\Sigma}_{i j}}\right)+\ln \left(\frac{\beta_{-}^{j}}{\beta_{-}^{j}+\boldsymbol{\Sigma}_{i j}}\right)+\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j} \beta_{-}^{j}}\left(\beta_{+}^{j}-\beta_{-}^{j}\right)\right\}
$$

or,

$$
w_{1}^{j}=c_{j}\left\{\left(\beta_{+}^{j}-\boldsymbol{\Sigma}_{i j}\right) \ln \left(\frac{\beta_{+}^{j}-\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)+\left(\beta_{-}^{j}+\boldsymbol{\Sigma}_{i j}\right) \ln \left(\frac{\beta_{-}^{j}+\boldsymbol{\Sigma}_{i j}}{\beta_{-}^{j}}\right)\right\}
$$

must hold, respectively.
Proof. See Appendix B

### 1.3.4 Kou Process

The Kou process is a finite activity process that has been proposed in Kou 32. A general one-dimensional Kou process reads $X^{\mathrm{Kou}}=\left(X_{t}^{\mathrm{Kou}}\right)_{t \geq 0}$ with,

$$
X_{t}^{\mathrm{Kou}}=\gamma t+\sigma W_{t}+\sum_{k=1}^{N_{t}} Y_{k}
$$

where $k \in \mathbb{N}, \gamma \in \mathbb{R}, \sigma>0, W=\left(W_{t}\right)_{t \geq 0}$ is a $\mathbb{R}$-valued BM, $N=\left(N_{t}\right)_{t \geq 0}$ is a Poisson process with intensity $\lambda>0$ and $\left\{Y_{k}\right\}_{k \in \mathbb{N}}$ are asymmetric double exponentially distributed with density,

$$
f_{Y}^{\mathrm{Kou}}(y)=p \eta_{+} e^{-\eta_{+}|z|} \mathbb{1}_{[z \geq 0]}+q \eta_{-} e^{-\eta_{-}|z|} \mathbb{1}_{[z<0]}
$$

with $p, q>0, p+q=1, \eta_{+}>1$ and $\eta_{-}>0$. Again we need $\eta_{+}>1$ to have finite exponential moment. Further, we have the Lévy measure,

$$
\begin{equation*}
\nu^{\mathrm{Kou}}(d z)=\lambda\left(p \eta_{+} e^{-\eta_{+} z} \mathbb{1}_{[z \geq 0]}+q \eta_{-} e^{\eta_{-} z} \mathbb{1}_{[z<0]}\right) d z \tag{1.11}
\end{equation*}
$$

Also here one can show that the Kou process fulfills the conditions in Assumptions 1.14, see Appendix A. Note that the Kou process has a finite variation jump part, i.e. $\int_{|z| \leq 1}|z| \nu^{\mathrm{Kou}}(d z)<\infty$. This gives us the opportunity to rewrite the process for the model in (1.2). For the Kou process we do this as follows,

$$
\begin{equation*}
X_{t}^{j}=\sigma_{j} W_{t}^{j}+\int_{0}^{t} \int_{\mathbb{R} \backslash\{0\}} z_{j} J_{X^{j}}\left(d s, d z_{j}\right) \tag{1.12}
\end{equation*}
$$

Recall that the martingale condition in 1.6 changes as well.

Proposition 1.17. Let $S=\left(S_{t}\right)_{t \geq 0}$ be defined as in (1.2), where $X$ is a twodimensional Kou process where the Lévy measure $\nu_{X_{j}}^{K o u}\left(\overline{d z_{j}}\right)$ of each independent component is defined as in 1.11 then for $i=1,2, e^{-r t} S^{i}$ is a martingale if and only if,

$$
w_{i}^{\mathrm{Kou}}:=-\sum_{j=1}^{2} \frac{\sigma_{j}^{2} \boldsymbol{\Sigma}_{i j}^{2}}{2}+\lambda_{j}\left[\frac{p_{j} \eta_{+}^{j}}{\eta_{+}^{j}-\boldsymbol{\Sigma}_{i j}}+\frac{q_{j} \eta_{-}^{j}}{\eta_{-}^{j}+\boldsymbol{\Sigma}_{i j}}-1\right]
$$

Proof. Since the jump part of $X_{t}^{j}$ has finite variation, we can split $X_{t}^{j}$ into two parts; a BM and a finite variation pure jump part. Using Corollary 1.7 for the jump part and the Lévy-Khinchin representation for 1.12 we get the following Lévy exponents for $X^{j}$,

$$
\begin{equation*}
\psi_{j}(u)=-\frac{1}{2} \sigma_{j}^{2} u^{2}+\int_{\mathbb{R} \backslash\{0\}}\left(e^{i u z_{j}}-1\right) \nu_{X^{j}}\left(d z_{j}\right) \tag{1.13}
\end{equation*}
$$

where by Sato [46, Theorem 25.17 (iii)] we may set $u=-i \boldsymbol{\Sigma}_{i j}$ and we get,

$$
\begin{align*}
& \int_{\mathbb{R} \backslash\{0\}}\left(e^{\boldsymbol{\Sigma}_{i j} z_{j}}-1\right) \nu_{X^{j}}(d z)=\lambda_{j} {\left[p_{j} \eta_{+}^{j} \int_{0}^{\infty}\left(e^{\boldsymbol{\Sigma}_{i j} z_{j}}-1\right) e^{-\eta_{+}^{j} z_{j}} d z\right.} \\
&\left.+q_{j} \eta_{-}^{j} \int_{-\infty}^{0}\left(e^{\boldsymbol{\Sigma}_{i j} z_{j}}-1\right) e^{\eta_{-}^{j} z_{j}} d z\right] \\
&=\lambda_{j} {\left[\frac{p_{j} \eta_{+}^{j}}{\eta_{+}^{j}-\boldsymbol{\Sigma}_{i j}}-p_{j}+\frac{q_{j} \eta_{-}^{j}}{\eta_{-}^{j}+\boldsymbol{\Sigma}_{i j}}-q_{j}\right] } \\
&=\lambda_{j}\left[\frac{p_{j} \eta_{+}^{j}}{\eta_{+}^{j}-\boldsymbol{\Sigma}_{i j}}+\frac{q_{j} \eta_{-}^{j}}{\eta_{-}^{j}+\boldsymbol{\Sigma}_{i j}}-1\right] . \tag{1.14}
\end{align*}
$$

The martingale property in 1.7 ) then prescribes that $w_{i}^{\mathrm{Kou}}=-\sum_{j=1}^{2} \psi_{j}\left(-i \boldsymbol{\Sigma}_{i j}\right)$. The proposition follows by plugging $u=-i \boldsymbol{\Sigma}_{i j}$ and (1.14) into (1.13).

### 1.4 Lévy Copulas

In this section, we review some results concerning Lévy copulas which we need for proofs and further analysis throughout the course of this paper. Lévy copulas are mathematical tools that "couple" the marginal LPs to form a multivariate LP. Arguably the most important result in the field of copulas is Sklar's theorem which we do not state here. For details about Sklar's theorem for Lévy processes see Kallsen and Tankov [27, Theorem 3.6]. The theorem allows us to find a (unique) Lévy copula for each multivariate LP and vice versa. This makes Lévy copulas an extremely beneficial tool for modelling and simulating multidimensional LPs. A useful reference for copulas in general is Nelson 41, where a detailed proof of Sklar's theorem can be found for general copulas. In this section, we follow the work of Winter [49] and Kallsen and Tankov [27]. First, we review some results on increasing functions. Let $\overline{\mathbb{R}}:=(-\infty, \infty]$ and $(\underline{a}, \underline{b}] \subset \overline{\mathbb{R}}^{d}$ with $(\underline{a}, \underline{b}]=\left(a_{1}, b_{1}\right] \times \cdots \times\left(a_{d}, b_{d}\right]$ with $a_{k} \leq b_{k}$. The $F$-volume of the orthotope $(\underline{a}, \underline{b}]$ for a function $F: S \rightarrow \overline{\mathbb{R}}$ with $S \subset \overline{\mathbb{R}}^{d}$ is defined by

$$
V_{F}((\underline{a}, \underline{b}]):=\sum_{u \in\left\{a_{1}, b_{1}\right\} \times \cdots \times\left\{a_{d}, b_{d}\right\}}(-1)^{N(\underline{u})} F(\underline{u}),
$$

where $N(\underline{u}):=\#\left\{k \in\{1, \ldots d\}: u_{k}=a_{k}\right\}$ is the number of $k \in\{1, \ldots d\}$ such that $u_{k}=a_{k}$. This and the following two definitions can be found in Kallsen and Tankov [27, Definition 2.1, 2.2 and 2.4].

Definition 1.18 ( $d$-increasing). A function $F: S \rightarrow \overline{\mathbb{R}}$ with $S \subset \overline{\mathbb{R}}^{d}$ is called $d$ increasing if the $F$-volume of any interval $(\underline{a}, \underline{b}] \subset S$ with $a_{k} \leq b_{k}$ for $k=1, \ldots, d$ and $\overline{(\underline{a}, \underline{b}]} \subset S$ is non-negative.

Now we can define the margins of the copula $F$ which we need in the main theorem (Theorem 1.21) of the section.

Definition 1.19. Suppose $F: \overline{\mathbb{R}}^{d} \rightarrow \overline{\mathbb{R}}$ is a $d$-increasing function satisfying $F(\underline{u})=0$ if $u_{k}=0$ for at least one $k \in\{1, \ldots, d\}$. For $\mathcal{I} \subset\{1, \ldots, d\}$ a non-empty index set, the $\mathcal{I}$-margin of $F$ is a function $F^{\mathcal{I}}: \overline{\mathbb{R}}^{\# \mathcal{I}} \rightarrow \overline{\mathbb{R}}$ defined by,

$$
F^{\mathcal{I}}\left(\underline{u}_{\mathcal{I}}\right):=\lim _{a \rightarrow \infty} \sum_{\underline{u}_{\mathcal{I}^{c}} \in\{a, \infty\} \# \mathcal{I}^{c}}\left(\prod_{j \in \mathcal{I}^{c}} \operatorname{sgn}\left(u_{j}\right)\right) F(\underline{u}),
$$

where $\underline{u}_{\mathcal{I}}:=\left(u_{j}\right)_{j \in \mathcal{I}}$ and $\mathcal{I}^{c}:=\{1, \ldots, d\} \backslash \mathcal{I}$.
Next we give the definition of a Lévy copula.
Definition 1.20 (Lévy copula). A function $F: \overline{\mathbb{R}}^{d} \rightarrow \overline{\mathbb{R}}$ is a Lévy copula if it satisfies the following properties for $\underline{u}=\left(u_{1}, \ldots, u_{d}\right)$,
(C1) $F$ is $d$-increasing,
(C2) $F(\underline{u}) \neq \infty$ for $\underline{u} \neq \underline{\infty}$,
(C3) $F(\underline{u})=0$ if $\exists i \in\{1, \ldots, d\}: u_{i}=0$,
(C4) For any $i \in\{1, \ldots, d\}: F^{i}(u)=u$ with $u \in \mathbb{R}$.
The definition of Lévy copulas can be found in Winter [49, Definition 2.2.3]. In the next theorem we elaborate on the integration of a function with respect to a multivariate Lévy measure.

Theorem 1.21. Let $X$ be a d-dimensional LP and $\nu_{X}$ its Lévy measure with Lévy copula $F$ and marginal Lévy measures $\nu_{X^{j}}$. Then, we have for a bounded function $f \in C^{\infty}\left(\mathbb{R}^{d}\right)$ that vanishes on a neighbourhood of the origin,

$$
\begin{aligned}
\int_{\mathbb{R}^{d}} f(\underline{z}) \nu_{X}(d \underline{z})= & \sum_{j=1}^{d} \int_{\mathbb{R}} f\left(\underline{0}+z_{j}\right) \nu_{X^{j}}\left(d z_{j}\right) \\
& +\sum_{j=2}^{d} \sum_{\substack{|\mathcal{I}|=j \\
\mathcal{I}^{1}<\cdots<\mathcal{I}^{j}}} \int_{\mathbb{R}^{j}} \partial_{\underline{z}_{\mathcal{I}}} f\left(\underline{0}+\underline{z}_{\mathcal{I}}\right) F^{\mathcal{I}}\left(\left(U_{k}\left(z_{k}\right)\right)_{k \in \mathcal{I}}\right) d \underline{z}_{\mathcal{I}},
\end{aligned}
$$

where $U_{k}\left(z_{k}\right):=\operatorname{sgn}\left(z_{k}\right) \nu_{X^{k}}\left(I_{z_{k}}\right)$ and $I_{z_{k}}:= \begin{cases}\left(z_{k}, \infty\right), & \text { if } z_{k} \geq 0, \\ \left(-\infty, z_{k}\right), & \text { if } z_{k}<0 .\end{cases}$
Proof. The theorem and its proof can be reviewed in Winter [49, Lemma 2.2.7].

One explicit copula is exhibited in the example below. The independent Lévy copula is employed at a later stage to simplify the system of inequalities for the option pricing problem.

Example 1.22 (Independent copula). The independent copula function for an LP with independent components reads,

$$
\begin{equation*}
F(\underline{u})=\sum_{j=1}^{d} u_{j} \prod_{i \neq j} \mathbb{1}_{[\{\infty\}]}\left(u_{i}\right) . \tag{1.15}
\end{equation*}
$$

In this section, we have presented some properties of Lévy copulas. These properties are useful at a later stage of this paper to derive the system of inequalities for the American option pricing problem with two underlying assets.

### 1.5 Lévy Processes as Markov Processes and their Semigroups

In this section, we introduce Markov processes and to some extent the connection to Lévy processes. The class of Markov processes is a generalisation of the class of Lévy processes. Markov processes are SPs which are characterised by the Markov property, which informally states that the future value of the process depends only on the past through its current value. Let $B_{b}\left(\mathbb{R}^{d}\right)$ denote the set of all bounded Borel-measurable functions on $\mathbb{R}^{d}$.

Definition 1.23 (Markov Process). An adapted SP $M=\left(M_{t}\right)_{t \geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a Markov process if for all $f \in B_{b}\left(\mathbb{R}^{d}\right)$ and $0 \leq s \leq t<\infty$,

$$
\begin{equation*}
\mathrm{E}\left[f\left(M_{t}\right) \mid \mathcal{F}_{s}\right]=\mathrm{E}\left[f\left(M_{t}\right) \mid M_{s}\right] \tag{1.16}
\end{equation*}
$$

This definition can be found in Applebaum [1 Definition 3.1.1]. In Definition 1.23 we call 1.16 the Markov property.

Proposition 1.24. Every $\mathbb{R}^{d}$-valued Lévy Process $L=\left(L_{t}\right)_{t \geq 0}$ is a Markov process.

Proof. For the proof we refer to Sato [46, Theorem 10.5(i)].
Definition 1.25 (Strong Markov Property). Let $Y=\left(Y_{t}\right)_{t \geq 0}$ be an adapted SP on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Then $Y$ satisfies the strong Markov property if for each stopping time $\tau$ with $\tau<\infty$ a.s., $Y_{\tau}$ is independent of $\mathcal{F}_{\tau}=\left\{A \in \mathcal{F} \mid A \cap\{\tau \leq t\} \in \mathcal{F}_{t} \forall t \geq 0\right\}$ and $\mathcal{L}\left(Y_{\tau+t}-Y_{\tau}\right)=\mathcal{L}\left(Y_{t}\right)$ for all $t \geq 0$.

Theorem 1.26. Let $L=\left(L_{t}\right)_{t \geq 0}$ be a d-dimensional LP then $L$ satisfies the strong Markov property.

Proof. See Applebaum [1, Theorem 2.2.11].
Now let us introduce the semigroups of Markov processes. The following definition can be found in Applebaum [1, Section 3.2].

Definition 1.27 (One-parameter Semigroup). Let $B$ be a Banach space and $L(B)$ the algebra of all bounded linear operators on $B$ with identity operator $I$. Then the family of bounded, linear operators $\left(T_{t} \mid t \geq 0\right)$ is a one-parameter semigroup of contraction if for $x \in B$ and $0 \leq s \leq t<\infty$,
(T1) $T_{0}=I$
(T2) $T_{t+s}=\left(T_{s} \circ T_{t}\right)$,
(T3) $\left\|T_{t}(x)\right\|_{B} \leq 1, \quad \forall x \in B$
(T4) $\left\|T_{t}(x)-x\right\|_{B} \longrightarrow 0$ as $t \rightarrow 0, \quad \forall x \in B$.
The semigroup $T_{t}^{M}$ of Markov process $M$ is given by conditioning the expected value, i.e. $T_{t}^{M}: B \rightarrow B$ with $\left(T_{t}^{M} f\right)(\underline{y})=\mathrm{E}\left[f\left(M_{t}\right) \mid M_{0}=\underline{y}\right]$, where $f \in B=B_{b}\left(\mathbb{R}^{d}\right), M=\left(M_{t}\right)_{t \geq 0}$ is a Markov process, $\underline{y} \in \mathbb{R}^{d}$. By $(L \overline{1})$ of Definition 1.1 we have the following semigroups $T_{t}^{L}$ associated with an $\mathbb{R}^{d}$-valued Lévy process $L=\left(L_{t}\right)_{t \geq 0}$,

$$
\left(T_{t}^{L} f\right)(\underline{x})=\mathrm{E}\left[f\left(L_{t}+\underline{x}\right)\right]
$$

with $f \in B_{d}\left(\mathbb{R}^{d}\right)$ and $\underline{x} \in \mathbb{R}^{d}$. These results can be found in Applebaum [1, Section 3.1].

In this chapter, we described a bivariate Lévy model for the spot price process $S$, partially analysed the dependence structure of the model and reviewed some of the preliminary theory of LPs and Lévy copulas. We proved which form the drift $w_{i}$ must have in order for the discounted asset price to be a martingale and gave explicit expressions for the two LP discussed in this paper, i.e. the CGMY and the Kou process. Moreover, we introduced the concept of semigroups associated with Lévy processes.

## Chapter 2

## Option Pricing in a Bivariate Setting with Dependent Asset Price Processes

In this chapter, we derive a pricing scheme for American style options with two underlying assets. The asset price process is given by the model in Section 1.3.1. Firstly, we deduce the system of inequalities for the option pricing problem in terms of the LP $X$. The advantage is that the integro-differential operator $\mathcal{A}$ in terms of $X$ has a nice form and the resulting finite elements matrices can all be tensorised, however, we will be dealing with a time-dependent obstacle. Secondly, we rewrite the PIDI in variational form and discuss the localised solution and the error due to the localisation. Thirdly, we discretise the variational formulation using finite elements and account for the inequality satisfied by the American option price by the Lagrangian multiplier method. Our final outcome is a fully-discretised scheme to price American style options within model (1.2).

Let us first give the definition of the optimal stopping time for American options. This can be reviewed in Cont and Tankov [10, Section 11.4].

Definition 2.1 (Optimal stopping). The value of an American option with payoff $g(\underline{s})$ at time $t$ as an optimal stopping problem is given by,

$$
V(t, \underline{s}):=\operatorname{ess} \sup _{\tau \in \mathcal{T}_{t, T}} \mathrm{E}\left[e^{-r(\tau-t)} g\left(S_{\tau}\right) \mid S_{t}=\underline{s}\right]
$$

where $\underline{s}=\left(s_{1}, s_{2}\right)^{\top}$ and $\mathcal{T}_{t, T}$ is the set of all stopping times in the interval $J:=[t, T]$.

Remark. The essential supremum in Definition 2.1 over an uncountable set of stopping times takes the smallest stopping time dominating all stopping times in $\mathcal{T}_{t, T}$ such that the set $\left\{\tilde{\tau} \in \mathcal{T}_{t, T} \mid \mathrm{E}\left[e^{-r(\tilde{\tau}-t)} g\left(S_{\tilde{\tau}}\right) \mid S_{t}=\underline{s}\right]>V(t, \underline{s})\right\}$ has measure zero.
As we have already mentioned above, we derive the PIDI in terms of the LP $X$
in 1.2). For this reason, we introduce the following notation.

$$
\begin{align*}
& \bar{u}(t, \underline{x}):=V\left(t,\binom{e^{\left(r+w_{1}\right) t+\boldsymbol{\Sigma}_{1} \underline{x}}}{e^{\left(r+w_{2}\right) t+\boldsymbol{\Sigma}_{2} \underline{x}}}\right)  \tag{2.1a}\\
& \bar{g}(t, \underline{x}):=g\left(\binom{e^{\left(r+w_{1}\right) t+\boldsymbol{\Sigma}_{1} \underline{x}}}{e^{\left(r+w_{2}\right) t+\boldsymbol{\Sigma}_{2} \underline{x}}}\right) \tag{2.1b}
\end{align*}
$$

where $\boldsymbol{\Sigma}_{i}=\left(\boldsymbol{\Sigma}_{i 1}, \boldsymbol{\Sigma}_{i 2}\right)$ for $i=1,2$ and $\underline{x}=\left(x_{1}, x_{2}\right)^{\top}$.

### 2.1 Derivation of the System of Inequalities for the American Option Pricing Problem

Here, we deduce the system of inequalities for the American option pricing problem using the model described in Section 1.3.1. We adopt some of the theories from El Karoui, et al. [15]. However, the model discussed here is different in that it has a sum of two LP in the exponent. This requires us to prove the following lemma to be able to use the results in El Karoui, et al. [15].
Lemma 2.2. Let $f \in C^{1,2}\left(J \times \mathbb{R}^{2}\right) \cap C^{0}\left(\bar{J} \times \mathbb{R}^{2}\right)$ with bounded derivatives and $\mathcal{A}$ denote the infinitesimal operator of a two-dimensional LP $X_{t}$ with independent components, given by,

$$
\begin{equation*}
(\mathcal{A} f)(t, \underline{x}):=\left(\mathcal{A}^{B S} f\right)(t, \underline{x})+\left(\mathcal{A}^{J} f\right)(t, \underline{x}) \tag{2.2}
\end{equation*}
$$

with

$$
\begin{align*}
\left(\mathcal{A}^{B S} f\right)(t, \underline{x}) & =\frac{1}{2} \sum_{j=1}^{2} \sigma_{j}^{2} \partial_{x_{j} x_{j}} f(t, \underline{x})  \tag{2.3a}\\
\left(\mathcal{A}^{J} f\right)(t, \underline{x}) & =\sum_{j=1}^{2} \int_{\mathbb{R} \backslash\{0\}}\left[f\left(t, \underline{x}+z_{j}\right)-f(t, \underline{x})-\partial_{x_{j}} f(t, \underline{x}) z_{j}\right] \nu_{X^{j}}\left(d z_{j}\right) . \tag{2.3b}
\end{align*}
$$

Then the process $P_{t}=e^{-r t} f\left(t, X_{t}\right)$ is a super-martingale for $t \in J$ if and only if $\partial_{t} f\left(s, X_{s}\right)+(\mathcal{A} f)\left(s, X_{s}\right)-r f\left(s, X_{s}\right) \leq 0$ for all $s \in J$.

Remark. Lemma 2.2 is given for processes with infinite variation jump part. If the process $X$ has components with finite variation jump part, we may use a different expression for $\mathcal{A}^{J}$. That is, we would have,

$$
\left(\mathcal{A}^{J} f\right)(t, \underline{x})=\sum_{j=1}^{2} \int_{\mathbb{R} \backslash\{0\}}\left[f\left(t, \underline{x}+z_{j}\right)-f(t, \underline{x})\right] \nu_{X^{j}}\left(d z_{j}\right) .
$$

Since we would have a process with $\underline{\gamma}_{0}=\underline{0}$ as in the remark in Section 1.3.1 for the model in 1.2 .

Proof. Let $Z_{t}=e^{-r t}$, then by the product rule and Itô's formula we have,

$$
\begin{align*}
\frac{d P_{t}}{Z_{t}}= & -r f\left(t, X_{t}\right) d t+d f\left(t, X_{t}\right)  \tag{2.4a}\\
= & \left(\partial_{t} f\left(t, X_{t}\right)-r f\left(t, X_{t}\right)\right) d t+\sum_{j=1}^{2} \partial_{x_{j}} f\left(t, X_{t}\right) d X_{t}^{j}  \tag{2.4b}\\
& +\frac{1}{2} \sum_{j, k=1}^{2} \partial_{x_{j} x_{k}} f\left(t, X_{t}\right) d\left\langle X^{j}, X^{k}\right\rangle_{t}^{c}+\Delta f\left(t, X_{t}\right)-\sum_{j=1}^{2} \partial_{x_{j}} f\left(t, X_{t}\right) \Delta X_{t}^{j} \tag{2.4c}
\end{align*}
$$

Resulting from the fact that $X^{j}$ for $j=1,2$ are independent, we get,

$$
\begin{aligned}
& d X_{t}^{j}=\sigma_{j} d W_{t}^{j}+\int_{\mathbb{R} \backslash\{0\}} z_{j} \tilde{J}_{X^{j}}\left(d t, d z_{j}\right) \\
& d\left\langle X^{j}, X^{k}\right\rangle_{t}^{c}=\delta_{j k} \sigma_{j} \sigma_{k} d t
\end{aligned}
$$

Hence (2.4) becomes,

$$
\begin{aligned}
\frac{d P_{t}}{Z_{t}}= & \left(\partial_{t} f\left(t, X_{t}\right)+\frac{1}{2} \sum_{j=1}^{2} \sigma_{j}^{2} \partial_{x_{j} x_{j}} f\left(t, X_{t}\right)-r f\left(t, X_{t}\right)\right) d t \\
& +\int_{\mathbb{R}^{2} \backslash\{0\}}\left(f\left(t, X_{t}+z\right)-f\left(t, X_{t}\right)-\sum_{j=1}^{2} \partial_{x_{j}} f\left(t, X_{t}\right) z_{j}\right) J_{X}(d t, d \underline{z}) \\
& +\sum_{j=1}^{2} \sigma_{j} \partial_{x_{j}} f\left(t, X_{t}\right) d W_{t}^{j}+\sum_{j=1}^{2} \partial_{x_{j}} f\left(t, X_{t}\right) \int_{\mathbb{R} \backslash\{0\}} z_{j} \tilde{J}_{X^{j}}\left(d t, d z_{j}\right)
\end{aligned}
$$

where $\underline{z}=\left(z_{1}, z_{2}\right)^{\top}$. Using 2.3a and rewriting $J_{X}(d t, d \underline{z})=\tilde{J}_{X}(d t, d \underline{z})+$ $\nu_{X}(d \underline{z}) d t$ gives,

$$
\begin{align*}
\frac{d P_{t}}{Z_{t}}= & \left(\partial_{t} f\left(t, X_{t}\right)+\left(\mathcal{A}^{B S} f\right)\left(t, X_{t}\right)-r f\left(t, X_{t}\right)\right) d t  \tag{2.5a}\\
& +\int_{\mathbb{R}^{2} \backslash\{0\}}\left(f\left(t, X_{t}+z\right)-f\left(t, X_{t}\right)-\sum_{j=1}^{2} \partial_{x_{j}} f\left(t, X_{t}\right) z_{j}\right) \nu_{X}(d \underline{z}) d t  \tag{2.5b}\\
& +\sum_{j=1}^{2} \partial_{x_{j}} f\left(t, X_{t}\right) \int_{\mathbb{R} \backslash\{0\}} z_{j} \tilde{J}_{X^{j}}\left(d t, d z_{j}\right)-\sum_{j=1}^{2} \partial_{x_{j}} f\left(t, X_{t}\right) \int_{\mathbb{R}^{2} \backslash\{0\}} z_{j} \tilde{J}_{X}(d t, d \underline{z})  \tag{2.5c}\\
& +\sum_{j=1}^{2} \partial_{x_{j}} f\left(t, X_{t}\right) \sigma_{j} d W_{t}^{j}+\int_{\mathbb{R}^{2} \backslash\{0\}}\left(f\left(t, X_{t}+\underline{z}\right)-f\left(t, X_{t}\right)\right) \tilde{J}_{X}(d t, d \underline{z}) \tag{2.5~d}
\end{align*}
$$

Let us now focus on the terms in $(2.5 \mathrm{c})$ in order to simplify the expression for the differential $d P_{t}$. We can rewrite $\mathbb{R}^{2} \backslash\{\underline{0}\}=I_{1} \cup I_{2} \cup I_{3}$ with $I_{1}=\mathbb{R} \backslash\{0\} \times \mathbb{R} \backslash\{0\}$,
$I_{2}=\mathbb{R} \backslash\{0\} \times\{0\}$ and $I_{3}=\{0\} \times \mathbb{R} \backslash\{0\}$ such that $I_{l} \cap I_{k}=\emptyset$ for $l, k=1,2,3$. Then we have for $j=1$,

$$
\begin{align*}
\int_{\mathbb{R}^{2} \backslash\{\underline{0}\}} z_{1} J_{X}(d t, d \underline{z})= & \int_{I_{1}} z_{1} J_{X}(d t, d \underline{z})+\int_{I_{2}} z_{1} J_{X}(d t, d \underline{z})+\int_{I_{3}} z_{1} J_{X}(d t, d \underline{z}) \\
= & \int_{\mathbb{R} \backslash\{0\}} z_{1} \int_{\mathbb{R} \backslash\{0\}} J_{X}(d t, d \underline{z})+\int_{\mathbb{R} \backslash\{0\}} z_{1} J_{X}\left(d t, d z_{1} \times\{0\}\right)  \tag{2.6~b}\\
& +\int_{\{0\}} z_{1} J_{X}\left(d t, d z_{1} \times \mathbb{R} \backslash\{0\}\right)  \tag{2.6c}\\
= & \int_{\mathbb{R} \backslash\{0\}} z_{1} J_{X}\left(d t, d z_{1} \times \mathbb{R} \backslash\{0\}\right)+\sum_{s \in d t} \mathbb{1}_{\left[\Delta X_{s} \in \mathbb{R} \backslash\{0\} \times\{0\}\right]} \quad(2.6 \mathrm{~d})  \tag{2.6d}\\
= & \int_{\mathbb{R} \backslash\{0\}} z_{1} J_{X^{1}}\left(d t, d z_{1}\right)
\end{align*}
$$

where the second term in 2.6 d vanishes a.s. since,

$$
\begin{aligned}
& \mathbb{P}\left[\Delta X_{s} \in I_{2}\right]=\mathbb{P}\left[\Delta X_{s}^{1} \in \mathbb{R} \backslash\{0\}, \Delta X_{s}^{2} \in\{0\}\right] \\
& =\mathbb{P}\left[\Delta X_{s}^{1} \in \mathbb{R} \backslash\{0\}\right] \mathbb{P}\left[\Delta X_{s}^{2} \in\{0\}\right]=0 .
\end{aligned}
$$

An equivalent reasoning holds for $j=2$ and also for the remaining terms $\int_{\mathbb{R}^{2} \backslash\{0\}} z_{j} \nu_{X}(d \underline{z}) d t$ for $j=1,2$. Hence the terms in 2.5 c cancel out. By Theorem 1.21 , we know that for $\phi_{t, \underline{x}}(\underline{z}):=f(t, \underline{x}+\underline{z})-f(t, \underline{x})-\sum_{j=1}^{2} \partial_{x_{j}} f(t, \underline{x}) z_{j}$ and independent copula function $F$ in 1.15,

$$
\begin{align*}
\int_{\mathbb{R}^{2}} \phi_{t, \underline{x}}(\underline{z}) \nu_{X}(d z)= & \sum_{j=1}^{2} \int_{\mathbb{R}} \phi_{t, \underline{x}}\left(\underline{0}+z_{j}\right) \nu_{X^{j}}\left(d z_{j}\right)  \tag{2.7a}\\
& +\int_{\mathbb{R}^{2}} \partial_{z_{1} z_{2}} \phi_{t, \underline{x}}(\underline{z}) F\left(U_{1}\left(z_{1}\right), U_{2}\left(z_{2}\right)\right) d \underline{z} \tag{2.7b}
\end{align*}
$$

