Analytic regularity and *hp*-Discontinuous Galerkin approximation of viscous, incompressible flow in a polygon

Master thesis
Yanchen He
MSc ETH Mathematics

Supervisors:
Prof. Dr. Christoph Schwab, Dr. Carlo Marcati

Seminar for Applied Mathematics, D-MATH
ETH Zürich
August 20th, 2021

Acknowledgement

Firstly, I would like to thank Prof. Christoph Schwab and Dr. Carlo Marcati for the valuable discussions and feedback, without which I could not finish this thesis. I really enjoy the time working with them on this interesting topic.

Also, I want to thank my friends for their support during my thesis writing. Special thanks to Zhenwei Xu for discussing the physical background of the equation with me, Jingzhi Hu for translating references written in German for me and Yuan Chen for usually buying me coffee.

Last but not least, I would like to thank my parents and my brother for their unconditional love and encouragement during my studies in Switzerland.

Abstract

This thesis studies the regularity property of the stationary incompressible Navier-Stokes equation (NSE) with various homogeneous boundary conditions in a polygonal domain $\Omega \subset \mathbb{R}^2$ and examines the performance of mixed hp-discontinuous Galerkin finite element method (hp-DGFEM) on the equation with only Dirichlet boundary condition. We will show that given sufficiently small and weighted analytic data there will be a unique and weighted analytic solution to the equation. Also, we justify that with the analyticity of the solution and with geometrically refined meshes following corresponding linearly increasing polynomial orders, hp-DGFEM leads to exponential convergence of the numerical solution.

Freedom does not consist in any dreamt-of independence from natural laws, but in the knowledge of these laws, and in the possibility this gives of systematically making them work towards definite ends.

Contents

Chapter 1

Introduction

This chapter serves as an introduction to the topics discussed after and to the structure of the thesis.

1.1 The stationary incompressible Navier-Stokes equation, analytic regularity

In Part 1(Chapter 2-5) of this thesis, we study the following stationary incompressible Navier-Stokes equation (NSE) in a 2-dimensional polygon domain Ω with homogeneous Dirichlet, slip and(or) Neumann boundary conditions (details of those conditions will be introduced in Chapter 3):

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} \qquad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \qquad \text{in } \partial\Omega. \end{aligned}$$

In this thesis we always assume that Dirichlet condition is on at least one side of the boundary $\partial\Omega$. This equation is frequently used in the modelling of viscous, incompressible flow.

The following Stokes equation, which can be considered as a linearized version of stationary NSE, is also of interest:

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + \nabla p &= \boldsymbol{f} \qquad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \qquad \text{in } \partial\Omega. \end{aligned}$$

One of the central topics in this thesis is to investigate on the (analytic) regularity of the solution to the stationary incompressible NSE in a polygon given analytic data. This is very important in the study of numerical methods (e.g. *hp*-DGFEM which will be studied later in this thesis) to this equation as higher regularity of the solution usually implies higher order of convergence with carefully chosen numerical methods and in the case of analytic regularity, exponential rate of convergence (See, for example, [15]). A systematic treatment of the NSE or Stokes problem could be found in [27, 43, 14, 12].

For the regularity of the solution to the stationary incompressible NSE or the Stokes equation, if the underlying domain Ω has smooth boundary $\partial\Omega$, then we have classical interior and boundary regularity-shift results for both equations, see [12, Chapter IV and IX] and the references therein. Moreover, analyticity in interior area or regular part of boundaries for both equations are possible given analytic data, see [31, Chapter 6]. We also refer to [29, 20, 13] for analyticity of the solution to the stationary incompressible NSE.

Things become more complicated when we deal with the polygonal domain Ω as standard regularity results of elliptic equations do not work even if data is regular ($\boldsymbol{f} \in L^2(\Omega)^2$ does not imply $(\boldsymbol{u}, p) \in H^2(\Omega)^2 \times H^1(\Omega)$) and the key reason for that is about the corner singularities. This issue triggered the study about the regularity theory of incompressible stationary NSE or Stokes equations in polygons (or polyhedral domains in 3-dimensional case) over the past several decades.

Some studies were conducted in the standard Sobolev spaces: In [22] it was shown that for the Stokes equation in convex polygon with only zero Dirichlet boundary condition $(\boldsymbol{u}, p) \in H^2(\Omega)^2 \times H^1(\Omega)$ can be ensured given $\boldsymbol{f} \in L^2(\Omega)^2$. [8] studies the H^s-regularity of the solution to Stokes equation with zero Dirichlet boundary condition where s is possibly a non-integer. See also [25] for the regularity results of the Stokes equation or NSE in 3-dimension.

For elliptic problems, the singularities near the corner usually contain power-logarithm form[24]. Therefore, it is easy to understand that a powerful tool to remedy the singularities is the *weighted Sobolev spaces*, which weights the derivatives with powers of the distance to the singular points. The Kondrat'ev spaces $W^k_\beta(\Omega)$, introduced in the pioneer work [23], are applied to Stokes equation and stationary incompressible NSE in [33, 32]. For the application of the Kondrat'ev spaces to other elliptic problems with corner singularities, see [30, 7].

Another type of weighted Sobolev spaces $H^{k,l}_\beta(\Omega)$, which was introduced in [4], has more flexibility in weighting derivatives than Kondrat'ev spaces and thus is a better choice for describing the regularity for elliptic problems. The weighted analytic function spaces $B^l_\beta(\Omega)$ was also defined in this reference based on $H^k_\beta(\Omega)$. These spaces were used to study the stationary Stokes problem in [17] and regularity-shift theorem as well as analyticity of the solution were obtained using $H^{k,l}_\beta(\Omega)$ and $B^l_\beta(\Omega)$ here. The analyticity of the solution to stationary incompressible NSE with zero no-slip boundary condition was justified recently in [28] using the weighted spaces $\mathcal{K}^{\varpi,s}_\gamma(\Omega)$ with $s > 2$, which can be regarded as the non-Hilbertian version of $B^l_\beta(\Omega)$. See also [16] for the usage of these spaces in elastic problem and [7] for general elliptic problems.

In Chapter 2-5, we try to extend the result in [17, 28] to the stationary incompressible NSE with mixed homogeneous boundary conditions using Hilbertian spaces $H^{k,l}_\beta(\Omega)$ and $B^l_\beta(\Omega)$ and Theorem 5.2.1 is the main result for this part.

1.2 The *hp*-DGFEM method

Part 2 (Chapter 6-9) of this thesis will be dedicated to the numerical analysis of the stationary incompressible NSE using the *hp*-DGFEM method.

hp-DGFEM combine *h*-refinement (reducing the size of specific elements towards singular points of the analytic solution), *p*-refinement (increasing the order of the polynomial used for approximation), and discontinuous approximation functions. Here the first two ingredients help to achieve exponential convergence in solving many modelling problems with singularities given that the solution exhibits analyticity or weighted analyticity (see, for example, [15] for general weighted analytic solutions or [40] for the Stokes equation in a polygon) while *h*-refinement or *p*-refinement alone only leads to algebraic convergence rates given strong regularity assumption (e.g. $(\boldsymbol{u}, p) \in H^k(\Omega)^2 \times H^{k-1}(\Omega)$ for $k \in \mathbb{N}_{\geq 2}$ in Stokes system). Moreover, the discontinuous approximation functions admit the possibility to use irregular mesh so that there are more choices for the mesh design. Therefore, *hp*-DGFEM is a perfect candidate for the resolution of the stationary incompressible NSE. Recently, [36] showed that *hp*-DGFEM achieves exponential convergence for the stationary incompressible NSE with zero no-slip condition.

In Chapter 6-9, we will study the well-posedness, quasioptimality the the exponential convergence of the *hp*-DGFEM discretization proposed in [36] on the stationary incompressible NSE with zero no-slip condition and Theorem 9.2.1 is the main result.

1.3 Outline of the thesis

In Chapter 2, we set related notations and define some useful function spaces, these notations and spaces follow mainly from [17, 36]. Chapter 3 studies the existence and uniqueness of the stationary incompressible NSE with mixed homogeneous boundary condition. We mainly follow [33] here. Chapter 4 examines the strain formulation of the Stokes equation in a sector. We extend the existence, uniqueness and regularity results in [17] to three possible boundary conditions. This chapter will serve as a preparation for the next chapter on the regularity analysis of NSE in a sector. Chapter 5 dedicates to the proof of the

weighted analyticity of the solution. Chapter 6 introduces a hp-DGFEM discretization to the stationary incompressible NSE with zero Dirichlet boundary condition. In Chapter 7 we present the proof of the existence and uniqueness of the numerical solution to the discretization proposed before. In Chapter 8 we derive abstract error analysis for piecewise analytic solutions. Chapter 9 justifies that the hp-DGFEM discretization proposed before achieves exponential convergence in solving NSE. And the final chapter, Chapter 10, will serve as a conclusion to this thesis, it also contains some further discussion to the main results.

Part I

Analytic regularity of the solution

Chapter 2

Preliminaries

2.1 Notations

We firstly introduce some notations that will be used below. The notations here mainly follow from [36]. We denote by $\mathbb{N}$ the set of all nonnegative integers and $\mathbb{N}_{\geq k}$ the set of all integers that are larger than or equal to k. For $\boldsymbol{v}, \boldsymbol{w} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ and $\underline{\sigma}, \underline{\tau} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$, we write $(\nabla \boldsymbol{v})_{ij} = \partial_j v_i$, $(\nabla \cdot \underline{\sigma})_i = \partial_1 \sigma_{i1} + \partial_2 \sigma_{i2}$, $\underline{\sigma} : \underline{\tau} = \sum_{i,j=1}^{2} \sigma_{ij}\tau_{ij}$ and $(\boldsymbol{v} \otimes \boldsymbol{w})_{ij} = v_i w_j$. Also, $\boldsymbol{v} \cdot \boldsymbol{w}$ corresponds to inner product. It is easy to establish the identity $(\underline{\sigma}\boldsymbol{w}) \cdot \boldsymbol{v} = \sum_{i,j=1}^{2} \boldsymbol{v}_i \sigma_{ij} \boldsymbol{w}_j = \underline{\sigma} : (\boldsymbol{v} \otimes \boldsymbol{w})$. For $\boldsymbol{v} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^3$, we define $(\boldsymbol{v})_{12} := (\boldsymbol{v}_1, \boldsymbol{v}_2)^t$ and $(\boldsymbol{v})_3 := \boldsymbol{v}_3$. For a multi-index $\alpha \in \mathbb{N}^2$, $\alpha = (\alpha_1, \alpha_2)$, we write $|\alpha| = \alpha_1 + \alpha_2$, $D^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}$ and $\mathscr{D}^\alpha = \partial_r^{\alpha_1} \partial_\theta^{\alpha_2}$ and for any $n \in \mathbb{N}_0$ we denote by $\sum_{|\alpha| \leq n} A(\alpha)$ the sum of all $A(\alpha)$ satisfying $|\alpha| \leq n$. For $n \in \mathbb{N}$, $\beta = (\beta_1, \beta_2, \cdots, \beta_n) \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$, we allow componentwise operation and we write $\gamma > (<) \beta$ if $\gamma > (<) \beta_i$. We denote the Euclidean distance between sets and (or) points by $d(\cdot, \cdot)$ and the diameter of a set A by $diam(A)$. For two quantities A and B, we write $A \simeq B$ if there exists a constant $C > 0$ which is independent of discretization parameters such that $CA \leq B \leq C^{-1}A$. For any set V, $|V|$ describes the cardinal number of V.

2.2 Underlying domain, function spaces

Let Ω be a polygon with n vertices A_i and n open edges Γ_i, those vertices and edges are placed in clockwise order with respect to the indices. For $1 \leq i \leq n$, Γ_i connects A_i and A_{i+1}. We always understand the index i modulo n in this thesis. Let $\mathcal{D}$, $\mathcal{G}$ and $\mathcal{N}$ be disjoint subsets of $\{1, 2, \cdots, n\}$ such that $\mathcal{D} \cup \mathcal{G} \cup \mathcal{N} = \{1, 2, \cdots, n\}$. We further assume throughout this thesis that $\mathcal{D} \neq \emptyset$. Set $\Gamma_D = \cup_{i \in \mathcal{D}} \overline{\Gamma_i}$, $\Gamma_G = \cup_{i \in \mathcal{G}} \overline{\Gamma_i}$ and $\Gamma_N = \cup_{i \in \mathcal{N}} \overline{\Gamma_i}$, then Γ_D, Γ_G and Γ_N are either empty or a finite union of one or several Γ_i and they together constitute the boundary of Ω. Moreover, $|\Gamma_D| > 0$. We denote by $\omega_i \in (0, 2\pi)$ the angle at A_i and assume that if $\omega_i = \pi$ then the two edges Γ_{i-1} and Γ_i cannot be in the same set Γ_D, Γ_G or Γ_N. Furthermore, we do not allow, for any $i \in \mathbb{N}_{\geq 1}$, that both $\{i, i+1\} \subset \mathcal{N} \cup \mathcal{G}$ and $\{i, i+1\} \not\subset \mathcal{G}$ hold.

We define the weight function

$$\Phi_{\beta+k} = \prod_{i=1}^{n} (r_i(x))^{\beta_i + k},$$

with $r_i(x) := d(x, A_i)$.

Given a domain $U \subset \Omega$ with Lipschitz boundary, we denote by $W^{n,p}(U)$ for $n \in \mathbb{N}$ and $p \geq 1$ the usual Sobolev space (if $p = 2$ we may write $H^n(U)$ alternatively) and by $C^{n,\gamma}(\overline{U})$ for $n \in \mathbb{N}$ and $\mu \in [0, 1]$ the Hölder space. We write $f \in C^{n,\mu}_{loc}(U)(H^1_{loc}(U))$ if $f|_{\tilde{U}} \in C^{n,\gamma}(\tilde{U})(H^1(\tilde{U})$, respectively) for any compact subset $\tilde{U} \subset \overline{U}$.

We now define some function spaces to describe the singularities near corners. These spaces were introduced in [23, 4, 16, 17].

Definition 2.2.1 (Function spaces on a finite or infinite sector)**.** For a sector $S_\delta := \{(r,\theta) : 0 < r < \delta, 0 < \theta < \omega\}$ where $\delta < \infty$ or $\delta = \infty$ and for two integers $k \geq l \geq 0$, we denote by $H^{k,l}_\beta(S_\delta)$ the weighted Sobolev spaces equipped with norm

$$\|u\|^2_{H^{k,l}_\beta(S_\delta)} := \|u\|^2_{H^{l-1}(S_\delta)} + \sum_{|\alpha|\geq l}^{k} \|r^{\beta+|\alpha|-l} D^\alpha u\|^2_{L^2(S_\delta)},$$

and we denote by $\mathscr{H}^{k,l}_\beta(S_\delta)$ the weighted Sobolev spaces equipped with the norm

$$\|u\|^2_{\mathscr{H}^{k,l}_\beta(S_\delta)} := \|u\|^2_{H^{l-1}(S_\delta)} + \sum_{|\alpha|\geq l}^{k} \|r^{\beta+\alpha_1-l} \mathscr{D}^\alpha u\|^2_{L^2(S_\delta)}.$$

In above norms the term $\|u\|^2_{H^{l-1}(\Omega)}$ shall be omitted if $l = 0$.

We denote by $W^k_\beta(S_\delta)$ the weighted Sobolev spaces equipped with the following norm:

$$\|u\|^2_{W^k_\beta(S_\delta)} = \sum_{|\alpha|\leq k} \|r^{\beta+\alpha_1-k} |\mathscr{D}^\alpha u|\|^2_{L^2(S_\delta)}.$$

Finally, given two fixed constants $C, d \geq 1$, we also define the countably normed spaces $B^l_\beta(\Omega, C, d)$ by

$$B^l_\beta(S_\delta, C, d) := \{u \in \cap_{k\geq l} H^{k,l}_\beta(S_\delta) : \|r^{\beta+k-l} D^\alpha u\|_{L^2(S_\delta)} \leq C d^{k-l}(k-l)! \text{ for } |\alpha| = k \geq l\}.$$

The spaces $\mathscr{B}^l_\beta(S_\delta, C, d)$ are defined similarly as

$$\mathscr{B}^l_\beta(S_\delta, C, d) := \{u \in \cap_{k\geq l} \mathscr{H}^{k,l}_\beta(S_\delta) : \|r^{\beta+\alpha_1-l} \mathscr{D}^\alpha u\|_{L^2(S_\delta)} \leq C d^{k-l}(k-l)! \text{ for } |\alpha| = k \geq l\}.$$

We usually omit C, d if they are not emphasized.

Definition 2.2.2 (Function spaces on an infinite strip)**.** For a strip $D = \{(t,\theta) : t \in \mathbb{R}, 0 < \theta < \omega\}$, a nonnegative integer k and any $h > 0$, we define $H^k_h(D) := \{u \in L^2(D), \|u\|_{H^k_h(D)} < +\infty\}$ with the norm defined by

$$\|u\|^2_{H^k_h(D)} := \sum_{|\alpha|\leq k} \int_D e^{2ht} |\mathscr{D}^\alpha u|^2 dt d\theta.$$

Definition 2.2.3 (Function spaces on a polygon)**.** Given a polygon $\Omega \subset \mathbb{R}^2$ with finite edges. For two integers $l \geq k \geq 0$, we denote by $H^{k,l}_\beta(\Omega)$ the weighted Sobolev spaces equipped with norm

$$\|u\|^2_{H^{k,l}_\beta(\Omega)} := \|u\|^2_{H^{l-1}(\Omega)} + \sum_{|\alpha|\geq l}^{k} \|\Phi_{\beta+|\alpha|-l} D^\alpha u\|^2_{L^2(\Omega)}$$

where the term $\|u\|^2_{H^{l-1}(\Omega)}$ shall be omitted if $l = 0$. We write also $L_\beta(\Omega)$ for $H^{0,0}_\beta(\Omega)$.

For two fixed constants $C, d \geq 1$, we also define the countably normed spaces $B^l_\beta(\Omega, C, d)$ by

$$B^l_\beta(\Omega, C, d) := \{u \in \cap_{k\geq l} H^{k,l}_\beta(\Omega) : \|\Phi_{\beta+k-l} D^\alpha u\|_{L^2(\Omega)} \leq C d^{k-l}(k-l)! \text{ for } |\alpha| = k \geq l\}.$$

C, d may be omitted if the dependence on them is not considered.

Definition 2.2.4 (Trace spaces)**.** For a sector $S_\delta := \{(r,\theta) : 0 < r < \delta, 0 < \theta < \omega\}$ where $\delta < \infty$ or $\delta = \infty$ and for an edge $\Gamma := \{r = 0 \text{ or } \omega\}$, we define the following trace spaces:

$$\|g\|_{W_\beta^{k-\frac{1}{2}}(\Gamma)} = \inf_{G|_\Gamma = g} \|G\|_{W_\beta^k(S_\delta)}.$$

We state here two useful lemmas about the relations between those function spaces defined above.

Lemma 2.2.5. *For $l = 0, 1, 2$ and $k \in \mathbb{N}$, if $\beta \in (0,1)$, then*

$$u \in H_\beta^{k,l}(S_\delta) \Longleftrightarrow u \in \mathscr{H}_\beta^{k,l}(S_\delta), u \in B_\beta^l(S_\delta) \Longleftrightarrow u \in \mathscr{B}_\beta^l(S_\delta).$$

This Lemma is shown in the proof of [4, Theorem 1.1, Theorem 2.1].

Lemma 2.2.6.

- $W_\beta^1(S_\delta) = H_\beta^{1,1}(S_\delta)$.
- $v \in H_\beta^{2,2}(S_\delta)$ *and* $v(0) = 0$ *imply* $v \in W_\beta^2(S_\delta)$.
- $W_\beta^2(S_\delta) \subset H_\beta^{2,2}(S_\delta) \subset C^0(\overline{S_\delta})$ *where* $\delta < +\infty$.

For proof see [4, Lemma A.2] and [5, Lemma 2.1]

Chapter 3

Weak formulation of stationary incompressible NSE in Ω

This chapter is mainly based on [33].

3.1 The incompressible NSE with homogeneous boundary conditions

We consider the following problem:

$$
\begin{aligned}
-\nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega,\\
\nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega,\\
\boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma_D,\\
\begin{cases} \boldsymbol{u}\cdot\boldsymbol{n} = 0 \\ (\underline{\sigma}(\boldsymbol{u},p)\boldsymbol{n})\cdot\boldsymbol{t} = 0 \end{cases} & && \text{on } \Gamma_G,\\
\underline{\sigma}(\boldsymbol{u},p)\boldsymbol{n} &= \boldsymbol{0} && \text{on } \Gamma_N.
\end{aligned}
\tag{3.1}
$$

Here ν is the kinematic velocity, $\boldsymbol{u}$ is the velocity field, p is the pressure, $\boldsymbol{f}$ is the source term, $\boldsymbol{n}$ is the normal vector and $\boldsymbol{t}$ is the tangential vector on the boundary, pointing in clockwise tangential direction. The stress tensor of the fluid is defined as

$$\underline{\sigma}(\boldsymbol{u},p) = -p\underline{I} + \nu(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^t).$$

Homogeneous Dirichlet condition, slip condition and Neumann boundary condition are prescribed correspondingly on Γ_D, Γ_G and Γ_N. For other possible boundary conditions, see [35, Chapter 10.1.1].

Remark 3.1.1 (Remark on boundary conditions)**.** *Our setting on the domain Ω and $\mathcal{D},\mathcal{G},\mathcal{N}$ in Chapter 2 implies that the following conditions on the boundary conditions of the NSE must hold:*

- ***Condition 1: Dirichlet boundary condition is on at least one edge.***
- ***Condition 2: Each corner A_i must have at least one touching edge with Dirichlet boundary condition or have both touching edges with slip boundary condition.***

It is easy to see that if $\mathcal{N} = \emptyset$, which means that we have only Dirichlet boundary condition and Slip boundary condition on $\partial\Omega$, then Condition 2 is satisfied.

The following spaces are needed for the analysis.

$$\mathbf{W} := \{\boldsymbol{u} \in H^1(\Omega)^2 : \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_D \text{ and } \boldsymbol{u}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_G\}, \text{ equipped with } H^1(\Omega)\text{-norm.}$$

$$L_0 = \{q \in L^2(\Omega), \int_\Omega q = 0\}, \text{ equipped with } L^2(\Omega)\text{-norm.}$$

We introduce the variational problem of (3.1). To this end, define the forms:

$$A(\boldsymbol{u},\boldsymbol{v}) = \frac{\nu}{2}\int_\Omega (\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^t) : (\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^t)\, d\boldsymbol{x}, \tag{3.2}$$

$$B(\boldsymbol{u},p) = -\int_\Omega p\nabla\cdot\boldsymbol{u}\, d\boldsymbol{x}, \tag{3.3}$$

$$O(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) = \int_\Omega ((\boldsymbol{w}\cdot\nabla)\boldsymbol{u})\cdot\boldsymbol{v}\, d\boldsymbol{x}. \tag{3.4}$$

The variational problem now reads: Find $(\boldsymbol{u},p) \in \mathbf{W}\times L^2(\Omega)$ such that for all $\boldsymbol{v}\in\mathbf{W}$ and $q\in L^2(\Omega)$,

$$\begin{aligned} A(\boldsymbol{u},\boldsymbol{v}) + O(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) + B(\boldsymbol{v},p) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, d\boldsymbol{x}, \\ B(\boldsymbol{u},q) &= 0. \end{aligned} \tag{3.5}$$

Here we assume that $\boldsymbol{f}\in\mathbf{W}^*$ is given.

Lemma 3.1.2. $\mathbf{W}$ *is a closed subspace of* $H^1(\Omega)^2$.

Proof. It is easy to show that $\mathbf{W}$ is a linear subspace of $H^1(\Omega)^2$. To prove that $\mathbf{W}$ is closed, we select a sequence $\{\boldsymbol{u}_i\}_i \subset \mathbf{W}$ such that $\boldsymbol{u}_i \to \boldsymbol{u}$ in H^1 norm. Then by the property of the trace operator, there exists a constant C such that

$$\|\boldsymbol{u}\cdot\boldsymbol{n} - \boldsymbol{u}_i\cdot\boldsymbol{n}\|_{L^2(\Gamma_D)} \le \|\boldsymbol{u}-\boldsymbol{u}_i\|_{L^2(\Gamma_D)} \le C\|\boldsymbol{u}-\boldsymbol{u}_i\|_{H^1(\Omega)} \to 0,$$

and

$$\|\boldsymbol{u}-\boldsymbol{u}_i\|_{L^2(\Gamma_G)} \le C\|\boldsymbol{u}-\boldsymbol{u}_i\|_{H^1(\Omega)} \to 0.$$

As $\boldsymbol{u}_i\in\mathbf{W}$, $\boldsymbol{u}=\boldsymbol{0}$ on Γ_D and $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on Γ_G and thus $\boldsymbol{u}\in\mathbf{W}$. □

Before we state the result about the existence and uniqueness of the weak solution to (3.5), we list the following inequalities which are useful for the proof.

There exists a positive constant $C_{coer} = C_{coer}(\nu)$ such that for any $\boldsymbol{u}\in\mathbf{W}$

$$A(\boldsymbol{u},\boldsymbol{u}) \ge C_{coer}\|\boldsymbol{u}\|^2_{H^1(\Omega)}. \tag{3.6}$$

This is the so-called *Korn's inequality*. For proof see [44, Theorem 2].

Another useful result is about the inf-sup property of the form $B(\cdot,\cdot)$: If $|\Gamma_N| = 0$, then there exists a constant η that

$$\inf_{0\neq q\in L_0}\sup_{\boldsymbol{0}\neq\boldsymbol{v}\in\mathbf{W}} \frac{|B(\boldsymbol{v},q)|}{\|\boldsymbol{v}\|_{H^1(\Omega)}\|q\|_{L^2(\Omega)}} \ge \eta. \tag{3.7}$$

In the case that $|\Gamma_N| > 0$, then there exists a constant η that

$$\inf_{0\neq q\in L^2(\Omega)} \sup_{\boldsymbol{0}\neq \boldsymbol{v}\in \mathbf{W}} \frac{|B(\boldsymbol{v},q)|}{\|\boldsymbol{v}\|_{H^1(\Omega)}\|q\|_{L^2(\Omega)}} \geq \eta. \tag{3.8}$$

This result is stated in [33, Section 3].

Now we state and prove the theorem about the existence of the weak solution. This result is an extension to [33, Theorem 3.2]

Theorem 3.1.3. *If* $\|\boldsymbol{f}\|_{\mathbf{W}^*} := \sup_{\boldsymbol{0}\neq \boldsymbol{v}\in \mathbf{W}} \frac{|\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}|}{\|\boldsymbol{v}\|_{H^1(\Omega)}} < \frac{C_{coer}^2}{4C_{conv}}$ *(Here* C_{conv} *is the constant appeared in the inequality* $O(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) \leq C_{conv}\|\boldsymbol{w}\|_{H^1(\Omega)}\|\boldsymbol{u}\|_{H^1(\Omega)}\|\boldsymbol{v}\|_{H^1(\Omega)}$*), then there exists a solution* $(\boldsymbol{u},p)$ *to the variational problem (3.5).*

Moreover, if $|\Gamma_N| > 0$*, then* $\boldsymbol{u}$ *is uniquely determined in* $\mathbf{M}_1 := \{\boldsymbol{v}\in\mathbf{W}, \|\boldsymbol{v}\|_{H^1(\Omega)} \leq (\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{C_{conv}}{C_{coer}^2}\|\boldsymbol{f}\|_{\mathbf{W}^*}})\frac{C_{coer}}{C_{conv}}\}$ *and* $p\in L^2(\Omega)$ *associated with that* $\boldsymbol{u}$ *is uniquely determined. Otherwise,* $\boldsymbol{u}$ *is uniquely determined in* $\mathbf{W}$*,* $\boldsymbol{u}\in\mathbf{M}_2 := \{\boldsymbol{v}\in\mathbf{W}, \|\boldsymbol{v}\|_{H^1(\Omega)} \leq \frac{\|\boldsymbol{f}\|_{\mathbf{W}^*}}{C_{coer}}\}$ *and* p *is uniquely determined in* $L_0(\Omega)$*.*

The condition

$$\|\boldsymbol{f}\|_{\mathbf{W}^*} < \frac{C_{coer}^2}{4C_{conv}} \tag{3.9}$$

is the *small data assumption.*

Proof. Consider the following problem which is equivalent to (3.5): Given $\boldsymbol{f}\in\mathbf{W}^*$, find a solution $\boldsymbol{u}\in\mathbf{V} := \{\boldsymbol{v}\in\mathbf{W} : \nabla\cdot\boldsymbol{v} = 0\}$ such that

$$A(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} - O(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}).$$

for any $\boldsymbol{v}\in\mathbf{V}$.

We firstly consider the following auxiliary problem: Given a fixed $\boldsymbol{u}_0\in\mathbf{V}$, find $\boldsymbol{u}\in\mathbf{W}$ such that

$$A(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} - O(\boldsymbol{u}_0;\boldsymbol{u}_0,\boldsymbol{v}), \tag{3.10}$$

for any $\boldsymbol{v}\in\mathbf{V}$. By (3.6) $A(\cdot,\cdot)$ is coercive on $\mathbf{V}$ and clearly it is also continuous with respect to both parameters and $\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} - O(\boldsymbol{u}_0;\boldsymbol{u}_0,\boldsymbol{v})$ is a bounded linear functional for $\boldsymbol{v}\in\mathbf{V}$. Therefore, by Lax-Milgram Lemma we could show that there exists a unique solution $\boldsymbol{u}\in\mathbf{V}$ to (3.10) and we could define a mapping $\Psi : \mathbf{V}\to\mathbf{V}, \boldsymbol{u}_0\mapsto\boldsymbol{u}$. If $\boldsymbol{u}_0\in\mathbf{M}$, then

$$\begin{aligned}
&C_{coer}\|\boldsymbol{u}\|_{\mathbf{W}}^2 \leq A(\boldsymbol{u},\boldsymbol{u}) \leq |\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}| + |O(\boldsymbol{u}_0;\boldsymbol{u}_0,\boldsymbol{u})| \\
&\leq \|\boldsymbol{f}\|_{\mathbf{W}^*}\|\boldsymbol{u}\|_{H^1(\Omega)} + \|\boldsymbol{u}_0\|_{H^1(\Omega)}^2\|\boldsymbol{u}\|_{H^1(\Omega)} \leq (\|\boldsymbol{f}\|_{\mathbf{W}^*} + ((\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{C_{conv}}{C_{coer}^2}\|\boldsymbol{f}\|_{\mathbf{W}^*}})\frac{C_{coer}}{C_{conv}})^2)\|\boldsymbol{u}\|_{H^1(\Omega)} \\
&= C_{coer}(\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{C_{conv}}{C_{coer}^2}\|\boldsymbol{f}\|_{\mathbf{W}^*}})\frac{C_{coer}}{C_{conv}})\|\boldsymbol{u}\|_{H^1(\Omega)},
\end{aligned}$$

so $\boldsymbol{u}\in\mathbf{M}$ and Ψ always maps $\mathbf{M}$ into $\mathbf{M}$.