For 2.7b we use the formula for the independence copula,

$$
\begin{aligned}
F\left(U_{1}\left(z_{1}\right), U_{2}\left(z_{2}\right)\right)= & \operatorname{sgn}\left(z_{1}\right) \nu_{X^{1}}\left(I_{z_{1}}\right) \mathbb{1}_{[\{\infty\}]}\left(\operatorname{sgn}\left(z_{2}\right) \nu_{X^{2}}\left(I_{z_{2}}\right)\right) \\
& +\operatorname{sgn}\left(z_{2}\right) \nu_{X^{2}}\left(I_{z_{2}}\right) \mathbb{1}_{[\{\infty\}]}\left(\operatorname{sgn}\left(z_{1}\right) \nu_{X^{1}}\left(I_{z_{1}}\right)\right) \\
= & \left\{\begin{array}{cl}
\operatorname{sgn}\left(z_{1}\right) \nu_{X^{1}}\left(I_{z_{1}}\right) \mathbb{1}_{\left[z_{2}=0\right]} \\
+\operatorname{sgn}\left(z_{2}\right) \nu_{X^{2}}\left(I_{z_{2}}\right) \mathbb{1}_{\left[z_{1}=0\right]}, & \text { for infinite activity models, } \\
0, & \text { for finite activity models. }
\end{array}\right.
\end{aligned}
$$

Thus in the case of the infinite activity we have,

$$
\begin{aligned}
\int_{\mathbb{R}^{2}} \phi_{t, \underline{x}}(\underline{z}) \nu_{X}(d \underline{z})= & \sum_{j=1}^{2} \int_{\mathbb{R}} \phi_{t, \underline{x}}\left(\underline{0}+z_{j}\right) \nu_{X^{j}}\left(d z_{j}\right) \\
& +\int_{\mathbb{R}} \int_{\mathbb{R}} \partial_{z_{1} z_{2}} \phi_{t, \underline{x}}(\underline{z}) \operatorname{sgn}\left(z_{1}\right) \nu_{X^{1}}\left(I_{z_{1}}\right) \mathbb{1}_{\left[z_{2}=0\right]} d z_{1} d z_{2} \\
& +\int_{\mathbb{R}} \int_{\mathbb{R}} \partial_{z_{1} z_{2}} \phi_{t, \underline{x}}(\underline{z}) \operatorname{sgn}\left(z_{2}\right) \nu_{X^{2}}\left(I_{z_{2}}\right) \mathbb{1}_{\left[z_{1}=0\right]} d z_{1} d z_{2} \\
= & \sum_{j=1}^{2} \int_{\mathbb{R}} \phi_{t, \underline{x}}\left(\underline{0}+z_{j}\right) \nu_{X^{j}}\left(d z_{j}\right) \\
& +\int_{\mathbb{R}} \partial_{z_{1} z_{2}} \phi_{t, \underline{x}}\left(\underline{0}+z_{1}\right) \operatorname{sgn}\left(z_{1}\right) \nu_{X^{1}}\left(I_{z_{1}}\right) d z_{1} \\
& +\int_{\mathbb{R}} \partial_{z_{1} z_{2}} \phi_{t, \underline{x}}\left(\underline{0}+z_{2}\right) \operatorname{sgn}\left(z_{2}\right) \nu_{X^{2}}\left(I_{z_{2}}\right) d z_{2} \\
= & \sum_{j=1}^{2} \int_{\mathbb{R}} \phi_{t, \underline{x}}\left(\underline{0}+z_{j}\right) \nu_{X^{j}}\left(d z_{j}\right)
\end{aligned}
$$

since $\partial_{z_{1} z_{2}} \phi_{t, \underline{x}}\left(\underline{0}+z_{j}\right)=0$ for $j=1,2$. By the fact that the two terms in 2.5d are martingales we have,

$$
\begin{aligned}
P_{t}= & P_{0}+\int_{0}^{t} Z_{s}\left(\partial_{t} f\left(s, X_{s}\right)+\left(\mathcal{A}^{B S} f\right)\left(s, X_{s}\right)-r f\left(s, X_{s}\right)\right) d s \\
& +\int_{0}^{t} Z_{s} \sum_{j=1}^{2} \int_{\mathbb{R} \backslash\{0\}}\left(f\left(s, X_{s}+z_{j}\right)-f\left(s, X_{s}\right)-\partial_{x_{j}} f\left(s, X_{s}\right) z_{j}\right) \nu_{X^{j}}\left(d z_{j}\right) d s \\
& +\int_{0}^{t} Z_{s} d M_{s} \\
= & \int_{0}^{t} Z_{s}(\partial_{t} f\left(s, X_{s}\right)+\underbrace{\left(\mathcal{A}^{B S} f\right)\left(s, X_{s}\right)+\left(\mathcal{A}^{J} f\right)\left(s, X_{s}\right)}_{=(\mathcal{A} f)\left(s, X_{s}\right)}-r f\left(s, X_{s}\right)) d s \\
& +\int_{0}^{t} Z_{s} d M_{s}
\end{aligned}
$$

For $P_{t}$ to be a super-martingale it suffices that the integrand of the first integral is non-positive on $J$. The fact that $Z_{t}$ is positive and bounded on $J$ gives the result,

$$
\begin{equation*}
P_{t} \text { is a super-martingale } \Leftrightarrow\left(\partial_{t} f\left(s, X_{s}\right)+\mathcal{A} f\left(s, X_{s}\right)-r f\left(s, X_{s}\right)\right) \leq 0 \forall s \in J \tag{2.8}
\end{equation*}
$$

Remark. If the inequality sign in 2.8 is an equality then $P_{t}$ is a martingale and vice versa.
Now, we can prove the following theorem.

Theorem 2.3. Let $v(t, \underline{x}) \in C^{1,2}\left(J \times \mathbb{R}^{2}\right) \cap C^{0}\left(\bar{J} \times \mathbb{R}^{2}\right)$ be a sufficiently smooth solution to the following system of inequalities,

$$
\begin{align*}
\partial_{t} v+\mathcal{A}^{B S} v+\mathcal{A}^{J} v-r v & \leq 0 & & \text { on } J \times \mathbb{R}^{2}  \tag{2.9a}\\
v(t, \underline{x}) & \geq \bar{g}(t, \underline{x}) & & \text { on } J \times \mathbb{R}^{2}  \tag{2.9b}\\
\left(\partial_{t} v+\mathcal{A}^{B S} v+\mathcal{A}^{J} v-r v\right)(v-\bar{g}) & =0 & & \text { on } J \times \mathbb{R}^{2}  \tag{2.9c}\\
v(T, \underline{x}) & =\bar{g}(T, \underline{x}) & & \text { on } \mathbb{R}^{2}, \tag{2.9d}
\end{align*}
$$

where $J=[0, T]$ for $T>0$. Then $v(t, \underline{x})=\bar{u}(t, \underline{x})$ is as in 2.1a and $V(t, \underline{s})$ is the value of an American option with payoff $\bar{g}(T, \underline{x})$ as defined in Definition 2.1.

Proof. $\Leftarrow$ : By the definition of $\bar{u}(t, \underline{x})$ the conditions in 2.9 b and 2.9 d are satisfied. Since $X$ satisfies the strong Markov property by Theorem 1.26, we have that $\left(T_{t} \bar{g}\right)(t, \underline{x})=\mathrm{E}\left[\bar{g}\left(t, X_{t}+\underline{x}\right)\right]$ is the semigroup associated to $X$. Moreover, assuming $\bar{g}(t, \cdot)$ is Borel-measurable on $\mathbb{R}^{2}$ for eacht $t \in J$, we can apply the result from El Karoui, et al. [15, Theorem 3.4] such that the process $U:=\left(e^{-r t} \bar{u}(t, \underline{x})\right)_{0 \leq t \leq T}$ is the Snell envelope with time horizon $T$ of $Z^{\bar{g}}=\left(\bar{g}\left(t, X_{t}+\underline{x}\right)\right)_{0 \leq t \leq T}$. By definition of the Snell envelope we know that $U$ is the smallest supermartingale larger than or equal to the process $Z^{\bar{g}}$, see El Karoui, et al. [15] Introduction]. By Lemma 2.2 we can conclude that $\bar{u}(t, \underline{x})$ satisfies 2.9a. From the fact that the optimal stopping time $\tau_{1}^{*}$ for the American option is given by $\tau_{1}^{*}=\inf \{t \geq 0: \bar{u}(t, \underline{x})=\bar{g}(t, \underline{x})\}$, we can deduce from Karoui [14, Theorem 2.31] that the stopped process $\left(U_{t \wedge \tau_{1}^{*}}\right)_{0 \leq t \leq T}$ is a martingale and therefore, $\bar{u}(t, \underline{x})$ satisfies 2.9c).
$\Rightarrow$ : Conversely, $v(t, \underline{x})$ satisfies 2.9a then by Lemma 2.2, $\bar{V}:=\left(e^{-r t} v(t, \underline{x})\right)_{0 \leq t \leq T}$ is a supermartingale. Since the Snell envelope $E=\left(E_{t}\right)_{0 \leq t \leq T}$ of $Z^{\bar{g}}$ is the smallest supermartingale dominating $Z^{\bar{g}}$ we have $\bar{V}_{t} \geq E_{t} \geq Z^{\bar{g}}$ for all $(t, \underline{x}) \in J \times \mathbb{R}^{2}$. By 2.9c we know that $\left(\bar{V}_{t \wedge \tau_{2}^{*}}\right)_{0 \leq t \leq T}$ with $\tau_{2}^{*}=\inf \{t \geq 0: v(t, \underline{x})=\bar{g}(t, \underline{x})\}$ is a martingale and hence $v(t, \underline{x})=\bar{u}(t, \underline{x})$.
Corollary 2.4. Define the functions $\tilde{u}: J \times \mathbb{R}^{2} \rightarrow \mathbb{R}$ and $\tilde{g}: J \times \mathbb{R}^{2} \rightarrow \mathbb{R}$ as follows,

$$
\begin{align*}
& \tilde{u}(t, \underline{x}):=e^{-r(T-t)} \bar{u}(T-t, \underline{x})  \tag{2.10a}\\
& \tilde{g}(t, \underline{x}):=e^{-r(T-t)} \bar{g}(T-t, \underline{x}) \tag{2.10b}
\end{align*}
$$

where $\underline{x}=\left(x_{1}, x_{2}\right)^{\top}$. Then $\tilde{u}(t, \underline{x})$ satisfies the following set of conditions,

$$
\begin{align*}
\partial_{t} \tilde{u}-\mathcal{A}^{B S} \tilde{u}-\mathcal{A}^{J} \tilde{u} & \geq 0 & & \text { on } J \times \mathbb{R}^{2}  \tag{2.11a}\\
\tilde{u}(t, \underline{x}) & \geq \tilde{g}(t, \underline{x}) & & \text { on } J \times \mathbb{R}^{2}  \tag{2.11b}\\
\left(\partial_{t} \tilde{u}-\mathcal{A}^{B S} \tilde{u}-\mathcal{A}^{J} \tilde{u}\right)(\tilde{u}-\tilde{g}) & =0 & & \text { on } J \times \mathbb{R}^{2}  \tag{2.11c}\\
\tilde{u}(0, \underline{x}) & =\tilde{g}(0, \underline{x}) & & \text { on } \mathbb{R}^{2}, \tag{2.11d}
\end{align*}
$$

Proof. Differentiating $\tilde{u}(t, x)$ towards $t$ gives,

$$
\begin{equation*}
\partial_{t} \tilde{u}(t, \underline{x})=e^{-r(T-t)}\left(-r \bar{u}(T-t, \underline{x})-\partial_{t} \bar{u}(T-t, \underline{x})\right) \tag{2.12}
\end{equation*}
$$

Plugging 2.12 and 2.10a into 2.11a and 2.11c gives 2.9a and 2.9c), respectively. Moreover using the definitions in 2.10), 2.11b and 2.11d follow directly from 2.9 b and 2.9 d , respectively.

### 2.2 Variational Formulation

Let $\mathcal{V} \subset \mathcal{H}$ be two Hilbert spaces with a continuous, dense embedding. The variational setting is founded on the real Gelfand triplet,

$$
\mathcal{V} \subset \mathcal{H} \equiv \mathcal{H}^{*} \subset \mathcal{V}^{*}
$$

where $\mathcal{H}^{*}$ is the dual space of $\mathcal{H}$. The choice of the spaces $\mathcal{V}$ and $\mathcal{H}$ depends on the operator $\mathcal{A}$ defined in $(2.2)$ and the parameters of the model. The bilinear forms associated with $\mathcal{A}^{B S}$ and $\mathcal{A}^{J}$ are obtained by multiplying a smooth test function $w \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$ with the respective operator and integrate over $\underline{x}$. In this section we adopted some of the outlines in Reich, et al. 43.
Next we deduce the bilinear form $a^{B S}(\cdot, \cdot)$ associated with $\mathcal{A}^{B S}$. Let $\phi, \psi \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$,

$$
\begin{align*}
a^{B S}(\phi, \psi) & =-\left(\left(\mathcal{A}^{B S} \phi\right)(x), \psi(x)\right)_{\mathcal{V}^{*}, \mathcal{V}}  \tag{2.13a}\\
& =-\frac{1}{2} \sum_{j=1}^{2} \sigma_{j}^{2} \int_{\mathbb{R}^{2}} \partial_{x_{j} x_{j}} \phi(\underline{x}) \psi(\underline{x}) d \underline{x}  \tag{2.13b}\\
& =\frac{1}{2} \sum_{j=1}^{2} \sigma_{j}^{2} \int_{\mathbb{R}^{2}} \partial_{x_{j}} \phi(\underline{x}) \partial_{x_{j}} \psi(\underline{x}) d \underline{x}, \tag{2.13c}
\end{align*}
$$

where in 2.13 c we used integration by parts. For the bilinear form $a^{J}(\cdot, \cdot)$ associated to the jump part, we assume $\nu_{X^{j}}\left(d z_{j}\right)=k_{X^{j}}\left(z_{j}\right) d z_{j}$ where $k_{X^{j}}\left(z_{j}\right)$ satisfies the $(A 1)-(A 3)$ in Assumptions 1.14 for $j=1,2$ and use the antiderivatives of the one-dimensional Lévy densities $k_{X^{j}}\left(z_{j}\right)$ for $j=1,2$ given by,

$$
k_{X^{j}}^{(-n)}\left(z_{j}\right)= \begin{cases}\int_{-\infty}^{z_{j}} k_{X^{j}}^{(-n+1)}(\zeta) d \zeta, & \text { if } z_{j}<0 \\ -\int_{z_{j}}^{\infty} k_{X^{j}}^{(-n+1)}(\zeta) d \zeta, & \text { if } z_{j}>0\end{cases}
$$

The $n^{\text {th }}$ antiderivative $k_{X^{j}}^{(-n)}\left(z_{j}\right)$ vanishes at $\pm \infty$. To obtain the bilinear form associated to the jump part we separate the integrals in the jump operator $\mathcal{A}^{J}$ for $j=1,2$. Let $\phi \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$ then,

$$
\begin{aligned}
& \int_{0}^{\infty}\left(\phi\left(\underline{x}+z_{j}\right)-\phi(\underline{x})-\partial_{x_{j}} \phi(\underline{x}) z_{j}\right) k_{X^{j}}\left(z_{j}\right) d z_{j} \\
& =\left[\left(\phi\left(\underline{x}+z_{j}\right)-\phi(\underline{x})-\partial_{x_{j}} \phi(\underline{x}) z_{j}\right) k_{X^{j}}^{(-1)}\left(z_{j}\right)\right]_{0}^{\infty} \\
& \quad-\int_{0}^{\infty}\left(\partial_{x_{j}} \phi\left(\underline{x}+z_{j}\right)-\partial_{x_{j}} \phi(\underline{x})\right) k_{X^{j}}^{(-1)}\left(z_{j}\right) d z_{j} \\
& =-\left[\left(\partial_{x_{j}} \phi\left(\underline{x}+z_{j}\right)-\partial_{x_{j}} \phi(\underline{x})\right) k_{X^{j}}^{(-2)}\left(z_{j}\right)\right]_{0}^{\infty}+\int_{0}^{\infty} \partial_{x_{j} x_{j}} \phi\left(\underline{x}+z_{j}\right) k_{X^{j}}^{(-2)}\left(z_{j}\right) d z_{j} .
\end{aligned}
$$

The same holds for the integral over negative half of the real line since the antiderivatives also vanish at $-\infty$. Let $\phi, \psi \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$ then the bilinear form
$a^{J}(\cdot, \cdot)$ is given by,

$$
\begin{align*}
a^{J}(\phi, \psi) & =-\left(\left(\mathcal{A}^{J} \phi\right)(\underline{x}), \psi(\underline{x})\right)_{\mathcal{V}^{*}, \mathcal{V}}  \tag{2.14a}\\
& =-\sum_{j=1}^{2} \int_{\mathbb{R}^{2}} \int_{\mathbb{R} \backslash\{0\}} \partial_{x_{j} x_{j}} \phi\left(\underline{x}+z_{j}\right) k_{X^{j}}^{(-2)}\left(z_{j}\right) d z_{j} \psi(\underline{x}) d \underline{x} .  \tag{2.14b}\\
& =\sum_{j=1}^{2} \int_{\mathbb{R}^{2}} \int_{\mathbb{R} \backslash\{0\}} \partial_{x_{j}} \phi\left(\underline{x}+z_{j}\right) \partial_{x_{j}} \psi(\underline{x}) k_{X^{j}}^{(-2)}\left(z_{j}\right) d z_{j} d \underline{x} . \tag{2.14c}
\end{align*}
$$

Remark. For processes $X$ with finite variation jump part we may write for $\phi, \psi \in C_{0}^{\infty}\left(\mathbb{R}^{2}\right)$,

$$
a^{J}(\phi, \psi)=\sum_{j=1}^{2} \int_{\mathbb{R}^{2}} \int_{\mathbb{R} \backslash\{0\}} \partial_{x_{j}} \phi\left(\underline{x}+z_{j}\right) \psi(\underline{x}) k_{X^{j}}^{(-1)}\left(z_{j}\right) d z_{j} d \underline{x}
$$

For the spaces $\mathcal{H}$ and $\mathcal{V}$ we take $L^{2}\left(\mathbb{R}^{2}\right)$ and,

$$
\mathcal{D} \underline{\rho}= \begin{cases}H^{1}\left(\mathbb{R}^{2}\right), & \text { if } \underline{\rho}=(1,1), \\ H \underline{\rho}\left(\mathbb{R}^{2}\right), & \text { if } \underline{\rho} \neq(1,1),\end{cases}
$$

respectively, where $\underline{\rho}=\left(\rho_{1}, \rho_{2}\right) \in(0,1]^{2}$. The space $H^{1}\left(\mathbb{R}^{2}\right)$ is an isotropic Sobolev space defined by,

$$
H^{1}\left(\mathbb{R}^{2}\right)=\left\{v \in L^{2}\left(\mathbb{R}^{2}\right): \partial_{x_{j}} v(\underline{x}) \in L^{2}\left(\mathbb{R}^{2}\right) \text { for } j=1,2\right\}
$$

and the space $H^{\rho}\left(\mathbb{R}^{2}\right)$ is an anisotropic Sobolev space of fractional order defined as follows,

$$
H \underline{\rho}\left(\mathbb{R}^{2}\right):=\left\{v \in L^{2}\left(\mathbb{R}^{2}\right):\|v\|_{H \underline{\underline{\rho}}}^{2}=\int_{\mathbb{R}^{2}} \sum_{j=1}^{2}\left(1+\xi_{j}^{2}\right)^{\rho_{j}}|\hat{v}(\underline{\xi})|^{2} d \underline{\xi}<\infty\right\}
$$

where $\hat{v}(\underline{\xi})$ is the Fourier transform of $v(\underline{x})$. We denote the dual space of $\mathcal{D} \underline{\rho}$ by $\mathcal{D}^{-\underline{\rho}}$. Moreover, the dual space of $H^{\underline{\rho}}\left(\mathbb{R}^{2}\right)$ is denoted by $H^{-\underline{\rho}}\left(\mathbb{R}^{2}\right)=\left(H^{\underline{\rho}}\left(\mathbb{R}^{2}\right)\right)^{*}$ and equivalently, $H^{-1}\left(\mathbb{R}^{2}\right)=\left(H^{1}\left(\mathbb{R}^{2}\right)\right)^{*}$ is the dual space of $H^{1}\left(\mathbb{R}^{2}\right)$. Before we specify $\rho$, for which the entries are closely related to the infinitesimal operator of $X^{j}$ for $j=1,2$, let us exploit the intersection property of the space $H^{\underline{\rho}}\left(\mathbb{R}^{2}\right)$ in the next proposition.

Proposition 2.5. Let $\underline{s}=\left(s_{1}, s_{2}\right) \in \mathbb{R}_{+}$. The space $H$ s $\left(\mathbb{R}^{2}\right)$ is an anisotropic Sobolev space that admits the following intersection structure,

$$
H^{\underline{s}}\left(\mathbb{R}^{2}\right)=\bigcap_{j=1,2} H_{j}^{s_{j}}\left(\mathbb{R}^{2}\right)
$$

where for $j=1,2$,

$$
H_{j}^{s_{j}}\left(\mathbb{R}^{2}\right):=\left\{v \in L^{2}\left(\mathbb{R}^{2}\right):\|v\|_{H_{j}^{s_{j}}}^{2}:=\int_{\mathbb{R}^{2}}\left(1+\xi_{j}^{2}\right)^{s_{j}}|\hat{v}(\underline{\xi})|^{2} d \underline{\xi}<\infty\right\}
$$

Remark. Proposition 2.5 also holds for the space $H^{1}\left(\mathbb{R}^{2}\right)$ such that $H^{1}\left(\mathbb{R}^{2}\right)=\bigcap_{j=1,2} H_{j}^{1}\left(\mathbb{R}^{2}\right)$.

Proof. See Nikol'skiï [42, Section 9.2] for details of the result in Proposition 2.5

Using Proposition 2.5 we have for $\underline{\rho} \in(0,1]^{2}$,

$$
\begin{equation*}
\mathcal{D}^{\rho}=\bigcap_{j=1,2} H_{j}^{\rho_{j}}\left(\mathbb{R}^{2}\right) \tag{2.15}
\end{equation*}
$$

as in Reich,et al. [43, Remark 4.9]. Now, the entries of $\underline{\rho}$ can be given as,

$$
\rho_{j}= \begin{cases}1, & \text { if } \sigma_{j}>0  \tag{2.16}\\ \frac{\alpha_{j}}{2}, & \text { if } \sigma_{j}=0\end{cases}
$$

where $\alpha_{j} \in(0,2)$ is the order of the infinitesimal operator of $X^{j}$ given in Assumptions 1.14 that is, if $X^{j}$ is a pure jump process, i.e. $\sigma_{j}=0$.
Remark. For the CGMY process described in Section 1.3.3 the order of the infinitesimal operator is given by the parameter $\alpha_{j}$ of the process. The Kou process, on the other hand, has a diffusion part implying $\mathcal{D}^{(1,1)}=H^{1}\left(\mathbb{R}^{2}\right)$.
Remark. We require $\tilde{g}(t, \cdot) \in L^{2}\left(\mathbb{R}^{2}\right)$ for the existence of a solution. This condition can be relaxed in localised form in the next section and eventually puts a growth condition on the payoff $g(\underline{s})$.
Due to 2.11b we can narrow the set of admissible functions for the solution of the variational form of 2.11 to the following set,

$$
\mathcal{K}_{\tilde{g}(t, \cdot)}:=\{v \in \mathcal{D} \underline{\rho} \mid v(\underline{x}) \geq \tilde{g}(t, \underline{x}) \text { a.e. } \underline{x}\} \subset \mathcal{D} \underline{\rho} \quad \text { a.e. } t \in J .
$$

Furthermore to attain the variational formulation of 2.11, we multiply 2.11a with $w \in \mathcal{D}^{\underline{\rho}}$ such that $w(\underline{x}) \geq 0$ a.e. and integrate over $\underline{x}$ to obtain,

$$
\begin{equation*}
\left(\partial_{t} \tilde{u}, w\right)_{\mathcal{D}^{-\varrho}, \mathcal{D}^{\rho}}+\underbrace{a(\tilde{u}, w)+a^{J}(\tilde{u}, w)}_{:=a^{B S}} \geq 0, \tag{2.17}
\end{equation*}
$$

where $a^{B S}: \mathcal{D}^{\rho} \times \mathcal{D}^{\rho} \rightarrow \mathbb{R}$ is given in 2.13c and $a^{J}: \mathcal{D}^{\rho} \times \mathcal{D}^{\rho} \rightarrow \mathbb{R}$ is stated in 2.14 c . Subsequently, from 2.11 c we acquire,

$$
\begin{equation*}
\left(\partial_{t} \tilde{u}, \tilde{u}-\tilde{g}\right)_{\mathcal{D}^{-\varrho}, \mathcal{D}^{\varrho}}+a(\tilde{u}, \tilde{u}-\tilde{g})=0 \tag{2.18}
\end{equation*}
$$

By subtracting (2.18 from 2.17, we obtain the following variational formulation of 2.11,

Find $\tilde{u} \in L^{2}\left(J ; \mathcal{D}^{\rho}\right) \cap H^{1}\left(J ; \mathcal{D}^{-\underline{\rho}}\right)$ s.t. $\tilde{u}(t, \cdot) \in \mathcal{K}_{\tilde{g}(t, \cdot)}$ and
$\left(\partial_{t} \tilde{u}, w-\tilde{u}\right)_{\mathcal{D}^{-\varrho}, \mathcal{D}^{\rho}}+a(\tilde{u}, w-\tilde{u}) \geq 0, \forall w \in \mathcal{K}_{\tilde{g}(t, \cdot)}$ a.e. $t \in J$,

$$
\begin{equation*}
\tilde{u}(0, \underline{x})=\tilde{g}(0, \underline{x}) . \tag{2.19b}
\end{equation*}
$$

Let us now define the function $u(t, \underline{x}):=\tilde{u}(t, \underline{x})-\tilde{g}(t, \underline{x})$ and the convex set $\mathcal{K}:=\left\{v \in \mathcal{D}^{\underline{\rho}} \mid v(\underline{x}) \geq 0\right.$ a.e. $\left.x\right\}$ with indicator function,

$$
\mathbb{I}_{\mathcal{K}}(v):= \begin{cases}0, & \text { if } v \in \mathcal{K},  \tag{2.20}\\ +\infty, & \text { if } v \notin \mathcal{K}\end{cases}
$$

Then we can rewrite (2.19) as follows, similar to Reich, et al. 43, Section 2.3],

$$
\begin{align*}
& \text { Find } u \in L^{2}\left(J ; \mathcal{D}^{\rho}\right) \cap H^{1}\left(J ; \mathcal{D}^{-\underline{\rho}}\right) \text { such that } u(t, \cdot) \in \mathcal{D}^{\rho} \text { and }  \tag{2.21a}\\
& \begin{array}{r}
\left(\partial_{t} u, w-u\right)_{\mathcal{D}^{-} \underline{\rho}, \mathcal{D}^{\rho}}+a(u, w-u) \\
\quad+\mathbb{I}_{\mathcal{K}}(w)-\mathbb{I}_{\mathcal{K}}(u) \geq f_{t}(w-u), \forall w \in \mathcal{D}^{\rho},
\end{array}
\end{align*}
$$

where,

$$
f_{t}(w-u):=-\left(\partial_{t} \tilde{g}, w-u\right)_{\mathcal{D}^{-\varrho}, \mathcal{D}^{\varrho}}-a(\tilde{g}, w-u) .
$$

The subscript $t$ in $f_{t}(w-u)$ indicates the time-dependence of the right hand side of 2.21 b . It is not to be confused with the derivative with respect to $t$. Another theoretical result which is of great importance is given in the following theorem. This theorem ensures that there is a unique solution to the variational formulation in 2.21 under some regularity assumptions. First the following lemma.

Lemma 2.6. The bilinear form $a: \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R}$ with $a(\cdot, \cdot)=a^{B S}(\cdot, \cdot)+a^{J}(\cdot, \cdot)$ where $a^{B S}(\cdot, \cdot)$ and $a^{J}(\cdot, \cdot)$ are given in (2.13) and (2.14), respectively, is continuous and satisfies the Gärding inequality, i.e. for all $\psi, \phi \in \mathcal{V}$,

$$
\begin{align*}
|a(\psi, \phi)| & \leq C_{1}\|\psi\|_{\mathcal{V}}\|\phi\|_{\mathcal{V}}  \tag{2.22a}\\
a(\psi, \psi) & \geq C_{2}\|\psi\|_{\mathcal{V}}^{2}-C_{3}\|\psi\|_{\mathcal{H}}^{2} \tag{2.22b}
\end{align*}
$$

Proof. Here we refer to the proof of Reich et al. [43, Theorem 4.8].
Remark. In Section 3.2 we need that the bilinear form is coercive, i.e. $C_{3}=0$ in 2.22 b . We may achieve this by shifting the solution $u(t, \underline{x})$ to $e^{-C_{3} t} u(t, \underline{x})$ where $C_{3}>0$ is the same constant as in 2.22b. We refer to Winter 49, Remark 1.3.3].

Theorem 2.7. Let $X$ be an LP with characteristic triplet $\left(0, \mathbf{Q}, \nu_{X}\right)$ where the Lévy measure $\nu_{X}$ satisfies the assumptions (A1)-(A3). Then the variational formulation in (2.21) admits a unique solution $u(t, x) \in L^{2}(J ; \mathcal{V}) \cap L^{\infty}(J ; \mathcal{H})$ for $\mathcal{H}=L^{2}\left(\mathbb{R}^{2}\right)$ and $\mathcal{V}=\mathcal{D}^{\underline{\rho}}$ given in 2.15 with $\underline{\rho}$ as in 2.16 under the following regularity assumptions,

$$
\begin{aligned}
\left.\partial_{t} \tilde{g}(t, \underline{x})\right|_{t=0} & \in L^{2}\left(\mathbb{R}^{2}\right), \\
-\partial_{t} \tilde{g}(t, \underline{x})-(\mathcal{A} \tilde{g})(t, \underline{x}) & \in L^{2}\left(J ; \mathcal{D}^{-\underline{\rho}}\right) \\
\text { and }-\partial_{t}\left(\partial_{t} \tilde{g}(t, \underline{x})+(\mathcal{A} \tilde{g})(t, \underline{x})\right) & \in L^{2}\left(J ; \mathcal{D}^{-\underline{\rho}}\right) .
\end{aligned}
$$

Proof. We know that for $v \in L^{2}(J ; \mathcal{K})$ we have $\left|\int_{0}^{T} \mathbb{I}_{\mathcal{K}}(v(t, \underline{x})) d t\right|=0<\infty$, where $\mathbb{I}_{\mathcal{K}}(v(t, \underline{x}))$ is defined in 2.20 . Now using the result in Lemma 2.6 and the fact that $\bar{g}(0, \underline{x}) \in \mathcal{K}$, we can apply the result in Glowinski, et al. 19, Theorem 6.2.1].

Remark. The well-posedness of a weaker form of 2.21 is provided by Savaré [47] under less restrictive regularity assumptions on $f_{t}(\phi)$ and $\bar{g}(t, \underline{x})$.

### 2.3 Localisation

We truncate the domain $\mathbb{R}^{2}$ of the variational problem in 2.19 to a bounded domain $G=(-R, R)^{2} \subset \mathbb{R}^{2}$ for $R \in \mathbb{R}_{+}$. This localisation is achieved by installing an artificial boundary condition as follows,

$$
\bar{u}_{R}(t, \underline{x}):=\sup _{\tau \in \mathcal{T}_{t, T}} \mathrm{E}\left[e^{-r(\tau-t)} \bar{g}\left(\tau, X_{\tau}\right) \mathbb{1}_{\left[\tau<\tau_{G}\right]} \mid X_{t}=\underline{x}\right],
$$

where $\tau_{G}:=\inf \left\{t \geq 0 \mid X_{t} \in \mathbb{R}^{2} \backslash G\right\}$ is the first hitting time of $X$ to hit the set $\mathbb{R}^{2} \backslash G$. In financial terms, we can view $\bar{u}_{R}(t, \underline{x})$ as the price of an American knock-out barrier option with zero rebate and payoff $\bar{g}(t, \underline{x})$. The price of this option is zero for $\underline{x} \in \mathbb{R}^{2} \backslash G$ and for $\left|x_{j}\right|$ with either $j=1,2$ approaching $R, \bar{u}_{R}(t, \underline{x})$ will decrease towards zero. This is intuitive for a knock-out barrier with zero rebate, but different for general American options. Their value process does not become zero, when the process $X$ hits the set $\mathbb{R}^{2} \backslash G$, meaning that we incur an error when only considering the bounded domain $G$. We define the localised function $u_{R}(t, \underline{x}):=e^{-r(T-t)} \bar{u}_{R}(T-t, \underline{x})-\left.\tilde{g}(t, \underline{x})\right|_{\underline{x} \in G}$ which is the excess of $\bar{u}_{R}(t, \underline{x})$ to the time-dependent obstacle $\left.\tilde{g}(t, \underline{x})\right|_{\underline{x} \in G}$ on $G$. Let us now state the localised variational formulation in terms of $u_{R}(t, \underline{x})$.

$$
\begin{align*}
& \text { Find } u_{R} \in L^{2}\left(J ; \mathcal{D}^{\rho}(G)\right) \cap H^{1}\left(J ; \mathcal{D}^{-\underline{\rho}}(G)\right) \text { s.t. } u_{R}(t, \cdot) \in \mathcal{D} \text { and }  \tag{2.23a}\\
& \begin{aligned}
\left(\partial_{t} u_{R}, w-u_{R}\right)_{\mathcal{D}^{-} \underline{\rho}(G), \mathcal{D}}(G) & +a\left(u_{R}, w-u_{R}\right)+\mathbb{I}_{\mathcal{K}(G)}\left(u_{R}\right) \\
& -\mathbb{I}_{\mathcal{K}(G)}(w) \geq f_{t}\left(w-u_{R}\right), \forall w \in \mathcal{D}^{\underline{\rho}}(G),
\end{aligned} \\
& \begin{array}{r}
u_{R}(0, \underline{x})=0, \quad \text { for } \underline{x} \in G,
\end{array} \tag{2.23b}
\end{align*}
$$

where $f_{t}$ is given by,

$$
\begin{equation*}
f_{t}\left(w-u_{R}\right):=-\left(\partial_{t} \tilde{g}, w-u_{R}\right)_{\mathcal{D}^{-\underline{\rho}}(G), \mathcal{D} \underline{\rho}(G)}-a\left(\tilde{g}, w-u_{R}\right), \tag{2.24}
\end{equation*}
$$

for $w \in \mathcal{D}^{\rho}(G)$ and for $\underline{\rho} \in(0,1]^{2}$ defined in (2.16) we have,

$$
\mathcal{D}^{\underline{\rho}}(G):= \begin{cases}H_{0}^{1}(G), & \text { if } \underline{\rho}=(1,1) \\ \widetilde{H}_{\underline{\rho}}(G), & \text { if } \underline{\rho} \neq(1,1)\end{cases}
$$

with,

$$
\begin{aligned}
\widetilde{H}^{\rho}(G) & :=\left\{\left.v\right|_{G}: v \in H^{\underline{\rho}}\left(\mathbb{R}^{2}\right) \text { and }\left.v\right|_{\mathbb{R}^{2} \backslash G}=0\right\}, \\
H_{0}^{1}(G) & :=\left\{\left.v\right|_{G}: v \in H^{1}\left(\mathbb{R}^{2}\right) \text { and }\left.v\right|_{\mathbb{R}^{2} \backslash G}=0\right\},
\end{aligned}
$$

and,

$$
\begin{aligned}
\mathcal{K}(G) & :=\left\{v \in \mathcal{D}^{\rho}(G) \mid v(\underline{x}) \geq 0 \text { a.e. } \underline{x} \in G\right\}, \\
\mathbb{I}_{\mathcal{K}(G)}(v) & := \begin{cases}0, & \text { if } v \in \mathcal{K}(G), \\
+\infty, & \text { if } v \notin \mathcal{K}(G) .\end{cases}
\end{aligned}
$$

The bilinear form in 2.23 is given by,

$$
\begin{align*}
a(\psi, \phi)=\frac{1}{2} & \sum_{j=1}^{2} \sigma_{j}^{2} \int_{G} \partial_{x_{j}} \psi(\underline{x}) \partial_{x_{j}} \phi(\underline{x}) d \underline{x} \\
& +\sum_{j=1}^{2} \int_{G} \int_{\mathbb{R} \backslash\{0\}}\left(\partial_{x_{j}} \psi\left(\underline{x}+z_{i}\right) \partial_{x_{j}} \phi(\underline{x}) k_{X^{j}}^{(-2)}\left(z_{j}\right)\right) d z_{j} d \underline{x}, \tag{2.25}
\end{align*}
$$

with $\psi, \phi \in \mathcal{D}^{\rho}(G)$. We note that the dual space of $\tilde{H}^{\underline{\rho}}(G)$ is $\left(\widetilde{H}^{\underline{\rho}}(G)\right)^{*}=H^{-\underline{\rho}}(G)$ with $\underline{\rho} \in(0,1]^{2} \backslash\{(1,1)\}$ for $j=1,2$. The tilde results from the fact that we force $v(\underline{x})=0$ for all $\underline{x} \in \mathbb{R}^{2} \backslash G$ if $v \in \widetilde{H} \underline{\rho}(G)$. The dual space of $H_{0}^{1}(G)$ is denoted by $H^{-1}(G)=\left(H_{0}^{1}(G)\right)^{*}$. For the localised form 2.23 the condition on $g(\underline{s})$ can be weakened to the growth condition,

$$
\begin{equation*}
g(\underline{s}) \leq C\left(\sum_{i=1}^{2} s_{i}+1\right)^{q} \quad \forall \underline{s} \in \mathbb{R}_{+}^{2} \tag{2.26}
\end{equation*}
$$

with $q \geq 1$ and some constant $C>0$ as in Reich, et al. [43, Section 4.5]. The well-posedness of 2.23 is ensured by Theorem 2.7. The next theorem gives us an estimate of the localisation error.

Theorem 2.8. Suppose $g: \mathbb{R}^{2} \rightarrow \mathbb{R}$ satisfies the growth condition in 2.26 for some $q \geq 1$. Further, let $X$ be a two-dimensional Lévy process with Lévy measure $\nu_{X}(\underline{z})$ such that for $j=1,2$ the marginal Lévy measures $\nu_{X^{j}}\left(d z_{j}\right)=k_{X^{j}}\left(z_{j}\right) d z_{j}$ satisfy (A1) in Assumptions 1.14 with $\zeta_{+}^{j}, \zeta_{-}^{j}>q$. Then for some constant $C>0$,

$$
\begin{array}{r}
\left|u(t, \underline{x})-u_{R}(t, \underline{x})\right| \leq C e^{-\gamma_{1} R+\gamma_{2}\|\underline{x}\|_{l} \infty}, \\
\text { with }-q<\gamma_{2}<\min _{j=1,2} \min \left(\zeta_{+}^{j}, \zeta_{-}^{j}\right) \text { and } \gamma_{1}=\gamma_{2}-q .
\end{array}
$$

Proof. The theorem and proof can be found in Reich, et al. [43, Theorem 4.15].

Theorem 2.8 states that the localisation error decreases exponentially in $R$. As mentioned above the value of the approximation $u_{R}(t, \underline{x})$ of $u(t, \underline{x})$ will decrease to zero as $\left|x_{j}\right|$ approaches the artificial barrier $R$ for $j=1,2$ whereas the value of $u(t, \underline{x})$ may not decrease to zero for $\left|x_{j}\right|$ close to $R$. Therefore the localisation error might be large for $\left|x_{j}\right|$ close to $R$ and it is wise to choose $R$ large enough, such that the area of interest for $u(t, \underline{x})$ is bounded away from $\partial G$. However, keep in mind that to have a grid with mesh size $h$ for a large $R$ requires more gird points then a grid with the same mesh size $h$ for a small $R$. Thus choosing $R$ is a trade-off between accuracy and computational time.

### 2.4 Discretisation and Lagrangian Multiplier Form

In this section we discretise the variational formulation in 2.23 using the continuous Garlakin method. First we will discretise in space using finite elements and then reformulate the problem using Lagrangian multipliers. We will introduce the Lagrangian multiplier space and some of its properties. Finally,
we discretise the semi-discrete form in time using finite difference obtaining a fully discrete representation of the variational problem (2.21) using Lagrangian multipliers. In this section, we denote the inner product in $L^{2}(G)$ by $(\cdot, \cdot)$.

### 2.4.1 Space Discretisation

Let us first introduce the notation for the discretisation in space. We denote $N$ as the number of inner nodes in each dimension, that is we work with an equidistant grid in both dimensions resulting in a total number of $\tilde{N}:=N^{2}$ inner nodes. Furthermore, we define $h:=\frac{2 R}{N+1}$ as the mesh size and the finite dimensional finite element space $V_{N}$ as follows,

$$
\begin{equation*}
V_{N}:=\operatorname{span}\left\{\psi_{j}(\underline{x}) \mid 1 \leq j \leq N^{2}\right\} \tag{2.27}
\end{equation*}
$$

where we ulilize products of two hat functions as basis for $V_{N}$, that is $\psi_{j}(\underline{x})=b_{j_{1}}\left(x_{1}\right) b_{j_{2}}\left(x_{2}\right)$ with $j=\left(j_{1}-1\right) N+j_{2}$ for $1 \leq j_{1}, j_{2} \leq N$. The hat functions are given by $b_{j_{l}}\left(x_{l}\right)=\max \left(0,1-h^{-1}\left|x_{l}-x_{j_{l}}\right|\right)$ for $l=1,2$. We depict the one-dimensional hat functions in Figure 2.1 Moreover, the dimension of $V_{N}$ is $\tilde{N}$. We approximate the solution $u(t, \underline{x})$ to $(2.21)$ with a piecewise linear function $u_{N} \in V_{N}$. Hence, the approximation $u_{N}(t, \underline{x})$ can be represented as,

$$
u_{N}(t, \underline{x})=\sum_{j_{1}, j_{2}=1}^{N} u_{N, j_{1}, j_{2}}(t) b_{j_{1}}\left(x_{1}\right) b_{j_{2}}\left(x_{2}\right)
$$

with $\underline{x}=\left(x_{1}, x_{2}\right)^{\top}$ and $\underline{u}_{N}(t)=\left(u_{N, j_{1}, j_{2}}(t)\right)_{1 \leq j_{1}, j_{2} \leq N}$ is the coefficient vector in $\mathbb{R}^{\tilde{N}}$ dependent on time $t$. By plugging this expression into the localised vari-


Figure 2.1: One-dimensional hat functions $b_{j}(x)$ for $j=1, \ldots, N$ on domain $G=(-R, R) \subset \mathbb{R}$.
ational formulation (2.23), we obtain for $w(\underline{x})=\sum_{i_{1}, i_{2}=1}^{N} w_{i_{1}, i_{2}} b_{i_{1}}\left(x_{1}\right) b_{i_{2}}\left(x_{2}\right)$ the following semi-discretised form where $\dot{v}(t)$ represents the derivative towards
time $t$ and $a(\cdot, \cdot)$ is defined in 2.25,

$$
\begin{array}{r}
\sum_{i_{1}, i_{2}=1}^{N}\left(w_{i_{1}, i_{2}}-u_{N, i_{1}, i_{2}}(t)\right)\left[\sum_{j_{1}, j_{2}=1}^{N} \dot{u}_{N, j_{1}, j_{2}}(t)\left(b_{j_{1}}\left(x_{1}\right) b_{j_{2}}\left(x_{2}\right), b_{i_{1}}\left(x_{1}\right) b_{i_{2}}\left(x_{2}\right)\right)\right. \\
\left.+\sum_{j_{1}, j_{2}=1}^{N} u_{N, j_{1}, j_{2}}(t) a\left(b_{j_{1}}\left(x_{1}\right) b_{j_{2}}\left(x_{2}\right), b_{i_{1}}\left(x_{1}\right) b_{i_{2}}\left(x_{2}\right)\right)\right] \\
\geq \sum_{i_{1}, i_{2}=1}^{N}\left(w_{i_{1}, i_{2}}-u_{N, i_{1}, i_{2}}(t)\right) f\left(b_{i_{1}}\left(x_{1}\right) b_{i_{2}}\left(x_{2}\right)\right), \quad(2.28)  \tag{2.28}\\
u_{N, j_{1}, j_{2}}(0)=0 \\
\text { for } j_{1}, j_{2}=1, \ldots, N .
\end{array}
$$

This results in the following matrix representation. Define,

$$
\mathcal{K}_{N}:=\left\{v \in V_{N}: v_{j_{1}, j_{2}} \geq 0 \text { for } j_{1}, j_{2}=1, \ldots, N\right\} \subset V_{N}
$$

then the semi-discrete problem can be stated as:

$$
\begin{align*}
& \text { Find } u_{N}(t, \cdot) \in \mathcal{K}_{N} \text { such that for a.e. } t \in J \text {, }  \tag{2.29a}\\
& \left(\underline{w}-\underline{u}_{N}(t)\right)^{\top}\left[\mathbf{M} \underline{\dot{u}}_{N}(t)+\mathbf{A} \underline{u}_{N}(t)\right] \geq\left(\underline{w}-\underline{u}_{N}(t)\right)^{\top} \underline{F}(t), \forall w \in V_{N}  \tag{2.29b}\\
& \underline{u}_{N}(0)=\underline{0} \tag{2.29c}
\end{align*}
$$

where from 2.28 it follows that the mass and stiffness matrix in $\mathbb{R}^{\tilde{N} \times \tilde{N}}$ are given by,

$$
\begin{align*}
\mathbf{M}_{\left(i_{1}, i_{2}\right),\left(j_{1}, j_{2}\right)} & :=\left(b_{j_{1}}\left(x_{1}\right) b_{j_{2}}\left(x_{2}\right), b_{i_{1}}\left(x_{1}\right) b_{i_{2}}\left(x_{2}\right)\right),  \tag{2.30a}\\
\mathbf{A}_{\left(i_{1}, i_{2}\right),\left(j_{1}, j_{2}\right)} & :=a\left(b_{j_{1}}\left(x_{1}\right) b_{j_{2}}\left(x_{2}\right), b_{i_{1}}\left(x_{1}\right) b_{i_{2}}\left(x_{2}\right)\right), \tag{2.30b}
\end{align*}
$$

respectively, and $\underline{F}_{i_{1}, i_{2}}(t):=f_{t}\left(b_{i_{1}} b_{i_{2}}\right) \in \mathbb{R}^{\tilde{N}}$ with $f_{t}$ given in (2.24) for $1 \leq i_{1}, i_{2}, j_{1}, j_{2} \leq N$. The matrices defined in 2.30 have the following tensor structure for which the precise calculations can be found in Appendix C.

$$
\begin{equation*}
\mathbf{M}=\mathbf{M}^{1} \otimes \mathbf{M}^{2} \text { and } \mathbf{A}=\frac{\sigma_{1}^{2}}{2} \mathbf{S}^{1} \otimes \mathbf{M}^{2}+\frac{\sigma_{2}^{2}}{2} \mathbf{M}^{1} \otimes \mathbf{S}^{2}+\mathbf{A}^{1} \otimes \mathbf{M}^{2}+\mathbf{M}^{1} \otimes \mathbf{A}^{2} \tag{2.31}
\end{equation*}
$$

where for $l=1,2$,

$$
\begin{array}{r}
\mathbf{M}_{i_{l}, j_{l}}^{l}=\int_{-R}^{R} b_{j_{l}}\left(x_{l}\right) b_{i_{l}}\left(x_{l}\right) d x_{l} \quad \mathbf{S}_{i_{l}, j_{l}}^{l}=\int_{-R}^{R} b_{j_{l}}^{\prime}\left(x_{l}\right) b_{i_{l}}^{\prime}\left(x_{l}\right) d x_{l} \\
\mathbf{A}_{i_{l}, j_{l}}^{l}=\int_{-R}^{R} \int_{-R}^{R} b_{j_{l}}^{\prime}\left(y_{l}\right) b_{i_{l}}^{\prime}\left(x_{l}\right) k_{X^{l}}^{(-2)}\left(y_{l}-x_{l}\right) d y_{l} d x_{l}
\end{array}
$$

Equation 2.29 is called the semi-discrete form resulting from the discretisation in space.

### 2.4.2 Semi-Discrete Form using Lagrangian Multiplier

In this section we reformulate 2.29 utilizing Lagrangian multipliers. The Lagrangian multiplier space $\mathcal{M}_{N}$ denotes the set of admissible functions for the Lagrangian multiplier applicable to the finite dimensional problem in 2.29. For the analysis in this section we refer to some of the results from Hager, et al. [20, Section 3]. The Lagrangian multiplier space is spanned by the dual basis of $V_{N}$,

$$
\mathcal{M}_{N}:=\left\{\mu_{N}(t, \underline{x})=\sum_{j=1}^{\tilde{N}} \underline{\mu}_{N, j}(t) \psi_{j}^{*}(\underline{x}): \underline{\mu}_{N, j} \geq 0\right\}
$$

where $\psi_{j}^{*}(\underline{x})$ is the dual function of $\psi_{j}(\underline{x})$ defined by,

$$
\begin{equation*}
\psi_{j}^{*}(\underline{x}):=3 \psi_{j}(\underline{x})-\frac{1}{3} \mathbb{1}_{\left[\operatorname{supp} \psi_{j}\right]}(\underline{x}) \quad \text { for } j=1, \ldots, \tilde{N} . \tag{2.32}
\end{equation*}
$$

We notice that supp $\psi_{j}^{*}=\operatorname{supp} \psi_{j}$ for $j=1, \ldots, \tilde{N}$. Another crucial property is shown in the following lemma.

Lemma 2.9. Let $\left\{\psi_{j}\right\}_{j=1}^{\tilde{N}}$ be the basis of $V_{N}$ and $\left\{\psi_{j}^{*}\right\}_{j=1}^{\tilde{N}}$ be the basis of $\mathcal{M}_{N}$ defined in 2.32 then we have,

$$
\int_{G} \psi_{j}(\underline{x}) \psi_{i}^{*}(\underline{x}) d \underline{x}=\delta_{i j} \int_{G} \psi_{j}(\underline{x}) d \underline{x} \quad \text { for } i, j=1 \ldots, \tilde{N},
$$

where $\delta_{i j}$ is the Kronecker's delta.
Proof. We know that $\int_{G} \psi_{j}(\underline{x}) d \underline{x}=h^{2}$. Further, we can infer that,

$$
\begin{aligned}
\int_{G} \psi_{j}(\underline{x}) \psi_{i}^{*}(\underline{x}) d \underline{x}= & 3 \int_{G_{1}} b_{j_{1}}\left(x_{1}\right) b_{i_{1}}\left(x_{1}\right) d x_{1} \int_{G_{2}} b_{j_{2}}\left(x_{2}\right) b_{i_{2}}\left(x_{2}\right) d x_{2} \\
& -\frac{1}{3} \int_{G_{1} \cap\left\{\operatorname{supp} b_{i_{2}}\right\}} \quad b_{j_{1}}\left(x_{1}\right) d x_{1} \int_{G_{2} \cap\left\{\operatorname{supp} b_{i_{2}}\right\}} b_{j_{2}}\left(x_{2}\right) d x_{2} \\
= & \begin{cases}h^{2}, & \text { if } i_{k}=j_{k} \text { for } k=1,2, \\
0, & \text { otherwise } .\end{cases}
\end{aligned}
$$

Due to the work in Kikuchi and Oden [28, Chapter 3.4] the semi-discrete formulation in 2.28) can be rewritten as:

$$
\begin{array}{ll}
\text { Find }\left(u_{N}(t, \cdot), \lambda_{N}(t, \cdot)\right) \in \mathcal{K}_{N} \times \mathcal{M}_{N} \text { s.t. for a.e. } t \in J, & \\
\left(\dot{u}_{N}, w_{N}\right)+a\left(u_{N}, w_{N}\right)-\left(\lambda_{N}, w_{N}\right)=f_{t}\left(w_{N}\right) & \forall w_{N} \in V_{N}, \\
\left(u_{N}, \mu_{N}-\lambda_{N}\right) \geq 0 & \forall \mu_{N} \in \mathcal{M}_{N},(2.33 \mathrm{a}) \\
u_{N}(0, \underline{x})=0, & \tag{2.33~d}
\end{array}
$$

where $f_{t}\left(w_{N}\right)=-\left(\partial_{t} \tilde{g}, w_{N}\right)-a\left(\tilde{g}, w_{N}\right)$ and $a(\cdot, \cdot)$ is given in 2.25). If 2.33b holds for all basis functions $w_{N}(\underline{x})=\psi_{i}(\underline{x})$, then it holds for all functions in $V_{N}$. Thus we get for $\lambda_{N}(t, \underline{x})=\sum_{j=1}^{\tilde{N}} \underline{\lambda}_{N, j}(t) \psi_{j}^{*}(\underline{x})$ :

Find $\left(u_{N}, \lambda_{N}\right) \in \mathcal{K}_{N} \times \mathcal{M}_{N}$ with $\underline{u}_{N}(t) \geq 0$ and $\underline{\lambda}_{N}(t) \geq 0 \forall t \in J$ s.t., (2.34a)

$$
\begin{align*}
& \sum_{j=1}^{\tilde{N}} \dot{u}_{N, j}(t)\left(\psi_{j}, \psi_{i}\right)+\sum_{j=1}^{\tilde{N}} u_{N, j}(t) a\left(\psi_{j}, \psi_{i}\right)-\sum_{j=1}^{\tilde{N}} \lambda_{N, j}(t)\left(\psi_{j}^{*}, \psi_{i}\right)=F_{i}(t),  \tag{2.34b}\\
& \sum_{j=1}^{\tilde{N}} u_{N, j}(t) \sum_{i=1}^{\tilde{N}}\left(\mu_{N, i}(t)-\lambda_{N, i}(t)\right)\left(\psi_{j}, \psi_{i}^{*}\right) \geq 0 \quad \forall \mu_{N} \in \mathcal{M}_{N}  \tag{2.34c}\\
& \underline{u}_{N}(0)=\underline{0} \tag{2.34~d}
\end{align*}
$$

where $F_{i}(t)=f_{t}\left(\psi_{i}\right)$ for $i=1, \ldots, \tilde{N}$. Using Lemma 2.9, we can acquire the following matrix representation of 2.34 b and rewrite 2.33 c as,

$$
\begin{align*}
& \mathbf{M} \underline{\dot{u}}_{N}(t)+\mathbf{A} \underline{u}_{N}(t)-\mathbf{D} \underline{\lambda}_{N}(t)=\underline{F}(t)  \tag{2.35a}\\
& \sum_{i=1}^{\tilde{N}} u_{N, i}(t)\left(\mu_{N, i}(t)-\lambda_{N, i}(t)\right) \geq 0 \quad \forall \mu_{N} \in \mathcal{M}_{N} \tag{2.35b}
\end{align*}
$$

where $\mathbf{D} \in \mathbb{R}^{\tilde{N} \times \tilde{N}}$ is a diagonal matrix given by $\mathbf{D}_{i j}:=\delta_{i j} h^{2}$. We may also rewrite 2.35 b into a nonlinear complementary condition. This is done in the next lemma.

Lemma 2.10. Equation 2.35b is equivalent to the complementary condition,

$$
\begin{equation*}
C\left(\underline{u}_{N}(t), \underline{\lambda}_{N}(t)\right):=\mathbf{D} \underline{\lambda}_{N}(t)-\max \left(0, \mathbf{D} \underline{\lambda}_{N}(t)-c \underline{u}_{N}(t)\right)=\underline{0}, \quad \forall t \in J \tag{2.36}
\end{equation*}
$$

for any $c>0$ and where the maximum is taken component-wise.
Proof. Let us prove that each condition is equivalent to the fact that for each $i=1, \ldots, \tilde{N}$ either $\lambda_{N, i}(t)$ or $u_{N, i}(t)$ is equal to zero for all $t \in J$. Let us fix an arbitrary $t \in J$. Since $\lambda_{N}(t, \cdot) \in \mathcal{M}_{N}$ we know that $\lambda_{N, i}(t) \geq 0$ for all $i=1, \ldots, \tilde{N}$. Assuming (2.36 holds and $h^{2} \lambda_{N, i}(t)>c u_{N, i}(t)$ for some $i \in\{1, \ldots, \tilde{N}\}$ then the $i^{\text {th }}$ component of $C\left(\underline{u}_{N}(t), \underline{\lambda}_{N}(t)\right)$ is equal to $c u_{N, i}(t)$ and thus, we must have $u_{N, i}(t)=0$. Equivalently, $0 \leq h^{2} \lambda_{N, i}(t) \leq c u_{N, i}(t)$ implies $\lambda_{N, i}(t)=0$. Conversely, if for every $i=1, \ldots, \tilde{N}$ either $u_{N, i}(t)=0$ or $\lambda_{N, i}(t)=0$ then 2.36 holds. Therefore, 2.36 holds if and only if for each $i=1, \ldots, \tilde{N}$ either $u_{N, i}(t)=0$ or $\lambda_{N, i}(t)=0$.

Now suppose $u_{N, i}(t)=0$, then the $i^{\text {th }}$ term in the sum in 2.35b is zero. If $\lambda_{N, i}(t)=0$, we have that $\mu_{N, i} u_{N, i}(t) \geq 0$ for all $\mu_{N, i} \geq 0$ by 2.34a. Hence, the sum in 2.35b is non-negative if either $u_{N, i}(t)=0$ or $\lambda_{N, i}(t)=0$ for $i=1, \ldots, \tilde{N}$. Conversely, if there is an $i \in\{1, \ldots, \tilde{N}\}$ such that $u_{N, i}(t)>0$ and $\lambda_{N, i}(t)>0$, we may choose $\mu_{N, j}=\lambda_{N, j}$ for all $j \neq i$ and $\mu_{N, i}=0$ implying a violation of 2.35b.

Finally, we can state the full problem in semi-discrete Lagrangian form as fol-
lows:

$$
\begin{align*}
& \text { Find }\left(u_{N}, \lambda_{N}\right) \in \mathcal{K}_{N} \times \mathcal{M}_{N} \text { with } \underline{u}_{N}(t) \geq 0 \text { and } \underline{\lambda}_{N}(t) \geq 0 \forall t \in J \text { s.t., } \\
& \mathbf{M} \underline{u}_{N}(t)+\mathbf{A} \underline{u}_{N}(t)-\mathbf{D} \underline{\lambda}_{N}(t)=\underline{F}(t),  \tag{2.37b}\\
& C\left(\underline{u}_{N}(t), \underline{\lambda}_{N}(t)\right)=\underline{0},  \tag{2.37c}\\
& \underline{u}_{N}(0)=\underline{0} . \tag{2.37~d}
\end{align*}
$$

The advantage of this conversion from an inequality to an equality is that it facilitates the implementation of a semi-smooth Newton algorithm in Chapter 3.

### 2.4.3 Time Discretisation

In the present section, we discretise the semi-discrete problem in time to obtain a fully discretised version of 2.23 ). We partition $J=[0, T]$ in $M+1$ equidistant grid points with,

$$
t_{m}=m k, \quad m=0,1, \ldots, M ; \quad k:=T / M .
$$

We define $\underline{u}_{N}^{m}:=\underline{u}_{N}\left(t_{m}\right), \underline{\lambda}_{N}^{m}:=\underline{\lambda}_{N}\left(t_{m}\right)$ and equivalently $\underline{F}^{m}:=\underline{F}\left(t_{m}\right)$. Applying the finite difference and the $\theta$-scheme to 2.37 yields,

$$
\mathbf{M} \frac{\underline{u}_{N}^{m+1}-\underline{u}_{N}^{m}}{k}+\mathbf{A}\left(\theta \underline{u}_{N}^{m+1}+(1-\theta) \underline{u}_{N}^{m}\right)-\mathbf{D} \underline{\lambda}_{N}^{m+1}=\theta \underline{F}^{m+1}+(1-\theta) \underline{F}^{m}
$$

Consequently, we obtain the fully discretised formulation of the American option pricing problem:

$$
\begin{align*}
& \text { Find }\left(\underline{u}_{N}^{m}, \underline{\lambda}_{N}^{m}\right) \text { with } \underline{u}_{N}^{m} \geq 0 \text { and } \underline{\lambda}_{N}^{m} \geq 0 \text { s.t. for } m=0, \ldots, M-1,  \tag{2.38a}\\
& \begin{array}{r}
(\mathbf{M}+k \theta \mathbf{A}) \underline{u}_{N}^{m+1}-k \mathbf{D} \underline{\lambda}_{N}^{m+1}=(\mathbf{M}-k(1-\theta) \mathbf{A}) \underline{u}_{N}^{m} \\
\\
\\
C\left(\underline{u}_{N}^{m}, \underline{\lambda}_{N}^{m}\right)=0
\end{array} \\
& \underline{u}_{N}^{0}=\underline{0} . \tag{2.38b}
\end{align*}
$$

Starting from the optimal stopping problem for the American options in model (1.2) we have established a system of inequalities that the American option price satisfies. We have derived that variational formulation and ensured that the variational inequality is well-posed and has a unique solution. Thereafter, we discretised the variational formulation using Lagrangian multipliers in space and time. In the fully-discretised formulation 2.38 we encounter a so-called linear complementary problem (LCP) for each time step $m=1, \ldots, M$. In the next chapter, we go into more detail about the semi-smooth Newton algorithm used to solve these LCPs.