Moreover, we set $\boldsymbol{u}_i\in\mathbf{M}$ and $\mathbf{M}\ni\hat{\boldsymbol{u}}_i = \Psi(\boldsymbol{u}_i)$ for $i = 1,2$. Then we have, for $i = 1,2$,

$$A(\hat{\boldsymbol{u}}_i, \hat{\boldsymbol{u}}_1 - \hat{\boldsymbol{u}}_2) = \int_\Omega \boldsymbol{f}\cdot(\hat{\boldsymbol{u}}_1 - \hat{\boldsymbol{u}}_i) - O(\boldsymbol{u}_i;\boldsymbol{u}_2,\hat{\boldsymbol{u}}_1 - \hat{\boldsymbol{u}}_2).$$

Therefore,

$$
\begin{aligned}
&\|\hat{\boldsymbol{u}}_1-\hat{\boldsymbol{u}}_2\|^2_{H^1(\Omega)} \le \frac{1}{C_{coer}}|A(\hat{\boldsymbol{u}}_1-\hat{\boldsymbol{u}}_2,\hat{\boldsymbol{u}}_1-\hat{\boldsymbol{u}}_2)| \le \frac{1}{C_{coer}}|A(\hat{\boldsymbol{u}}_1,\hat{\boldsymbol{u}}_1-\hat{\boldsymbol{u}}_2)-A(\hat{\boldsymbol{u}}_2,\hat{\boldsymbol{u}}_1-\hat{\boldsymbol{u}}_2)|\\
&\le \frac{1}{C_{coer}}|-O(\boldsymbol{u}_1;\boldsymbol{u}_1,\hat{\boldsymbol{u}}_1-\hat{\boldsymbol{u}}_2)+O(\boldsymbol{u}_2;\boldsymbol{u}_2,\hat{\boldsymbol{u}}_1-\hat{\boldsymbol{u}}_2)|\\
&\le \frac{1}{C_{coer}}|O(\boldsymbol{u}_1;\boldsymbol{u}_1-\boldsymbol{u}_2,\hat{\boldsymbol{u}}_1-\hat{\boldsymbol{u}}_2)|+\frac{1}{C_{coer}}|O(\boldsymbol{u}_1-\boldsymbol{u}_2;\boldsymbol{u}_2,\hat{\boldsymbol{u}}_1-\hat{\boldsymbol{u}}_2)|\\
&\le \frac{C_{conv}}{C_{coer}}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{H^1(\Omega)}\cdot(\|\boldsymbol{u}_0\|_{H^1(\Omega)}+\|\boldsymbol{u}_1\|_{H^1(\Omega)})\cdot\|\hat{\boldsymbol{u}}_1-\hat{\boldsymbol{u}}_2\|_{H^1(\Omega)}\\
&\le 2(\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{C_{conv}}{C^2_{coer}}\|\boldsymbol{f}\|_{\mathbf{W}^*}})\frac{C_{coer}}{C_{conv}}\cdot\frac{C_{conv}}{C_{coer}}\cdot\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{H^1(\Omega)}\cdot\|\hat{\boldsymbol{u}}_1-\hat{\boldsymbol{u}}_2\|_{H^1(\Omega)}\\
&= (1-\sqrt{1-\frac{4C_{conv}}{C^2_{coer}}\|\boldsymbol{f}\|_{\mathbf{W}^*}})\cdot\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_{H^1(\Omega)}\cdot\|\hat{\boldsymbol{u}}_1-\hat{\boldsymbol{u}}_2\|_{H^1(\Omega)}.
\end{aligned}
$$

It is clear now that Ψ is a contraction mapping in $\mathbf{M}_1$. With the Banach fixed point theorem we know the existence and uniqueness of solution $\boldsymbol{u}$ in $\mathbf{M}_1$.

If $|\Gamma_N|>0$, then the existence and uniqueness of p associated with $\boldsymbol{u}$ obtained above follow from the Lion-Lax-Milgram theorem and from (3.8).

Now assume that $|\Gamma_N|=0$. Then for any solution $(\boldsymbol{u},p)\in\mathbf{V}\times L^2(\Omega)$ to (3.5), we have $\boldsymbol{u}\cdot\boldsymbol{n}=0$ on $\partial\Omega$. By [14, Chapter IV, Lemma 2.2], $O(\boldsymbol{u};\boldsymbol{u},\boldsymbol{u})=0$. Therefore,

$$
C_{coer}\|\boldsymbol{u}\|^2_{H^1(\Omega)} \le A(\boldsymbol{u},\boldsymbol{u}) \le |\int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}|+|O(\boldsymbol{u};\boldsymbol{u},\boldsymbol{u})| \le \|\boldsymbol{f}\|_{\mathbf{W}^*}\|\boldsymbol{u}\|_{H^1(\Omega)} \le \|\boldsymbol{f}\|_{\mathbf{W}^*}\|\boldsymbol{u}\|_{H^1(\Omega)},
$$

which implies that any solution $(\boldsymbol{u},p)\in\mathbf{V}\times L^2(\Omega)$ must satisfy that

$$
\|\boldsymbol{u}\|_{H^1(\Omega)} \le \frac{\|\boldsymbol{f}\|_{\mathbf{W}^*}}{C_{coer}} \le (\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{C_{conv}}{C^2_{coer}}\|\boldsymbol{f}\|_{\mathbf{W}^*}})\frac{C_{coer}}{C_{conv}}.
$$

Therefore we must have $\boldsymbol{u}\in\mathbf{M}_2\subset\mathbf{M}_1$ and thus $\boldsymbol{u}$ is uniquely determined. The existence and uniqueness of p in $L_0(\Omega)$ then follow from the Lion-Lax-Milgram theorem and from (3.7). □

Remark 3.1.4. *We assume before that $|\Gamma_D|>0$ throughout this thesis as the cases that no Dirichlet boundary condition is applied on boundary are rarely seen in physical application. However, we can still study these cases mathematically and rework on the existence, uniqueness and regularity results of the solution to NSE. To examine the existence and uniqueness, we shall introduce the following spaces*

$$
\begin{aligned}
\mathbf{R} &= span\{(1,0)^t,(0,1)^t,(-x_2,x_1)^t\},\\
\mathbf{W}_0 &= \{\boldsymbol{v}\in\mathbf{W}:\int_\Omega \boldsymbol{v}\cdot\boldsymbol{w}=0 \qquad \forall \boldsymbol{w}\in\mathbf{W}\cap\mathbf{R}\}.
\end{aligned}
$$

Note that $A(\boldsymbol{u},\boldsymbol{v})=B(\boldsymbol{v},q)=0$ for any $(\boldsymbol{u},\boldsymbol{v},q)\in H^1(\Omega)^2\times\mathbf{R}\times L^2(\Omega)$. If $\Gamma_D=\emptyset$, then (3.6) does not hold for functions in $\mathbf{W}$. But it still holds for function space $\mathbf{W}_0$[44, Theorem 2]. Therefore, we can prove the existence and uniqueness in a similar way as is in the proof of Theorem 3.1.3. Moreover, it is easy to see later that the weighted analytic regularity established in Theorem 5.2.1 is still correct for the case that $\Gamma_D=\emptyset$(Due to 3.1.1, here we must have $\Gamma_N=\emptyset$).

Remark 3.1.5. *In this thesis, no compatibility condition is required on the data $\boldsymbol{f}$. On the contrary, it is worthy mentioning that compatibility conditions are usually required for the Stokes problem.*

For the following strain formulation of the Stokes problem in a polygon

$$
\begin{aligned}
-\nabla\cdot\underline{\sigma}[\boldsymbol{u},p] &= \boldsymbol{f},\\
\nabla\cdot\boldsymbol{u} &= 0,
\end{aligned}
$$

with the same boundary conditions as (3.1), the weak formulation then reads: Find $(\boldsymbol{u}, p) \in \mathbf{W} \times L^2(\Omega)$ *such that for all* $\boldsymbol{v} \in \mathbf{W}$ *and* $q \in L^2(\Omega)$,

$$\begin{aligned} A(\boldsymbol{u}, \boldsymbol{v}) + B(\boldsymbol{v}, p) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, d\boldsymbol{x}, \\ B(\boldsymbol{u}, q) &= 0. \end{aligned}$$

From the first equation here we obtain that for any $\boldsymbol{v} \in \mathbf{W} \cap \mathbf{R}$,

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, d\boldsymbol{x} = 0.$$

This is the compatibility condition on $\boldsymbol{f}$ *for the Stokes problem.*

Our general strategy to study the regularity of the solution of (3.1) is to transform (3.1) into a linearized Stokes problem by moving the nonlinear term to the right-hand side and then analyze the regularity of the solution to this new Stokes problem. Namely, we will often study the following alternative problem:

$$\begin{aligned} -\nu \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} - (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} \qquad & \text{in } \Omega, \\ \nabla \cdot \boldsymbol{u} = 0 \qquad & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} \qquad & \text{on } \Gamma_D, \\ \left\{ \begin{aligned} \boldsymbol{u} \cdot \boldsymbol{n} &= 0 \\ (\underline{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n}) \cdot \boldsymbol{t} &= 0 \end{aligned} \right. \qquad & \text{on } \Gamma_G, \\ \underline{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n} = \boldsymbol{0} \qquad & \text{on } \Gamma_N. \end{aligned} \tag{3.11}$$

To analyze the regularity of the solution to (3.11), we apply the technique used in [17]. The first step is to study the Stokes problem in a sector.

Chapter 4

Stokes problem in the sector S_∞

This chapter mainly follows Section 4 in [17].

Given the underlying domain as a sector $S_\infty = \{(r,\theta) : 0 < r < +\infty, 0 < \theta < \omega\}$, we denote by Γ_{D,S_∞}, Γ_{G,S_∞} and Γ_{N,S_∞} three subsets of the collection of edges of S_∞ on which Dirichlet condition, slip boundary condition or Neumann condition are prescribed. We study the following Stokes problem in a sector, written in components.

$$\begin{aligned} -\nu(2\partial^2_{x_1}u_1 + \partial_{x_2}(\partial_{x_2}u_1 + \partial_{x_1}u_2)) + \partial_{x_1}p &= f_1, \\ -\nu(2\partial^2_{x_2}u_2 + \partial_{x_1}(\partial_{x_2}u_1 + \partial_{x_1}u_2)) + \partial_{x_2}p &= f_2, \\ \partial_{x_1}u_1 + \partial_{x_2}u_2 &= h, \end{aligned} \tag{4.1}$$

with three possible boundary conditions on two edges $\theta = 0$ and $\theta = \omega$:

- $\boldsymbol{u}|_{\Gamma_{D,S_\infty}} = \boldsymbol{g}^0 = (g_0^0, g_1^0)^t$. (Dirichlet condition)
- $\underline{\sigma}[\boldsymbol{u}, p]\boldsymbol{n}|_{\Gamma_{N,S_\infty}} = \boldsymbol{g}^1 = (g_0^1, g_1^1)^t$. (Neumann boundary condition)
- $\boldsymbol{u}\cdot\boldsymbol{n}|_{\Gamma_{G,S_\infty}} = g_2^0$
  $(\underline{\sigma}[\boldsymbol{u}, p]\boldsymbol{n})\cdot\boldsymbol{t}|_{\Gamma_{G,S_\infty}} = g_2^1$. (Slip boundary condition)

Remark 4.0.1. *(4.1) is equivalent to the strain formulation of the Stokes problem:*

$$\begin{aligned} -\nabla\cdot\underline{\sigma}[\boldsymbol{u}, p] &= \boldsymbol{f}, \\ \nabla\cdot\boldsymbol{u} &= h. \end{aligned} \tag{4.2}$$

If $h = 0$, (4.2) is equivalent to:

$$\begin{aligned} -\nu\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{f}, \\ \nabla\cdot\boldsymbol{u} &= 0. \end{aligned} \tag{4.3}$$

Remark 4.0.2. *If we assume $h = 0$, then with this incompressiblity the Stokes equation can be equipped with an alternative Neumann boundary condition:*

$$-p\boldsymbol{n} + 2\nu\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}} = \boldsymbol{g}^1.$$

This is due to different weak formulations and the Green formula. Also, these two conditions are equivalent with the incompressibility: We write $\boldsymbol{n} = (n_1, n_2)^t$ *and note that*

$$\begin{aligned}
&\underline{\sigma}[\boldsymbol{u}, p] \cdot \boldsymbol{n} = -p\boldsymbol{n} + \nu(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t) \cdot \boldsymbol{n} \\
&= -p\boldsymbol{n} + \nu(2\partial_x u_1 n_1 + (\partial_x u_2 + \partial_y u_1)n_2, 2\partial_y u_2 n_2 + (\partial_x u_2 + \partial_y u_1)n_1)^t \\
&= -p\boldsymbol{n} + \nu(2\partial_x u_1 n_1, 2\partial_y u_2 n_2)^t \\
&= -p\boldsymbol{n} + 2\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}}.
\end{aligned}$$

This boundary condition is studied in [26].

4.1 Polar-coordinate form in the sector S_∞

We introduce the polar coordinates (r, θ) and the polar components $\overline{\boldsymbol{u}} = (u_r, u_\theta)^t$, $\overline{\boldsymbol{f}} = (f_r, f_\theta)^t$ defined by:

$$\overline{\boldsymbol{u}} = Au = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \boldsymbol{u},$$

and

$$\overline{\boldsymbol{f}} = A\boldsymbol{f}.$$

Lemma 4.1.1 ([16], Corollary 4.2)**.** *For finite or infinite sector S_δ and $0 < \beta < 1$, $\boldsymbol{u} \in W^2_\beta(S_\delta)^2 \Longleftrightarrow \overline{\boldsymbol{u}} \in W^2_\beta(S_\delta)^2$.*

Lemma 4.1.2. *Let $\beta \in (0,1)$, $\delta \in (0, +\infty)$. Then $\boldsymbol{f} \in B^0_\beta(S_\delta)^2 \Longleftrightarrow \overline{\boldsymbol{f}} \in \mathscr{B}^0_\beta(S_\delta)^2$.*

Proof. Lemma 2.2.5 implies that $\boldsymbol{f} \in B^0_\beta(S_\delta)^2 \Longleftrightarrow \boldsymbol{f} \in \mathscr{B}^0_\beta(S_\delta)^2$.

It suffices to show that $\boldsymbol{f} \in \mathscr{B}^0_\beta(S_\delta)^2 \Longleftrightarrow \overline{\boldsymbol{f}} \in \mathscr{B}^0_\beta(S_\delta)^2$. We prove that $\boldsymbol{f} \in B^0_\beta(S_\delta)^2 \Longrightarrow \overline{\boldsymbol{f}} \in \mathscr{B}^0_\beta(S_\delta)^2$ and the reverse direction could be proved by a similar argument. We have that there exists $A_0 > 1$ such that, for all $|\alpha| \geq 1$,

$$\begin{aligned}
&\|r^{\alpha_1+\beta}\mathscr{D}^\alpha f_r\|_{L^2(S_\delta)} \\
&\leq \sum_{j=0}^{\alpha_2} \binom{\alpha_2}{j} \|\partial_\theta^j \cos\theta\|_{L^\infty(S_\delta)} \|r^{\alpha_1+\beta}\partial_r^{\alpha_1}\partial_\theta^{\alpha_2-j} f_1\|_{L^2(S_\delta)} + \sum_{j=0}^{\alpha_2} \binom{\alpha_2}{j} \|\partial_\theta^j \sin\theta\|_{L^\infty(S_\delta)} \|r^{\alpha_1+\beta}\partial_r^{\alpha_1}\partial_\theta^{\alpha_2-j} f_2\|_{L^2(S_\delta)} \\
&\leq 2A_0^{|\alpha|}|\alpha|! \sum_{j=0}^{\alpha_2} A_0^{-j} \binom{\alpha_2}{j} \leq 2(2A_0)^{|\alpha|}|\alpha|!.
\end{aligned}$$

The estimate for f_θ follows by the same argument.

□

Lemma 4.1.3 ([16], Lemma 5.1)**.** *Let $\beta \in (0,1)$, $\delta \in (0, +\infty)$ and $\boldsymbol{u}(\boldsymbol{0}) = \boldsymbol{0}$. Then $\boldsymbol{u} \in \mathscr{B}^2_\beta(S_\delta)^2 \Longleftrightarrow \overline{\boldsymbol{u}} \in \mathscr{B}^2_\beta(S_\delta)^2$.*

We remark here that $\mathscr{B}^2_\beta(S_\delta) \subset \mathscr{H}^{2,2}_\beta(S_\delta) = H^{2,2}_\beta(S_\delta) \subset C^0(\overline{S_\delta})$ by Definition 2.2.1, Lemma 2.2.6 and Lemma 2.2.5. Therefore the value of $\boldsymbol{u} \in \mathscr{B}^2_\beta(S_\delta)^2$ at the corner is well-defined.