## Chapter 3

## Convergence of the Semi-smooth Newton Algorithm


#### Abstract

In this chapter, we describe the semi-smooth Newton algorithm that is needed to solve the LCPs arising in the fully discrete pricing scheme developed in the previous chapter. Moreover, we state some convergence results for elliptic and parabolic variational inequalities and check numerical convergence rates on different problems. We analyse convergence behaviour of the elliptic variational inequality for one-dimensional perpetual American options. For the one-dimensional Kou model there exists a closed-form solution, see Kou and Wang [33, Theorem 3]. For this elliptic problem we observe optimal convergence of order $\mathcal{O}\left(N^{-2}\right)$ for the semi-smooth Newton algorithm. Thereafter, we adopt the one-dimensional tent problem from Moon, et al. [39, Section 5.2 and 5.3]. The obstacle in this parabolic problem is not differentiable in one particular point. In the case where the spatial grid matches this singularity, meaning there is an $i \in \mathcal{N}$ such that the obstacle $\tilde{g}(x)$ has its singularity at $x_{i}$, we can achieve optimal convergence. However, when the spatial grid does not incorporate the singularity, we notice that the convergence reduces to an order of $\mathcal{O}\left(N^{-\frac{1}{2}}\right)$ similar to the findings in Moon, et al. [39, Section 5.2 and 5.3]. Finally, we employ the so-called overkill method to check convergence of the two-dimensional American basket put option using the model in Section 1.3.1.


### 3.1 The Semi-smooth Newton Algorithm

In this section, we present the semi-smooth Newton algorithm and delineate its procedure to solve LCPs. Here, we follow the outline of Hager, et al. [20, Section 4.1]. For notational simplicity we examine more general LCPs in this section.

Linear Complementary Problem 3.1. Find $\underline{x} \in \mathbb{R}^{n}$ such that,

$$
\begin{aligned}
& \mathbf{B} \underline{x} \geq \underline{b}, \\
& \underline{x} \geq \underline{c}, \\
& (\underline{x}-\underline{c})^{\top}(\mathbf{B} \underline{x}-\underline{b})=\underline{0},
\end{aligned}
$$

for $\mathbf{B} \in \mathbb{R}^{n \times n}$ and $\underline{b}, \underline{c} \in \mathbb{R}^{n}$.
Although the well-established projected successive over-relaxation (PSOR) algorithm is a widespread method for solving LCPs we proceed with the semi-smooth Newton algorithm to solve the LCP for each time step of the fully discretised formulation of the American option pricing problem in 2.38). One of the advantages of the semi-smooth Newton method is its faster convergence, see Section 3.4.1. More details about the PSOR method can be found in Cryer [9. The semi-smooth Newton method exploits the so-called primal-dual active set strategy (PDASS). Hintermüller, et al. [25, Chapter 2] proved that the PDASS is equivalent to a semi-smooth Newton algorithm. The PDASS splits the space domain into several parts and, hence, divides the computational work to these parts. In our case, the stopping and continuation region decouple the space domain into two disjoint sets. To use appropriate terminology we denote the set of grid points $i \in \mathcal{N}$ for which $\sum_{j=1}^{n} \mathbf{B}_{i j} x_{j}=b_{i}$ and where $x_{i}=c_{i}$ by $\mathcal{I}$ (inactive set) and $\mathcal{A}$ (active set), respectively. Hence $\mathcal{N}=\mathcal{I} \cup \mathcal{A}$ where $\mathcal{N}$ is the set of all grid points. Reformulating the Linear Complementary Problem 3.1in terms of the Lagrangian multiplier $\underline{\lambda} \in \mathbb{R}_{+}^{n}$ reads:

Find $\underline{x}, \underline{\lambda} \in \mathbb{R}^{n}$ such that,

$$
\begin{align*}
& \mathbf{B} \underline{x}-\underline{\lambda}=\underline{b},  \tag{3.1b}\\
& C(\underline{x}, \underline{\lambda})=\underline{\lambda}-\max (\underline{0}, \underline{\lambda}-\omega(\underline{x}-\underline{c}))=0,
\end{align*}
$$

where $\omega>0$ is a penalty constant and the constraint $C$ is the resulting semismooth equation where the maximum is taken component-wise. The function $C(x, \lambda)$ is depicted in Figure 3.1 where we observe that the red line is the constraint in 3.1 c . Using the complementary function $C(x, \lambda)$ in 3.1c and the decomposition of all the grid points $\mathcal{N}=\mathcal{I} \cup \mathcal{A}$ with $\mathcal{I} \cap \mathcal{A}=\emptyset$, we can state the problem separately on the active and inactive set and using the knowledge of the values of either $\underline{x}_{\mathcal{A}}:=\left(x_{i}\right)_{i \in \mathcal{A}}$ or $\underline{\lambda}_{\mathcal{I}}:=\left(\lambda_{i}\right)_{i \in \mathcal{I}}$, that is $\underline{x}_{\mathcal{A}}=\underline{c}_{\mathcal{A}}$ and $\underline{\lambda}_{\mathcal{I}}=\underline{0}$, to solve the problem on the complementary set. Hence, the idea is to compute $\underline{x}_{\mathcal{I}}$ by applying the information about $\underline{\lambda}$ on $\mathcal{I}$ and $\underline{\lambda}_{\mathcal{A}}$ using the information about $\underline{x}$ on $\mathcal{A}$. This is described in Table 3.1 where $\# \mathcal{I}$ denotes the number of elements in set $\mathcal{I}$.

In Figure 3.1, we observe that the function $C(x, \lambda)$ is not continuously differentiable on the line $\lambda-\omega(x-c)=0$ for $x, \lambda \in \mathbb{R}$. For this reason, we use the generalized derivative to explain the variation of the complementary function $C(x, \lambda)$. Let $D_{C}$ denote the generalised derivative of $C(x, \lambda)$ in (3.1c) then the variation of $C$ at point $(x, \lambda)$ in direction $(\Delta x, \Delta \lambda)$ is given by,

$$
D_{C(x, \lambda)}(\Delta x, \Delta \lambda)=\Delta \lambda-\chi_{\mathcal{A}}(\Delta \lambda-\omega \Delta x)
$$

where $\Delta x, \Delta \lambda$ denote small variations in $x$ and $\lambda$, respectively, and $\chi_{\mathcal{A}}$ is defined as the characteristic function of the set $\mathcal{A}$,

$$
\chi_{\mathcal{A}}:= \begin{cases}1, & \text { if } \lambda-\omega(x-c)>0 \\ 0, & \text { if } \lambda-\omega(x-c) \leq 0\end{cases}
$$

For the semi-smooth Newton algorithm, we denote the $k^{\text {th }}$ Newton step by the iteration $\left(\underline{x}^{k}, \underline{\lambda}^{k}\right)=\left(\underline{x}^{k-1}, \underline{\lambda}^{k-1}\right)+\left(\Delta \underline{x}^{k-1}, \Delta \underline{\lambda}^{k-1}\right)$ for $k \in \mathbb{N}$ such that $D_{C\left(\underline{x}^{k-1}, \underline{\lambda}^{k-1}\right)}\left(\Delta \underline{x}^{k-1}, \Delta \underline{\lambda}^{k-1}\right)=\underline{0}$ where $\left(\Delta \underline{x}^{k-1}, \Delta \underline{\lambda}^{k-1}\right)$ is the change from

| Steps | Inactive set | Active set |
| :---: | :---: | :---: |
| 1. Apply the definition of $\mathcal{I}$ and $\mathcal{A}$ to the LCP | $\begin{aligned} & \forall i \in \mathcal{I}: \lambda_{i}=0 \\ & \sum_{j \in \mathcal{N}} \mathbf{B}_{i j} x_{j}=b_{i} . \end{aligned}$ | $\begin{aligned} & \forall i \in \mathcal{A}: x_{i}=c_{i} \\ & \sum_{j \in \mathcal{I}} \mathbf{B}_{i j} x_{j}-\lambda_{i}=b_{i}-\sum_{j \in \mathcal{A}} \mathbf{B}_{i j} c_{j} . \end{aligned}$ |
|  | Resulting in $\# \mathcal{I}$ equations to solve for $\# \mathcal{N}$ unknowns. | Resulting in $\# \mathcal{A}$ equations to solve for $\# \mathcal{N}$ unknowns. |
| 2. Apply the definition of set $\mathcal{A}$ to the LCP on set $\mathcal{I}$. | $\begin{aligned} & \hline \forall i \in \mathcal{I}: \lambda_{i}=0 \\ & \sum_{j \in \mathcal{I}} \mathbf{B}_{i j} x_{j}=f_{i} \\ & \text { with } \\ & f_{i}=b_{i}-\sum_{j \in \mathcal{A}} \mathbf{B}_{i j} c_{i} . \end{aligned}$ |  |
|  | Resulting in $\# \mathcal{I}$ equations to solve for $\# \mathcal{I}$ unknowns. |  |
| 3. Apply the solution $\underline{x}_{\mathcal{I}}^{\mathcal{I}}$ on set $\mathcal{I}$ to the LCP on set $\mathcal{A}$. |  | $\begin{aligned} & \hline \forall i \in \mathcal{A}: x_{i}=c_{i} \\ & \lambda_{i}=f_{i} \\ & \text { with } \\ & f_{i}=\sum_{j \in \mathcal{I}} \mathbf{B}_{i j} x_{j}^{\mathcal{I}}+\sum_{j \in \mathcal{A}} \mathbf{B}_{i j} c_{j}-b_{i} \end{aligned}$ |
|  |  | Resulting in $\# \mathcal{A}$ equations to solve for $\# \mathcal{A}$ unknowns. |

Table 3.1: Primal-Dual Active Set Strategy for Linear Complementary Problem 3.1.


Figure 3.1: Complementary Function $C(x, \lambda)=\lambda-\max (0, \lambda-\omega(x-c))$ with $\omega=.9$ and $c=2$. The red line delineates the constraint in 3.1c).
the $(k-1)^{\text {th }}$ to the $k^{\text {th }}$ iterate. More about this can be found in Hintermüller, et al. [25, Chapter 2] and Hager, et al. [20, Section 4.1]. The detailed description of the semi-smooth Newton algorithm is displayed in Table 3.2 where we use the following notation, $\mathbf{B}_{\left(\mathcal{A}^{k}, \mathcal{A}^{k}\right)}:=\left(\mathbf{B}_{i j}\right)_{i, j \in \mathcal{A}^{k}}$.

Choose initial guess $\underline{x}^{0} \geq \underline{c}$ and $\underline{\lambda}^{0} \geq \underline{0}$,
Choose $\omega>0$.
For $k=1,2, \ldots$ do:

$$
\begin{aligned}
& \mathcal{A}^{k}=\left\{i \in \mathcal{N}: \underline{\lambda}^{k-1}-\omega\left(\underline{x}^{k-1}-\underline{c}\right) \geq 0\right\} \\
& \mathcal{I}^{k}=\left\{i \in \mathcal{N}: \underline{\lambda}^{k-1}-\omega\left(\underline{x}^{k-1}-\underline{c}\right)<0\right\} \\
& \text { If } k \geq 2 \text { and } \mathcal{A}^{k}=\mathcal{A}^{k-1} \text { and } \mathcal{I}^{k}=\mathcal{I}^{k-1} \rightarrow \text { stop. Else: }
\end{aligned}
$$

$$
\begin{aligned}
& \underline{\lambda}_{\mathcal{I}^{k}}^{k}=\underline{0} \\
& \underline{x}_{\mathcal{A}^{k}}^{k}=\underline{\mathcal{A}}^{k} \\
& \underline{x}_{\mathcal{I}^{k}}^{k}=\mathbf{B}_{\left(\mathcal{I}^{k}, \mathcal{I}^{k}\right)}^{-1}\left(\underline{b}_{\mathcal{I}^{k}}-\mathbf{B}_{\left(\mathcal{I}^{k}, \mathcal{A}^{k}\right)} \underline{\mathcal{A}}_{\mathcal{A}^{k}}\right) \\
& \underline{\lambda}_{\mathcal{A}^{k}}^{k}=\mathbf{B}_{\left(\mathcal{A}^{k}, \mathcal{I}^{k}\right)} \underline{x}_{\mathcal{I}^{k}}^{k}+\mathbf{B}_{\left(\mathcal{A}^{k}, \mathcal{A}^{k}\right) \underline{\mathcal{A}}^{k}}-\underline{b}_{\mathcal{A}^{k}}
\end{aligned}
$$

Next $k$.

Table 3.2: Semi-smooth Newton Algorithm for Linear Complementary Problem 3.1.

Remark. Within the PDASS, the free boundary is a direct outcome of the algorithm whereas with the well-known PSOR algorithm this is not the case.

Remark. The PDASS can also be applied to European style options by enforcing $\mathcal{A}=\emptyset$ for all iterations $k \in \mathbb{N}$ and all time steps $m=1, \ldots, M$. Therefore, only one algorithm is needed to price American as well as European style options.

In the next section, we focus on the uniqueness of the solution to the Linear Complementary Problem 3.1 and the convergence of the semi-smooth Newton algorithm in Table 3.2.

### 3.2 Solution Uniqueness and Convergence

In this section, we address the convergence of the semi-smooth Newton algorithm. First, we give some theoretical facts and then apply these results to the LCPs for the American option pricing problem solved by the semi-smooth Newton algorithm in Table 3.2. The following definition can be found in Berman and Plemmons [3, Section 10.2].

Definition 3.2. An $n \times n$ matrix is a P-matrix if all its principal minors are positive.

A subset of all P-matrices are M-matrices for which the definition can be found in Berman and Plemmons [3, Definition 6.1.2].

Definition 3.3. An $n \times n$ matrix is an M-matrix if it is a P-matrix and all its off-diagonal entries are non-positive.

The following theorem characterises the uniqueness of a solution to the LCP above.

Theorem 3.4. The LCP in (3.1) has a unique solution if and only if $\mathbf{B}$ is a $P$-matrix.

Proof. For the proof we refer to Murty [40, Theorem 3.1].
Imposing the condition that the matrix $\mathbf{B}$ in (3.1) is a P-matrix does not only imply uniqueness of the solution but also local convergence as we can observe from the following theorem.

Theorem 3.5. Let B be a P-matrix, then the semi-smooth Newton method given in Table 3.2 converges superlinearly to $\left(\underline{x}^{*}, \underline{\lambda}^{*}\right)$, provided that $\left\|\underline{x}_{0}-\underline{x}^{*}\right\|_{l^{2}}$ and $\left\|\lambda_{0}-\underline{\lambda}^{*}\right\|_{l^{2}}$ are sufficiently small.

Proof. The theorem and proof is given in Hintermüller, et al. 25] Theorem 3.1].

Global convergence can be achieved by imposing the M-matrix condition on matrix B.

Theorem 3.6. If $\mathbf{B}$ is an M-matrix, then the semi-smooth Newton method converges to $\left(\underline{x}^{*}, \underline{\lambda}^{*}\right)$ for arbitrary initial data $\underline{x}_{0}$ and $\underline{\lambda}_{0}$.

Proof. The theorem can be found in Hintermüller, et al. [25, Theorem 3.2] and the proof is given in Hintermüller, et al. [25, Appendix A].

Now that we have established the groundwork for the uniqueness and convergence of the LCP in 3.1, we turn to the specific LCP that we need to solve for every time step $m=0, \ldots, M-1$ to obtain the price of the American option. For this we recall the fully discretised formulation in 2.38. First, we need the following lemma.

Lemma 3.7. Let $w_{N}(\underline{x})=\sum_{i=1}^{\tilde{N}} w_{i} \psi_{i}(\underline{x}) \in V_{N}$ where $V_{N}$ is the finite element space given in 2.27). Then there exist constants $C_{\min }, C_{\max }>0$ such that,

$$
C_{\min } \underline{w}^{\top} \underline{w} \leq\left\|w_{N}\right\|_{L^{2}(G)}^{2} \leq C_{\max } \underline{w}^{\top} \underline{w} .
$$

Proof. First let us rewrite $\left\|w_{N}\right\|_{L^{2}(G)}^{2}$ using $w_{N}(\underline{x})=\sum_{j=1}^{\tilde{N}} w_{j} \psi_{j}(\underline{x})$ as follows,

$$
\begin{aligned}
\left\|w_{N}\right\|_{L^{2}(G)}^{2} & =\sum_{i=1}^{\tilde{N}} \sum_{j=1}^{\tilde{N}} w_{i} w_{j} \int_{G} \psi_{j}(\underline{x}) \psi_{i}(\underline{x}) d x \\
& =\sum_{i=1}^{\tilde{N}} \sum_{j=1}^{\tilde{N}} w_{i} w_{j} \mathbf{M}_{i, j} \\
& =\underline{w}^{\top} \mathbf{M} \underline{w},
\end{aligned}
$$

where $\mathbf{M}=\mathbf{M}^{1} \otimes \mathbf{M}^{2}$ is given in (2.31). Furthermore by Gershgorin's Theorem, we know that all eigenvalues $\tilde{\lambda}_{l}^{i}$ of $\mathbf{M}^{i}$ lie in some disc $D\left(\mathbf{M}_{l l}^{i}, \tilde{R}_{l}^{i}\right)$ with
radius $\tilde{R}_{l}^{i}:=\sum_{k \neq l}\left|\mathbf{M}_{l k}^{i}\right|$ for $l, k=1, \ldots, N$ and $i=1,2$. As we know $\mathbf{M}^{i}$ explicitly from C.2, we have $\mathbf{M}_{l l}^{i}+\tilde{R}_{l}^{i}=h$ and $\mathbf{M}_{l l}^{i}-\tilde{R}_{l}^{i}=\frac{h}{3}>0$ for all $l=1, \ldots, N$ and $i=1,2$. Hence, all eigenvalues $\lambda_{l}^{i}$ of $\mathbf{M}^{i}$ are positive. By Magnus and Neudecker [35, Theorem 2.3.1], the eigenvalues of $\mathbf{M}$ are given by $\tilde{\lambda}_{l(N-1)+k}:=\tilde{\lambda}_{l}^{1} \tilde{\lambda}_{k}^{2} \in\left[\frac{h^{2}}{9}, h^{2}\right]$ for $l, k=1, \ldots, N$ and thus for all $\underline{w} \in \mathbb{R}^{\tilde{N}}$,

$$
C_{\min } \underline{w}^{\top} \underline{w} \leq \tilde{\lambda}_{\min } \underline{w}^{\top} \underline{w} \leq \underline{w}^{\top} \mathbf{M} \underline{w} \leq \tilde{\lambda}_{\max } \underline{w}^{\top} \underline{w} \leq C_{\max } \underline{w}^{\top} \underline{w}
$$

where,

$$
\tilde{\lambda}_{\min }=\min _{j \in\{1, \ldots, \tilde{N}\}} \tilde{\lambda}_{j}, \quad \tilde{\lambda}_{\max }=\max _{j \in\{1, \ldots, \tilde{N}\}} \tilde{\lambda}_{j},
$$

$C_{\min }:=\frac{h^{2}}{9}$ and $C_{\max }:=h^{2}$.
Turning to the LCP in the discretised option pricing scheme in 2.38, we can use Lemma 3.7 to prove uniqueness and local convergence.

Proposition 3.8. Let $\mathbf{B}=\mathbf{M}+k \theta \mathbf{A}$ with $k \in \mathbb{R}_{+}$the time mesh size, $\theta \in[0,1]$ the $\theta$-scheme parameter and $\mathbf{M}, \mathbf{A}$ the matrices given as in 2.31. Further assume that the bilinear form $a(\cdot, \cdot)$ given in 2.25 associated to the matrix $\mathbf{A}$ is coercive, then $\mathbf{B}$ is a P-matrix.

Proof. By Theorem 3.4, it is equivalent to show that the LCP has a unique solution. Using Stampacchia's Theorem (e.g. Kinderlehrer and Stampacchia [29, Theorem 2.2.1]) we need to show that the bilinear form $b: \mathbb{R}^{\tilde{N}} \times \mathbb{R}^{\tilde{N}} \rightarrow \mathbb{R}$ with $b(\underline{v}, \underline{w}):=\left(v_{N}, w_{N}\right)+k \theta a\left(v_{N}, w_{N}\right)$ is continuous and coercive for $v_{N}, w_{N} \in V_{N}$ and $a(\cdot, \cdot)$ given in 2.25 with $V_{N}$ defined in 2.27. Since we have $v_{N}(\underline{x})=\sum_{j=1}^{\tilde{N}} v_{i} \psi_{j}(\underline{x})$ for $v_{N} \in V_{N}$ with $\underline{v} \in \mathbb{R}^{\tilde{N}}$ it follows that,

$$
b(\underline{v}, \underline{w})=\sum_{j=1}^{\tilde{N}} \sum_{i=1}^{\tilde{N}} v_{j} w_{i}\left[\left(\psi_{j}, \psi_{i}\right)+k \theta a\left(\psi_{j}, \psi_{i}\right)\right]=\underline{v}^{\top} \mathbf{B} \underline{w} .
$$

Hence, continuity of the bilinear form $b(\cdot, \cdot)$ follows from

$$
\begin{align*}
\underline{v}^{\top} \mathbf{B} \underline{w} & =\underline{v}^{\top} \mathbf{M} \underline{w}+k \theta \underline{v}^{\top} \mathbf{A} \underline{w}  \tag{3.2a}\\
& =\left\|v_{N}\right\|_{L^{2}(G)}\left\|w_{N}\right\|_{L^{2}(G)}+k \theta a\left(v_{N}, w_{N}\right)  \tag{3.2b}\\
& \leq\left\|v_{N}\right\|_{L^{2}(G)}\left\|w_{N}\right\|_{L^{2}(G)}+k \theta C_{1}\left\|v_{N}\right\|_{\tilde{H}^{\rho}(G)}\left\|w_{N}\right\|_{\tilde{H}^{\rho}(G)}  \tag{3.2c}\\
& \leq\left\|v_{N}\right\|_{L^{2}(G)}\left\|w_{N}\right\|_{L^{2}(G)}+k \theta C_{1}\left\|v_{N}\right\|_{H_{0}^{1}(G)}\left\|w_{N}\right\|_{H_{0}^{1}(G)}  \tag{3.2d}\\
& \leq\left\|v_{N}\right\|_{L^{2}(G)}\left\|w_{N}\right\|_{L^{2}(G)}\left(1+k \theta \frac{C_{1} \tilde{C}^{2}}{h^{2}}\right)  \tag{3.2e}\\
& \leq C_{\max }\left(1+k \theta \frac{C_{1} \tilde{C}^{2}}{h^{2}}\right)\|\underline{v}\|_{l^{2}}\|\underline{w}\|_{l^{2}}=\underbrace{\left(h^{2}+k \theta C_{1} \tilde{C}^{2}\right)}_{=: \tilde{C}_{1}>0}\|\underline{v}\|_{l^{2}}\|\underline{w}\|_{l^{2}}, \tag{3.2f}
\end{align*}
$$

where $a(\cdot, \cdot)$ is given in 2.25 . In 3.2 C we use the continuity property of $a(\cdot, \cdot)$ in $2.22 \mathrm{a}, 3.2 \mathrm{~d})$ follows from the fact that $\rho_{j} \leq 1$ for $j=1,2$, for 3.2 e we use the inverse norm inequality in Brenner and Scott [6, Lemma 4.5.3] and 3.2e)
results from Lemma 3.7. Assuming that the bilinear form $a(\cdot, \cdot)$ is coercive we derive in a similar way to 3.2,

$$
\begin{aligned}
\underline{w}^{\top} \mathbf{B} \underline{w} & \geq\left\|w_{N}\right\|_{L^{2}(G)}^{2}+k \theta\left(C_{2}\left\|w_{N}\right\|_{\tilde{H}^{\rho}(G)}^{2}\right) \\
& \geq\left\|w_{N}\right\|_{L^{2}(G)}^{2}\left(1+k \theta C_{2}\right) \\
& \geq C_{\min }\left(1+k \theta C_{2}\right) \underline{w}^{\top} \underline{w}=\underbrace{\frac{h^{2}}{9}\left(1+k \theta C_{2}\right)}_{=: \bar{C}_{2}>0}\|\underline{w}\|_{l^{2}}^{2} .
\end{aligned}
$$

Under the assumption that the bilinear form $a(\cdot, \cdot)$ is coercive, the result of Proposition 3.8 does not only ensure a unique solution but also convergence of the semi-smooth Newton algorithm for specific initial data due to Theorem 3.5. In order to achieve convergence for arbitrary initial data we need $\mathbf{B}$ to be an M-matrix. By Definition 3.2, it only remains to show that all off-diagonal entries are non-positive. This, however, will prove to be a non trivial task for the matrix $\mathbf{B}=\mathbf{M}+k \theta \mathbf{A}$. For the one-dimensional case we refer to Reichmann [44, Lemma 11.2.7] where it is proven that the one-dimensional stiffness matrix is an M-matrix.

Let us now look at the stiffness matrix $\mathbf{A}=\mathbf{A}^{B S}+\mathbf{A}^{J}$ defined in 2.31.
Proposition 3.9. Let $\mathbf{A}=\mathbf{A}^{B S}+\mathbf{A}^{J}$ defined in 2.31. Assume that the bilinear form $a(\cdot, \cdot)$ given in 2.25 associated to the matrix $\mathbf{A}$ is coercive and that the following inequalities hold,

$$
\begin{align*}
{\left[\sigma_{1}^{2}-2 \sigma_{2}^{2}+\left(2 k_{0}^{1}-k_{1,+}^{1}-k_{1,-}^{1}\right)+4\left(2 k_{1,-}^{2}-k_{2,-}^{2}-k_{0}^{2}\right)\right] \leq 0, }  \tag{3.3a}\\
{\left[\sigma_{1}^{2}-2 \sigma_{2}^{2}+\left(2 k_{0}^{1}-k_{1,+}^{1}-k_{1,-}^{1}\right)+4\left(2 k_{1,+}^{2}-k_{2,+}^{2}-k_{0}^{2}\right)\right] \leq 0 }  \tag{3.3b}\\
{\left[-2 \sigma_{1}^{2}+\sigma_{2}^{2}+4\left(2 k_{1,-}^{1}-k_{2,--}^{1}-k_{0}^{1}\right)+\left(2 k_{0}^{2}-k_{1,+}^{2}-k_{1,--}^{2}\right)\right] \leq 0 }  \tag{3.3c}\\
{\left[-2 \sigma_{1}^{2}+\sigma_{2}^{2}+4\left(2 k_{1,+}^{1}-k_{2,+}^{1}-k_{0}^{1}\right)+\left(2 k_{0}^{2}-k_{1,+}^{2}-k_{1,--}^{2}\right)\right] \leq 0, } \tag{3.3d}
\end{align*}
$$

where the definitions of $k_{l, \pm}^{j}$ for $j=1,2$ and $l=0,1,2$ can be found in Appendix C. Then $\mathbf{A}$ is an M-matrix.

Proof. Since $a(\cdot, \cdot)$ is continuous by Lemma 2.6 and we assume that $a(\cdot, \cdot)$ is coercive it follows that $\mathbf{A}$ is a P-matrix by Kinderlehrer and Stampacchia 29, Theorem 2.2.1] and Theorem 3.5. Now it only remains to show that the offdiagonal entries of $\mathbf{A}$ are non-positive by Berman and Plemmons [3, Theorem 6.2.4(A1)]. By (C.3) and Reichmann [44, Lemma 11.2.7] we know that $\mathbf{S}^{l}$ and $\mathbf{A}^{l}$ are M-matrices for $l=1,2$. This gives us the following positive off-diagonal entries of each of the Kronecker products in 2.31),

$$
\begin{aligned}
& \mathbf{S}^{1} \otimes \mathbf{M}^{2}: \mathbf{S}_{i_{1} i_{1}}^{1} \mathbf{M}_{i_{2}+1 i_{2}}^{2}, \\
& \mathbf{M}^{1} \otimes \mathbf{S}^{2}: \mathbf{M}_{i_{1} i_{1}}^{1} \mathbf{M}_{i_{2} i_{2}+1 i_{1}}^{2} \mathbf{S}_{i_{2} i_{2}}^{2}, \\
& \mathbf{M}_{i_{1} i_{1}+1}^{1} \mathbf{S}_{i_{2} i_{2}}^{2}>0, \\
& \mathbf{A}^{1} \otimes \mathbf{M}^{2}: \mathbf{A}_{i_{1} i_{1}}^{1} \mathbf{M}_{i_{2}+1 i_{2}}^{2}, \\
& \mathbf{A}_{i_{1} i_{1}} \mathbf{M}_{i_{2} i_{2}+1}^{2}>0, \\
& \mathbf{M}^{1} \otimes \mathbf{A}^{2}: \mathbf{M}_{i_{1}+1 i_{1}}^{1} \mathbf{A}_{i_{2} i_{2}}^{2},
\end{aligned} \mathbf{M}_{i_{1} i_{1}+1} \mathbf{A}_{i_{2} i_{2}}^{2}>0, ~ \$
$$

where $i_{1}, i_{2}=1 \ldots, N-1$. These individual entries compose the following entries of matrix $\mathbf{A}$,

$$
\begin{aligned}
& \mathbf{A}_{\left(i_{1}, i_{2}+1\right)\left(i_{1}, i_{2}\right)}=\frac{\sigma_{1}^{2}}{2} \mathbf{S}_{i_{1} i_{1}}^{1} \mathbf{M}_{i_{2}+1 i_{2}}^{2}+\frac{\sigma_{2}^{2}}{2} \mathbf{M}_{i_{1} i_{1}}^{1} \mathbf{S}_{i_{2}+1 i_{2}}^{2}+\mathbf{A}_{i_{1} i_{1}}^{1} \mathbf{M}_{i_{2}+1 i_{2}}^{2}+\mathbf{M}_{i_{1} i_{1}}^{1} \mathbf{A}_{i_{2}+1 i_{2}}^{2}, \\
& \mathbf{A}_{\left(i_{1}, i_{2}\right)\left(i_{1}, i_{2}+1\right)}=\frac{\sigma_{1}^{2}}{2} \mathbf{S}_{i_{1} i_{1}}^{1} \mathbf{M}_{i_{2} i_{2}+1}^{2}+\frac{\sigma_{2}^{2}}{2} \mathbf{M}_{i_{1} i_{1}}^{1} \mathbf{S}_{i_{2} i_{2}+1}^{2}+\mathbf{A}_{i_{1} i_{1}}^{1} \mathbf{M}_{i_{2} i_{2}+1}^{2}+\mathbf{M}_{i_{1} i_{1}}^{1} \mathbf{A}_{i_{2} i_{2}+1}^{2} \\
& \mathbf{A}_{\left(i_{1}+1, i_{2}\right)\left(i_{1}, i_{2}\right)}=\frac{\sigma_{1}^{2}}{2} \mathbf{S}_{i_{1}+1 i_{1}}^{1} \mathbf{M}_{i_{2} i_{2}}^{2}+\frac{\sigma_{2}^{2}}{2} \mathbf{M}_{i_{1}+1 i_{1}}^{1} \mathbf{S}_{i_{2} i_{2}}^{2}+\mathbf{A}_{i_{1}+1 i_{1}}^{1} \mathbf{M}_{i_{2} i_{2}}^{2}+\mathbf{M}_{i_{1}+1 i_{1}}^{1} \mathbf{A}_{i_{2} i_{2}}^{2}, \\
& \mathbf{A}_{\left(i_{1}, i_{2}\right)\left(i_{1}+1, i_{2}\right)}=\frac{\sigma_{1}^{2}}{2} \mathbf{S}_{i_{1} i_{1}+1}^{1} \mathbf{M}_{i_{2} i_{2}}^{2}+\frac{\sigma_{2}^{2}}{2} \mathbf{M}_{i_{1} i_{1}+1}^{1} \mathbf{S}_{i_{2} i_{2}}^{2}+\mathbf{A}_{i_{1} i_{1}+1}^{1} \mathbf{M}_{i_{2} i_{2}}^{2}+\mathbf{M}_{i_{1} i_{1}+1}^{1} \mathbf{A}_{i_{2} i_{2}}^{2},
\end{aligned}
$$

From the results in Appendix C we can retrieve the following four inequalities that are the conditions for $\mathbf{A}$ to be an M-matrix,

$$
\begin{aligned}
{\left[\sigma_{1}^{2}-2 \sigma_{2}^{2}+\left(2 k_{0}^{1}-k_{1,+}^{1}-k_{1,-}^{1}\right)+4\left(2 k_{1,-}^{2}-k_{2,--}^{2}-k_{0}^{2}\right)\right] \leq 0 } \\
{\left[\sigma_{1}^{2}-2 \sigma_{2}^{2}+\left(2 k_{0}^{1}-k_{1,+}^{1}-k_{1,-}^{1}\right)+4\left(2 k_{1,+}^{2}-k_{2,+}^{2}-k_{0}^{2}\right)\right] \leq 0 } \\
{\left[-2 \sigma_{1}^{2}+\sigma_{2}^{2}+4\left(2 k_{1,--}^{1}-k_{2,-}^{1}-k_{0}^{1}\right)+\left(2 k_{0}^{2}-k_{1,+}^{2}-k_{1,--}^{2}\right)\right] \leq 0 } \\
{\left[-2 \sigma_{1}^{2}+\sigma_{2}^{2}+4\left(2 k_{1,+}^{1}-k_{2,+}^{1}-k_{0}^{1}\right)+\left(2 k_{0}^{2}-k_{1,+}^{2}-k_{1,--}^{2}\right)\right] \leq 0 . }
\end{aligned}
$$

Remark. To prove that $\mathbf{B}$ is an M-matrix the validity of the four inequalities in (3.3) must be verified. That is, one has to show for which parameters of the model the four inequalities in 3.3 hold. Since for the Kou and the CGMY model we have ten and eight parameters, respectively, we cannot solve this explicitly. Finding bounds for these parameters is a focus of further research.

For the Brownian motion case, i.e. $\mathbf{A}=\mathbf{A}^{B S}$ we can prove that $\mathbf{A}$ is an M-matrix for specific values of the constant volatilities $\sigma_{j}$ of the processes $X^{j}$ for $j=1,2$.

Corollary 3.10. Let $\mathbf{A}=\frac{\sigma_{1}^{2}}{2} \mathbf{S}^{1} \otimes \mathbf{M}^{2}+\frac{\sigma_{2}^{2}}{2} \mathbf{M}^{1} \otimes \mathbf{S}^{2}$ and assume that $a^{B S}(\cdot, \cdot)$ in (2.13) is coercive, then $\mathbf{A}$ is an M-matrix if and only if $\frac{\sigma_{1}}{\sigma_{2}} \in\left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$ for $\sigma_{1}, \sigma_{2}>0$.

Proof. With the same reasoning as in Proposition 3.9 we obtain the following two inequalities similar to those in (3.3), namely,

$$
\begin{aligned}
\sigma_{1}^{2}-2 \sigma_{2}^{2} & \leq 0 \\
-2 \sigma_{1}^{2}+\sigma_{2}^{2} & \leq 0
\end{aligned}
$$

This gives $\frac{1}{\sqrt{2}} \leq \frac{\sigma_{1}}{\sigma_{2}} \leq \sqrt{2}$.
The next proposition finds conditions on $k, h \in \mathbb{R}_{+}$such that $\mathbf{B}=\mathbf{M}+k \theta \mathbf{A}^{B S}$ is an M-matrix.
Corollary 3.11. Suppose $a^{B S}(\cdot, \cdot)$ in 2.13 is coercive and that $\frac{\sigma_{1}}{\sigma_{2}} \in\left(\frac{1}{\sqrt{2}}, \sqrt{2}\right)$ with $\sigma_{i}>0$ for $i=1,2$. If the following two conditions hold for $k, h \in \mathbb{R}_{+}$and
$\theta \in[0,1]$,

$$
\begin{aligned}
& h^{2} \leq \frac{3 k \theta}{2}\left(2 \sigma_{1}^{2}-\sigma_{2}^{2}\right), \\
& h^{2} \leq \frac{3 k \theta}{2}\left(2 \sigma_{2}^{2}-\sigma_{1}^{2}\right)
\end{aligned}
$$

Then $\mathbf{B}=\mathbf{M}+k \theta \mathbf{A}^{B S}$ is an M-matrix.
Remark. Observe that the minimum and the maximum of the interval for $\frac{\sigma_{1}}{\sigma_{2}}$ in Proposition 3.10 are not feasible anymore since $h>0$.

Proof. By Proposition 3.8, we know that B is a P-matrix. Hence it remains to show that all off-diagonal entries of $\mathbf{B}$ are non-positive. The matrix $\mathbf{M}$ has several positive values $\mathbf{M}_{\left(i_{1}, i_{2}\right)\left(j_{1} j_{2}\right)}$ where for $l=1,2,\left|i_{l}-j_{l}\right| \leq 1$. This gives us the following conditions for the off-diagonal entries,

$$
\mathbf{M}_{i_{1} j_{1}}^{1} \mathbf{M}_{i_{2} j_{2}}^{2}+k \theta\left(\frac{\sigma_{1}^{2}}{2} \mathbf{S}_{i_{1} j_{1}}^{1} \mathbf{M}_{i_{2} j_{2}}^{2}+\frac{\sigma_{2}^{2}}{2} \mathbf{M}_{i_{1} j_{1}}^{1} \mathbf{S}_{i_{2} j_{2}}^{2}\right) \leq 0
$$

where $\left|i_{l}-j_{l}\right| \leq 1$ except for $\left|i_{1}-j_{1}\right|=\left|i_{2}-j_{2}\right|=0$. From this, we retrieve the following three conditions,

$$
\begin{align*}
& \left|i_{1}-j_{1}\right|=1 \text { and }\left|i_{2}-j_{2}\right|=1 \text { gives } \frac{h^{2}}{3} \leq k \theta\left(\sigma_{1}^{2}+\sigma_{2}^{2}\right)  \tag{3.5a}\\
& \left|i_{1}-j_{1}\right|=1 \text { and }\left|i_{2}-j_{2}\right|=0 \text { gives } \frac{h^{2}}{3} \leq \frac{k \theta}{2}\left(2 \sigma_{1}^{2}-\sigma_{2}^{2}\right),  \tag{3.5b}\\
& \left|i_{1}-j_{1}\right|=0 \text { and }\left|i_{2}-j_{2}\right|=1 \text { gives } \frac{h^{2}}{3} \leq \frac{k \theta}{2}\left(2 \sigma_{2}^{2}-\sigma_{1}^{2}\right) . \tag{3.5c}
\end{align*}
$$

By adding (3.5b) to (3.5c) we infer that 3.5a) is obsolete. Since $\frac{\sigma_{1}}{\sigma_{2}} \notin\left\{\frac{1}{\sqrt{2}}, \sqrt{2}\right\}$ we have that $h>0$.

This concludes the section on the convergence of the semi-smooth Newton algorithm. We found that for the general matrix $\mathbf{B}=\mathbf{M}+k \theta \mathbf{A}$ where $\mathbf{M}$ and $\mathbf{A}$ are defined in 2.31 , we can only prove that the semi-smooth Newton method converges for specific initial data. In the Brownian motion case, we found conditions for $\sigma_{j}$ with $j=1,2$ such that $\mathbf{B}=\mathbf{M}+k \theta \mathbf{A}^{B S}$ is an M-matrix and thus the semi-smooth Newton algorithm in Table 3.2 converges for arbitrary initial data $\underline{x}_{0}$ and $\underline{\lambda}_{0}$ by Theorem 3.6.

### 3.3 A Priori Error Estimates

In this section we state some of the a priori error estimates for elliptic and parabolic variational inequalities. First we discuss the optimal $L^{2}$ convergence for a particular class of elliptic variational inequalities. Thereafter, we discuss a priori error estimates for parabolic variational inequalities in two space dimensions presented in Johnson [26], Fetter [17] and Vuik [50]. Finally, we give the a priori error estimates for the American option pricing problem for onedimensional Lévy models discussed in Matache, et al. [36]. For this we introduce isotropic Sobolev spaces of fractional order defined by,

$$
H^{s}\left(\mathbb{R}^{2}\right):=\left\{v \in L^{2}\left(\mathbb{R}^{2}\right):\|v\|_{H^{s}}^{2}=\int_{\mathbb{R}^{2}}\left(1+\|\underline{\xi}\|_{l^{2}}\right)^{2 s}|\hat{v}(\underline{\xi})|^{2} d \underline{\xi}<\infty\right\}
$$

for $s \in \mathbb{R}_{+}$, see Winter [49, Section 1.2].

### 3.3.1 Elliptic Variational Inequalities

For the a priori error estimates of discretised elliptic variational inequalities with a non-degenerate linear operator $\mathcal{A}: \mathcal{V} \rightarrow \mathcal{V}^{*}$ of order two we refer to Falk [16]. The following localised elliptic variational inequality is treated in Falk [16, Section 3],

$$
\begin{align*}
& \text { Find } u \in \mathcal{K}(G)=\left\{v \in H_{0}^{1}(G): v(\underline{x}) \geq \tilde{g}(\underline{x}) \text { a.e. } x \in G\right\} \text { such that, }  \tag{3.6a}\\
& a(u, u-v) \geq(f, u-v)_{H_{0}^{-1}(G), H_{0}^{1}(G)}, \quad \forall v \in H_{0}^{1}(G) \tag{3.6~b}
\end{align*}
$$

where $a(\phi, \psi):=\sum_{i, j=1}^{2} \int_{G} a_{i j}(\underline{x}) \partial_{x_{i}} \phi(\underline{x}) \partial_{x_{j}} \psi(\underline{x}) d \underline{x}+\int_{G} c(\underline{x}) \phi(\underline{x}) \psi(\underline{x}) d \underline{x}$ with $\underline{x}=\left(x_{1}, x_{2}\right)^{\top}, f(\underline{x}) \in L^{2}(G), \tilde{g}(\underline{x}) \in H^{2}(G)$ and $G \subset \mathbb{R}^{2}$ is bounded. For the bilinear form $a(\cdot, \cdot)$ to be continuous and coercive we assume that $c \in L^{\infty}(G)$, $a(\underline{x})$ is symmetric, i.e. $a_{i j}(\underline{x})=a_{j i}(\underline{x}), a_{i j}(\underline{x}) \in C^{1}(\bar{G})$ for $i, j=1,2$ and that there exists a constant $C>0$ such that,

$$
\sum_{i, j=1}^{2} a_{i j}(\underline{x}) \xi_{i} \xi_{j} \geq C\|\underline{\xi}\|_{l^{2}}^{2} \quad \forall \underline{\xi} \in \mathbb{R}^{2}
$$

Falk [16, Lemma 3] then states that if $u \in H_{0}^{2}(G)$ is a solution to (3.6) and $u_{I}(\underline{x})=\sum_{i=1}^{\tilde{N}} u\left(\underline{x}^{i}\right) \psi_{i}(\underline{x})$ is its linear interpolation where $\underline{x}^{i}=\left(x_{1}^{i}, x_{2}^{i}\right)^{\top}$ and $\psi_{i}(\underline{x})$ are piecewise linear continuous functions, e.g. product of hat functions (see Figure 2.1, then,

$$
\left\|u(\underline{x})-u_{I}(\underline{x})\right\|_{L^{2}(G)} \leq C h^{2}\|u\|_{H^{2}(G)}
$$

where $C>0$ is a constant independent on the mesh size $h$. This indicates $\mathcal{O}\left(h^{2}\right)$ is the optimal convergence order for continuous, linear finite elements in $L^{2}$.

### 3.3.2 Parabolic Variational Inequalities

A priori error estimates of fully discretised parabolic variational inequalities with elliptic, non-degenerate operator $\mathcal{A}$ of order two have been studied in, amongst others, Johnson [26], Fetter [17] and Vuik [50]. The parabolic variational inequality focused on in these papers is of the form,

Find $u(t, \cdot) \in \mathcal{K}(G)$ such that on $J \times G$,

$$
\begin{align*}
& \left(\partial_{t} u, u-v\right)_{H_{0}^{-1}(G), H_{0}^{1}(G)}+a(u, u-v) \geq(f, u-v)_{H_{0}^{-1}(G), H_{0}^{1}(G)},  \tag{3.7b}\\
& u(0, \underline{x})=u_{0}(\underline{x}) \quad \text { on } G,
\end{align*}
$$

with $a(\phi, \psi)=\int_{G} \partial_{x_{1}} \phi(\underline{x}) \partial_{x_{1}} \psi(\underline{x})+\partial_{x_{2}} \phi(\underline{x}) \partial_{x_{2}} \psi(\underline{x}) d \underline{x}$ for $\phi, \psi \in H_{0}^{1}(G)$ with $G \subset \mathbb{R}^{2}$ and $J=[0, T]$. Johnson [26] uses continuous, linear finite elements in space and the implicit Euler scheme in time. He imposes regularity conditions on $f$ and $u_{0}$ to ensure a unique solution and enforces a condition on the propagation of the free boundary through time, namely,

$$
\begin{equation*}
\sum_{m=0}^{M-1} \mu\left(D_{m}\right) \leq C \tag{3.8}
\end{equation*}
$$

where $C>0$ is a constant, $D_{m}:=\bigcup_{t \in\left(t_{m}, t_{m+1}\right)}\left(\mathcal{I}_{t_{m+1}} \cup \mathcal{I}_{t}\right) \backslash \overline{\left(\mathcal{I}_{t_{m+1}} \cap \mathcal{I}_{t}\right)}$, $\mathcal{I}_{t}:=\{\underline{x} \in G: u(t, \underline{x})>0\}$ is the inactive set at time $t \in J$ and $\mu$ is the Lebesgue measure on $\mathbb{R}^{2}$. Under these conditions Johnson [26] proves the following error estimate. First define,

$$
W^{2, \infty}(G):=\left\{v(\underline{x}) \in L^{\infty}(G): \partial_{x_{i}} v(\underline{x}), \partial_{x_{i} x_{j}} v(\underline{x}) \in L^{\infty}(G) \text { for } i, j=1,2\right\} .
$$

Theorem 3.12. Let $u(t, \underline{x})$ be a solution to (3.7) and $\left(u_{N}^{m}(\underline{x})\right)_{m=0, \ldots, M}$ be a solution to the fully discretised scheme using continuous, linear finite elements in space and the implicit Euler scheme in time, i.e. $\theta=1$. Further suppose that,

$$
\begin{equation*}
f(t, \underline{x}) \in C\left(J ; L^{\infty}(G)\right), \partial_{t} f(t, \underline{x}) \in L^{2}\left(J ; L^{\infty}(G)\right), u_{0}(\underline{x}) \in W^{2, \infty}(G) \cap \mathcal{K}(G) \tag{3.9}
\end{equation*}
$$

and that there exists a constant $C>0$ such that (3.8) holds then,

$$
\begin{aligned}
\max _{m=1, \ldots, M} & \left\|u\left(t_{m}, \underline{x}\right)-u_{N}^{m}(\underline{x})\right\|_{L^{2}(G)} \\
& +\left(\sum_{m=1}^{M} k\left\|u\left(t_{m}, \underline{x}\right)-u_{N}^{m}(\underline{x})\right\|_{H_{0}^{1}(G)}^{2}\right)^{\frac{1}{2}} \leq \tilde{C}\left\{\left(\ln \left(k^{-1}\right)\right)^{\frac{1}{4}} k^{\frac{3}{4}}+h\right\}
\end{aligned}
$$

for some constant $\tilde{C}>0$ independent of temporal mesh size $k>0$ and spatial mesh size $h>0$.

Proof. For the proof of the above result we refer to Johnson [26, Section 2].
Without the assumption (3.8) on the free boundary, Johnson [26] remarks that the above error can be bounded by $\tilde{C}\left(h+k^{\frac{1}{2}}\right)$. Vuik [50, Theorem 5.1] extends the result of Johnson [26] in Theorem 3.12 for general $\theta$-schemes where the constant $\tilde{C}$ in Theorem 3.12 now depends on $\theta$. However the error bound is of the same order in terms of $k$ and $h$, namely $\mathcal{O}\left(\left(\ln \left(\frac{T}{k}\right)\right)^{\frac{1}{4}} k^{\frac{3}{4}}+h\right)$. From the numerical experiments in Vuik [50, Chapter 6] one notices that the error estimates in Theorem 3.12 or Vuik 50, Theorem 5.16 ] are not optimal since the convergence rate of their experiment behaves like $\mathcal{O}\left(h^{2}+k^{\frac{3}{4}}\right)$ for the backward Euler scheme and $\mathcal{O}\left(h^{2}+k\right)$ for the Crank-Nicolson scheme. In Fetter [17, Section 3] an error estimate in the $L^{\infty}$-norm is proved. Fetter [17, Section 1] also employs continuous, linear finite elements in space and the implicit Euler scheme in time for the discretisation of (3.7). The result in Fetter [17, Section 1] is given in the next theorem.

Theorem 3.13. Let $u(t, \underline{x})$ be a solution to (3.7) and $\left(u_{N}^{m}(\underline{x})\right)_{m=0, \ldots, M}$ be a solution to the fully discretised scheme using linear, continuous finite elements in space and the implicit Euler scheme in time, i.e. $\theta=1$. Suppose that the conditions in (3.9) hold and additionally assume $\partial_{t t} u \in L^{2}\left(J ; L^{2}(G)\right)$. Moreover, let $c_{1}, c_{2}>0$ be constants such that no angle of each element of the spatial domain $G_{\tilde{C}}$ exceeds $\pi / 2-c_{1}$ and $k \geq c_{2} h$. Then for every $\epsilon>0$ there exists a constant $\tilde{C}>0$ such that,

$$
\max _{m=1, \ldots, M}\left\|u\left(t_{m}, \underline{x}\right)-u_{N}^{m}(\underline{x})\right\|_{L^{\infty}(G)} \leq \tilde{C} k^{-\epsilon} h^{-\epsilon}\left(h^{2}+k\right)
$$

Proof. The proof can be found in Fetter [17, Section 2 and 3].
A priori error estimates for parabolic variational inequalities with a linear operator of order $\underline{\alpha}=\left(\alpha_{1}, \alpha_{2}\right) \in(0,2)^{2}$ in two dimensions have to our knowledge not yet been established. However for the one-dimensional case, the parabolic variational inequality for the American option pricing problem with one underlying is treated in Matache et al. 36, there the a priori error estimates for a general one-dimensional exponential jump diffusion model are analysed. The error estimates contain not only the truncation and discretisation errors but also an error stemming from the matrix compression; see Matache et al. [36, Section 4.5]. For the discretisation continuous, linear finite elements is used in space and the finite difference and implicit Euler scheme in time. Let $u(t, x)$ be the exact solution to the variational inequality in Matache et al. [36, Section 3.4] and $\left(u_{N}^{m}\right)_{m=1, \ldots, M}$ the finite elements approximation such that $u \in C^{0}\left(J ; \widetilde{H}^{\rho}(G) \cap \widetilde{H}^{s}(G)\right)$ and $\partial_{t} u \in C^{\gamma}\left(J ; L^{2}(G)\right)$ with $G \subset \mathbb{R}, \rho<s \leq 2$, $\gamma \in(0,1]$ and

$$
\rho= \begin{cases}\frac{\alpha}{2}, & \text { if } \sigma=0 \\ 1, & \text { if } \sigma>0\end{cases}
$$

where $\alpha$ is the order of the linear operator and $\sigma$ is the volatility term; see Matache et al. [36, Assumtions in Section 2.1 and Section 3.1]. Then the following error estimate is established in Matache et al. [36, Section 4.6],

$$
\begin{aligned}
& \max _{m=1, \ldots, M}\left\|u\left(t_{m}, x\right)-u_{N}^{m}(x)\right\|_{L^{2}(G)} \\
& \quad+\left(\sum_{m=1}^{M} k\left\|u\left(t_{m}, x\right)-u_{N}^{m}(x)\right\|_{\widetilde{H}^{\rho}(G)}^{2}\right)^{\frac{1}{2}} \leq C\left(k^{\gamma}+\sigma h^{s-1}+h^{\min \left(\frac{s}{2}, s-\frac{\alpha}{2}\right)}\right),
\end{aligned}
$$

where $k$ and $h$ are the temporal and spatial mesh sizes and $C>0$ is a constant independent of $h$ and $k$. Now that we have analysed the a priori error estimates of various variational inequalities we proceed in the subsequent section with numerical convergence tests of the semi-smooth Newton method for elliptic and parabolic variational inequalities.

### 3.4 Numerical Experiments

In this section we discuss some numerical experiments for the convergence analysis of the semi-smooth Newton method. First, we treat the perpetual American option which is an elliptic problem as discussed in Section 3.3.1. For this problem we obtain optimal convergence rates. Second, we consider the one-dimensional tent problem where we investigate the convergence of the algorithm with respect to an obstacle that is not continuously differentiable in a specific point. In the case where this singularity is absorbed by the spatial grid we achieve an optimal convergence, however, if the singularity does not lie on the spatial grid the order of convergence in $H^{1}$ is significantly lower, namely $\mathcal{O}\left(N^{-\frac{1}{2}}\right)$. This problem is adopted from Moon, et al. [39, Section 5.2 and 5.3]. Third, we depict the convergence rates of the two-dimensional problem covered in Chapter 2 using the model in Section 1.3.1. Here the overkill method is employed, as a closed-form solution is not available. For this problem we analyse three different
cases. In the first case we modify the obstacle such that it is time-independent by removing the time-dependent terms $\left(r-w_{i}\right)(T-t)$ for $i=1,2$ from the model described in Section 1.3.1. Further we take $\tilde{g}(\underline{x}) \in V_{N}$. The second case deals with the American basket put option where the time-dependent obstacle $\tilde{g}(t, \underline{x})$ is not in $V_{N}$. In the third case, we analyse the effect of the violation of the smooth pasting condition discussed in Lamberton and Mikou [34] on the convergence of the American basket put option. Finally, we give an analysis of the speed of the semi-smooth Newton method by comparing the algorithm to the well-known PSOR method adopting the one-dimensional tent problem from Moon, et al. [39, Section 5.2 and 5.3]. For the numerical experiments in this chapter we use a $16 \times$ Quad-Core AMD Opteron ${ }^{\text {TM }}$ processor with MATLAB® $®$ version 7.14.0.739 (R2012a).

### 3.4.1 Elliptic Problem with Order Two Differential Operator

For the elliptic problem we consider the perpetual American put option. Under the Kou model, also known as the double exponential jump diffusion model, Kou and Wang [33, Theorem 3] proved that a closed-form solution can be derived. It is also shown that the value of the perpetual American option satisfies an elliptic PIDI in Kou and Wang [33, Lemma 5]. If $\left(S_{t}\right)_{t \geq 0}:=\left(e^{r t+X_{t}}\right)_{t \geq 0}$ is the underlying asset price process with $\left(X_{t}\right)_{t \geq 0}$ a one-dimensional Kou process as in Section 1.3.4 or Kou and Wang [33, Section 2.1] then the PIDI reads,

$$
\begin{aligned}
\mathcal{A} v(x)-r v(x) & \leq 0 \\
v(x) & \geq\left(K-e^{x}\right)_{+} \\
(\mathcal{A} v(x)-r v(x))\left(v(x)-\left(K-e^{x}\right)_{+}\right) & =0
\end{aligned}
$$

and $v(x)=\sup _{\tau} \mathrm{E}\left[e^{-r \tau}\left(K-e^{X_{\tau}}\right)_{+}\right]$where the supremum is taken over all stopping times $\tau \in[0, \infty)$; see Kou and Wang [33, Section 4.4].

For the discretisation we use finite elements (FE) with piecewise linear basis functions, i.e. the hat functions as in Figure 2.1 on a bounded domain $G=(-R, R)$. We consider a range of levels $L$ for the discretisation to test the convergence. This means the number of hat functions is given by $N=2^{L}-1$ where $L \in\{7, \ldots, 10\}$. This implies that we have $N+2=2^{L}+1$ grid points and that $h=\frac{2 R}{N+1}$ is the mesh size. The parameters for the convergence test are set equal to,
$\sigma=.15, \lambda=.5, p=.35, \eta_{+}=\eta_{-}=5, r=.05, K=1, R=7$ and $L \in\{5, \ldots, 11\}$.
The convergence results and the finite elements approximation of the perpetual American put option value are depicted in Figure 3.2. Notice that we procure a convergence rate of order $\mathcal{O}\left(N^{-2}\right)$ in the $L^{2}(I)$ and the $L^{\infty}(I)$ norm where $I:=\left\{s \in \mathbb{R}_{+}:|s-K|<.85 K\right\}$. Since $h=\frac{2 R}{N+1}$ we observe that this convergence is optimal due to Falk [16, Lemma 3] stated in Section 3.3.1.


Figure 3.2: Value (upper) and convergence (lower) of a perpetual American put option under the Kou model with parameters given in 3.10.

### 3.4.2 Parabolic Problem with Order Two Differential Operator

In this section we analyse the convergence of the semi-smooth Newton algorithm for a parabolic problem. We consider the one-dimensional tent problem in Moon et al. [39, Section 5.2 and 5.3]. The system of inequalities is given by,

$$
\begin{align*}
(\mathcal{B} w)(t, x):=\partial_{t} w(t, x)-\partial_{x x} w(t, x) & \geq f(t, x), & & \text { for }(t, x) \in J \times \mathbb{R},  \tag{3.11a}\\
w(t, x) & \geq \tilde{g}(x), & & \text { for }(t, x) \in J \times \mathbb{R},  \tag{3.11b}\\
(\mathcal{B} w)(t, x)(w(t, x)-\tilde{g}(x)) & =0, & & \text { for }(t, x) \in J \times \mathbb{R},  \tag{3.11c}\\
w(.5, x) & =w_{0}(x), & & \text { for } x \in \mathbb{R}, \tag{3.11d}
\end{align*}
$$

where $J:=[.5,1], \tilde{g}(x):=1-3|x|$ and the forcing function $f$ is given by,

$$
f(t, x)= \begin{cases}-72 t^{-2}, & \text { if } x \in \mathcal{A}_{t} \\ -12 t^{-2}\left(6 t^{-1} x^{2}-|x|+6\right), & \text { if } x \in \mathcal{I}_{t}\end{cases}
$$

where $\mathcal{A}_{t}$ and $\mathcal{I}_{t}$ denote the active and inactive set, respectively. The exact solution to (3.11) is,

$$
w(t, x)= \begin{cases}1-3|x|, & \text { if } x \in \mathcal{A}_{t} \\ 36 t^{-2} x^{2}-\left(3+12 t^{-1}\right)|x|+2, & \text { if } x \in \mathcal{I}_{t}\end{cases}
$$

where the exact active and inactive sets are given by $\mathcal{A}_{t}=\left\{x \in G:|x| \leq \frac{t}{6}\right\}$ and $\mathcal{I}_{t}=\left\{x \in G:|x|>\frac{t}{6}\right\}$, respectively. From this we can inquire the initial condition $w_{0}(x)$ at time $t=.5$,

$$
w_{0}(x)= \begin{cases}1-3|x|, & \text { if } x \in\left[-\frac{1}{12}, \frac{1}{12}\right] \\ 144 x^{2}-27|x|+2, & \text { otherwise }\end{cases}
$$

For the numerical computation we discretise (3.11) using finite elements in space with piecewise linear hat functions as basis for $V_{N}=\operatorname{span}\left\{b_{i}(x): i=1, \ldots, N\right\}$ where $b_{i}(x)$ is defined in Section 2.4.1 and the $\theta$-scheme in time. As in Section 3.4.1, we consider a range of levels $L$ for the discretisation to test the convergence. Moreover, we take $k=\mathcal{O}(h)$ where $k=\frac{T}{M}$ is the mesh size in time. For the finite element method we use the following parameters,

$$
\begin{equation*}
\theta=.5, R=7 \text { and } L \in\{9, \ldots, 12\} \tag{3.12}
\end{equation*}
$$

We investigate the convergence in two scenarios. One where the obstacle $\tilde{g}(x)$ is not continuously differentiable in one particular point on the spatial grid, meaning $\tilde{g}(x)=\tilde{g}_{h}(x) \in V_{N}$ and where this point is not absorbed by the spatial grid, meaning $\tilde{g}_{h}(x) \neq \tilde{g}(x)$ or $\tilde{g}(x) \notin V_{N}$. The function $\tilde{g}_{h}(x)$ is the projection of $\tilde{g}(x)$ to the finite element space $V_{N}$.

## One-dimensional Tent Problem with $\tilde{g}(x) \in V_{N}$

Here the point $x=0$, where the tent function $\tilde{g}(x)=1-3|x|$ is not continuously differentiable, is part of the partition of the spatial domain $G=(-R, R)$. The domain $G$ is symmetric around the singularity and we have $N+2=2^{L}+1$ grid points, placing $x=0$ on the grid at $x_{\frac{N+1}{2}}=0$. In this case, we procure an optimal convergence rate of order $\mathcal{O}\left(N^{-2}\right)$ as seen in the middle two plots in Figure 3.3. In this case the area of interest $I$ is defined as $I:=\{x \in \mathbb{R}:|x| \leq 1\}$. Moreover, we show the finite elements and the exact solution with the initial condition and the tent function $\tilde{g}(x)$ in the upper left plot of Figure 3.3. On the top right, we observe the free boundary plot, which in this case is twosided. The lower plots are the log-log plots of the $L^{2}(J)$ - and $L^{\infty}(J)$-errors of the free boundary on the set $J=[.5,1]$. Notice that the convergence of the free boundary is in both norms of order $\mathcal{O}\left(N^{-1}\right)$. In Moon et al. 39, Section 5.2 and 5.3], the optimal convergence of order $\mathcal{O}\left(N^{-1}\right)$ for the implicit Euler scheme is obtained using the PSOR algorithm. Moon et al. 39] separate the space error in three different parts and find that two error estimators show super convergence, i.e. convergence rates larger than optimal, and one estimator exhibits an optimal convergence rate. This kind of detailed error analysis for the semi-smooth Newton algorithm remains a topic of further research.

## One-dimensional Tent Problem with $\tilde{g}(x) \notin V_{N}$

As in Moon, et al. [39, Section 5.3], we shift the problem to achieve $\tilde{g}(x) \neq \tilde{g}_{h}(x)$ and do not change the partition of the spatial domain $G=(-R, R)$. Therefore, we define $v(t, x):=w\left(t, x-\frac{1}{3}\right)$ and $\tilde{g}_{v}(x):=\tilde{g}\left(x-\frac{1}{3}\right)$ for $x \in G$ which places the singularity at $x=\frac{1}{3}$. Moreover, the exact active and inactive sets are described by $\mathcal{A}_{t}=\left\{x \in G:\left|x-\frac{1}{3}\right| \leq \frac{t}{6}\right\}$ and $\mathcal{I}_{t}=\left\{x \in G:\left|x-\frac{1}{3}\right|>\frac{t}{6}\right\}$, respectively, and $I:=\left\{x \in \mathbb{R}:\left|x-\frac{1}{3}\right| \leq 1\right\}$. In Figure 3.4 we observe that the convergence rate in $H^{1}(I)$ has drastically decreased. In the $H^{1}(I)$-norm we only procure a convergence of order $\mathcal{O}\left(N^{-\frac{1}{2}}\right)$. The other rates of convergence remain identical to the results in the previous experiment. Also the convergence rates of the free boundary remains of order $\mathcal{O}\left(N^{-1}\right)$ as seen in the two lower plots of Figure 3.4 .