With polar coordinates and polar components of $\boldsymbol{u}$ we rewrite (4.1) as:

$$\begin{aligned}
-\nu(\overline{\Delta} u_r - \frac{1}{r^2}u_r - \frac{2}{r^2}\partial_\theta u_\theta + \partial_r(\overline{\nabla} \cdot \overline{\boldsymbol{u}})) + \partial_r p &= f_r, \\
-\nu(\overline{\Delta} u_\theta - \frac{1}{r^2}u_\theta + \frac{2}{r^2}\partial_\theta u_r + \frac{1}{r}\partial_\theta(\overline{\nabla} \cdot \overline{\boldsymbol{u}})) + \frac{1}{r}\partial_\theta p &= f_\theta, \\
\partial_r u_r + r^{-1}u_r + r^{-1}\partial_\theta u_\theta &= h.
\end{aligned} \tag{4.4}$$

where $\overline{\Delta} := \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2$, $\overline{\nabla}\cdot\overline{\boldsymbol{u}} := \partial_r u_r + r^{-1}u_r + r^{-1}\partial_\theta u_\theta$ and the three possible boundary conditions are:

- $\overline{\boldsymbol{u}}|_{\Gamma_{D,S_\infty}} = (g_r^0, g_\theta^0)^t = \overline{\boldsymbol{g}}^0$. (Dirichlet condition)
- $\pm\begin{pmatrix} \nu(r^{-1}\partial_\theta u_r + \partial_r u_\theta - r^{-1}u_\theta \\ -p + 2\nu r^{-1}(\partial_\theta u_\theta + u_r)) \end{pmatrix}|_{\Gamma_{N,S_\infty}} = (g_r^1, g_\theta^1)^t = \overline{\boldsymbol{g}}^1$. (Neumann boundary condition)
- $\begin{pmatrix} \pm u_\theta \\ \nu(\partial_r u_\theta + \frac{1}{r}\partial_\theta u_r - \frac{1}{r}u_\theta) \end{pmatrix}|_{\Gamma_{G,S_\infty}} = (g_2^0, g_2^1)^t = \overline{\boldsymbol{g}}^2$. (Slip boundary condition)

To justify the polar-coordinate form of the Stokes equation shown above, we note that by elementary calculus,

$$
\begin{aligned}
\partial_{x_1} &= \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta,\\
\partial_{x_2} &= \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta,\\
\partial_{x_1}^2 &= \cos^2\theta\partial_r^2 + \frac{2\cos\theta\sin\theta}{r^2}\partial_\theta + \frac{\sin^2\theta}{r}\partial_r - \frac{2\cos\theta\sin\theta}{r}\partial_{r\theta} + \frac{\sin^2\theta}{r^2}\partial_\theta^2,\\
\partial_{x_2}^2 &= \sin^2\theta\partial_r^2 - \frac{2\cos\theta\sin\theta}{r^2}\partial_\theta + \frac{\cos^2\theta}{r}\partial_r + \frac{2\cos\theta\sin\theta}{r}\partial_{r\theta} + \frac{\cos^2\theta}{r^2}\partial_\theta^2,\\
\partial_{x_1x_2} &= \cos\theta\sin\theta\partial_r^2 + \frac{\sin^2\theta - \cos^2\theta}{r^2}\partial_\theta + \frac{\cos^2\theta - \sin^2\theta}{r}\partial_{r\theta} - \frac{\sin\theta\cos\theta}{r}\partial_r - \frac{\sin\theta\cos\theta}{r^2}\partial_\theta^2.
\end{aligned}
$$

Therefore we rewrite (4.1) as:

$$
\begin{pmatrix}
\begin{array}{c}
-\nu((\cos^2\theta\partial_r^2 + \frac{2\cos\theta\sin\theta}{r^2}\partial_\theta + \frac{\sin^2\theta}{r}\partial_r - \frac{2\cos\theta\sin\theta}{r}\partial_{r\theta} + \frac{\sin^2\theta}{r^2}\partial_\theta^2 + \overline{\Delta})(\cos\theta u_r - \sin\theta u_\theta)\\
+(\cos\theta\sin\theta\partial_r^2 + \frac{\sin^2\theta-\cos^2\theta}{r^2}\partial_\theta + \frac{\cos^2\theta-\sin^2\theta}{r}\partial_{r\theta} - \frac{\sin\theta\cos\theta}{r}\partial_r - \frac{\sin\theta\cos\theta}{r^2}\partial_\theta^2)(\sin\theta u_r + \cos\theta u_\theta))\\
+(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta)p
\end{array}\\
\begin{array}{c}
-\nu((\sin^2\theta\partial_r^2 - \frac{2\cos\theta\sin\theta}{r^2}\partial_\theta + \frac{\cos^2\theta}{r}\partial_r + \frac{2\cos\theta\sin\theta}{r}\partial_{r\theta} + \frac{\cos^2\theta}{r^2}\partial_\theta^2 + \overline{\Delta})(\sin\theta u_r + \cos\theta u_\theta)\\
+(\cos\theta\sin\theta\partial_r^2 + \frac{\sin^2\theta-\cos^2\theta}{r^2}\partial_\theta + \frac{\cos^2\theta-\sin^2\theta}{r}\partial_{r\theta} - \frac{\sin\theta\cos\theta}{r}\partial_r - \frac{\sin\theta\cos\theta}{r^2}\partial_\theta^2)(\cos\theta u_r - \sin\theta u_\theta))\\
+(\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta)p
\end{array}
\end{pmatrix}
= A^{-1}\overline{\boldsymbol{f}}
\tag{4.5}
$$

and

$$
(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta)(\cos\theta u_r - \sin\theta u_\theta) + (\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta)(\sin\theta u_r + \cos\theta u_\theta) = h. \tag{4.6}
$$

(4.5) is equivalent to

$$
\begin{pmatrix}
\begin{array}{c}
-\nu((2\cos\theta\partial_r^2 - \frac{\sin\theta}{r}\partial_{r\theta} + \frac{\cos\theta}{r^2}\partial_\theta^2 + \frac{2\cos\theta}{r}\partial_r - \frac{3\sin\theta}{r}\partial_\theta - \frac{2\cos\theta}{r^2})u_r\\
+(-\sin\theta\partial_r^2 + \frac{\cos\theta}{r}\partial_{r\theta} - \frac{2\sin\theta}{r^2}\partial_\theta^2 - \frac{\sin\theta}{r}\partial_r - \frac{3\cos\theta}{r}\partial_\theta + \frac{\sin\theta}{r^2})u_\theta\\
+(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta)p
\end{array}\\
\begin{array}{c}
-\nu((2\sin\theta\partial_r^2 + \frac{\cos\theta}{r}\partial_{r\theta} + \frac{\sin\theta}{r^2}\partial_\theta^2 + \frac{2\sin\theta}{r}\partial_r + \frac{3\cos\theta}{r^2}\partial_\theta - \frac{2\sin\theta}{r^2})u_r\\
+(\cos\theta\partial_r^2 + \frac{\sin\theta}{r}\partial_{r\theta} + \frac{2\cos\theta}{r^2}\partial_\theta^2 + \frac{\cos\theta}{r}\partial_r - \frac{3\sin\theta}{r^2}\partial_\theta - \frac{\cos\theta}{r^2})u_\theta)\\
+(\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta)p
\end{array}
\end{pmatrix}
= A^{-1}\overline{\boldsymbol{f}}.
\tag{4.7}
$$

The first and the second equations of (4.4) will then be obtained by multiplying both sides of (4.7) by A and the third equation of (4.4) is clearly equivalent to (4.6).

Now we check the polar-component form of the boundary conditions. Without loss of generality we only consider the edge $\theta = \omega$, then $\boldsymbol{n} = (-\sin\omega, \cos\omega)^t$ and $\boldsymbol{t} = (\cos\omega, \sin\omega)^t$. The polar-component form of Dirichlet boundary condition is straightforward to be obtained. To transform Neumann boundary condition to polar-component form, we write

$$
\begin{aligned}
&\underline{\sigma}[\boldsymbol{u}, p]\boldsymbol{n}|_{\theta=\omega} = \begin{pmatrix} -p\sin\theta \\ p\cos\theta \end{pmatrix}|_{\theta=\omega} \\
&+\nu \begin{pmatrix} 2(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta)(\cos\theta u_r - \sin\theta u_\theta) & (\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta)(\cos\theta u_r - \sin\theta u_\theta) \\ & +(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta)(\sin\theta u_r + \cos\theta u_\theta) \\ (\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta)(\cos\theta u_r - \sin\theta u_\theta) & 2(\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta)(\sin\theta u_r + \cos\theta u_\theta) \\ +(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta)(\sin\theta u_r + \cos\theta u_\theta) & \end{pmatrix}|_{\theta=\omega}\boldsymbol{n} \\
&= \begin{pmatrix} -p\sin\theta \\ p\cos\theta \end{pmatrix}|_{\theta=\omega} \\
&+\nu \begin{pmatrix} 2(\cos^2\theta\partial_r u_r - \cos\theta\sin\theta\partial_r u_\theta & 2\sin\theta\cos\theta\partial_r u_r + (\cos^2\theta - \sin^2\theta)\partial_r u_\theta \\ -\frac{\sin\theta\cos\theta}{r}\partial_\theta u_r + \frac{\sin^2\theta}{r}u_r & +\frac{\cos^2\theta-\sin^2\theta}{r}\partial_\theta u_r - \frac{2\sin\theta\cos\theta}{r}u_r \\ -\frac{\sin^2\theta}{r}\partial_\theta u_\theta - \frac{\sin\theta\cos\theta}{r}u_\theta) & -\frac{2\sin\theta\cos\theta}{r}\partial_\theta u_\theta - \frac{\cos^2\theta-\sin^2\theta}{r}u_\theta \\ 2\sin\theta\cos\theta\partial_r u_r + (\cos^2\theta - \sin^2\theta)\partial_r u_\theta & 2(\sin^2\theta\partial_r u_r + \cos\theta\sin\theta\partial_r u_\theta \\ +\frac{\cos^2\theta-\sin^2\theta}{r}\partial_\theta u_r - \frac{2\sin\theta\cos\theta}{r}u_r & +\frac{\sin\theta\cos\theta}{r}\partial_\theta u_r + \frac{\cos^2\theta}{r}u_r \\ -\frac{2\sin\theta\cos\theta}{r}\partial_\theta u_\theta - \frac{\cos^2\theta-\sin^2\theta}{r}u_\theta & +\frac{\cos^2\theta}{r}\partial_\theta u_\theta - \frac{\sin\theta\cos\theta}{r}u_\theta) \end{pmatrix}|_{\theta=\omega}\boldsymbol{n}
\end{aligned}
$$

The polar-component version of Neumann boundary condition can then be derived by multiplying A on both side of the above equality. Finally, to derive the Slip boundary condition in polar-component form, we note that condition $\boldsymbol{u}\cdot\boldsymbol{n} = g_2^0$ is equivalent to $u_\theta = g_2^0$. Also,

$$
\begin{aligned}
&(\underline{\sigma}[\boldsymbol{u}, p]\boldsymbol{n})\cdot\boldsymbol{t}|_{\theta=\omega} \\
&= (\nu \begin{pmatrix} 2(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta)(\cos\theta u_r - \sin\theta u_\theta) & (\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta)(\cos\theta u_r - \sin\theta u_\theta) \\ & +(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta)(\sin\theta u_r + \cos\theta u_\theta) \\ (\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta)(\cos\theta u_r - \sin\theta u_\theta) & 2(\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta)(\sin\theta u_r + \cos\theta u_\theta) \\ +(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta)(\sin\theta u_r + \cos\theta u_\theta) & \end{pmatrix}|_{\theta=\omega}\boldsymbol{n})\cdot\boldsymbol{t} \\
&= (\nu \begin{pmatrix} 2(\cos^2\theta\partial_r u_r - \cos\theta\sin\theta\partial_r u_\theta & 2\sin\theta\cos\theta\partial_r u_r + (\cos^2\theta - \sin^2\theta)\partial_r u_\theta \\ -\frac{\sin\theta\cos\theta}{r}\partial_\theta u_r + \frac{\sin^2\theta}{r}u_r & +\frac{\cos^2\theta-\sin^2\theta}{r}\partial_\theta u_r - \frac{2\sin\theta\cos\theta}{r}u_r \\ -\frac{\sin^2\theta}{r}\partial_\theta u_\theta - \frac{\sin\theta\cos\theta}{r}u_\theta) & -\frac{2\sin\theta\cos\theta}{r}\partial_\theta u_\theta - \frac{\cos^2\theta-\sin^2\theta}{r}u_\theta \\ 2\sin\theta\cos\theta\partial_r u_r + (\cos^2\theta - \sin^2\theta)\partial_r u_\theta & 2(\sin^2\theta\partial_r u_r + \cos\theta\sin\theta\partial_r u_\theta \\ +\frac{\cos^2\theta-\sin^2\theta}{r}\partial_\theta u_r - \frac{2\sin\theta\cos\theta}{r}u_r & +\frac{\sin\theta\cos\theta}{r}\partial_\theta u_r + \frac{\cos^2\theta}{r}u_r \\ -\frac{2\sin\theta\cos\theta}{r}\partial_\theta u_\theta - \frac{\cos^2\theta-\sin^2\theta}{r}u_\theta & +\frac{\cos^2\theta}{r}\partial_\theta u_\theta - \frac{\sin\theta\cos\theta}{r}u_\theta) \end{pmatrix}|_{\theta=\omega}\boldsymbol{n})\cdot\boldsymbol{t}.
\end{aligned}
$$

This leads to

$$((\underline{\sigma}[\boldsymbol{u}, p]\boldsymbol{n})\cdot\boldsymbol{t})|_{\theta=\omega} = \nu(\partial_r u_\theta + \frac{1}{r}\partial_\theta u_r - \frac{1}{r}u_\theta).$$

Remark 4.1.4. *The symbol "$\pm$" will be omitted in the following, we just need to change the sign of the boundary value according to the edge chosen.*

Remark 4.1.5. *We look at the following Laplacian form of Stokes problem in a sector:*

$$-\nu\Delta u_1 + \partial_{x_1}p = f_1, \tag{4.8}$$

$$-\nu\Delta u_2 + \partial_{x_2}p = f_2, \tag{4.9}$$

$$\partial_{x_1}u_1 + \partial_{x_2}u_2 = h, \tag{4.10}$$

with three possible boundary conditions on two edges $\theta = 0$ and $\theta = \omega$:

- $\boldsymbol{u}|_{\Gamma_{D,S_\infty}} = \boldsymbol{g}^0 = (g_0^0, g_1^0)^t$. *(Dirichlet condition)*
- $\underline{\sigma}[\boldsymbol{u}, p]\boldsymbol{n}|_{\Gamma_{N,S_\infty}} = \boldsymbol{g}^1 = (g_0^1, g_1^1)^t$. *(Neumann boundary condition)*
- $\boldsymbol{u}\cdot\boldsymbol{n}|_{\Gamma_{G,S_\infty}} = g_2^0$, $(\underline{\sigma}[\boldsymbol{u}, p]\boldsymbol{n})\cdot\boldsymbol{t}|_{\Gamma_{G,S_\infty}} = g_2^1$. *(Slip boundary condition)*

We are interested in the polar-component form of this Stokes problem. The polar-component forms of the boundary conditions are clearly the same as those of (4.1). Now we rewrite (4.8)-(4.10) as

$$\begin{pmatrix} -\nu\overline{\Delta}(\cos\theta u_r - \sin\theta u_\theta) + (\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta)p \\ -\nu\overline{\Delta}(\cos\theta u_r - \sin\theta u_\theta) + (\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta)p \end{pmatrix} = A^{-1}\overline{\boldsymbol{f}} \tag{4.11}$$

and

$$(\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta)(\cos\theta u_r - \sin\theta u_\theta) + (\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta)(\sin\theta u_r + \cos\theta u_\theta) = h. \tag{4.12}$$

(4.12) is equivalent to (4.10). Moreover, (4.11) could be rewritten as

$$\begin{pmatrix} -\nu((\cos\theta\partial_r^2 + \frac{\cos\theta}{r}\partial_r + \frac{\cos\theta}{r^2}\partial_\theta^2 - \frac{2\sin\theta}{r^2}\partial_\theta - \frac{\cos\theta}{r^2})u_r \\ -(\sin\theta\partial_r^2 + \frac{\sin\theta}{r}\partial_r + \frac{\sin\theta}{r^2}\partial_\theta^2 + \frac{2\cos\theta}{r^2}\partial_\theta - \frac{\sin\theta}{r^2})u_\theta) + (\cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta)p \\ -\nu((\sin\theta\partial_r^2 + \frac{\sin\theta}{r}\partial_r + \frac{\sin\theta}{r^2}\partial_\theta^2 + \frac{2\cos\theta}{r^2}\partial_\theta - \frac{\sin\theta}{r^2})u_r \\ -(\cos\theta\partial_r^2 + \frac{\cos\theta}{r}\partial_r + \frac{\cos\theta}{r^2}\partial_\theta^2 - \frac{2\sin\theta}{r^2}\partial_\theta - \frac{\cos\theta}{r^2})u_\theta) + (\sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta)p \end{pmatrix} = A^{-1}\overline{\boldsymbol{f}}.$$

Multiply both side by A and we have

$$\begin{pmatrix} -\nu(\partial_r^2 u_r + \frac{1}{r}\partial_r u_r + \frac{1}{r^2}\partial_\theta^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\partial_\theta u_\theta) + \partial_r p \\ -\nu(\partial_r^2 u_\theta + \frac{1}{r}\partial_r u_\theta + \frac{1}{r^2}\partial_\theta^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\partial_\theta u_r) + \frac{1}{r}\partial_\theta p \end{pmatrix} = \overline{\boldsymbol{f}}.$$

Therefore, if we define the differential operator $\overline{L}_{st,1}(\cdot,\cdot)$ as

$$\overline{L}_{st,1}(\boldsymbol{u}, p) = \begin{pmatrix} -\nu(\partial_r^2 u_r + \frac{1}{r}\partial_r u_r + \frac{1}{r^2}\partial_\theta^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\partial_\theta u_\theta) + \partial_r p \\ -\nu(\partial_r^2 u_\theta + \frac{1}{r}\partial_r u_\theta + \frac{1}{r^2}\partial_\theta^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2}\partial_\theta u_r) + \frac{1}{r}\partial_\theta p \\ \partial_r u_r + r^{-1}u_r + r^{-1}\partial_\theta u_\theta \end{pmatrix} \tag{4.13}$$

and the boundary operator $\overline{B}(\cdot,\cdot)$ as

- $\overline{B}(\boldsymbol{u}, p)|_{\Gamma_{D,S_\infty}} = \overline{\boldsymbol{u}}$.
- $\overline{B}(\boldsymbol{u}, p)|_{\Gamma_{N,S_\infty}} = \begin{pmatrix} \nu(r^{-1}\partial_\theta u_r + \partial_r u_\theta - r^{-1}u_\theta) \\ (-p + 2\nu r^{-1}(\partial_\theta u_\theta + u_r)) \end{pmatrix}$.
- $\overline{B}(\boldsymbol{u}, p)|_{\Gamma_{G,S_\infty}} = \begin{pmatrix} u_\theta \\ \nu(\partial_r u_\theta + \frac{1}{r}\partial_\theta u_r - \frac{1}{r}u_\theta) \end{pmatrix}$.