Figure 3.3: Convergence of the semi-smooth Newton algorithm for the one-dimensional tent problem with $\tilde{g}(x) \in V_{N}$. The upper left plot shows the finite element solution and the exact solution. The upper right plot shows the corresponding free boundaries. The middle plots are the convergence plots of the solution for different norms and the lower plots are the convergence plots of the free boundaries. The parameters are given in 3.12 .

### 3.4.3 Parabolic Problem with Order $\underline{\alpha}$ Integro-differential Operator

In this section, we discuss the numerical convergence of the finite element solution of the parabolic variational problem in 2.21. For the Lévy process $X$ in (1.2) we employ a two-dimensional GGMY process with independent components. Therefore, the integro-differential operator in $\sqrt{2.2}$ then has order $\underline{\alpha}:=\left(\alpha_{1}, \alpha_{2}\right)^{\top}$, where $\alpha_{j}$ are the parameters for the LPs $X^{j}$ with $j=1,2$. We do not have an exact solution to the PIDI in (2.11), consequently we exploit the overkill method to proceed with the numerical convergence tests. For the overkill solution we set the total number of inner grid points equal to $\tilde{N}=\left(2^{L}-1\right)^{2}$ with level $L=10$. Since we are now dealing with a multidimensional problem, the discussion about the 'curse of dimensionality' arises. The optimal convergence for the scheme proposed in Section 2.4 is $\mathcal{O}\left(\tilde{N}^{-\frac{2}{d}}\right)$ with $d=2$ dimensions and $\tilde{N}=N^{2}$ since the degrees of freedom for the equidistant mesh size $h$ in dimension $d$ grows with rate $\mathcal{O}\left(h^{-d}\right)$ as $h \rightarrow 0$. The 'curse of dimensionality' can be avoided by using a spline wavelet basis for the finite elements discretisation, however this is beyond the scope of this paper and we refer to Hilber 22,


Figure 3.4: Convergence of the semi-smooth Newton algorithm for the one-dimensional tent problem with $\tilde{g}(x) \notin V_{N}$. The upper left plot displays the finite element solution and the exact solution. The upper right plot shows the corresponding free boundaries. The middle plots are the convergence plots of the solution for different norms and the lower plots are the convergence plots of the free boundaries. The parameters are given in 3.12.

Section 4.2]. Although the solution $u(t, \underline{x})$ in this case only belongs to $H \underline{\underline{\rho}}$, we do estimate the $H^{1}$-error of the finite element solution to say something about the derivative of $u(t, \underline{x})$. As in Section 3.4 .2 , we first establish the numerical convergence results when $\tilde{g} \in V_{N}$ and then for the case $\tilde{g} \notin V_{N}$, where $V_{N}$ is defined in 2.27). In both types of problems we observe the 'curse of dimensionality' whereas in the latter case the convergence rate in $H^{1}$ is additionally reduced by the fact that the singularities are not absorbed by the spatial grid. Finally, we briefly examine the smooth pasting condition given in Lamberton and Mikou [34] and report on the convergence rates under the violation of the smooth pasting property in Lamberton and Mikou [34, Theorem 4.2] for the model 1.2 .

## Problem with $\tilde{g}(\underline{x}) \in V_{N}$

In order to achieve that $\tilde{g}_{h}(\underline{x})=\tilde{g}(\underline{x})$ we remove the time-dependent terms $\left(r-w_{i}\right)(T-t)$ for $i=1,2$ from the model in 1.2 and use the identity matrix $\mathbf{I}_{2}$ for $\boldsymbol{\Sigma}$. The consequence of this is that the discounted stock price processes
$e^{-r t} S_{t}^{i}$ for $i=1,2$ do not fulfil the martingale property any longer, however, for the numerical experiment this is not necessary. We use the following expression for $\tilde{g}(x)$,

$$
\begin{equation*}
\tilde{g}(\underline{x})=\left(K-e^{\sum_{j=1}^{2} \boldsymbol{\Sigma}_{1 j} x_{j}}\right)_{+}, \tag{3.13}
\end{equation*}
$$

where $\underline{x}=\left(x_{1}, x_{2}\right)^{\top}$ and set the parameters equal to,

$$
\begin{align*}
& c_{1}=0.8, \beta_{-}^{1}=4, \beta_{+}^{1}=5, \alpha_{1}=1.1, c_{2}=1.2, \beta_{-}^{2}=5, \beta_{+}^{2}=8, \alpha_{2}=1.1 \\
& T=1, K=1, a=.5, b=.5, r=.03, R_{-}=R_{+}=7, \theta=.5, L \in\{4, \ldots, 8\} \text { and } \\
& \boldsymbol{\Sigma}=\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \tag{3.14}
\end{align*}
$$

Additionally, we define the set $I$ as follows where $\underline{x}=\left(x_{1}, x_{2}\right)^{\top}$,

$$
I:=\left\{\underline{s}=\left(e^{x_{1}}, e^{x_{2}}\right)^{\top} \in \mathbb{R}_{+}^{2} \mid \ln (0.25) \leq x_{1} \leq \ln (2.75) \text { and } \ln (0.6) \leq x_{2} \leq \ln (2.4)\right\}
$$

In Figure 3.5. we observe that the optimal convergence rate of $\mathcal{O}\left(\tilde{N}^{-\frac{2}{d}}\right)=\mathcal{O}\left(\tilde{N}^{-1}\right)$ for $d=2$ in $L^{2}(I)$ and $L^{\infty}(I)$. The convergence in $\tilde{H}^{1}(I)$ is slightly below the expected convergence rate of order $\mathcal{O}\left(\tilde{N}^{-1}\right)$. The reason for this might be the low regularity of the solution $u(t, \underline{x}) \in \tilde{H}^{\underline{\rho}}(G)$. In Figure 3.6 , we exhibit the number


Figure 3.5: Convergence of the semi-smooth Newton algorithm for the two-dimensional parabolic variational problem using the overkill method for time-independent obstacle $\tilde{g}(\underline{x})$ in 3.13 where all singularities of $\tilde{g}(\underline{x})$ are absorbed by the space grid. The parameters are given in (3.14).
of iterations for level $L=5, \ldots, 8$. We observe that for each level the number of iterations is larger for the first time steps. This can perhaps be avoided by properly graded time meshes, however, this is not further investigated here.

Problem with $\tilde{g}(t, \underline{x}) \notin V_{N}$
Here, we audit the convergence of the American basket put option with payoff $g(\underline{s})=\left(K-\left(a s_{1}+b s_{2}\right)\right)_{+}$where $a, b \in[0,1]$ such that $a+b=1$. The parabolic


Figure 3.6: Number of iterations for the GMRES solver on active (Aset) and inactive (Iset) sets and the number of while iterations for the semi-smooth Newton algorithm for the finite element solution of the problem in 2.38 with time-independent obstacle $\tilde{g}(\underline{x})$ in (3.13). The maximum while iteration is set to 50 . The parameters are given in (3.14) with $\boldsymbol{\Sigma}=\mathbf{I}_{2}$.

PIDI for this problem is treated in Chapter 2. By 2.10b the time-dependent obstacle is given by,

$$
\begin{align*}
& \tilde{g}(t, \underline{x})= \\
& e^{-r(T-t)}\left(K-\left(a e^{\left(r+w_{1}\right)(T-t)+\sum_{j=1}^{2} \boldsymbol{\Sigma}_{1 j} x_{j}}+b e^{\left(r+w_{2}\right)(T-t)+\sum_{j=1}^{2} \boldsymbol{\Sigma}_{2 j} x_{j}}\right)\right)_{+}, \tag{3.15}
\end{align*}
$$

and for this experiment we specify the following parameters,
$c_{1}=0.8, \beta_{-}^{1}=4, \beta_{+}^{1}=5, \alpha_{1}=1.1, c_{2}=1.2, \beta_{-}^{2}=5, \beta_{+}^{2}=8, \alpha_{2}=1.1$,
$T=1, K=1, a=.5, b=.5, r=.03, R_{-}=7, R_{+}=2, \theta=.5, L \in\{4, \ldots, 8\}$ and
$\boldsymbol{\Sigma}=\left(\begin{array}{cc}1 & 0.048 \\ 0.048 & 1\end{array}\right)$.
Moreover we define the area of interest $I$ as,
$I:=\left\{s(0, \underline{x}) \in \mathbb{R}_{+}^{2} \mid 0.3 \leq s_{1}\left(0,\left(x_{1},-R\right)^{\top}\right), s_{2}\left(0,\left(-R, x_{2}\right)^{\top}\right) \leq 2.7\right.$ with $\left.\underline{x} \in G\right\}$,
where $s_{i}(0, \underline{x}):=\exp \left(\sum_{j=1}^{2} \boldsymbol{\Sigma}_{i j} x_{j}\right)$ for $i=1,2$. In this case we are dealing not only with the 'curse of dimension' but also with the fact that the obstacle is timedependent and hence the spatial grid does not incorporate the non-continuously differentiable points of $\tilde{g}(t, \underline{x})$ for all $t \in J$. From the experiments in the


Figure 3.7: Convergence of the semi-smooth Newton algorithm for the American basket put option value using the overkill method. The parameters are given in 3.16.
previous section we expect a convergence of order $\mathcal{O}\left(\tilde{N}^{-1}\right)$ in $L^{2}(I)$ and $L^{\infty}(I)$ and in $H^{1}(I)$ a convergence of order less than one in terms of $\tilde{N}$. In Figure 3.7. we observe the convergence rates. In $L^{2}(I)$ the convergence is indeed of order $\mathcal{O}\left(\tilde{N}^{-1}\right)$, however, in $H^{1}(I)$ as well as in $L^{\infty}(I)$ we observe an order of $\mathcal{O}\left(\tilde{N}^{-\frac{1}{2}}\right)$. In Figure 3.8 we exhibit the number of iterations the semi-smooth Newton algorithm uses to converge for the levels $L=5, \ldots, 8$. We observe that the GMRES function constantly needs more iterations on the inactive set solving for $u(t, \underline{x})$ than on the active set solving for the Lagrangian multiplier $\lambda(t, \underline{x})$. This is a result of the nice characteristic of the matrix $\mathbf{D}$ explained in Lemma 2.9

Plots for the value and the free boundary of the American basket put option can be found in the last chapter, Chapter 4


Figure 3.8: Number of iterations for the GMRES solver on active and inactive sets and the number of while iterations for the semi-smooth Newton algorithm for the finite element solution of the American basket option. The maximum while iteration is set to 50 . The parameters are given in (3.16).

## Convergence under Marginal Violation of the Smooth Pasting Conditions

In a one-dimensional setting the smooth pasting property of the American option price problem is the continuous differentiability of the option price with respect to the spot price $s$ at the free boundary $s^{*}(t)$, i.e. if $V(t, s)$ is the value of an American option with a single underlying and free boundary $s^{*}(t)$, the smooth pasting property is equivalent to,

$$
\begin{equation*}
\lim _{s \uparrow s^{*}(t)} \partial_{s} V(t, s)=\lim _{s \downarrow s^{*}(t)} \partial_{s} V(t, s), \quad \forall t \in J=[0, T], \tag{3.17}
\end{equation*}
$$

where $s \in \mathbb{R}_{+}$. Sufficient conditions such that the smooth pasting property fails for a plain vanilla American put option in a one-dimensional exponential Lévy model can be found in Lamberton and Mikou [34, Theorem 4.2]. We state it here as a proposition.
Proposition 3.14. Let $X=\left(X_{t}\right)_{t \geq 0}$ be a one-dimensional Lévy process with Lévy measure $\nu_{X}(d z)$. If $d_{+}:=r-\int_{\mathbb{R}}\left(e^{z}-1\right)_{+} \nu_{X}(d z) \geq 0$ and $X$ is of finite variation then the value of the plain vanilla American put option $V(t, s)$ does not satisfy the smooth pasting property in 3.17).

Proof. See Lamberton and Mikou [34, Theorem 4.2].
We restate the above proposition for a CGMY process in the following corollary.
Corollary 3.15. Let $X=\left(X_{t}\right)_{t \geq 0}$ be a one-dimensional CGMY process with parameters $c, \beta_{-}>0, \beta_{+}>1$ and $\alpha \in(0,2)$ as in Section 1.3.3. If for $r \in \mathbb{R}_{+}$ such that $r-\frac{c}{\alpha}\left(\beta_{+}^{\alpha}-\left(\beta_{+}-1\right)^{\alpha}\right) \Gamma(1-\alpha) \geq 0$ and $\alpha \in(0,1)$ then the smooth pasting property for the plain vanilla American put is violated.
Proof. Let $\nu_{X}^{\mathrm{CGMY}}(d z)$ be the Lévy measure of a one-dimensional LP as in 1.10 then,

$$
\begin{aligned}
\int_{\mathbb{R}}\left(e^{z}-1\right)_{+} \nu_{X}(d z)= & c \int_{0}^{\infty} \frac{e^{-\left(\beta_{+}-1\right) z}-e^{-\beta_{+} z}}{z^{1+\alpha}} d z \\
= & c\left[\frac{e^{-\left(\beta_{+}-1\right) z}-e^{-\beta_{+} z}}{-\alpha z^{\alpha}}\right]_{0}^{\infty}-\frac{c}{\alpha}\left(\beta_{+}-1\right) \int_{0}^{\infty} \frac{e^{-\left(\beta_{+}-1\right) z}}{z^{\alpha}} d z \\
& +\frac{c}{\alpha} \beta_{+} \int_{0}^{\infty} \frac{e^{-\beta_{+} z}}{z^{\alpha}} d z \\
= & -\frac{c}{\alpha}\left(\beta_{+}-1\right)^{\alpha} \int_{0}^{\infty} \frac{e^{-z}}{z^{\alpha}} d z+\frac{c}{\alpha} \beta_{+}^{\alpha} \int_{0}^{\infty} \frac{e^{-z}}{z^{\alpha}} d z \\
= & \frac{c}{\alpha}\left(\beta_{+}^{\alpha}-\left(\beta_{+}-1\right)^{\alpha}\right) \Gamma(1-\alpha),
\end{aligned}
$$

and $d_{+}=r-\frac{c}{\alpha}\left(\beta_{+}^{\alpha}-\left(\beta_{+}-1\right)^{\alpha}\right) \Gamma(1-\alpha)$. From Table 1.1. we know that $X$ is of finite variation if $\alpha \in(0,1)$. Consequently, the corollary follows from Proposition 3.14

In Figure 3.9. we show for which parameter values $\beta_{+}, \alpha$, the smooth pasting property fails in case of a plain vanilla American put option according to Corollary 3.15. The parameter values above each surface imply a violation of the smooth pasting property. To our knowledge, sufficient conditions for a twodimensional Lévy model have not been established yet. Here we will examine numerically the convergence of the American basket option when each component of the process $X=\left(X^{1}, X^{2}\right)^{\top}$ in 1.2 fulfils the conditions in Proposition 3.14. For this we use the following parameters for the CGMY process $X$,

$$
\begin{align*}
& c_{1}=.5, \beta_{-}^{1}=10, \beta_{+}^{1}=12, \alpha_{1}=.4, c_{2}=.25, \beta_{-}^{2}=8, \beta_{+}^{2}=10, \alpha_{2}=.3 \\
& T=1, K=1, a=.5, b=.5, r=.2, R_{-}=7, R_{+}=2, \theta=.5, L \in\{4, \ldots, 8\} \text { and } \\
& \boldsymbol{\Sigma}=\left(\begin{array}{cc}
1 & 0.048 \\
0.048 & 1
\end{array}\right) . \tag{3.18}
\end{align*}
$$



Figure 3.9: Parameter values of $\beta_{+}$and $\alpha$ for which the CGMY process does not satisfy the smooth pasting conditions in Corollary 3.15 for $c=1$ (upper) and $c=7$ (lower).

Moreover we define the set $I$ as follows,
$I:=\left\{s(0, \underline{x}) \in \mathbb{R}_{+}^{2} \mid 0.3 \leq s_{1}\left(0,\left(x_{1},-R\right)^{\top}\right), s_{2}\left(0,\left(-R, x_{2}\right)^{\top}\right) \leq 2.7\right.$ with $\left.\underline{x} \in G\right\}$, where $s_{i}(0, \underline{x})=\exp \left(\sum_{j=1}^{2} \boldsymbol{\Sigma}_{i j} x_{j}\right)$ for $i=1,2$.


Figure 3.10: Convergence of the semi-smooth Newton algorithm for the American basket put option value using the overkill method where the components of the process $X$ fulfil the conditions in Corollary 3.15. The parameters are given in 3.18.

In Figure 3.10 we depict the numerical convergence rate for the American basket put option as in the previous subsection, however the parameters of the model (1.2) and the CGMY process $X$ are now adjusted such that the smooth pasting property fails. We notice that the finite element solution does not converge using the overkill method. Regarding the number of iterations of the semismooth Newton algorithm, we observe that the number of iterations of the GMRES solver for $u$ on the inactive set are substantially larger than in the above experiments. This is depicted in Figure 3.11.

### 3.4.4 Speed of the Semi-smooth Newton Algorithm

In this section we want to delineate the difference in speed between the semismooth Newton method and the well-established PSOR algorithm to solve LCPs. For this we again adopt the one-dimensional tent problem analysed in Section 3.4.2 such that $\tilde{g}(x)=1-3|x| \in V_{N}$ with $x \in G=(-R, R)$ is the time-independent obstacle. We use the following parameters in this section,

$$
\begin{equation*}
R=7, \theta=.5, L \in\{6, \ldots, 11\} \tag{3.19}
\end{equation*}
$$

In terms of convergence, the PSOR algorithm performs almost identically to the semi-smooth Newton, i.e. in all the numerical examples above the convergence rates of both algorithms are indistinguishable. For the one-dimensional tent problem this is depicted in Figure 3.12. In the first two plots of Figure 3.12 the


Figure 3.11: Number of iterations for the GMRES solver on active and inactive sets and the number of while iterations for the semi-smooth Newton algorithm for the finite element solution of the American basket option where the components of the process $X$ fulfil the conditions in Corollary 3.15. The parameters are given in (3.18).
$L^{\infty}(I)$ - and $L^{2}(I)$-error of both algorithms are identical. The set $I$ is defined as $I=\{x \in G:|x| \leq 1\}$. Moreover, the absolute value of the difference of the $L^{\infty}(I)$ - and $L^{2}(I)$-error is displayed in the last plot of Figure 3.12 . We see that these differences decay with increasing number of inner grid points $N$ and that it is less than $8 \times 10^{-5}$ for all discretisation levels $L$ in (3.19). In Figure 3.13. we show the speed of both algorithms measured in seconds. In this first plot of Figure 3.13 , we depict the MATLAB® implementation of both algorithms and observe that the semi-smooth Newton algorithm is much faster than the PSOR algorithm obtaining the same accuracy as seen in Figure 3.12. The semi-smooth Newton algorithm stays under 15 seconds for $L \leq 10$ whereas the MATLAB® $®$


Figure 3.12: Convergence of the semi-smooth Newton and the PSOR algorithm solving the one-dimensional tent problem with time-independent obstacle $\tilde{g}(x)=1-3|x| \in V_{N}$ as in Section (3.4.2).
implementation of PSOR takes almost 200 seconds for level $L=9$ and 2000 seconds for $L=10$. However, if we use the mex-file implementation of the PSOR algorithm we observe that the PSOR algorithm is faster than the semi-smooth Newton method for the one-dimensional tent problem as we can see in the second plot of Figure 3.13. Here the PSOR algorithm stays below 5 seconds for all discretisation levels in (3.19). Although, this is not a fair comparison since one can also create a mex-file implement for the semi-smooth Newton method. This, however, if beyond the scope of this paper and subject to further research.

In this chapter, we have discussed the semi-smooth Newton algorithm to solve the American option pricing problem. We have analysed the uniqueness and convergence of the solution for the LCP in (3.1) using the semi-smooth



Figure 3.13: Time in seconds of the semi-smooth Newton and the PSOR algorithm solving the one-dimensional tent problem with time-independent obstacle $\tilde{g}(x)=1-3|x| \in V_{N}$ as in Section 3.4.2.

Newton algorithm. Thereafter, we have applied this to the case of the American option pricing problem covered in Chapter 2. We found that there exists a unique solution and that we have local convergence, that is if for each time step $m=0, \ldots, M-1$, the initial guess $u_{N}\left(t_{m}, \underline{x}\right)$ is close enough to the unique solution $u\left(t_{m+1}, \underline{x}\right)$ then the semi-smooth Newton method converges. Additionally, we have laid out conditions for global convergence of the semi-smooth Newton algorithm. However, further analysis of these conditions is beyond the scope of this paper. Also we have treated various numerical experiments to assess the convergence of the semi-smooth Newton algorithm numerically. In the next chapter we give numerical examples for the pricing scheme developed in Chapter 2.

## Chapter 4

## Numerical Examples and Conclusion

In this chapter, we present some numerical examples of American options with two underlying assets using the bivariate Lévy model for dependent assets in (1.2). In particular, we cover the American basket put option and the best-of American put option. For the American basket option we use the model in 1.2 with an underlying Kou process $X$ and in the case of the best-of option the process $X$ follows a CGMY process. Additional to approximating their option value and free boundary, we calculate their argument Greeks, i.e. $\Theta, \Delta$ and $\Gamma$. We apply the finite difference method for the computation of these Greeks. This method is discussed in Appendix D. In this chapter, we perform all the numerical computations on 2.7 GHz Intel Core ${ }^{\mathrm{TM}}$ i7 with 8 GB 1333 MHz DDR3 and MATLAB® ${ }^{\text {® }}$ version 7.12.0.635 (R2011a).

### 4.1 American Basket Put Option

As already mentioned above the payoff of the American basket put option is given by $g(\underline{s})=\left(K-\left(a s_{1}+b s_{2}\right)\right)_{+}$where $K \in \mathbb{R}_{+}$is the strike and $a, b \in[0,1]$ with $a+b=1$ are the weights for each asset and $\underline{s}=\left(s_{1}, s_{2}\right)^{\top}$. The expression of the time-dependent obstacle can be found in (3.15). For the numerical example in this section, the process $X$ is a Kou process and we use the following parameters,

$$
\begin{array}{r}
\sigma_{1}=.45, \lambda_{1}=.75, p_{1}=.35, \eta_{+}^{1}=2.9, \eta_{-}^{1}=2.6, \\
\sigma_{2}=.25, \lambda_{2}=.5, p_{2}=.25, \eta_{+}^{2}=2.6, \eta_{-}^{2}=2.2, \\
T=1, K=1, a=.75, b=.25, r=0.03, R=5, \theta=.5, L=7,  \tag{4.1}\\
\text { and } \boldsymbol{\Sigma}=\left(\begin{array}{cc}
.8 & .0675 \\
.0675 & .65
\end{array}\right) .
\end{array}
$$

### 4.1.1 Price and Free Boundary

In this section, we present the value and free boundary of an American basket put option for two dependent assets. In Figure 4.1 we show the values of the


Figure 4.1: Value of an American basket put option. The parameters are given in 4.1.
option at time $t=0$. We observe that the free boundary in Figure 4.1 is downward sloping in terms of the option price from $s_{2} \approx 0$ to $s_{2} \approx 3$. This is the result of the low volatility term $\sigma_{2}$ and the low fraction of $X^{2}$ represented in the matrix $\Sigma$. From the off-diagonal entries in the matrix $\Sigma$ in 4.1, we identify the positive correlation among the assets and this gives us the oval ordering of the grid points, see 1.2 . In the subsequent example we consider negative correlation among the assets which results in a different shape. In Figure 4.2, we show the entire free boundary of the American basket options. We display the free boundary in three dimensions as a 'hill-shaped' boundary between the exercise and continuation region. The z-axis represents the time-to-maturity and the space underneath the boundary is the exercise region. We observe that the range of $s_{2}$ is much wider than that of $s_{1}$. This is the result of the high weight of $s_{1}$ in our portfolio, or 'basket'.

In the next two tables, we compare the price of the American basket option at time $t=0$ solved by two algorithms; the PSOR and the semi-smooth Newton algorithm. In Table 4.1 we give values of the option price at specific points $s$ where the PSOR algorithm is employed to solve the LCP for each time step. Due to the transformation of the grid from $x$ to $s$, the values of $s_{1}$ and $s_{2}$ in the first row and column of Table 4.1 are not on the spatial grid. That is, there do not exist indices $i_{1}, i_{2} \in\{1, \ldots, N\}$ such that the grid points $\underline{s}_{i_{1}, i_{2}}=\left(s_{i_{1}}, s_{i_{2}}\right)^{\top}$ are equal to the values $\left(\tilde{s}_{1}, \tilde{s}_{2}\right)$ where $\tilde{s}_{i} \in\{0, .5,1,1.5,2,2.5,3,3.5\}$ for $i=1,2$. For this reason we take the value of the option at the closed grid point in the Euclidean norm. That is,

$$
\left\|\binom{s_{1}}{s_{2}}-\binom{\tilde{s}_{1}}{\tilde{s}_{2}}\right\|_{l^{2}}=\inf _{s_{i_{1}}, s_{i_{2}}} \text { for } 1 \leq i_{1}, i_{2} \leq N ~\left\|\binom{s_{i_{1}}}{s_{i_{2}}}-\binom{\tilde{s}_{1}}{\tilde{s}_{2}}\right\|_{l^{2}}
$$

where $\left(s_{i_{1}}, s_{i_{2}}\right)$ for $1 \leq i_{1}, i_{2} \leq N$ are points on the spatial grid. The exact values of $\underline{s}=\left(s_{1}, s_{2}\right)^{\top}$ at the grid points is depicted underneath each option value in


Figure 4.2: Free boundary for an American put basket option. The parameters are given in 4.1.

| $\tilde{s}_{1} \backslash \tilde{s}_{2}$ | 0 | .5 | 1 | 1.5 | 2 | 2.5 | 3 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | .943 | .788 | .665 | .529 | .362 | .264 | .168 |
|  | $(.055, .060)$ | $(.107, .526)$ | $(.095,1.055)$ | $(.100,1.584)$ | $(.149,2.107)$ | $(.105,2.632)$ | $(.084,3.157)$ |
| .5 | .600 | .486 | .313 | .193 | .143 | .058 | .047 |
|  | $(.504, .089)$ | $(.505, .543)$ | $(.576,1.056)$ | $(.601,1.585)$ | $(.514,2.115)$ | $(.633,2.633)$ | $(.504,3.159)$ |
| 1 | .275 | .190 | .121 | .069 | .040 | .016 | .011 |
|  | $(1.007, .099)$ | $(1.010, .605)$ | $(1.011,1.107)$ | $(1.049,1.579)$ | $(1.017,2.131)$ | $(1.177,2.638)$ | $(1.061,3.199)$ |
| 1.5 | .115 | .077 | .043 | .025 | .012 | .008 | .004 |
|  | $(1.513, .138)$ | $(1.539, .539)$ | $(1.557,1.092)$ | $(1.526,1.630)$ | $(1.668,2.112)$ | $(1.511,2.694)$ | $(1.739,3.171)$ |
| 2 | .054 | .032 | .018 | .012 | .007 | .003 | .002 |
|  | $(2.014, .110)$ | $(2.019, .675)$ | $(2.020,1.234)$ | $(2.074,1.591)$ | $(2.011,2.146)$ | $(2.187,2.643)$ | $(2.098,3.222)$ |
| 2.5 | .029 | .020 | .011 | .006 | .004 | .002 | .001 |
|  | $(2.519, .087)$ | $(2.555, .534)$ | $(2.554,1.082)$ | $(2.515,1.700)$ | $(2.583,2.2192)$ | $(2.638,2.686)$ | $(2.531,3.273)$ |
| 3 | .017 | .012 | .007 | .004 | .002 | .001 | .001 |
|  | $(3.022, .084)$ | $(3.046, .543)$ | $(3.080,1.100)$ | $(3.018,1.642)$ | $(3.099,2.117)$ | $(3.182,2.728)$ | $(3.037,3.161)$ |
| 3.5 | .009 | .007 | .004 | .003 | .002 | .001 | .001 |
|  | $(3.522, .367)$ | $(3.525, .672)$ | $(3.527,1.230)$ | $(3.622,1.586)$ | $(3.530,2.250)$ | $(3.605,2.757)$ | $(3.663,3.211)$ |

Table 4.1: Values of the American Basket put option using the PSOR algorithm at specified grid points. The free boundary lies between the red coloured values and the black coloured values of the option. The parameters are given in 4.1. CPU time: 53.79 seconds.

| $\tilde{s}_{1} \backslash \tilde{s}_{2}$ | 0 | .5 | 1 | 1.5 | 2 | 2.5 | 3 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | .943 | .788 | .665 | .529 | .362 | .264 | .168 |
|  | $(.055,0.060)$ | $(.107,0.526)$ | $(.095,1.055)$ | $(.100,1.584)$ | $(.149,2.107)$ | $(.105,2.632)$ | $(.084,3.157)$ |
| .5 | .600 | .486 | .313 | .193 | .143 | .058 | .047 |
|  | $(.504,0.089)$ | $(.505,0.543)$ | $(.576,1.056)$ | $(.601,1.585)$ | $(.514,2.115)$ | $(.633,2.633)$ | $(.504,3.159)$ |
| 1 | .275 | .190 | .121 | .069 | .040 | .016 | .011 |
|  | $(1.007,0.099)$ | $(1.010,0.605)$ | $(1.011,1.107)$ | $(1.049,1.579)$ | $(1.017,2.131)$ | $(1.177,2.638)$ | $(1.061,3.199)$ |
| 1.5 | .115 | .077 | .043 | .025 | .012 | .008 | .004 |
|  | $(1.513,0.0138)$ | $(1.539,0.539)$ | $(1.557,1.092)$ | $(1.526,1.630)$ | $(1.668,2.112)$ | $(1.511,2.694)$ | $(1.739,3.171)$ |
| 2 | .054 | .032 | .018 | .012 | .007 | .003 | .002 |
|  | $(2.014,0.110)$ | $(2.019,0.675)$ | $(2.020,1.234)$ | $(2.074,1.591)$ | $(2.011,2.146)$ | $(2.187,2.643)$ | $(2.098,3.222)$ |
| 2.5 | .029 | .020 | .011 | .006 | .004 | .002 | .001 |
|  | $(2.519,0.087)$ | $(2.525,0.534)$ | $(2.554,1.082)$ | $(2.515,1.700)$ | $(2.583,2.192)$ | $(2.638,2.686)$ | $(2.531,3.273)$ |
| 3 | .017 | .012 | .007 | .004 | .002 | .001 | .001 |
|  | $(3.022,0.084)$ | $(3.046,0.543)$ | $(3.080,1.100)$ | $(3.018,1.642)$ | $(3.099,2.117)$ | $(3.182,2.728)$ | $(3.037,3.161)$ |
| 3.5 | .009 | .007 | .004 | .003 | .002 | .001 | .001 |
|  | $(3.522,0.367)$ | $(3.525,0.672)$ | $(3.527,1.230)$ | $(3.622,1.586)$ | $(3.530,2.250)$ | $(3.605,2.757)$ | $(3.663,3.211)$ |

Table 4.2: Values of the American Basket put option using the semi-smooth Newton algorithm at specified grid points. The free boundary lies between the red coloured values and the black coloured values of the option. The parameters are given in 4.1. CPU time: 7.14 seconds.
the table. The red coloured values indicate the exercise region and show that the free boundary lies between the red coloured values and the next black coloured values. In Table 4.2, we present the option values that are computed by the semi-smooth Newton method. Comparing Tables 4.1 and 4.2 we cannot identify the differences between the prices engendered by either algorithm. Calculating the $L^{\infty}$-distance and $L^{2}$-distance of the two prices at time $t=0$ on the set,
$I:=\left\{s(0, \underline{x}) \in \mathbb{R}_{+}^{2} \mid 0.05 \leq s_{1}\left(0,\left(x_{1},-R\right)^{\top}\right), s_{2}\left(0,\left(-R, x_{2}\right)^{\top}\right) \leq 2.95\right.$ with $\left.\underline{x} \in G\right\}$, where $s_{i}(0, \underline{x}):=\exp \left(\sum_{j=1}^{2} \boldsymbol{\Sigma}_{i j} x_{j}\right)$ for $i=1,2$ we find that,

$$
\begin{aligned}
\left\|V^{\mathrm{SSN}}(0, \underline{s})-V^{\mathrm{PSOR}}(0, \underline{s})\right\|_{L^{\infty}(I)} & \approx 7.8010 \times 10^{-11} \\
\left\|V^{\mathrm{SSN}}(0, \underline{s})-V^{\mathrm{PSOR}}(0, \underline{s})\right\|_{L^{2}(I)} & \approx 2.1711 \times 10^{-11}
\end{aligned}
$$

where $V^{\mathrm{SSN}}(0, \underline{s})$ and $V^{\mathrm{PSOR}}(0, \underline{s})$ are the prices computed by the semi-smooth Newton and the PSOR, respectively. Finally, we want to point out the difference in computational time of both methods. The PSOR algorithm produces a price in approximately 54 seconds and the semi-smooth Newton in approximately 7 seconds as we can see in the captions of Table 4.1 and 4.2 .