Then the polar-component form of (4.8)-(4.10) could be represented in the following concise form:

$$\overline{L}_{st,1}(\boldsymbol{u}, p) = ((\overline{\boldsymbol{f}})^t, h)^t$$

with boundary conditions

- $\overline{B}(\boldsymbol{u}, p)|_{\Gamma_{D,S_\infty}} = \overline{\boldsymbol{g}}^0$.
- $\overline{B}(\boldsymbol{u}, p)|_{\Gamma_{N,S_\infty}} = \overline{\boldsymbol{g}}^1$.
- $\overline{B}(\boldsymbol{u}, p)|_{\Gamma_{G,S_\infty}} = (g_2^0, g_2^1)^t$.

4.2 Parametric boundary value problem on the strip D and the interval $(0,\omega)$

Now we introduce new variable $t = \log\frac{1}{r}$ in (4.4) and set $\tilde{\boldsymbol{u}}(t,\theta) := \overline{\boldsymbol{u}}(e^{-t},\theta)$, $\tilde{p} := e^{-t}p(e^{-t},\theta)$, $\tilde{\boldsymbol{f}}(t,\theta) := e^{-2t}\overline{\boldsymbol{f}}(e^{-t},\theta)$, $\tilde{h}(t,\theta) = e^{-t}h(e^{-t},\theta)$, $\tilde{\boldsymbol{g}}^l(t,\theta) = e^{-lt}\overline{\boldsymbol{g}}(e^{-t},\theta)$ for $l = 0,1$ and $\tilde{\boldsymbol{g}}^2(t,\theta) = (g_2^0(e^{-t},\theta), e^{-t}g_2^1(e^{-t},\theta))^t$. Let $\hat{\Gamma}_D$, $\hat{\Gamma}_G$ and $\hat{\Gamma}_N$ be the image of the variable transformation $t = \log\frac{1}{r}$ applied on Γ_{D,S_∞}, Γ_{G,S_∞} and Γ_{N,S_∞}. The resulting equation (4.14) is now in the infinite strip $D = \{(t,\theta) : -\infty < t < +\infty, 0 < \theta < \omega\}$:

$$\begin{aligned} -\nu(2(\partial_t^2\tilde{u}_t - \tilde{u}_t) + \partial_\theta^2\tilde{u}_t - \partial_{t\theta}\tilde{u}_\theta - 3\partial_\theta\tilde{u}_\theta) - (\partial_t\tilde{p} + \tilde{p}) &= \tilde{f}_t, \\ -\nu(-\partial_{t\theta}\tilde{u}_r + 3\partial_\theta\tilde{u}_t + \partial_t^2\tilde{u}_\theta + 2\partial_\theta^2\tilde{u}_\theta - \tilde{u}_\theta) + \partial_\theta\tilde{p} &= \tilde{f}_\theta, \\ -\partial_t\tilde{u}_t + \tilde{u}_t + \partial_\theta\tilde{u}_\theta &= \tilde{h}. \end{aligned} \tag{4.14}$$

The three types of boundary conditions now become:

- $\tilde{\boldsymbol{u}}|_{\hat{\Gamma}_D} = \tilde{\boldsymbol{g}}^0$. (Dirichlet condition)
- $\begin{pmatrix} \nu(\partial_\theta\tilde{u}_t + \partial_t\tilde{u}_\theta - \tilde{u}_\theta) \\ -\tilde{p} + 2\nu(\partial_\theta\tilde{u}_\theta + \tilde{u}_t) \end{pmatrix}|_{\hat{\Gamma}_N} = \tilde{\boldsymbol{g}}^1$. (Neumann boundary condition)
- $\begin{pmatrix} \tilde{u}_\theta \\ \nu(-\partial_t\tilde{u}_\theta + \partial_\theta\tilde{u}_t - \tilde{u}_\theta) \end{pmatrix}|_{\hat{\Gamma}_G} = (\tilde{g}_2^0, \tilde{g}_2^1)^t = \tilde{\boldsymbol{g}}^2$. (Slip boundary condition)

We finally apply Fourier transform with respect to t: for any $\lambda = \xi + i\eta \in \mathbb{C}$, we set $[\hat{\boldsymbol{u}}, \hat{p}] = \mathcal{F}[\tilde{\boldsymbol{u}}, \tilde{p}] := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\lambda t}[\tilde{\boldsymbol{u}}, \tilde{p}]\, dt$. We also set $\hat{h} = \mathcal{F}(\tilde{h})$ and $\hat{\boldsymbol{g}}^l = \mathcal{F}\tilde{\boldsymbol{g}}^l$ for $l = 0,1,2$. The equation now becomes a parametric two-point boundary problem on $I = (0,\omega)$:

$$\begin{aligned} -\nu(\partial_\theta^2\hat{u}_t - 2(1+\lambda^2)\hat{u}_t - (3+i\lambda)\partial_\theta\hat{u}_\theta) - (1+i\lambda)\hat{p} &= \hat{f}_t, \\ -\nu(2\partial_\theta^2\hat{u}_\theta - (1+\lambda^2)\hat{u}_\theta + (3-i\lambda)\partial_\theta\hat{u}_t) + \partial_\theta\hat{p} &= \hat{f}_\theta, \\ (1-i\lambda)\hat{u}_t + \partial_\theta\hat{u}_\theta &= \hat{h}. \end{aligned} \tag{4.15}$$

Denote by V_D, V_G and V_N the collection of boundary points corresponding to Dirichlet, Neumann and slip boundary conditions. The three types of boundary conditions now become:

- $\hat{\boldsymbol{u}}|_{V_D} = (\hat{g}_0^0, \hat{g}_0^1)^t =: \hat{\boldsymbol{g}}^0$. (Dirichlet condition)
- $\begin{pmatrix} \nu(\partial_\theta\hat{u}_t - (1+i\lambda)\hat{u}_\theta) \\ -\hat{p} + 2\nu(\partial_\theta\hat{u}_\theta + \hat{u}_t) \end{pmatrix}|_{V_N} = (\hat{g}_1^0, \hat{g}_1^1)^t =: \hat{\boldsymbol{g}}^1$. (Neumann boundary condition)
- $\begin{pmatrix} \hat{u}_\theta \\ \nu(-\partial_t\tilde{u}_\theta + \partial_\theta\tilde{u}_t - \tilde{u}_\theta) \end{pmatrix}|_{V_G} = (\hat{g}_2^0, \hat{g}_2^1)^t =: \hat{\boldsymbol{g}}^2$. (Slip boundary condition)

We may rewrite (4.15) using operator pencil notation. Denote $\partial_\theta = iD$. Then

$$\begin{aligned} \hat{L}(D,\lambda)(\hat{\boldsymbol{u}}, \hat{p}) &= (\hat{\boldsymbol{f}}, h) \qquad \text{on} \quad (0,\omega), \\ \hat{B}(D,\lambda)(\hat{\boldsymbol{u}}, \hat{p}) &= (\hat{\boldsymbol{g}}^0, \hat{\boldsymbol{g}}^1, \hat{\boldsymbol{g}}^2) \qquad \text{on} \quad \{0,\omega\}. \end{aligned} \tag{4.16}$$

Here

$$\hat{L}(D,\lambda) = \begin{pmatrix} \nu D^2 + 2\nu(1+\lambda^2) & \nu(3+i\lambda)iD & -(1+i\lambda) \\ -\nu(3-i\lambda)iD & \nu 2D^2 + \nu(1+\lambda^2) & iD \\ 1-i\lambda & iD & 0 \end{pmatrix}$$

and

$$\hat{B}(D,\lambda)|_{V_D} = A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\hat{B}(D,\lambda)|_{V_N} = A_2 = \begin{pmatrix} \nu i D & -\nu(1+i\lambda) & 0 \\ 2\nu & 2\nu i D & -1 \end{pmatrix},$$
$$\hat{B}(D,\lambda)|_{V_G} = A_3 = \begin{pmatrix} 0 & 1 & 0 \\ iD & -(1+i\lambda) & 0 \end{pmatrix}.$$

The principal parts $\hat{L}_0(D,\lambda)$ and $\hat{B}_0(D,\lambda)$ of those operators are

$$\hat{L}_0(D,\lambda) = \begin{pmatrix} \nu D^2 + 2\nu\lambda^2 & -\nu\lambda D & -i\lambda \\ -\nu\lambda D & 2\nu D^2 + \nu\lambda^2 & iD \\ -i\lambda & iD & 0 \end{pmatrix}$$

and

$$\hat{B}_0(D,\lambda)|_{V_D} = A_1,$$
$$\hat{B}_0(D,\lambda)|_{V_N} = \tilde{A}_2 = \begin{pmatrix} \nu i D & -\nu i\lambda & 0 \\ 2\nu & 2\nu i D & -1 \end{pmatrix},$$
$$\hat{B}_0(D,\lambda)|_{V_G} = \tilde{A}_3 = \begin{pmatrix} 0 & 1 & 0 \\ iD & -i\lambda & 0 \end{pmatrix}.$$

4.3 Parametric Norms $|||\cdot|||^2_{H^k(I)}$

To analyze this boundary value problem and establish a priori estimates for the solution to (4.16), we introduce norms on I: For any natural number k and any $\lambda \in \mathbb{C}$, we set,

$$|||u|||^2_{H^k(I)} := \sum_{l=0}^{k} |\lambda|^{2l} \|u\|^2_{H^{k-l}(I)}.$$

It is easy to verify that there exists a constant C which is independent of λ but depends on k such that

$$\|u\|^2_{H^k(I)} + |\lambda|^{2k}\|u\|^2_{L^2(I)} \le |||u|||^2_{H^k(I)} \le C(\|u\|^2_{H^k(I)} + |\lambda|^{2k}\|u\|^2_{L^2(I)}). \tag{4.17}$$

4.4 A priori estimate on the entire line $\mathbb{R}$

All results stated in this subsection could be found in [17, Section 4.2], for completeness we give proofs to all of them. For any $\phi_0 \in (0, \frac{\pi}{2})$, we set $\Sigma_{\phi_0} := \{\lambda \in \mathbb{C} || \arg\lambda| < \phi_0 \text{ or } |\pi - \arg\lambda| < \phi_0\}$.

We consider the principal part of the system (4.15) defined on $(-\infty, +\infty)$:

$$\hat{L}_0(D,\lambda)(\hat{\boldsymbol{u}}, \hat{p}) = (\hat{\boldsymbol{f}}, \hat{h}) \qquad \text{for } \lambda = \xi + i\eta \text{ with fixed } \eta, \qquad \eta \in \mathbb{R}. \tag{4.18}$$

By using Fourier transform with respect to θ,

$$\hat{\hat{u}}(\xi,\lambda) = \hat{\mathcal{F}}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-i\xi\theta)\hat{\boldsymbol{u}}(\theta,\lambda)\, d\theta,$$

(4.18) is transformed to a parametric linear system

$$\hat{L}_0(\xi,\lambda)(\hat{\hat{\boldsymbol{u}}}, \hat{\hat{p}}) = (\hat{\hat{\boldsymbol{f}}}, \hat{\hat{h}}). \tag{4.19}$$

Here,

$$\hat{L}_0(\xi,\lambda)=\begin{pmatrix}\nu\xi^2+2\nu\lambda^2 & -\nu\lambda\xi & -i\lambda\\ -\nu\lambda\xi & \nu\xi^2+2\nu\lambda^2 & i\xi\\ -i\lambda & i\xi & 0\end{pmatrix}.$$

It is easy to check that $\det(\hat{L}_0(\xi,\lambda))=\nu(\lambda^2+\xi^2)^2$ and

$$(\hat{L}_0(\xi,\lambda))^{-1}=\frac{1}{\nu(\lambda^2+\xi^2)^2}\begin{pmatrix}\xi^2 & \xi\lambda & i\nu\lambda(\xi^2+\lambda^2)\\ \xi\lambda & \lambda^2 & -i\nu\xi(\xi^2+\lambda^2)\\ i\nu\lambda(\xi^2+\lambda^2) & -i\nu\xi(\xi^2+\lambda^2) & 2\nu^2(\xi^2+\lambda^2)^2\end{pmatrix}. \tag{4.20}$$

Theorem 4.4.1. *For arbitrary $\lambda_0>0$ and for any λ satisfying $\lambda\in\Sigma_{\phi_0}$, $|\lambda|>\lambda_0$ and $(\hat{\boldsymbol{f}},\hat{h})\in H^{k-2}(\mathbb{R})^2\times H^{k-1}(\mathbb{R})$, the following system*

$$\hat{L}_0(D,\lambda)(\hat{\boldsymbol{u}},\hat{p})=(\hat{\boldsymbol{f}},\hat{h}) \tag{4.21}$$

has a unique solution $(\hat{\boldsymbol{u}},\hat{p})\in H^k(\mathbb{R})^2\times H^{k-1}(\mathbb{R})$ and we have the following estimate:

$$|||\hat{\boldsymbol{u}}|||^2_{H^k(I)}+|||\hat{p}|||^2_{H^{k-1}(I)}\le C(|||\hat{\boldsymbol{f}}|||^2_{H^{k-2}(I)}+|||\hat{h}|||^2_{H^{k-1}(I)}). \tag{4.22}$$

The constant C here depends only on λ_0,ϕ_0,k.

Proof. For real ξ and for any $\lambda\in\Sigma_{\phi_0}$ with $|\lambda|\ge\lambda_0>0$, $\det(\hat{L}_0)\ne 0$. Therefore (4.19) is uniquely solvable and due to (4.20) we have, for any $k\ge 2$,

$$(1+|\lambda|+|\xi|)^{2k}|\hat{\hat{\boldsymbol{u}}}|^2+(1+|\lambda|+|\xi|)^{2k-2}|\hat{\hat{p}}|^2\le C((1+|\lambda|+|\xi|)^{2k-4}|\hat{\hat{f}}|^2+(1+|\lambda|+|\xi|)^{2k-2}|\hat{\hat{h}}|^2). \tag{4.23}$$

Furthermore, $(\hat{\boldsymbol{u}},\hat{p})=(\hat{\mathcal{F}})^{-1}(\hat{\hat{u}},\hat{\hat{p}})$ is the solution of (4.18), and (4.23) gives (4.22). □

Lemma 4.4.2. *For any integer $k\ge 2$ and for $|\lambda|\ge\lambda_0>0$ with sufficiently large $\lambda_0>0$, it holds for $I=\mathbb{R},(0,+\infty),(0,\omega)$ that*

$$|||(\hat{L}(D,\lambda)(\hat{\boldsymbol{u}},\hat{p})-\hat{L}_0(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}))_{12}|||^2_{H^{k-2}(I)}\le C(|||\hat{\boldsymbol{u}}|||^2_{H^{k-1}(I)}+|||\hat{p}|||^2_{H^{k-2}(I)}),$$
$$|||(\hat{L}(D,\lambda)(\hat{\boldsymbol{u}},\hat{p})-\hat{L}_0(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}))_{3}|||^2_{H^{k-1}(I)}\le C(|||\hat{\boldsymbol{u}}|||^2_{H^{k-1}(I)}+|||\hat{p}|||^2_{H^{k-2}(I)}).$$

Proof. This follows from the definition of the principal operator $\hat{L}_0(D,\lambda)$. □

Theorem 4.4.3. *There exists a constant λ_0 such that for any λ satisfying $\lambda\in\Sigma_{\phi_0}$, $|\lambda|>\lambda_0$ and $(\hat{\boldsymbol{f}},\hat{h})\in H^{k-2}(\mathbb{R})^2\times H^{k-1}(\mathbb{R})$, the following system*

$$\hat{L}(D,\lambda)(\hat{\boldsymbol{u}},\hat{p})=(\hat{\boldsymbol{f}},\hat{h}) \tag{4.24}$$

has a unique solution $(\hat{\boldsymbol{u}},\hat{p})\in H^k(\mathbb{R})^2\times H^{k-1}(\mathbb{R})$ and we have the following estimate:

$$|||\hat{\boldsymbol{u}}|||^2_{H^k(I)}+|||\hat{p}|||^2_{H^{k-1}(I)}\le C(|||\hat{\boldsymbol{f}}|||^2_{H^{k-2}(I)}+|||\hat{h}|||^2_{H^{k-1}(I)}). \tag{4.25}$$

The constant C here depends only on λ_0,ϕ_0,k.

Proof. We have $\det(\hat{L}(\xi,\lambda))=\nu(\lambda^2+(\xi+1)^2)(\lambda^2+(\xi-1)^2)$. Therefore, for real ξ and $\lambda\in\Sigma_{\phi_0}$ with $|\lambda|\ge\lambda_0>0$ where λ_0 is arbitrary, $\det(\hat{L}(\xi,\lambda))\ne 0$ and thus (4.24) has a unique solution $(\hat{\boldsymbol{u}},\hat{p})\in H^k(\mathbb{R})^2\times H^{k-1}(\mathbb{R})$. For this pair of solution, by (4.22),

$$\begin{aligned}&|||\hat{\boldsymbol{u}}|||^2_{H^k(\mathbb{R})}+|||\hat{p}|||^2_{H^{k-1}(\mathbb{R})}\\ &\le C(|||(\hat{L}(D,\lambda)(\hat{\boldsymbol{u}},\hat{p})-\hat{L}_0(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}))_{12}|||^2_{H^{k-2}(\mathbb{R})}+|||(\hat{L}(D,\lambda)(\hat{\boldsymbol{u}},\hat{p})-\hat{L}_0(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}))_{3}|||^2_{H^{k-1}(\mathbb{R})}\\ &+|||\hat{f}|||^2_{H^{k-2}(\mathbb{R})}+|||h|||^2_{H^{k-1}(\mathbb{R})}).\end{aligned}$$

Lemma 4.4.2 yields

$$|||\hat{\boldsymbol{u}}|||^2_{H^k(\mathbb{R})} + |||\hat{p}|||^2_{H^{k-1}(\mathbb{R})} \leq C(|||\hat{f}|||^2_{H^{k-2}(\mathbb{R})} + |||h|||^2_{H^{k-1}(\mathbb{R})} + |||\hat{\boldsymbol{u}}|||^2_{H^{k-1}(\mathbb{R})} + |||\hat{p}|||^2_{H^{k-2}(\mathbb{R})}). \tag{4.26}$$

By definition of the norm $|||\cdot|||^2_{H^k(I)}$, for $|\lambda| > \lambda_0$ with $\lambda_0 > 2C$,

$$|||\hat{\boldsymbol{u}}|||^2_{H^{k-1}(\mathbb{R})} + |||\hat{p}|||^2_{H^{k-2}(\mathbb{R})} \leq \frac{1}{2C}(|||\hat{\boldsymbol{u}}|||^2_{H^k(\mathbb{R})} + |||\hat{p}|||^2_{H^{k-1}(\mathbb{R})})$$

which, with (4.26), lead to the result. □

4.5 A priori estimate on $\mathbb{R}^+$

We consider the principal system on the half-line:

$$\begin{aligned} &\hat{L}_0(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}) = (\hat{\boldsymbol{f}},\hat{h}) \qquad \text{on } I = \mathbb{R}_+ = (0,+\infty) \\ &\hat{B}_0(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}) = (\hat{\boldsymbol{g}}^0,\hat{\boldsymbol{g}}^1,\hat{\boldsymbol{g}}^2) \end{aligned} \tag{4.27}$$

Firstly we assume that $\hat{\boldsymbol{f}} = \boldsymbol{0}$ and $\hat{h} = 0$, in this case (4.27) becomes a homogeneous system. Then the fundamental solutions can be written in the form $e^{b\theta}\underline{E}$ with b satisfying $\det(\hat{L}_0(-ib,\lambda)) = \nu(\lambda^2 - b^2)^2 = 0$.

If $\lambda = 0$, then $b = 0$ and all solutions to (4.27) could not be integrable on I. We now assume that $\operatorname{Re}\lambda < 0$. To obtain solutions which are integrable on I, we should choose $b = \lambda$ and then the two appropriate fundamental solutions are $\boldsymbol{w}_1 = e^{\lambda\theta}(1,i,0)^t$ and $\boldsymbol{w}_2 = e^{\lambda\theta}(1+\lambda\theta, i\lambda\theta, 2i\nu\lambda)^t$. We seek for solutions of the form $(\hat{\boldsymbol{u}},\hat{p})^t = c_1\boldsymbol{w}_1 + c_2\boldsymbol{w}_2$.

For the Dirichlet condition $\hat{B}_0(\hat{\boldsymbol{u}},\hat{p})|_{\theta=0} = \hat{\boldsymbol{g}}^0 = (\hat{g}^0_0,\hat{g}^1_0)^t$, we have $c_1 = -i\hat{g}^1_0$ and $c_2 = \hat{g}^0_0 + i\hat{g}^1_0$. Therefore, for any $l \in \mathbb{N}$

$$|\partial^l_\theta \hat{\boldsymbol{u}}|^2 \leq C\exp(2\theta\operatorname{Re}\lambda)|\lambda|^{2l}(1+\theta^2|\lambda|^2)|\hat{\boldsymbol{g}}^0|^2, \tag{4.28}$$

and

$$|\partial^l_\theta \hat{p}|^2 \leq C\exp(2\theta\operatorname{Re}\lambda)|\lambda|^{2l+2}|\hat{\boldsymbol{g}}^0|^2. \tag{4.29}$$

Note that for fixed $\lambda_0 > 0$, $\lambda \in \Sigma_{\phi_0}$ and any $m \in \mathbb{N}_{\geq 1}$,

$$\int_0^\infty \theta^m \exp(2\theta\operatorname{Re}\lambda)d\theta \leq C_m|\lambda|^{-m-1}. \tag{4.30}$$

Here the constant C_m depends on λ_0.

(4.28), (4.29) with (4.30) imply that for $k \in \mathbb{N}_{\geq 2}$,

$$\int_0^\infty \sum_{l=0}^{k} |\frac{d^l\hat{\boldsymbol{u}}}{d\theta^l}|^2|\lambda|^{2(k-l)} + \sum_{l=0}^{k-1}|\frac{d^l\hat{p}}{d\theta^l}|^2|\lambda|^{2(k-1-l)}\ d\theta \leq C|\lambda|^{2k-1}|\hat{\boldsymbol{g}}^0|^2. \tag{4.31}$$

For Neumann boundary condition $\hat{B}_0(\hat{\boldsymbol{u}},\hat{p})|_{\theta=0} = \hat{\boldsymbol{g}}^1 = (\hat{g}^0_1,\hat{g}^1_1)^t$, we have $c_1 = -\frac{i\nu}{2\lambda}\hat{g}^1_1$ and $c_2 = \frac{\nu}{2\lambda}(\hat{g}^0_1 + i\hat{g}^1_1)$. Therefore,

$$|\partial^l_\theta \hat{\boldsymbol{u}}|^2 \leq C\exp(2\theta\operatorname{Re}\lambda)|\lambda|^{2l-2}(1+\theta^2|\lambda|^2)|\hat{\boldsymbol{g}}^1|^2, \tag{4.32}$$

and

$$|\partial^l_\theta \hat{p}|^2 \leq C\exp(2\theta\operatorname{Re}\lambda)|\lambda|^{2l}|\hat{\boldsymbol{g}}^1|^2. \tag{4.33}$$

The bounds (4.32), (4.33) and (4.30) give, for Neumann boundary condition,

$$\int_0^\infty \sum_{l=0}^{k} |\frac{d^l \hat{\boldsymbol{u}}}{d\theta^l}|^2 |\lambda|^{2(k-l)} + \sum_{l=0}^{k-1} |\frac{d^l \hat{p}}{d\theta^l}|^2 |\lambda|^{2(k-1-l)} \, d\theta \le C|\lambda|^{2k-3}|\hat{\boldsymbol{g}}^1|^2. \tag{4.34}$$

We finally consider the case for slip boundary condition $\hat{B}_0(\hat{\boldsymbol{u}}, \hat{p})|_{\theta=0} = \hat{\boldsymbol{g}}^2 = (\hat{g}_2^0, \hat{g}_2^1)^t$. We have $c_1 = -i\hat{g}_2^0$ and $C_2 = \frac{\hat{g}_2^1}{2\lambda\nu} + i\hat{g}_2^0$. Now:

$$|\frac{d^l \hat{\boldsymbol{u}}}{d\theta^l}|^2 \le C|\lambda|^{2l}(1 + |\lambda|^2\theta^2)(|\lambda|^{-2}|\hat{g}_2^1|^2 + |\hat{g}_2^0|^2)\exp(2\,\mathrm{Re}\,\lambda\theta) \tag{4.35}$$

and

$$|\frac{d^l \hat{p}}{d\theta^l}|^2 \le C|\lambda|^{2l+2}(|\lambda|^{-2}|\hat{g}_2^1|^2 + |\hat{g}_2^0|^2)\exp(2\,\mathrm{Re}\,\lambda\theta). \tag{4.36}$$

With (4.35), (4.36) and (4.30) we obtain that for the slip boundary condition,

$$\int_0^\infty \sum_{l=0}^{k} |\frac{d^l \hat{\boldsymbol{u}}}{d\theta^l}|^2 |\lambda|^{2(k-l)} + \sum_{l=0}^{k-1} |\frac{d^l \hat{p}}{d\theta^l}|^2 |\lambda|^{2(k-1-l)} \, d\theta \le C(|\lambda|^{2k-1}|\hat{g}_2^0|^2 + |\lambda|^{2k-3}|\hat{g}_2^1|^2). \tag{4.37}$$

This bound together with (4.31) and (4.34) help us to derive the following results:

Theorem 4.5.1. *There exists a $\lambda_0>0$ such that, for any $\lambda \in \Sigma_{\phi_0}$ with $|\lambda| \ge \lambda_0>0$, the principal system (4.27) admits, for any $\hat{\boldsymbol{f}} \in H^{k-2}(\mathbb{R}^+)^2$, $\hat{h} \in H^{k-1}(\mathbb{R}^+)$, $k>1$, and any initial data $\hat{\boldsymbol{g}}^l \in \mathbb{C}^2$, $l = 0, 1, 2$, a unique solution $(\hat{\boldsymbol{u}}, \hat{p}) \in H^k(\mathbb{R}^+)^2 \times H^{k-1}(\mathbb{R}^+)$ and satisfies the following estimate: there exists a constant C depending on λ_0, ϕ_0, k such that for $l = 0, 1$, we have*

$$|||\hat{\boldsymbol{u}}|||^2_{H^k(\mathbb{R}^+)} + |||\hat{p}|||^2_{H^{k-1}(\mathbb{R}^+)} \le C(|||\hat{\boldsymbol{f}}|||^2_{H^{k-2}(\mathbb{R}^+)} + |||\hat{h}|||^2_{H^{k-1}(\mathbb{R}^+)} + |\lambda|^{2k-1-2l}|\hat{\boldsymbol{g}}^l|^2) \tag{4.38}$$

and for $l = 2$:

$$|||\hat{\boldsymbol{u}}|||^2_{H^k(\mathbb{R}^+)} + |||\hat{p}|||^2_{H^{k-1}(\mathbb{R}^+)} \le C(|||\hat{\boldsymbol{f}}|||^2_{H^{k-2}(\mathbb{R}^+)} + |||\hat{h}|||^2_{H^{k-1}(\mathbb{R}^+)} + |\lambda|^{2k-1}|\hat{g}_2^0|^2 + |\lambda|^{2k-3}|\hat{g}_2^1|^2). \tag{4.39}$$

Proof. If $\hat{\boldsymbol{f}} = \boldsymbol{0}$ and $\hat{h} = 0$, we can construct the explicit solution as above and (4.31), (4.34) with (4.37) lead to the estimates. Otherwise, we extend $\hat{\boldsymbol{f}}$ and $\hat{h}$ to $\mathbb{R}$ preserving there norms. Theorem 4.4.1 implies that there exist a solution $(\hat{\boldsymbol{u}}_0, \hat{p}_0) \in H^k(\mathbb{R})^2 \times H^{k-1}(\mathbb{R})$ to (4.21) and the estimate (4.22) holds for $(\hat{\boldsymbol{u}}_0, \hat{p}_0)$. We consider the following problem

$$\begin{aligned} &\hat{L}_0(D, \lambda)(\hat{\boldsymbol{v}}, \hat{q}) = (\hat{\boldsymbol{f}}, \hat{h}) \qquad \text{on } I = \mathbb{R}_+ = (0, +\infty), \\ &\hat{B}_0(D, \lambda)(\hat{\boldsymbol{v}}, \hat{q}) = (\hat{\boldsymbol{g}}^0, \hat{\boldsymbol{g}}^1, \hat{\boldsymbol{g}}^2) - \hat{B}_0(D, \lambda)(\hat{\boldsymbol{u}}_0, \hat{p}_0). \end{aligned}$$

Clearly $(\hat{\boldsymbol{u}}, p) := (\hat{\boldsymbol{u}}_0 + \hat{\boldsymbol{v}}, \hat{p}_0 + \hat{q})$ is a solution to (4.27). As in the case that $\hat{\boldsymbol{f}} = \boldsymbol{0}$ and $\hat{h} = 0$, we have, for $l = 0, 1$,

$$|||\hat{\boldsymbol{v}}|||^2_{H^k(\mathbb{R}^+)} + |||\hat{q}|||^2_{H^{k-1}(\mathbb{R}^+)} \le C|\lambda|^{2k-1-2l}|\hat{\boldsymbol{g}}^l - \hat{B}_0(D, \lambda)(\hat{\boldsymbol{u}}_0, \hat{p}_0)|^2, \tag{4.40}$$

and for $l = 2$,

$$|||\hat{\boldsymbol{u}}|||^2_{H^k(\mathbb{R}^+)} + |||\hat{p}|||^2_{H^{k-1}(\mathbb{R}^+)} \le C(|\lambda|^{2k-1}|\hat{g}_2^0 - (\hat{B}_0(D, \lambda)(\hat{\boldsymbol{u}}_0, \hat{p}_0))_1|^2 + |\lambda|^{2k-3}|\hat{g}_2^1 - (\hat{B}_0(D, \lambda)(\hat{\boldsymbol{u}}_0, \hat{p}_0))_2|^2). \tag{4.41}$$

By the definition of $|||\cdot|||_{H^k(I)}$, Lemma 4.5.2 (this will be proved later) and (4.22), we have

$$|\lambda|^{2k-1}|\hat{\boldsymbol{u}}_0(0)|^2 \le C|||\hat{\boldsymbol{u}}_0|||^2_{H^k(\boldsymbol{R})} \le C(|||\hat{\boldsymbol{f}}|||^2_{H^{k-2}(\boldsymbol{R}^+)} + |||h|||^2_{H^{k-1}(\boldsymbol{R}^+)}), \tag{4.42}$$

with

$$|\lambda|^{2k-3}|\hat{\boldsymbol{u}}_0'(0)|^2 \le C|||\hat{\boldsymbol{u}}_0|||^2_{H^k(\boldsymbol{R})} \le C(|||\hat{\boldsymbol{f}}|||^2_{H^{k-2}(\boldsymbol{R}^+)} + |||h|||^2_{H^{k-1}(\boldsymbol{R}^+)}), \tag{4.43}$$

and

$$|\lambda|^{2k-3}|\hat{p}_0(0)|^2 \le C|||\hat{p}_0|||^2_{H^{k-1}(\boldsymbol{R})} \le C(|||\hat{\boldsymbol{f}}|||^2_{H^{k-2}(\boldsymbol{R}^+)} + |||h|||^2_{H^{k-1}(\boldsymbol{R}^+)}). \tag{4.44}$$

By bounding the right-hand side of (4.40) and (4.41) using (4.42)-(4.44) we have (4.38) and (4.39). □

Lemma 4.5.2. *Given $v \in H^1(I)$ with $I = \mathbb{R}, \mathbb{R}^+$ or $(0,\omega)$. Then for any $\lambda_0 > 0$ the following inequality holds for $|\lambda| > \lambda_0$*

$$|\lambda||v(0)|^2 \le C|||v|||^2_{H^1(I)}.$$

Here C depends only on λ_0.