### 4.1.2 Greeks

In the present section, we depict the argument Greeks for the American basket put option for two dependent assets. In Figure 4.3 we have the $\Theta(t, \underline{s}):=\partial_{t} V(T-t, \underline{s})$ of the option. We present two plots; one for $s_{1}=K$ and


Figure 4.3: Theta for an American basket put option. The parameters are given in 4.1).
the other for $s_{2}=K$. In the left plot of Figure 4.3, we observe that for each time $t \in J$ the $\Theta\left(t,\left(K, s_{2}\right)^{\top}\right)$ is much steeper with respect to $s_{2}$ than $\Theta\left(t,\left(s_{1}, K\right)^{\top}\right)$ with respect to $s_{1}$ in the right plot. This is caused by the large volatility term $\sigma_{1}$. In Figure 4.4, we illustrate the $\Delta^{i}(t, \underline{s})=\partial_{s_{i}} V(T-t, \underline{s})$ for $i=1,2$ where time $t=T$. Here, we notice that the delta for $s_{1}$ give larger negative values than the delta for $s_{2}$. Implying that we need to sell more of $s_{1}$ and less of $s_{2}$ to $\Delta$-hedge the American basket put option. This is intuitive since our portfolio is strongly overweight in $s_{1}$. Furthermore in both plots of Figure 4.4, we can see the exercise region where $\Delta^{i}(T, \underline{s})$ are constant for $i=1,2$. The fact that $\Delta^{i}(T, \underline{s})$ is constant results from the fact that the value $V(0, \underline{s})$ is a linear function in terms of $s_{1}$ and $s_{2}$. One should be careful using these deltas for $\Delta$-hedge since the values in the exercise region are non zero. If the option is exercised there is no further need for any hedging position. Thus for hedging purposes the short position in the assets should be neutralised if the option hits the exercise barrier, or free boundary. In Figure 4.5 , we show $\Gamma^{l k}(t, \underline{s}):=\partial_{s_{l} s_{k}} V(T-t, \underline{s})$ for $l, k=1,2$ at time $t=0$.

### 4.2 Best-of American Put Option

The payoff of the best-of American put option is given by $g(\underline{s})=\left(K-\min \left(s_{1}, s_{2}\right)\right)_{+}$with $K \in \mathbb{R}_{+}$. This implies that we have the following time-dependent obstacle

$$
\begin{aligned}
\tilde{g}(t, \underline{x})= & e^{-r(T-t)} \\
& \left(K-\min \left(e^{\left(r+w_{1}\right)(T-t)+\sum_{j=1}^{2} \boldsymbol{\Sigma}_{1 j} x_{j}}, e^{\left(r+w_{2}\right)(T-t)+\sum_{j=1}^{2} \boldsymbol{\Sigma}_{2 j} x_{j}}\right)\right)_{+} .
\end{aligned}
$$



Figure 4.4: Delta for an American basket put option. The parameters are given in 4.1).

In this section, we utilize the CGMY process as underlying process $X$ to the model in 1.2. The parameters used for this numerical example are,

$$
\begin{align*}
& c_{1}=0.8, \beta_{-}^{1}=8, \beta_{+}^{1}=10, \alpha_{1}=1.1, c_{2}=1.2, \beta_{-}^{2}=16, \beta_{+}^{2}=17, \alpha_{2}=1.2 \\
& T=1, K=1, r=.03, R=5, \theta=.5, L=7 \text { and } \\
& \boldsymbol{\Sigma}=\left(\begin{array}{cc}
.8 & -.14 \\
-.14 & .65
\end{array}\right) \tag{4.2}
\end{align*}
$$

The components of $X$ do not satisfy the conditions in Proposition 3.14 which imply the violation of the smooth pasting property for the plain vanilla American put option.

### 4.2.1 Price and Free Boundary

In this section, we show the value and free boundary of a best-of American put option for two negatively dependent assets. In Figure 4.6 we exhibit the value of the option. We notice that, unlike in the previous example, the grid no longer has an oval shape. This is due to the negative correlation among the assets. Moreover, it seems that the payoff is curved, however, this is only an optical illusion created by the transformation of the grid from $\underline{x}$ to $\underline{s}$ using $\boldsymbol{\Sigma}$. In Figure 4.7. we depict the free boundary for the best-of American put option. Here we exercise for low values for either underlying.

In Table 4.3 and 4.4 we find the specific option values of the best-of American put option solved by the PSOR and the semi-smooth Newton algorithm in Table 3.2 As in the previous example, the red option values belong to the exercise region and the black option values to the continuation region. The free boundary thus lies in between the red and black coloured values. Notice that the exact grid points for the option values now differ from the example in Section 4.1. This results from the fact that we used a different matrix $\boldsymbol{\Sigma}$ for


Figure 4.5: Gamma for an American basket put option. The parameters are given in 4.1).


Figure 4.6: Value of a best-of American put option. The parameters are given in (4.2).


Figure 4.7: Free boundary of a best-of American put option. The parameters are given in 4.2).

| $\tilde{s}_{1} \backslash \tilde{s}_{2}$ | 0 | .25 | .5 | .75 | 1 | 1.25 | 1.5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | .913 | .909 | .934 | .939 | .939 | .945 | .940 |
|  | $(.094,0.094)$ | $(.091,0.264)$ | $(.066,0.527)$ | $(.061,0.792)$ | $(.061,1.063)$ | $(.055,1.316)$ | $(.060,1.579)$ |
| .25 | .912 | .780 | .751 | .723 | .723 | .750 | .728 |
|  | $(.249,0.088)$ | $(.253, .273)$ | $(.249, .544)$ | $(.277, .792)$ | $(.277,1.063)$ | $(.250,1.316)$ | $(.272,1.579)$ |
| .5 | .921 | .734 | .580 | .525 | .501 | .487 | .501 |
|  | $(.496,0.079)$ | $(.526,0.267)$ | $(.517,0.533)$ | $(.507,0.792)$ | $(.508,1.063)$ | $(.514,1.356)$ | $(.499,1.579)$ |
| .75 | .914 | .722 | .519 | .379 | .314 | .278 | .284 |
|  | $(.745,0.086)$ | $(.750,0.278)$ | $(.744,0.527)$ | $(.770,0.816)$ | $(.771,1.096)$ | $(.788,1.329)$ | $(.757,1.628)$ |
| 1 | .933 | .692 | .488 | .300 | .202 | .177 | .147 |
|  | $(.996,0.067)$ | $(.994,0.308)$ | $(1.007,0.527)$ | $(1.041,0.816)$ | $(1.043,1.096)$ | $(1.002,1.342)$ | $(1.024,1.628)$ |
| 1.25 | .925 | .656 | .423 | .280 | .164 | .102 | .074 |
|  | $(1.241,0.075)$ | $(1.238,0.344)$ | $(1.254,0.590)$ | $(1.256,0.792)$ | $(1.258,1.063)$ | $(1.273,1.356)$ | $(1.301,1.645)$ |
| 1.5 | .862 | .666 | .405 | .251 | .128 | .075 | .045 |
|  | $(1.486,0.138)$ | $(1.493,0.334)$ | $(1.498,0.602)$ | $(1.500,0.808)$ | $(1.502,1.085)$ | $(1.536,1.316)$ | $(1.570,1.596)$ |
| 1.75 | .914 | .719 | .306 | .205 | .105 | .054 | .030 |
|  | $(1.733,0.086)$ | $(1.744,0.281)$ | $(1.735,0.715)$ | $(1.773,0.867)$ | $(1.794,1.107)$ | $(1.834,1.342)$ | $(1.761,1.645)$ |

Table 4.3: Values of the best-of American put option using the PSOR algorithm at specified grid points. The free boundary lies between the red coloured values and the next black coloured values of the option. The parameters are given in 4.2. CPU time: 56.71 seconds.
this example. To compute the $L^{\infty}$ - and $L^{2}$-distance between the two prices for $t=0$, we use a different region as before, namely,
$I:=\left\{s(0, \underline{x}) \in \mathbb{R}_{+}^{2} \mid 0.1 \leq s_{1}\left(0,\left(x_{1},-R\right)^{\top}\right), s_{2}\left(0,\left(-R, x_{2}\right)^{\top}\right) \leq 4.9\right.$ with $\left.\underline{x} \in G\right\}$, where $s_{i}(0, \underline{x}):=\exp \left(\sum_{j=1}^{2} \boldsymbol{\Sigma}_{i j} x_{j}\right)$ for $i=1,2$. Subsequently we find that,

$$
\begin{aligned}
\left\|V^{\mathrm{SSN}}(0, \underline{s})-V^{\mathrm{PSOR}}(0, \underline{s})\right\|_{L^{\infty}(I)} & \approx 5.8426 \times 10^{-11} \\
\left\|V^{\mathrm{SSN}}(0, \underline{s})-V^{\mathrm{PSOR}}(0, \underline{s})\right\|_{L^{2}(I)} & \approx 3.6277 \times 10^{-11}
\end{aligned}
$$

where $V^{\mathrm{SSN}}(0, \underline{s})$ and $V^{\mathrm{PSOR}}(0, \underline{s})$ are the prices computed by the semi-smooth Newton and the PSOR, respectively. Comparing the computational time of both methods we see in Table 4.3 and 4.4 that the semi-smooth Newton's computational time is approximately $\frac{1}{10}$ of that of the PSOR algorithm.

### 4.2.2 Greeks

Here we illustrate the Greeks for the best-of American put option. In Figure 4.8 , we display the $\Theta(t, \underline{s})$ of the option where in the left plot we set $s_{1}=K$ and in the left plot $s_{2}=K$. The $\Theta(t, s)$ is related to the time value of the option. For this reason, we have the lowest value of $\Theta(t, \underline{s})$ for each $t \in J$ at $\underline{s}=(K, K)^{\top}$. Furthermore, in Figure 4.9 we exhibit the two deltas, i.e. $\Delta^{i}(t, \underline{s})$ for time $t=0$ with $i=1,2$. We observe a symmetry property which results from the symmetric payoff function $g(\underline{s})$.
In Figure 4.10 we display the $\Gamma^{l k}(t, \underline{s})$ for $l, k=1,2$ at time $t=0$. We only

| $\tilde{s}_{1} \backslash \tilde{s}_{2}$ | 0 | .25 | .5 | .75 | 1 | 1.25 | 1.5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0 | .913 | .909 | .934 | .939 | .939 | .945 | .940 |
|  | $(.094, .094)$ | $(.091, .264)$ | $(.066, .527)$ | $(.061, .792)$ | $(.061,1.063)$ | $(.055,1.316)$ | $(.060,1.579)$ |
| .25 | .912 | .780 | .751 | .723 | .723 | .750 | .728 |
|  | $(.249, .088)$ | $(.253, .273)$ | $(.249, .544)$ | $(.277, .792)$ | $(.277,1.063)$ | $(.250,1.316)$ | $(.272,1.579)$ |
| .5 | .921 | .734 | .580 | .525 | .501 | .487 | .501 |
|  | $(.496, .079)$ | $(.526, .267)$ | $(.517, .533)$ | $(.507, .792)$ | $(.508,1.063)$ | $(.514,1.356)$ | $(.499,1.579)$ |
| .75 | .914 | .722 | .519 | .379 | .314 | .278 | .284 |
|  | $(.745, .086)$ | $(.750, .278)$ | $(.744, .527)$ | $(.770, .816)$ | $(.771,1.096)$ | $(.788,1.329)$ | $(.757,1.1628)$ |
| 1 | .933 | .692 | .488 | .300 | .202 | .177 | .147 |
|  | $(.996, .067)$ | $(.994, .308)$ | $(1.007, .527)$ | $(1.041, .816)$ | $(1.043,1.096)$ | $(1.002,11342)$ | $(1.024,1.628)$ |
| 1.25 | .925 | .656 | .423 | .280 | .164 | .102 | .074 |
|  | $(1.241, .075)$ | $(1.238, .344)$ | $(1.254, .590)$ | $(1.256, .792)$ | $(1.258,1.063)$ | $(1.273,1.356)$ | $(1.301,1.645)$ |
| 1.5 | .862 | .666 | .405 | .251 | .128 | .075 | .045 |
|  | $(1.486, .138)$ | $(1.493, .334)$ | $(1.498, .602)$ | $(1.500, .808)$ | $(1.502,1.085)$ | $(1.536,1.316)$ | $(1.570,1.596)$ |
| 1.75 | .914 | .719 | .306 | .205 | .105 | .054 | .030 |
|  | $(1.733, .086)$ | $(1.744, .281)$ | $(1.735, .715)$ | $(1.773, .867)$ | $(1.794,1.107)$ | $(1.834,1.342)$ | $(1.761,1.645)$ |

Table 4.4: Values of the best-of American put option using the semi-smooth Newton algorithm at specified grid points. The free boundary lies between the red coloured values and the next black coloured values of the option. The parameters are given in 4.2 . CPU time: 5.22 seconds.


Figure 4.8: Theta of a best-of American put option. The parameters are given in 4.2.


Figure 4.9: Delta of a best-of American put option. The parameters are given in 4.2.
show three plots since $\Gamma^{12}(t, \underline{s})=\Gamma^{21}(t, \underline{s})$; see Appendix $D$
This concludes the numerical examples. We have seen two options, i.e. the American basket put option and the best-of American put option and their argument Greeks. We have discussed the effect of the transformation of the grid from $\underline{x}$ to $\underline{s}$ using the matrix $\boldsymbol{\Sigma}$. Moreover, we have compared the value of the options computed by the semi-smooth Newton algorithm to the values reproduced by the PSOR algorithm and concluded that there is only a small difference between them, however, the semi-smooth Newton method is much faster for each example.

### 4.3 Conclusion and Outlook

In this section, we give an overview of possible further research. We have proposed a bivariate Lévy model for dependent asset price processes in Section 1.3.1. A special feature of the model is that the dependence is created by a matrix-vector multiplication of a constant matrix $\boldsymbol{\Sigma}$ and a two-dimensional Lévy process $X=\left(X_{t}\right)_{t \geq 0}$ with independent components. We presented the necessary theoretical results on Lévy processes and gave two examples for the process $X$, namely the CGMY and Kou process in Section 1.3 .3 and 1.3.4 respectively. Thereafter, we developed a pricing scheme for American options with two underlying assets using model 1.2 by using the continuous Galerkin finite element method with products of piecewise linear hat functions as basis for the finite element space. The resulting scheme is equivalent to solving linear complementary problems for each time step $m=0, \ldots, M-1$. We explored the primal-dual active set strategy, which is equivalent to a semi-smooth Newton method, to solve these linear complementary problems. We established the uniqueness and local convergence of the solutions by showing that $\mathbf{B}=\mathbf{M}+k \theta \mathbf{A}$ as defined in (2.31) is a P-matrix. In order to achieve global convergence we need


Figure 4.10: Gamma of a best-of American put option. The parameters are given in 4.2).
to show that $\mathbf{B}$ to be an M-matrix, however, this is non-trivial and we only state sufficient conditions in Proposition 3.9 such that $\mathbf{A}$ is an M-matrix. Proposition 3.9 places restrictions on the volatility terms $\sigma_{j}$ and the Lévy densities $k_{X^{j}}\left(z_{j}\right)$ of the LPs $X^{j}$ for $j=1,2$. Further research is needed to find parameter bounds of specific LPs, such as the Kou or CGMY process. In Section 3.3, we stated an a priori error estimate for elliptic and parabolic variational inequalities. For the parabolic case we give error estimates for a two-dimensional case with a nondegenerate differential operator of order two and for a one-dimensional case with integro-differential operator of order $\alpha \in(0,2]$. However these error estimates do not seem to be sharp, see Matache, et al. 36. In Section 3.4. we numerically tested the convergence of the semi-smooth Newton algorithm in various variational problems. First, we treated the elliptic variational inequality of the perpetual American put option in the one-dimensional Kou model. Here the non-degenerate integro-differential operator is of order two. Our results show that optimal convergence in $L^{2}$ and $L^{\infty}$ can be achieved for this problem using the semi-smooth Newton algorithm. Thereafter, we discuss a one-dimensional parabolic case with a non-degenerate operator of order two. We found an optimal convergence rate of order $\mathcal{O}\left(N^{-2}\right)$ for the Crank-Nicolson scheme under the condition that the obstacle is in the finite element space. The convergence tests for the pricing scheme developed in Chapter 2 we find a convergence rate of almost order $\mathcal{O}\left(N^{-1}\right)$ in $L^{2}$ and of order $\mathcal{O}\left(N^{-\frac{1}{2}}\right)$ in $L^{\infty}$ and $H^{1}$ for the American basket put option with underlying CGMY process for $X$. Under the violation of the smooth pasting property for plain vanilla American puts described in Lamberton and Mikou [34] the pricing algorithm does not converge. In the last section of Chapter 3, we discuss the speed of the semi-smooth Newton algorithm and compare it to the PSOR algorithm. We found that the semi-smooth Newton algorithm is faster than the MATLAB® implementation of the PSOR algorithm.
As already mentioned in Section 1.3.1, further research has to be devoted to the assessment of the dependence structure created by the model in 1.2 . We have already analysed the two-dimensional LP $Y$ in 1.1) and its Lévy measure $\nu_{Y}(d \underline{y})$ to some extent in Section 1.3.1. However, the limitations of the dependence structure created by the LP $Y$ must be discussed further and a proper analysis of the model's ability to mimic the dependence among assets is important. Nevertheless, the model in (1.2) also yields very nice properties for numerical option pricing as we have seen in Chapter 2. There is no need for complicated Lévy copulas and we may completely tensorise the stiffness matrix in the finite elements discretisation due to the independence of the components of $X$ as seen in 2.31. Furthermore, the smooth pasting property for American option with two underlying assets is subject to further examination. Lamberton and Mikou [34] provide a good understanding of the univariate case. Similarly, a priori error estimates for two-dimensional variational inequalities with integro-differential operator of order $\underline{\alpha}=\left(\alpha_{1}, \alpha_{2}\right)^{\top}$ have, to our knowledge, not been derived yet. Matache, et al. [36] give a priori error estimates for the one-dimensional case, but the two-dimensional case also remains a topic of further research.

## Appendix A

## Market Model Assumptions

Here we check if the two processes underlying the bivariate model are consistent with the conditions in Assumptions 1.14. First, we validate the CGMY model and then the Kou model. The models fulfilling the properties in Assumptions 1.14 are called admissible market models. Since all the conditions are in terms of the one-dimensional Lévy measures, we only treat one-dimensional Lévy processes in this appendix.

## A. 1 CGMY Process

Let us recall the Lévy measure of the CGMY process in 1.10 in Section 1.3.3.

$$
\nu^{\mathrm{CGMY}}(d z)=\left(c \frac{e^{-\beta_{+}|z|}}{|z|^{1+\alpha}} \mathbb{1}_{[z>0]}+c \frac{e^{-\beta_{-}|z|}}{|z|^{1+\alpha}} \mathbb{1}_{[z<0]}\right) d z,
$$

where $c>0, \beta_{+}>1, \beta_{-}>0$ and $\alpha \in(0,2)$.
(A1) Let $\zeta_{-}=\beta_{-}>0, \zeta_{+}=\beta_{+}>1$ and $C:=c$ then,

$$
k^{\mathrm{CGMY}}(z) \leq \begin{cases}c e^{-\beta_{-}|z|}, & \text { if } z<-1, \\ c e^{-\beta_{+}|z|}, & \text { if } z>1,\end{cases}
$$

since $|z|^{-1-\alpha} \leq 1$ for $z \in \mathbb{R} \backslash(-1,1)$ and $\alpha \in(0,2)$.
(A2) Since $e^{-\beta_{ \pm}|z|}<1$ for $1>|z|>0$ we have for $C_{+}:=c$,

$$
k^{\mathrm{CGMY}}(z) \leq \frac{C_{+}}{|z|^{1+\alpha}} \text { for } 1>|z|>0
$$

(A3) Let $C_{-}:=\frac{c}{2}\left(e^{-\beta_{+}}+e^{-\beta_{-}}\right)$. Since $e^{-\beta_{ \pm}|z|} \geq e^{-\beta_{ \pm}}$for $0<|z|<1$ we have,

$$
\begin{aligned}
\frac{1}{2}\left(k_{X^{j}}\left(z_{j}\right)+k_{X^{j}}\left(-z_{j}\right)\right) & =\frac{c}{2}\left(\frac{e^{-\beta_{+}|z|}}{|z|^{1+\alpha}} \mathbb{1}_{[z>0]}+\frac{e^{-\beta_{-}|z|}}{|z|^{1+\alpha}} \mathbb{1}_{[z<0]}\right) \\
& =\frac{c}{2|z|^{1+\alpha}}\left(e^{-\beta_{+}|z|}+e^{-\beta_{-}|z|}\right) \mathbb{1}_{[z \in \mathbb{R} \backslash\{0\}]} \\
& \geq \frac{C_{-}}{|z|^{1+\alpha}}
\end{aligned}
$$

## A. 2 Kou Process

The Lévy measure of the Kou process in (1.11) is given by,

$$
\nu^{\mathrm{Kou}}(d z)=\lambda\left(p \eta_{+} e^{-\eta_{+} z} \mathbb{1}_{[z \geq 0]}+q \eta_{-} e^{\eta_{-} z} \mathbb{1}_{[z<0]}\right) d z
$$

where $\lambda>0, \eta_{+}>1, \eta_{-}>0$ and $p, q \in(0,1)$ such that $p+q=1$.
(A1) Let $\zeta_{-}=\eta_{-}>0, \zeta_{+}=\eta_{+}>1$ and $C:=\lambda \max \left(p \eta_{+}, q \eta_{-}\right)$then,

$$
k^{\mathrm{Kou}}(z) \leq \begin{cases}C e^{-\eta_{-}|z|}, & \text { if } z<-1 \\ C e^{-\eta_{+}|z|}, & \text { if } z>1\end{cases}
$$

(A2) Since $e^{-\eta_{ \pm}|z|}<1$ and $|z|^{-1-\alpha}>1$ for $1>|z|>0$ with $\alpha \in(0,2)$ we have for $C_{+}:=\lambda \max \left(p \eta_{+}, q \eta_{-}\right)$,

$$
k^{\mathrm{Kou}}(z) \leq C_{+} \leq \frac{C_{+}}{|z|^{1+\alpha}} \text { for } 1>|z|>0 .
$$

(A3) The Kou process does not have to satisfy this condition since we have $\sigma_{j}>0$ for $j=1,2$.

## Appendix B

## Martingale Property for the CGMY Process

Here we calculate the drift term $w_{i}$ needed for the martingale condition in Theorem 1.15 in case the LP $X$ follows a CGMY process. As the CGMY process is a pure jump process, we set $\sigma_{j}=0$ for $j=1,2$. Further we remark that by (1.6), we then have,

$$
w_{i}^{\mathrm{CGMY}}=-\sum_{j=1}^{2} \int_{\mathbb{R}}\left(e^{\boldsymbol{\Sigma}_{i j} z}-1-\boldsymbol{\Sigma}_{i j} z\right) \nu_{X^{j}}(d z)=-\sum_{j=1}^{2} w_{\alpha_{j}}^{j}
$$

We distinguish three cases of parameter values for $\alpha_{j}$ to calculate $w_{\alpha_{j}}^{j}$ as in Cont and Tankov [10, Proposition 4.2].
Case I: $\alpha_{j} \in(0,1) \cup(1,2)$
Using $e^{x}=\sum_{k=0}^{\infty} \frac{x^{k}}{k!}$ and the Gamma function $\Gamma(x)=\int_{0}^{\infty} e^{-y} y^{x-1} d y$ we have,

$$
\begin{aligned}
& \int_{0}^{\infty}\left(e^{\boldsymbol{\Sigma}_{i j} z}-1-\boldsymbol{\Sigma}_{i j} z\right) \nu_{X^{j}}(d z) \\
= & c_{j} \int_{0}^{\infty}\left(e^{\boldsymbol{\Sigma}_{i j} z}-1-\boldsymbol{\Sigma}_{i j} z\right) \frac{e^{-\beta_{+}^{j} z}}{z^{1+\alpha_{j}}} d z \\
= & c_{j} \sum_{k=2}^{\infty} \frac{\left(\boldsymbol{\Sigma}_{i j}\right)^{k}}{k!} \int_{0}^{\infty} z^{k-1-\alpha_{j}} e^{-\beta_{+}^{j} z} d z \\
= & c_{j} \sum_{k=2}^{\infty} \frac{\left(\boldsymbol{\Sigma}_{i j}\right)^{k}}{k!}\left(\beta_{+}^{j}\right)^{\alpha_{j}-k} \Gamma\left(k-\alpha_{j}\right) \\
= & c_{j}\left(\beta_{+}^{j}\right)^{\alpha_{j}} \Gamma\left(2-\alpha_{j}\right)\left\{\frac{1}{2!}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{2}+\frac{\left(2-\alpha_{j}\right)}{3!}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{3}\right. \\
& \left.+\frac{\left(2-\alpha_{j}\right)\left(3-\alpha_{j}\right)}{4!}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{4}+\ldots\right\}
\end{aligned}
$$

$$
\begin{aligned}
= & c_{j}\left(\beta_{+}^{j}\right)^{\alpha_{j}} \frac{\Gamma\left(2-\alpha_{j}\right)}{\alpha_{j}\left(\alpha_{j}-1\right)}\left\{\frac{\alpha_{j}\left(\alpha_{j}-1\right)}{2!}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{2}-\frac{\alpha_{j}\left(\alpha_{j}-1\right)\left(\alpha_{j}-2\right)}{3!}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{3}\right. \\
& \left.+\frac{\alpha_{j}\left(\alpha_{j}-1\right)\left(\alpha_{j}-2\right)\left(\alpha_{j}-3\right)}{4!}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{4}+\ldots\right\} \\
= & c_{j}\left(\beta_{+}^{j}\right)^{\alpha_{j}} \frac{\Gamma\left(2-\alpha_{j}\right)}{\alpha_{j}\left(\alpha_{j}-1\right)}\left\{\alpha_{j}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)-1+1-\alpha_{j}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)+\frac{\alpha_{j}\left(\alpha_{j}-1\right)}{2!}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{2}\right. \\
& \left.-\frac{\alpha_{j}\left(\alpha_{j}-1\right)\left(\alpha_{j}-2\right)}{3!}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{3}+\frac{\alpha_{j}\left(\alpha_{j}-1\right)\left(\alpha_{j}-2\right)\left(\alpha_{j}-3\right)}{4!}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{4}+\ldots\right\} \\
= & c_{j}\left(\beta_{+}^{j}\right)^{\alpha_{j}} \frac{\Gamma\left(2-\alpha_{j}\right)}{\alpha_{j}\left(\alpha_{j}-1\right)}\left\{\left(\frac{\beta_{+}^{j}-\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{\alpha_{j}}-1+\frac{\alpha_{j} \boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right\}
\end{aligned}
$$

where the last equality follows from the Binomial power series representation found in Knopp [30, Section 5.8],

$$
(1-x)^{q}=1-q x+q(q-1) \frac{x^{2}}{2!}-q(q-1)(q-2) \frac{x^{3}}{3!}+\ldots
$$

for which we set $x=\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}$ and $q=\alpha_{j}$. For the series to converge, we need $|x|<1$ or $\left|\boldsymbol{\Sigma}_{i j}\right|<\beta_{+}^{j}$ for $i, j=1,2$. In an equivalent way,

$$
\begin{aligned}
& \int_{-\infty}^{0}\left(e^{\boldsymbol{\Sigma}_{i j} z}-1-\boldsymbol{\Sigma}_{i j} z\right) \nu_{X^{j}}(d z) \\
& =c_{j} \int_{0}^{\infty}\left(e^{-\boldsymbol{\Sigma}_{i j} z}-1+\boldsymbol{\Sigma}_{i j} z\right) \frac{e^{-\beta_{-}^{j} z}}{z^{1+\alpha_{j}}} d z \\
& =c_{j}\left(\beta_{-}^{j}\right)^{\alpha_{j}} \Gamma\left(-\alpha_{j}\right)\left\{\left(\frac{\beta_{-}^{j}+\boldsymbol{\Sigma}_{i j}}{\beta_{-}^{j}}\right)^{\alpha_{j}}-1-\frac{\alpha_{j} \boldsymbol{\Sigma}_{i j}}{\beta_{-}^{j}}\right\}
\end{aligned}
$$

Hence the drift imposed by the martingale condition is,

$$
\begin{array}{r}
w_{\alpha_{j}}^{j}:=c_{j} \Gamma\left(-\alpha_{j}\right)\left[\left(\beta_{+}^{j}\right)^{\alpha_{j}}\left\{\left(\frac{\beta_{+}^{j}-\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{\alpha_{j}}+\frac{\alpha_{j} \boldsymbol{\Sigma}_{i j}-\beta_{+}^{j}}{\beta_{+}^{j}}\right\}\right. \\
\left.+\left(\beta_{-}^{j}\right)^{\alpha_{j}}\left\{\left(\frac{\beta_{-}^{j}+\boldsymbol{\Sigma}_{i j}}{\beta_{-}^{j}}\right)^{\alpha_{j}}-\frac{\alpha_{j} \boldsymbol{\Sigma}_{i j}+\beta_{-}^{j}}{\beta_{-}^{j}}\right\}\right]
\end{array}
$$

Case II: $\alpha_{j}=0$
For $\alpha_{j}=0$ we have as above using integration by parts,

$$
\begin{aligned}
& \int_{0}^{\infty}\left(e^{\boldsymbol{\Sigma}_{i j} z}-1-\boldsymbol{\Sigma}_{i j} z\right) \nu_{X^{j}}(d z) \\
& =c_{j} \sum_{k=2}^{\infty} \frac{\left(\boldsymbol{\Sigma}_{i j}\right)^{k}}{k!} \int_{0}^{\infty} z^{k-1} e^{-\beta_{+}^{j} z} d z \\
& =c_{j} \sum_{k=2}^{\infty} \frac{\left(\boldsymbol{\Sigma}_{i j}\right)^{k}}{k!} \frac{(k-1)!}{\left(\beta_{+}^{j}\right)(k-1)} \int_{0}^{\infty} e^{-\beta_{+}^{j} z} d z \\
& =c_{j} \sum_{k=2}^{\infty}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{k} k^{-1} \\
& =c_{j}\left[-\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}-\ln \left(1-\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)\right] \\
& =c_{j}\left[\ln \left(\frac{\beta_{+}^{j}}{\beta_{+}^{j}-\boldsymbol{\Sigma}_{i j}}\right)-\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right]
\end{aligned}
$$

where note that $\ln (1-x)=-\sum_{k=1}^{\infty} \frac{x^{k}}{k}$ which follows from integrating the Binomial series above with $q=-1$, see in Königsberger 31, Example 6.1]. Here we also need the condition that $\left|\boldsymbol{\Sigma}_{i j}\right|<\beta_{+}^{j}$ for $i, j=1,2$ to ensure the convergence of the series. Finally we get,

$$
w_{0}^{j}:=c_{j}\left\{\ln \left(\frac{\beta_{+}^{j}}{\beta_{+}^{j}-\boldsymbol{\Sigma}_{i j}}\right)+\ln \left(\frac{\beta_{-}^{j}}{\beta_{-}^{j}+\boldsymbol{\Sigma}_{i j}}\right)+\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j} \beta_{-}^{j}}\left(\beta_{+}^{j}-\beta_{-}^{j}\right)\right\}
$$

Case III: $\alpha_{j}=1$
Here, we obtain using the same steps as above,

$$
\begin{aligned}
& \int_{0}^{\infty}\left(e^{\boldsymbol{\Sigma}_{i j} z}-1-\boldsymbol{\Sigma}_{i j} z\right) \nu_{X^{j}}(d z) \\
& =c_{j} \beta_{+}^{j} \sum_{k=2}^{\infty}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{k} \frac{1}{k(k-1)} \\
& =c_{j} \beta_{+}^{j}\left\{\sum_{k=2}^{\infty}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{k}\left(\frac{1}{k-1}-\frac{1}{k}\right)\right\} \\
& =c_{j} \beta_{+}^{j}\left\{\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}} \sum_{k=1}^{\infty}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{k} k^{-1}-\sum_{k=2}^{\infty}\left(\frac{\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)^{k} \frac{1}{k}\right\} \\
& =-c_{j} \boldsymbol{\Sigma}_{i j} \ln \left(\frac{\beta_{+}^{j}-\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)+c_{j} \beta_{+}^{j} \ln \left(\frac{\beta_{+}^{j}-\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)+c_{j} \boldsymbol{\Sigma}_{i j} \\
& =c_{j} \boldsymbol{\Sigma}_{i j}+c_{j}\left(\beta_{+}^{j}-\boldsymbol{\Sigma}_{i j}\right) \ln \left(\frac{\beta_{+}^{j}-\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)
\end{aligned}
$$

Thus,

$$
w_{1}^{j}:=c_{j}\left\{\left(\beta_{+}^{j}-\boldsymbol{\Sigma}_{i j}\right) \ln \left(\frac{\beta_{+}^{j}-\boldsymbol{\Sigma}_{i j}}{\beta_{+}^{j}}\right)+\left(\beta_{-}^{j}+\boldsymbol{\Sigma}_{i j}\right) \ln \left(\frac{\beta_{-}^{j}+\boldsymbol{\Sigma}_{i j}}{\beta_{-}^{j}}\right)\right\} .
$$

## Appendix C

## Finite Element Calculations

Here we depict the calculations for the finite element matrices resulting from the discretisation of variational formulation (2.23). First we give the definition of a Kronecker product.