Proof. The Sobolev embedding $H^1(I) \hookrightarrow C^0(\overline{I})$ implies that $v \in C^0(\overline{I})$, and

$$(v(0))^2 = (v(x))^2 + 2\int_0^x v'(t)v(t)\ dt.$$

Therefore, there exists $C(\lambda_0) > 0$ such that for $|\lambda| > \lambda_0$,

$$|\lambda|v(0)^2 \le |\lambda|\int_I |v(t)|^2\ dt + \int_I (|v'(t)|^2 + |\lambda|^2|v(t)|^2)\ dt \le C(|\lambda|^2\|v\|^2_{L^2(I)} + \|v'\|^2_{L^2(I)}) = C|||v|||^2_{H^1(I)}.$$

□

Theorem 4.5.3. *There exists a λ>0 such that, for any $\lambda \in \Sigma_{\phi_0}$ with $|\lambda| \ge \lambda_0$>0, the following system*

$$\begin{aligned} &\hat{L}(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}) = (\hat{\boldsymbol{f}},\hat{h}) \qquad \text{on } I = \mathbb{R}_+ = (0,+\infty) \\ &\hat{B}(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}) = (\hat{\boldsymbol{g}}^0,\hat{\boldsymbol{g}}^1,\hat{\boldsymbol{g}}^2) \end{aligned} \tag{4.45}$$

admits, for any $\hat{\boldsymbol{f}} \in H^{k-2}(\mathbb{R}^+)^2$, $\hat{h} \in H^{k-1}(\mathbb{R}^+)$, $k > 1$, and any initial data $\hat{\boldsymbol{g}}^l \in \mathbb{C}^2$, $l = 0,1,2$, a unique solution $(\hat{\boldsymbol{u}},\hat{p}) \in H^k(\mathbb{R}^+)^2 \times H^{k-1}(\mathbb{R}^+)$ and satisfies all estimates in Theorem 4.5.1.

Proof. (4.45) can be solved in the same way as (4.27). To justify the estimate we claim that there exists $\lambda_0 > 0$ such that for $\lambda \in \Sigma_{\phi_0}$ with $|\lambda| > \lambda_0$,

$$\begin{aligned} &|||((\hat{L}(D,\lambda) - \hat{L}_0(D,\lambda))(\hat{\boldsymbol{u}},\hat{p}))_{12}|||^2_{H^{k-2}(\mathbb{R}^+)} + |||((\hat{L}(D,\lambda) - \hat{L}_0(D,\lambda))(\hat{\boldsymbol{u}},\hat{p}))_3|||^2_{H^{k-1}(\mathbb{R}^+)} \\ &+ |\lambda|^{2k-3}|(\hat{B}(D,\lambda) - \hat{B}_0(D,\lambda))(\hat{\boldsymbol{u}},\hat{p})|^2 \le \frac{1}{2C}(|||\hat{\boldsymbol{u}}|||^2_{H^k(\mathbb{R}^+)} + |||\hat{p}|||^2_{H^{k-1}(\mathbb{R}^+)}). \end{aligned} \tag{4.46}$$

To prove (4.46), we observe by Lemma 4.4.2 that

$$\begin{aligned} &|||((\hat{L}(D,\lambda) - \hat{L}_0(D,\lambda))(\hat{\boldsymbol{u}},\hat{p}))_{12}|||^2_{H^{k-2}(\mathbb{R}^+)} + |||((\hat{L}(D,\lambda) - \hat{L}_0(D,\lambda))(\hat{\boldsymbol{u}},\hat{p}))_3|||^2_{H^{k-1}(\mathbb{R}^+)} \\ &\le C(|||\hat{\boldsymbol{u}}|||^2_{H^{k-1}(\mathbb{R}^+)} + |||\hat{p}|||^2_{H^{k-2}(\mathbb{R}^+)}). \end{aligned} \tag{4.47}$$

Also, it is easy to check that for any boundary conditions,

$$|(\hat{B}(D,\lambda) - \hat{B}_0(D,\lambda))(\hat{\boldsymbol{u}},\hat{p})|^2 \le \max(\nu,1)|\hat{\boldsymbol{u}}(0)|^2.$$

Therefore, by Lemma 4.5.2,

$$|\lambda|^{2k-3}|(\hat{B}(D,\lambda)-\hat{B}_0(D,\lambda))(\hat{\boldsymbol{u}},\hat{p})|^2 \le C|\lambda|^{2k-4}(|\lambda|^2\|\hat{\boldsymbol{u}}\|^2_{L^2(\mathbb{R}^+)}+\|\hat{\boldsymbol{u}}\|^2_{H^1(\mathbb{R}^+)}) \le C|||\hat{\boldsymbol{u}}|||^2_{H^{k-1}(\mathbb{R}^+)}.$$

So, for $|\lambda| \ge \lambda_0 > 0$ with sufficiently large λ_0,

$$\begin{aligned}
&|||((\hat{L}(D,\lambda)-\hat{L}_0(D,\lambda))(\hat{\boldsymbol{u}},\hat{p}))_{12}|||^2_{H^{k-2}(\mathbb{R}^+)}+|||((\hat{L}(D,\lambda)-\hat{L}_0(D,\lambda))(\hat{\boldsymbol{u}},\hat{p}))_3|||^2_{H^{k-1}(\mathbb{R}^+)}\\
&+|\lambda|^{2k-3}|(\hat{B}(D,\lambda)-\hat{B}_0(D,\lambda))(\hat{\boldsymbol{u}},\hat{p})|^2 \le \tilde{C}(|||\hat{\boldsymbol{u}}|||^2_{H^{k-1}(\mathbb{R}^+)}+|||\hat{p}|||^2_{H^{k-2}(\mathbb{R}^+)})\\
&\le \frac{1}{2C}(|||\hat{\boldsymbol{u}}|||^2_{H^k(\mathbb{R}^+)}+|||\hat{p}|||^2_{H^{k-1}(\mathbb{R}^+)}).
\end{aligned} \tag{4.48}$$

(4.47) and (4.48) imply (4.46). □

4.6 A priori estimate on the interval $I = (0, \omega)$

The following theorem holds:

Theorem 4.6.1. *There exists $\lambda_0 > 0$ such that for any $\lambda \in \Sigma_{\phi_0}$, $|\lambda| > \lambda_0$* and any *$k \ge 2$, and for $(\hat{\boldsymbol{f}}, \hat{h}) \in H^{k-2}(I)^2 \times H^{k-1}(I)$ the parametric two-point boundary value problem*

$$\begin{aligned}
&\hat{L}(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}) = (\hat{\boldsymbol{f}},\hat{h}) \qquad \text{on } I=(0,\omega),\\
&\hat{B}(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}) = (\hat{\boldsymbol{g}}^0,\hat{\boldsymbol{g}}^1,\hat{\boldsymbol{g}}^2) \qquad \text{on } \partial I
\end{aligned} \tag{4.49}$$

has a unique solution $(\hat{\boldsymbol{u}}, \hat{p}) \in H^k(I)^2 \times H^{k-1}(I)$.

Moreover, there exists $C = C(\lambda_0, \phi_0) > 0$ such that for all $\lambda \in \Sigma_{\phi_0}$, $|\lambda| > \lambda_0$ we have the a priori estimate

$$\begin{aligned}
&|||\hat{\boldsymbol{u}}|||^2_{H^k(I)}+|||\hat{p}|||^2_{H^{k-1}(I)} \le C(|||\hat{\boldsymbol{f}}|||^2_{H^{k-2}(I)}+|||\hat{h}|||^2_{H^{k-1}(I)}+\\
&|V_D|\cdot|\lambda|^{2k-1}|\hat{\boldsymbol{g}}^0|^2+|V_N|\cdot|\lambda|^{2k-3}|\hat{\boldsymbol{g}}^1|^2+|V_G|\cdot(|\lambda|^{2k-1}|\hat{g}^0_2|^2+|\lambda|^{2k-3}|\hat{g}^1_2|^2)).
\end{aligned}$$

Proof. We set $V_{D,0} = V_D \cap \{0\}$, $V_{N,0} = V_N \cap \{0\}$, $V_{G,0} = V_G \cap \{0\}$, $V_{D,\omega} = V_D \setminus V_{D,0}$, $V_{N,\omega} = V_N \setminus V_{N,0}$ and $V_{G,\omega} = V_G \setminus V_{G,0}$. Let $\{I_i\}_{i=1}^n$ be a covering of $\overline{I} = [0,\omega]$ and $\{\phi_i\}_{i=1}^n$ be a subordinate smooth partition of unity, which means that $\mathrm{supp}(\phi_i) \subset I_i$ and $\sum_{i=1}^n \phi_i = 1$ on $[0,\omega]$. Set $(\hat{\boldsymbol{u}}_i, \hat{p}_i) := \phi_i(\hat{\boldsymbol{u}}, \hat{p})$, $i = 1, \cdots, n$. Then $(\hat{\boldsymbol{u}}_i, \hat{p}_i)$ satisfies,

$$\begin{aligned}
&\hat{L}(D,\lambda)(\hat{\boldsymbol{u}}_i,\hat{p}_i)+\hat{L}_i(D,\lambda)(\hat{\boldsymbol{u}}_i,\hat{p}_i) = (\phi_i\hat{\boldsymbol{f}},\phi_i\hat{h}) \qquad \text{on } I=(0,\omega),\\
&\hat{B}(D,\lambda)(\hat{\boldsymbol{u}}_i,\hat{p}_i)+\hat{B}_i(D,\lambda)(\hat{\boldsymbol{u}}_i,\hat{p}_i) = (\hat{\boldsymbol{g}}^0_i,\hat{\boldsymbol{g}}^1_i,\hat{\boldsymbol{g}}^2_i) \qquad \text{on } \partial I,
\end{aligned} \tag{4.50}$$

where $\hat{L}_i$ are differential operators of one degree lower than $\hat{L}$ and

$$|||(\hat{L}_i(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}))_{12}|||^2_{H^{k-2}(I)}+|||(\hat{L}_i(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}))_3|||^2_{H^{k-1}(I)} \le C(|||\hat{\boldsymbol{u}}|||^2_{H^{k-1}(I)}+|||\hat{p}|||^2_{H^{k-2}(I)}).$$

Moreover, $\hat{B}_i$ are boundary operators that are one order lower than $\hat{B}$ at both endpoints $\{0,\omega\}$. We also have $\hat{\boldsymbol{g}}^0_1 = |V_{D,0}|\hat{\boldsymbol{g}}^0$, $\hat{\boldsymbol{g}}^1_1 = |V_{N,0}|\hat{\boldsymbol{g}}^1$, $\hat{\boldsymbol{g}}^2_1 = |V_{G,0}|\hat{\boldsymbol{g}}^2$, $\hat{\boldsymbol{g}}^0_n = |V_{D,\omega}|\hat{\boldsymbol{g}}^0$, $\hat{\boldsymbol{g}}^1_n = |V_{N,\omega}|\hat{\boldsymbol{g}}^1$ and $\hat{\boldsymbol{g}}^2_n = |V_{G,\omega}|\hat{\boldsymbol{g}}^2$. Furthermore, $\hat{\boldsymbol{g}}^0_i = \boldsymbol{0}$ for $i = 2, \cdots, n-1$ and $\hat{\boldsymbol{g}}^2_i = \boldsymbol{0}$ for $i = 2, \cdots, n-1$. We write $\hat{\boldsymbol{g}}^2_i := (\hat{g}^0_{2,i}, \hat{g}^1_{2,i})^t$. By Lemma 4.5.2 and the definition of the parametric norm $|||\cdot|||_{H^k(I)}$, there exists λ_0 such that for any λ satisfying $|\lambda| > \lambda_0$,

$$\begin{aligned}
&|\lambda|^{2k-3}|\hat{B}_i(D,\lambda)(\hat{\boldsymbol{u}}_i,\hat{p}_i)|^2 \le C|\lambda|^{2k-3}|\hat{\boldsymbol{u}}_i(\omega)|^2\\
&\le \tilde{C}(|\lambda|^{2k-4}\|\hat{\boldsymbol{u}}_i\|^2_{H^1(I)}+|\lambda|^{2k-2}\|\hat{\boldsymbol{u}}_i\|^2_{L^2(I)})\\
&\le \tilde{C}|||\hat{\boldsymbol{u}}_i|||^2_{H^{k-1}(I)}.
\end{aligned}$$

If $i = 1, n$, then (4.50) can be extended to the half-line $\mathbb{R}^+$ and (4.50) can be extended to the whole real line $\mathbb{R}$ otherwise. According to Theorem 4.4.3 and Theorem 4.5.3, there exists a sufficiently large $\lambda_0 > 0$ such that for any $\lambda \in \Sigma_{\phi_0}$ and $|\lambda| > \lambda_0$, the following estimate holds for all $i = 1, 2, \cdots, n$,

$$
\begin{aligned}
&|||\hat{\boldsymbol{u}}_i|||^2_{H^k(I)} + |||\hat{p}_i|||^2_{H^{k-1}(I)} \\
&\leq C(|||\hat{\boldsymbol{f}}_i|||^2_{H^{k-2}(I)} + |||\hat{h}_i|||^2_{H^{k-1}(I)} + |||(\hat{L}_i(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}))_{12}|||^2_{H^{k-2}(I)} + |||(\hat{L}_i(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}))_3|||^2_{H^{k-1}(I)} \\
&+ |V_D| \cdot |\lambda|^{2k-1}|\hat{\boldsymbol{g}}^0_i|^2 + |V_N| \cdot |\lambda|^{2k-3}|\hat{\boldsymbol{g}}^1_i|^2 + |V_G| \cdot (|\lambda|^{2k-1}|\hat{g}^0_{2,i}|^2 + |\lambda|^{2k-3}(|\hat{g}^1_{2,i}|^2 + |\hat{B}_i(D,\lambda)(\hat{\boldsymbol{u}}_i,\hat{p}_i)|^2))).
\end{aligned}
$$

By summing up estimates above with respect to i and noticing that for $|\lambda| \geq \lambda_0 > 2C$, $|||\hat{\boldsymbol{u}}|||^2_{H^{k-1}(\mathbb{R})} + |||\hat{p}|||^2_{H^{k-2}(\mathbb{R})} \leq \frac{1}{2C}(|||\hat{\boldsymbol{u}}|||^2_{H^k(\mathbb{R})} + |||\hat{p}|||^2_{H^{k-1}(\mathbb{R})})$, we have

$$
\begin{aligned}
&|||\hat{\boldsymbol{u}}|||^2_{H^k(\mathbb{R})} + |||\hat{p}|||^2_{H^{k-1}(\mathbb{R})} \\
&\leq C\sum_{i=1}^{n}(|||\hat{\boldsymbol{u}}_i|||^2_{H^k(\mathbb{R})} + |||\hat{p}_i|||^2_{H^{k-1}(\mathbb{R})}) \\
&\leq C(|||\hat{\boldsymbol{f}}|||^2_{H^{k-2}(I)} + |||\hat{h}|||^2_{H^{k-1}(I)} + |\lambda|^{2k-1}|\hat{\boldsymbol{g}}^0|^2 + |\lambda|^{2k-3}|\hat{\boldsymbol{g}}^1|^2 + (|\lambda|^{2k-1}|\hat{g}^0_2|^2 + |\lambda|^{2k-3}|\hat{g}^1_2|^2)).
\end{aligned}
$$

□

4.7 Analysis on the operator pencil $[\hat{L}, \hat{B}]$

The operator pencil $\mathfrak{U}(\lambda) = [\hat{L}(D,\lambda), \hat{B}(D,\lambda)] : H^k(I)^2 \times H^{k-1}(I) \to H^{k-2}(I)^2 \times H^{k-1}(I) \times \mathbb{C}^2 \times \mathbb{C}^2$ depends polynomially on λ. We justify, firstly, the Agranovich and Vishik condition I and II (see [2, Chapter 6]) for this operator pencil.