Definition C.1. Let $\mathbf{A} \in \mathbb{R}^{n \times m}$ and $\mathbf{B} \in \mathbb{R}^{p \times q}$ be matrices then its Kronecker product reads,

$$
\mathbf{A} \otimes \mathbf{B}=\left(\begin{array}{cccc}
\mathbf{A}_{11} \mathbf{B} & \mathbf{A}_{12} \mathbf{B} & \ldots & \mathbf{A}_{1 m} \mathbf{B} \\
\mathbf{A}_{21} \mathbf{B} & \ddots & & \vdots \\
\vdots & & \ddots & \mathbf{A}_{n-1 m} \mathbf{B} \\
\mathbf{A}_{n 1} \mathbf{B} & \cdots & \mathbf{A}_{n m-1} \mathbf{B} & \mathbf{A}_{n m} \mathbf{B}
\end{array}\right) \in \mathbb{R}^{n q \times m p}
$$

Due to the structure of $V_{N}$, we can split the integrals resulting from the variational formulation (2.34) in each dimension, therefore the mass matrix in 2.31) consists of the Kronecker product of two matrices $\mathbf{M}=\mathbf{M}^{1} \otimes \mathbf{M}^{2}$. Let $G=(-R, R)^{2}$ and $G_{l}=(-R, R) \subset \mathbb{R}$ for $l=1,2$ then,

$$
\begin{align*}
\mathbf{M}_{\left(i_{1}, j_{1}\right),\left(i_{2}, j_{2}\right)} & =\int_{G} b_{j_{1}}\left(x_{1}\right) b_{j_{2}}\left(x_{2}\right) b_{i_{1}}\left(x_{1}\right) b_{i_{2}}\left(x_{2}\right) d x \\
& =\int_{G_{1}} b_{j_{1}} b_{i_{1}}\left(x_{1}\right) d x_{1} \int_{G_{2}} b_{j_{2}}\left(x_{2}\right) b_{i_{2}}\left(x_{2}\right) d x_{2}  \tag{C.1}\\
& =\mathbf{M}_{i_{1} j_{1}}^{1} \mathbf{M}_{i_{2} j_{2}}^{2}
\end{align*}
$$

Moreover the matrices $\mathbf{M}^{l}$ with $l=1,2$ can be specified as,

$$
\mathbf{M}_{i_{l} j_{l}}^{l}=\int_{G_{l}} b_{j_{l}}\left(x_{l}\right) b_{i_{l}}\left(x_{l}\right) d x_{l}= \begin{cases}\frac{2 h}{3}, & \text { if } i_{l}=j_{l},  \tag{C.2}\\ \frac{h}{6}, & \text { if }\left|i_{l}-j_{l}\right|=1, \\ 0, & \text { if }\left|i_{l}-j_{l}\right|>1 .\end{cases}
$$

In general, the stiffness matrix can be separated into two parts, one part resulting from the diffusion and one jump part. Let us first look at the diffusion part
which corresponds to the bilinear form $a^{B S}(\cdot, \cdot)$ in 2.13,

$$
\begin{aligned}
\mathbf{A}_{\left(i_{1}, j_{1}\right),\left(i_{2}, j_{2}\right)}^{B S} & :=a^{B S}\left(b_{j_{1}}\left(x_{1}\right) b_{j_{2}}\left(x_{2}\right), b_{i_{1}}\left(x_{1}\right) b_{i_{2}}\left(x_{2}\right)\right) \\
& \frac{\sigma_{1}^{2}}{2} \mathbf{S}^{1} \otimes \mathbf{M}^{2}+\frac{\sigma_{2}^{2}}{2} \mathbf{M}^{1} \otimes \mathbf{S}^{2}
\end{aligned}
$$

where we employed the same approach as in C.1. By calculating the integral we get,

$$
\mathbf{S}_{i_{l} j_{l}}^{l}=\int_{G_{k}} b_{j_{l}}^{\prime}\left(x_{l}\right) b_{i_{l}}^{\prime}\left(x_{l}\right) d x_{l}= \begin{cases}\frac{2}{h}, & \text { if } i_{l}=j_{l}  \tag{C.3}\\ -\frac{1}{h}, & \text { if }\left|i_{l}-j_{l}\right|=1, \\ 0, & \text { if }\left|i_{l}-j_{l}\right|>1,\end{cases}
$$

for $l=1,2$. Let us continue with the jump part of the stiffness matrix. By the definition of $a^{J}(\cdot, \cdot)$ and the substitution $y_{l}=x_{l}+z_{l}$ for $l=1,2$ we acquire,

$$
\begin{aligned}
\mathbf{A}_{\left(i_{1}, j_{1}\right),\left(i_{2}, j_{2}\right)}^{J}:= & a^{J}\left(b_{j_{1}}\left(x_{1}\right) b_{j_{2}}\left(x_{2}\right), b_{i_{1}}\left(x_{1}\right) b_{i_{2}}\left(x_{2}\right)\right) \\
= & \int_{G} \int_{\mathbb{R} \backslash\{0\}} b_{j_{1}}^{\prime}\left(x_{1}+z_{1}\right) b_{j_{2}}\left(x_{2}\right) b_{i_{1}}^{\prime}\left(x_{1}\right) b_{i_{2}}\left(x_{2}\right) k_{X^{1}}^{(-2)}\left(z_{1}\right) d z_{1} d x \\
& +\int_{G} \int_{\mathbb{R} \backslash\{0\}} b_{j_{1}}\left(x_{1}\right) b_{j_{2}}^{\prime}\left(x_{2}+z_{2}\right) b_{i_{1}}\left(x_{1}\right) b_{i_{2}}^{\prime}\left(x_{2}\right) k_{X^{2}}^{(-2)}\left(z_{2}\right) d z_{2} d x \\
= & \int_{G_{1}} \int_{G_{1}} b_{j_{1}}^{\prime}\left(y_{1}\right) b_{i_{1}}^{\prime}\left(x_{1}\right) k_{X^{1}}^{(-2)}\left(y_{1}-x_{1}\right) d y_{1} d x_{1} \mathbf{M}_{i_{2} j_{2}}^{2} \\
& +\mathbf{M}_{i_{1} j_{1}}^{1} \int_{G_{2}} \int_{G_{2}} b_{j_{2}}^{\prime}\left(y_{2}\right) b_{i_{2}}^{\prime}\left(x_{2}\right) k_{X^{2}}^{(-2)}\left(y_{2}-x_{2}\right) d y_{2} d x_{2} \\
= & \mathbf{A}_{i_{1}, j_{1}}^{1} \mathbf{M}_{i_{2} j_{2}}^{2}+\mathbf{M}_{i_{1} j_{1}}^{1} \mathbf{A}_{i_{2} j_{2}}^{2} .
\end{aligned}
$$

Hence, $\mathbf{A}^{J}=\mathbf{A}^{1} \otimes \mathbf{M}^{2}+\mathbf{M}^{1} \otimes \mathbf{A}^{2}$. Let us denote the inner grid points of $G_{l}$ by $x_{i_{l}}^{l}$ for $i_{l}=1, \ldots, N$ then we get for the matrices $\mathbf{A}^{l}$ with $l=1,2$,

$$
\begin{align*}
& \mathbf{A}_{i_{l} j_{l}}^{l}=\frac{1}{h}\left[\int_{x_{i_{l}-1}^{l}}^{x_{i_{l}}^{l}} \int_{x_{j_{l}-1}^{l}}^{x_{j_{l}^{l}}^{l}} k_{X^{l}}^{(-2)}\left(y_{l}-x_{l}\right) d y_{l} d x_{l}-\int_{x_{i_{l}-1}^{l}}^{x_{i_{l}}^{l}} \int_{x_{j_{l}}^{l}}^{x_{j_{l}}^{l}+1} k_{X^{l}}^{l}-2\right) \\
&\left.x_{l}-x_{l}\right) d y_{l} d x_{l}  \tag{C.4}\\
&\left.=-\int_{x_{i_{l}}^{l}}^{x_{i_{l}+1}^{l}} \int_{x_{j_{l}-1}^{l}}^{x_{j_{l}}^{l}} k_{X^{l}}^{(-2)}\left(y_{l}-x_{l}\right) d y_{l} d x_{l}+\int_{x_{i_{l}}^{l}}^{x_{i_{l}+1}^{l}} \int_{x_{j_{l}}^{l}}^{x_{j_{l}}^{l}+1} k_{X^{l}}^{(-2)}\left(y_{l}-x_{l}\right) d y_{l} d x_{l}\right] .
\end{align*}
$$

The antiderivative $k_{X^{l}}^{(-2)}\left(x_{l}\right)$ might still have a singularity at 0 as we can see in Figure C. 1 Let us first introduce a change of variables. Let us define for $l=1,2$ and $j_{l}=0,1, \ldots, N-1$,

$$
\begin{align*}
k_{j_{l},+}^{l} & :=\int_{0}^{h} \int_{j_{l} h}^{\left(j_{l}+1\right) h} k_{X^{l}}^{(-2)}\left(y_{l}-x_{l}\right) d y_{l} d x_{l},  \tag{C.5a}\\
k_{j_{l},-}^{l} & :=\int_{j_{l} h}^{\left(j_{l}+1\right) h} \int_{0}^{h} k_{X^{l}}^{(-2)}\left(y_{l}-x_{l}\right) d y_{l} d x_{l} . \tag{C.5b}
\end{align*}
$$

In the expression C.5 we only integrate over the singularity for $j_{l}=0$ for which $k_{0,+}^{l}=k_{0,-}^{l}=: k_{0}^{l}$. We proceed by separating the inner integral at $x_{l}$ and


Figure C.1: $2^{\text {nd }}$ Antiderivative of the CGMY (upper) and the Kou (lower) Lévy densities for parameter values: $C=.5, G=2, M=15, Y=1.2$ and $\lambda=1.2, p=.35, q=.65, \eta_{+}=10, \eta_{-}=7$.
integrating from 0 to $x_{l}$ and from $x_{l}$ to $h$. Then we take the limit from either below or above approaching zero as follows, then we have for $j_{l}=0$,

$$
\begin{align*}
k_{0}^{l} & :=k_{0,+}^{l}=k_{0,-}^{l}=\int_{0}^{h} \int_{0}^{h} k_{X^{l}}^{(-2)}\left(y_{l}-x_{l}\right) d y_{l} d x_{l}  \tag{C.6a}\\
= & \int_{0}^{h} \int_{0}^{x_{l}} k_{X^{l}}^{(-2)}\left(y_{l}-x_{l}\right) d y_{l}+\int_{x_{l}}^{h} k_{X^{l}}^{(-2)}\left(y_{l}-x_{l}\right) d y_{l} d x_{l}  \tag{C.6b}\\
= & \int_{0}^{h} k_{X^{l}}^{(-3)}\left(0^{-}\right)-k_{X^{l}}^{(-3)}\left(-x_{l}\right)+k_{X^{l}}^{(-3)}\left(h-x_{l}\right)-k_{X^{l}}^{(-3)}\left(0^{+}\right) d x_{l}  \tag{C.6c}\\
= & \left(k_{X^{l}}^{(-3)}\left(0^{-}\right)-k_{X^{l}}^{(-3)}\left(0^{+}\right)\right) h \\
& \quad-k_{X^{l}}^{(-4)}\left(0^{-}\right)+k_{X^{l}}^{(-4)}(-h)-k_{X^{l}}^{(-4)}\left(0^{+}\right)+k_{X^{l}}^{(-4)}(h) \tag{C.6d}
\end{align*}
$$

where $k_{X^{l}}^{(-3)}\left(0^{-}\right)=\lim _{x_{l} \uparrow 0} k_{X^{l}}^{(-3)}\left(x_{l}\right)$ and $k_{X^{l}}^{(-3)}\left(0^{+}\right)=\lim _{x_{l} \downarrow 0} k_{X^{l}}^{(-3)}\left(x_{l}\right)$. Moreover, for $j_{l}=1, \ldots, N-1$ we have,

$$
\begin{align*}
k_{j_{l},+}^{l} & =\int_{0}^{h} k_{X^{l}}^{(-3)}\left(\left(j_{l}+1\right) h-x_{l}\right)-k_{X^{l}}^{(-3)}\left(j_{l} h-x_{l}\right) d x_{l}  \tag{C.7a}\\
& =-2 k_{X^{l}}^{(-4)}\left(j_{l} h\right)+k_{X^{l}}^{(-4)}\left(\left(j_{l}+1\right) h\right)+k_{X^{l}}^{(-4)}\left(\left(j_{l}-1\right) h\right),  \tag{C.7b}\\
k_{j_{l},-}^{l} & =\int_{j_{l} h}^{\left(j_{l}+1\right) h} k_{X^{l}}^{(-3)}\left(h-x_{l}\right)-k_{X^{l}}^{(-3)}\left(-x_{l}\right) d x_{l},  \tag{C.7c}\\
& =-2 k_{X^{l}}^{(-4)}\left(-j_{l} h\right)+k_{X^{l}}^{(-4)}\left(-\left(j_{l}-1\right) h\right)+k_{X^{l}}^{(-4)}\left(-\left(j_{l}+1\right) h\right) . \tag{C.7d}
\end{align*}
$$

Using (C.6) and C.7 the expression for (C.4 can be further simplified as,

$$
\begin{array}{rlrl}
\mathbf{A}_{i_{l} i_{l}}^{l} & =\frac{1}{h}\left(2 k_{0}^{l}-k_{1,+}^{l}-k_{1,-}^{l}\right), \\
\mathbf{A}_{i_{l} i_{l}+j}^{l} & =\frac{1}{h}\left(2 k_{j,+}^{l}-k_{j+1,+}^{l}-k_{j-1,+}^{l}\right), & & j=1, \ldots, N-i_{l} \\
\mathbf{A}_{i_{l} i_{l}-j}^{l} & =\frac{1}{h}\left(2 k_{j,-}^{l}-k_{j+1,-}^{l}-k_{j-1,-}^{l}\right), & & j=1, \ldots, i_{l}-1
\end{array}
$$

where $i_{l}=1, \ldots, N$. The results for the one-dimensional case can be found in Hilber, et al. [24, Section 10.6].

## Appendix D

## Computation of Greeks

To compute the Greeks we apply finite differences as discussed in Braess [5] Section 1.3]. The aspect of finite difference is based on the Taylor expansion of a function. The approximation of the first and second derivatives of a function $f(x)$ using Taylor expansions can be found in Smith 48, Section 1.3 and 1.4]. We give the approximation of the partial derivative $\partial_{x y} f(x, y)$ in the next proposition.

Proposition D.1. Let $f \in C^{4}\left(\mathbb{R}^{2}\right)$ then,

$$
\begin{aligned}
\partial_{x y} f(x, y)=\frac{1}{2 h^{2}}[ & f(x+h, y+h)-f(x+h, y)-f(x, y+h)+2 f(x, y) \\
& +f(x-h, y-h)-f(x-h, y)-f(x, y-h)]+\mathcal{O}\left(h^{2}\right)
\end{aligned}
$$

as $h \rightarrow 0$.
Proof. The Taylor expansion of $f$ in $(x+h, y+h)$ around $(x, y)$ reads,

$$
\begin{aligned}
f(x+h, y+h)=f( & x, y)+h\left[\partial_{x} f+\partial_{y} f\right]+\frac{h^{2}}{2!}\left[\partial_{x x} f+2 \partial_{x y} f+\partial_{y y} f\right] \\
& +\frac{h^{3}}{3!}\left[\partial_{x x x} f+3 \partial_{x x y} f+3 \partial_{x y y} f+\partial_{y y y} f\right] \\
& +\frac{h^{4}}{4!}\left[\partial_{x x x x} f\left(\xi_{1}^{1}, \xi_{1}^{2}\right)+4 \partial_{x x x y} f\left(\xi_{1}^{1}, \xi_{1}^{2}\right)+6 \partial_{x x y y} f\left(\xi_{1}^{1}, \xi_{1}^{2}\right)\right. \\
& \left.+4 \partial_{x y y y} f\left(\xi_{1}^{1}, \xi_{1}^{2}\right)+\partial_{y y y y} f\left(\xi_{1}^{1}, \xi_{1}^{2}\right)\right]
\end{aligned}
$$

and,

$$
\begin{aligned}
f(x-h, y-h)=f( & x, y)-h\left[\partial_{x} f+\partial_{y} f\right]+\frac{h^{2}}{2!}\left[\partial_{x x} f+2 \partial_{x y} f+\partial_{y y} f\right] \\
& -\frac{h^{3}}{3!}\left[\partial_{x x x} f+3 \partial_{x x y} f+3 \partial_{x y y} f+\partial_{y y y} f\right] \\
& +\frac{h^{4}}{4!}\left[\partial_{x x x x} f\left(\xi_{2}^{1}, \xi_{2}^{2}\right)+4 \partial_{x x x y} f\left(\xi_{2}^{1}, \xi_{2}^{2}\right)+6 \partial_{x x y y} f\left(\xi_{2}^{1}, \xi_{2}^{2}\right)\right. \\
& \left.+4 \partial_{x y y y} f\left(\xi_{2}^{1}, \xi_{2}^{2}\right)+\partial_{y y y y} f\left(\xi_{2}^{1}, \xi_{2}^{2}\right)\right]
\end{aligned}
$$

with $\left(\xi_{1}^{1}, \xi_{1}^{2}\right) \in(x, x+h) \times(y, y+h)$ and $\left(\xi_{2}^{1}, \xi_{2}^{2}\right) \in(x-h, x) \times(y-h, y)$. Adding up the two expressions above and dividing by $2 h^{2}$ gives Proposition D.1.

Due to the discretisation we only use specific values for the option price, i.e. those on the grid $\left\{\left(x_{p_{1}}^{1}, x_{p_{2}}^{2}\right)\right\}_{p_{1}, p_{2}=0}^{N+1}$ and $\left\{t_{m}\right\}_{m=0}^{M}$. Thus, using the results in Smith [48, Section 1.3 and 1.4] and Proposition D.1 above for the discretised solution $u_{p_{1}, p_{2}}^{m}=u\left(t_{m}, x_{p}\right)$ where $\underline{x}_{p}=\left(x_{p_{1}}^{1}, x_{p_{2}}^{2}\right)$ and $p=(N-1) p_{1}+p_{2}$ with $1 \leq p_{1}, p_{2} \leq N$, we have the following approximations of the first derivatives,

$$
\begin{aligned}
\left.\partial_{x_{j}} u(t, \underline{x})\right|_{(t, \underline{x})=\left(t_{m}, \underline{x}_{p}\right)} \approx \frac{u\left(t_{m}, \underline{x}_{p}+h_{j}\right)-u\left(t_{m}, \underline{x}_{p}-h_{j}\right)}{2 h} & =:\left(\delta_{x_{j}} u\right)_{p}^{m} \\
\left.\partial_{t} u(t, \underline{x})\right|_{(t, \underline{x})=\left(t_{m}, \underline{x}_{p}\right)} \approx \frac{u\left(t_{m+1}, \underline{x}_{p}\right)-u\left(t_{m}, \underline{x}_{p}\right)}{k} & =:\left(\delta_{t}^{+} u\right)_{p}^{m}
\end{aligned}
$$

and second derivatives,

$$
\begin{aligned}
\left.\partial_{x_{j} x_{j}} u(t, \underline{x})\right|_{(t, \underline{x})=\left(t_{m}, \underline{x}_{p}\right)} \approx & \frac{u\left(t_{m}, \underline{x}_{p}+h_{j}\right)-2 u\left(t_{m}, \underline{x}_{p}\right)+u\left(t_{m}, \underline{x}_{p}-h_{j}\right)}{h^{2}} \\
= & \left(\delta_{x_{j} x_{j}}^{2} u\right)_{p}^{m} \\
\left.\partial_{x_{j} x_{l}} u(t, \underline{x})\right|_{(t, \underline{x})=\left(t_{m}, \underline{x}_{p}\right)} \approx & \frac{1}{2 h^{2}}\left[u\left(t_{m}, \underline{x}_{p}+\underline{h}\right)-u\left(t_{m}, \underline{x}_{p}+h_{j}\right)\right. \\
& -u\left(t_{m}, \underline{x}_{p}+h_{l}\right)+2 u\left(t_{m}, \underline{x}_{p}\right)+u\left(t_{m}, \underline{x}_{p}-\underline{h}\right) \\
& \left.-u\left(t_{m}, \underline{x}_{p}-h_{j}\right)-u\left(t_{m}, \underline{x}_{p}-h_{l}\right)\right]=:\left(\delta_{x_{j} x_{l}}^{2} u\right)_{p}^{m}
\end{aligned}
$$

where $j, l=1,2$ and $\underline{h}=(h, h)^{\top}$. Now we have to account for the change in variable to obtain the Greeks. We rewrite $\underline{x}=\left(x_{1}, x_{2}\right)^{\top}$ as follows,

$$
\begin{equation*}
\underline{x}=|\boldsymbol{\Sigma}|^{-1}\binom{\boldsymbol{\Sigma}_{22}\left(\ln \left(s_{1}\right)-\left(r-w_{1}\right)(T-t)\right)-\boldsymbol{\Sigma}_{12}\left(\ln \left(s_{2}\right)-\left(r-w_{2}\right)(T-t)\right)}{\boldsymbol{\Sigma}_{11}\left(\ln \left(s_{2}\right)-\left(r-w_{2}\right)(T-t)\right)-\boldsymbol{\Sigma}_{21}\left(\ln \left(s_{1}\right)-\left(r-w_{1}\right)(T-t)\right)} . \tag{D.1}
\end{equation*}
$$

Using D.1 we find the following expression for $\Delta^{i}(t, \underline{s}):=\partial_{s_{i}} V(T-t, \underline{s})$ with $i=1,2$,

$$
\begin{aligned}
& \Delta^{1}\left(t_{m}, \underline{s}\left(t_{m}, \underline{x}_{p}\right)\right) \approx e^{r\left(T-t_{m}\right)}\left(|\boldsymbol{\Sigma}| s_{1}\left(t_{m}, \underline{x}_{p}\right)\right)^{-1}\left[\boldsymbol{\Sigma}_{22}\left(\delta_{x_{1}} u\right)_{p}^{m}-\boldsymbol{\Sigma}_{21}\left(\delta_{x_{2}} u\right)_{p}^{m}\right] \\
& \Delta^{2}\left(t_{m}, \underline{s}\left(t_{m}, \underline{x}_{p}\right)\right) \approx e^{r\left(T-t_{m}\right)}\left(|\boldsymbol{\Sigma}| s_{2}\left(t_{m}, \underline{x}_{p}\right)\right)^{-1}\left[\boldsymbol{\Sigma}_{11}\left(\delta_{x_{2}} u\right)_{p}^{m}-\boldsymbol{\Sigma}_{12}\left(\delta_{x_{1}} u\right)_{p}^{m}\right]
\end{aligned}
$$

where $s_{i}\left(t_{m}, \underline{x}_{p}\right)=e^{\left(r+w_{i}\right)\left(T-t_{m}\right)+\boldsymbol{\Sigma}_{i} \underline{x}_{p}}$ with $\boldsymbol{\Sigma}_{i}=\left(\boldsymbol{\Sigma}_{i 1}, \boldsymbol{\Sigma}_{i 2}\right)$. Moreover, for $\Gamma^{i l}(t, \underline{s}):=\partial_{s_{i} s_{l}} V(T-t, \underline{s})$ with $i, l=1,2$ we get,

$$
\begin{aligned}
& \Gamma^{11}\left(t_{m}, \underline{s}\left(t_{m}, \underline{x}_{p}\right)\right) \approx e^{r\left(T-t_{m}\right)}\left(|\boldsymbol{\Sigma}| s_{2}\left(t_{m}, \underline{x}_{p}\right)\right)^{-2}\left\{\boldsymbol{\Sigma}_{22}^{2}\left(\delta_{x_{1} x_{1}}^{2} u\right)_{p}^{m}\right. \\
& \left.-\boldsymbol{\Sigma}_{22} \boldsymbol{\Sigma}_{21}\left(\delta_{x_{1} x_{2}}^{2} u\right)_{p}^{m}+\boldsymbol{\Sigma}_{21}^{2}\left(\delta_{x_{2} x_{2}}^{2} u\right)_{p}^{m}\right\}-\left(s_{1}\left(t_{m}, \underline{x}_{p}\right)\right)^{-1} \Delta^{1}\left(t_{m}, \underline{s}\left(t_{m}, \underline{x}_{p}\right)\right) \\
& \Gamma^{22}\left(t_{m}, \underline{s}\left(t_{m}, \underline{x}_{p}\right)\right) \approx e^{r\left(T-t_{m}\right)}\left(|\boldsymbol{\Sigma}| s_{2}\left(t_{m}, \underline{x}_{p}\right)\right)^{-2}\left\{\boldsymbol{\Sigma}_{12}^{2}\left(\delta_{x_{1} x_{1}}^{2} u\right)_{p}^{m}\right. \\
& \left.-\boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{11}\left(\delta_{x_{1} x_{2}}^{2} u\right)_{p}^{m}+\boldsymbol{\Sigma}_{11}^{2}\left(\delta_{x_{2} x_{2}}^{2} u\right)_{p}^{m}\right\}-\left(s_{2}\left(t_{m}, \underline{x}_{p}\right)\right)^{-1} \Delta^{2}\left(t_{m}, \underline{s}\left(t_{m}, \underline{x}_{p}\right)\right), \\
& \Gamma^{21}\left(t_{m}, \underline{s}\left(t_{m}, \underline{x}_{p}\right)\right) \approx e^{r\left(T-t_{m}\right)}|\boldsymbol{\Sigma}|^{-2}\left(s_{1}\left(t_{m}, \underline{x}_{p}\right) s_{2}\left(t_{m}, \underline{x}_{p}\right)\right)^{-1} \\
& \left\{-\boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{22}\left(\delta_{x_{1} x_{1}}^{2} u\right)_{p}^{m}+\left[\boldsymbol{\Sigma}_{11} \boldsymbol{\Sigma}_{22}+\boldsymbol{\Sigma}_{12} \boldsymbol{\Sigma}_{21}\right]\left(\delta_{x_{1} x_{2}}^{2} u\right)_{p}^{m}-\boldsymbol{\Sigma}_{21} \boldsymbol{\Sigma}_{11}\left(\delta_{x_{2} x_{2}}^{2} u\right)_{p}^{m}\right\} .
\end{aligned}
$$

The same holds for $\Gamma^{12}\left(t_{m}, \underline{s}\left(t_{m}, \underline{x}_{p}\right)\right)$ since $\Gamma^{21}\left(t_{m}, \underline{s}\left(t_{m}, \underline{x}_{p}\right)\right)=\Gamma^{12}\left(t_{m}, \underline{s}\left(t_{m}, \underline{x}_{p}\right)\right)$. The theta $\Theta(t, \underline{s}):=\partial_{t} V(T-t, \underline{s})$ is approximated by,

$$
\begin{aligned}
& \Theta\left(t_{m}, \underline{s}\left(t_{m}, \underline{x}_{p}\right)\right) \approx e^{r(T-t)}\left\{-r u\left(t_{m}, \underline{x}_{p}\right)+\left(\delta_{t}^{+} u\right)_{p}^{m}\right\} \\
&+\sum_{i=1}^{2}\left(r+w_{i}\right) s_{i}\left(t_{m}, \underline{x}_{p}\right) \Delta^{i}\left(t_{m}, \underline{s}\left(t_{m}, \underline{x}_{p}\right)\right)
\end{aligned}
$$

for $m=0, \ldots, M-1$. From these finite difference approximation we can conclude that the cases where $|\boldsymbol{\Sigma}|=0$ are problematic since for instance we cannot calculate the Greeks with finite difference. More about the errors that we incur with this finite difference approximation of the derivatives of the option value can be found in Hilber, et al. 21 and Reiss and Wyst 45, where additionally closed-form solutions for Greeks of European style options in the Black-Scholes model can be found.

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