Lemma 4.7.1 (Condition I)**.** *Given any $\eta \in \mathbb{R}$ and $\lambda \in \Sigma_{\phi_0}$ with $|\eta|+|\lambda| \neq 0$, we have $\det(\hat{L}_0(\eta,\lambda)) \neq 0$. Moreover, $\det(\hat{L}_0(\eta,\lambda)) = 0$ as an equation of η, for any $\lambda \in \Sigma_{\phi_0}$ and $\lambda \neq 0$, has equal number of roots in upper and lower half-planes.*

Proof. We have $\det(\hat{L}_0(\eta,\lambda)) = \nu(\eta^2+\lambda^2)^2$. Since $\lambda \in \Sigma_{\phi_0}$, λ^2 is either a nonnegative real number or a complex number with nonzero imaginary part, in both cases $\nu(\eta^2+\lambda^2)^2 \neq 0$.

To justify the second part of the lemma, we only need to notice that for the equation $\det(\hat{L}_0(\eta,\lambda)) = 0$, $\eta = \pm\lambda$ with double multiplicities. □

Lemma 4.7.2 (Condition II)**.** *For any $\lambda \neq 0$, there exists a unique solution $(\hat{\boldsymbol{u}}, \hat{p})$ to (4.27) such that the solution tends to $[\boldsymbol{0}, 0]$ as $\theta \to +\infty$.*

This is justified in Section 4.5.

By the argument used in [2, Chapter 6], the operator pencil $\mathfrak{U}(\lambda)$ has the *Fredholm property* and thus, by [24, Theorem 1.1.1], the spectrum of this operator pencil consists of infinite isolated eigenvalues (thus at most countable) with finite algebraic multiplicities which do not have any accumulation point in $\mathbb{C}$. Hence, the resolvent $\mathfrak{R}(\lambda) = \mathfrak{U}^{-1}(\lambda)$ is an operator-valued, meromorphic function λ with (at most countable) poles of finite multiplicity. See Appendix A for results on the distribution of eigenvalues.

Theorem 4.7.3. *Let $\mathfrak{L}_h = \{\lambda \in \mathbb{C} : \operatorname{Im}\lambda = h\}$. If $\mathfrak{R}(\lambda)$ has no poles on $\mathfrak{L}_h$, then (4.16) admits a unique solution $(\hat{\boldsymbol{u}}, \hat{p}) \in H^k(I)^2 \times H^{k-1}(I)$ provided $(\hat{\boldsymbol{f}}, \hat{h}, \hat{\boldsymbol{g}}^l) \in H^{k-2}(I)^2 \times H^{k-1}(I) \times \mathbb{C}^2$ for any $l = 0, 1, 2$, and it holds for all $\lambda \in \mathfrak{L}_h$:*

$$
\begin{aligned}
&|||\hat{\boldsymbol{u}}|||^2_{H^k(I)} + |||\hat{p}|||^2_{H^{k-1}(I)} \leq C(|||\hat{\boldsymbol{f}}|||^2_{H^{k-2}(I)} + |||\hat{h}|||^2_{H^{k-1}(I)} + \\
&|V_D| \cdot |\lambda|^{2k-1}|\hat{\boldsymbol{g}}^l|^2 + |V_N| \cdot |\lambda|^{2k-3}|\hat{\boldsymbol{g}}^l|^2 + |V_G| \cdot (|\lambda|^{2k-1}|\hat{g}^0_2|^2 + |\lambda|^{2k-3}|\hat{g}^1_2|^2)),
\end{aligned}
\tag{4.51}
$$

with C independent of $\operatorname{Re}\lambda$.

Proof. As $\mathfrak{R}(\lambda)$ has no poles on $\mathfrak{L}_h$, the solution $(\hat{\boldsymbol{u}}, \hat{p}) \in H^k(I)^2 \times H^{k-1}(I)$ exists if $\lambda \in \mathfrak{L}_h$. Theorem 4.6.1 implies that there exists $\lambda_0 > 0$ such that for $\lambda \in \mathfrak{L}_h$ with $|\lambda| > \lambda_0$, (4.51) holds. If $\lambda \in \mathfrak{L}_h$ and $|\lambda| \leq \lambda_0$, then by assumption $\mathfrak{R}(\lambda)$ is a bounded operator and

$$\|\hat{\boldsymbol{u}}\|^2_{H^k(I)} + \|\hat{p}\|^2_{H^{k-1}(I)} \leq C(\|\hat{\boldsymbol{f}}\|^2_{H^{k-2}(I)} + \|\hat{h}\|^2_{H^{k-1}(I)} +$$
$$|V_D| \cdot |\lambda|^{2k-1}|\hat{\boldsymbol{g}}^l|^2 + |V_G| \cdot |\lambda|^{2k-3}|\hat{\boldsymbol{g}}^l|^2 + |V_N| \cdot (|\lambda|^{2k-1}|\hat{g}_2^0|^2 + |\lambda|^{2k-3}|\hat{g}_2^1|^2)),$$

with

$$|\lambda|^{2k}\|\hat{\boldsymbol{u}}\|^2_{H^k(I)} + |\lambda|^{2k-2}\|\hat{p}\|^2_{H^{k-1}(I)} \leq \tilde{C}(|\lambda|^{2k-4}\|\hat{\boldsymbol{f}}\|^2_{H^{k-2}(I)} + |\lambda|^{2k-2}\|\hat{h}\|^2_{H^{k-1}(I)} +$$
$$|V_D| \cdot |\lambda|^{2k-1}|\hat{\boldsymbol{g}}^l|^2 + |V_G| \cdot |\lambda|^{2k-3}|\hat{\boldsymbol{g}}^l|^2 + |V_N| \cdot (|\lambda|^{2k-1}|\hat{g}_2^0|^2 + |\lambda|^{2k-3}|\hat{g}_2^1|^2)).$$

Here $\tilde{C}$ depends on λ_0 but not on λ. By combining the above two inequalities we have (4.51) and the proof is finished. □

4.8 Regularity of the Stokes problem in the infinite sector

We now transform the regularity result Theorem 4.7.3 back to the strip D and to the sector Q. We need the following lemmas:

Lemma 4.8.1. *If $v(r,\theta) \in W^{k,\beta}(Q), k \geq 0$, then $\overline{v} := v(e^{-t}, \theta) \in H^k_h(D)$ with $h = k - 1 - \beta$ and*

$$C_1\|\overline{v}\|_{H^k_h(D)} \leq \|v\|_{W^k_\beta(Q)} \leq C_2\|\overline{v}\|_{H^k_h(D)}. \tag{4.52}$$

Moreover, for $0 \leq l \leq 1$, $\tilde{v}_l(r,\theta) := e^{(l-2)t}v(e^{-t},\theta) \in H^k_h(D)$ with $h = k + 1 - 1 - \beta$ and

$$C_1\|\tilde{v}_l\|_{H^k_h(D)} \leq \|v\|_{W^k_\beta(Q)} \leq C_2\|\tilde{v}_l\|_{H^k_h(D)}. \tag{4.53}$$

Here all constants are independent of v.

Lemma 4.8.2. *Let $\tilde{v} \in H^k_h(D)$ for any $k \geq 0$, then $\hat{v} = \mathcal{F}(\tilde{v}) \in H^k(I)$, and*

$$C_1\|\tilde{v}\|_{H^k_h(D)} \leq \int_{-\infty+ih}^{\infty+ih} |\|\hat{v}\||_{H^k(I)}\, d\lambda \leq C_2\|\tilde{v}\|_{H^k_h(D)} \tag{4.54}$$

Here all constants are independent of $\tilde{v}$.

These two lemmas can be found in [4].

Lemma 4.8.3. *For $l = 0, 1$, let $\overline{\boldsymbol{G}}^l(r,\theta) \in W^{k-l}_\beta(Q)^2$ with $\overline{\boldsymbol{G}}^0|_{\Gamma_{D,Q}} = \overline{\boldsymbol{g}}^0$ and $\overline{\boldsymbol{G}}^1|_{\Gamma_{N,Q}} = \overline{\boldsymbol{g}}^0$. Also, let $\overline{G}^0_2(r,\theta), \overline{G}^1_2(r,\theta) \in W^{k-l}_\beta(Q)$ with $\overline{G}^0_2|_{\Gamma_{G,Q}} = \overline{g}^0_2$ and $\overline{G}^1_2|_{\Gamma_{G,Q}} = \overline{g}^1_2$. Set, for $l = 0, 1$, $\hat{\boldsymbol{G}}^l = \mathcal{F}(\tilde{\boldsymbol{G}}^l)$ with $\tilde{\boldsymbol{G}}^l = e^{-lt}\overline{\boldsymbol{G}}^l(e^{-t},\theta)$ and $\hat{G}^l_2 = \mathcal{F}(\tilde{G}^l_2)$ with $\tilde{G}^l_2 = e^{-lt}\overline{G}^l_2(e^{-t},\theta)$.*

Then there exists a constant $C > 0$ such that we have, for $l = 0, 1$, $k \geq 2$

$$|\lambda|^{2(k-l-1/2)}|\hat{\boldsymbol{g}}^l|^2 \leq C|\|\hat{\boldsymbol{G}}^l\||^2_{H^{k-l}(I)}, \tag{4.55}$$

and

$$|\lambda|^{2(k-l-1/2)}|\hat{g}^l_2|^2 \leq C|\|\hat{G}^l_2\||^2_{H^{k-l}(I)}. \tag{4.56}$$

Proof. By Lemma 4.8.1 and Lemma 4.8.2, for $l = 0, 1$ we have $\hat{\boldsymbol{G}}^l \in H^{2-l}(I)^2 \subset C^0(\overline{I})^2$ and $\hat{G}_2^l \in H^{2-l}(I) \subset C^0(\overline{I})$. By Lemma 4.5.2, the following inequalities hold for $l = 0, 1$,

$$|\lambda|^{2k-2l-1}|\hat{\boldsymbol{g}}^l|^2 \leq C|\lambda|^{2k-2l-2}|||\hat{\boldsymbol{G}}^l|||^2_{H^1(I)} \leq C|||\hat{\boldsymbol{G}}^l|||^2_{H^{k-1}(I)},$$

and

$$|\lambda|^{2k-2l-1}|\hat{g}_2^l|^2 \leq C|\lambda|^{2k-2l-2}|||\hat{G}^l|||^2_{H^1(I)} \leq C|||\hat{G}^l|||^2_{H^{k-1}(I)},$$

which leads to the result. □

Theorem 4.8.4. *Given $k \geq 2$, let $\overline{\boldsymbol{f}} \in W^{k-2}_\beta(S_\infty)^2, h \in W^{k-1}_\beta(S_\infty), \overline{\boldsymbol{g}}^0 \in W^{k-1/2}_\beta(\Gamma_{D,S_\infty})^2, \overline{\boldsymbol{g}}^1 \in W^{k-3/2}_\beta(\Gamma_{N,S_\infty})^2$. For $l = 0, 1$, let $g_2^l \in W^{k-l-1/2}_\beta(\Gamma_{G,S_\infty})$.*

Then if $\mathfrak{R}(\lambda)$ has no pole on the line $\mathfrak{L}_h = \{\lambda : \operatorname{Im}\lambda = k - 1 - \beta\}$, then the Stokes problem (4.4) has a solution $(\overline{\boldsymbol{u}}, \overline{p}) \in W^k_\beta(S_\infty)^2 \times W^{k-1}_\beta(S_\infty)$ and there holds the a-priori bound

$$\begin{aligned}\|\overline{\boldsymbol{u}}\|_{W^k_\beta(S_\infty)} + \|\overline{p}\|_{W^{k-1}_\beta(S_\infty)} \leq & C(\|\overline{\boldsymbol{f}}\|_{W^{k-2}_\beta(S_\infty)} + \|h\|_{W^{k-1}_\beta(S_\infty)} + \\ & \|\overline{\boldsymbol{g}}^0\|_{W^{k-1/2}_\beta(\Gamma_{D,S_\infty})} + \|\overline{\boldsymbol{g}}^1\|_{W^{k-3/2}_\beta(\Gamma_{N,S_\infty})} + \sum_{l=0,1} \|g_2^1\|_{W^{k-l-1/2}_\beta(\Gamma_{G,S_\infty})}).\end{aligned} \tag{4.57}$$

Proof. The definition of trace space implies that there exist $\overline{\boldsymbol{G}}^l \in W^{k-l}_\beta(S_\infty)^2$ for $l = 0, 1$ and $\overline{\boldsymbol{G}}^2 = (\overline{G}_2^0, \overline{G}_2^1)^t \in W^k_\beta(S_\infty) \times W^{k-1}_\beta(S_\infty)$ such that $\overline{\boldsymbol{G}}^0|_{\Gamma_{D,S_\infty}} = \overline{\boldsymbol{g}}^0$, $\overline{\boldsymbol{G}}^1|_{\Gamma_{N,S_\infty}} = \overline{\boldsymbol{g}}^1$, $\overline{\boldsymbol{G}}^2|_{\Gamma_{G,S_\infty}} = (g_2^0, g_2^1)^t$ and the following relations holds,

$$\begin{aligned}
&\frac{1}{2}\|\overline{\boldsymbol{G}}^0\|_{W^k_\beta(S_\infty)} \leq \|\overline{\boldsymbol{g}}^0\|_{W^{k-\frac{1}{2}}_\beta(\Gamma_{D,S_\infty})} \leq \|\overline{\boldsymbol{G}}^0\|_{W^k_\beta(S_\infty)}, \\
&\frac{1}{2}\|\overline{\boldsymbol{G}}^1\|_{W^{k-1}_\beta(S_\infty)} \leq \|\overline{\boldsymbol{g}}^1\|_{W^{k-\frac{3}{2}}_\beta(\Gamma_{N,S_\infty})} \leq \|\overline{\boldsymbol{G}}^1\|_{W^{k-1}_\beta(S_\infty)}, \\
&\frac{1}{2}\|\overline{G}_2^0\|_{W^k_\beta(S_\infty)} \leq \|g_2^0\|_{W^{k-\frac{1}{2}}_\beta(\Gamma_{G,S_\infty})} \leq \|\overline{G}_2^0\|_{W^k_\beta(S_\infty)}, \\
&\frac{1}{2}\|\overline{G}_2^1\|_{W^{k-1}_\beta(S_\infty)} \leq \|g_2^1\|_{W^{k-\frac{3}{2}}_\beta(\Gamma_{G,S_\infty})} \leq \|\overline{G}_2^1\|_{W^{k-1}_\beta(S_\infty)}.
\end{aligned}$$

We define

$$\tilde{\boldsymbol{f}}(t,\theta) := e^{-2t}\overline{\boldsymbol{f}}(e^{-t},\theta), \qquad \tilde{h}(t,\theta) := e^{-t}h(e^{-t},\theta)$$

with

$$\tilde{\boldsymbol{G}}^l(t,\theta) := e^{-lt}\overline{\boldsymbol{f}}(e^{-t},\theta), \qquad \tilde{G}_2^l(t,\theta) := e^{-lt}\overline{G}_2^l(e^{-t},\theta)$$

for $l = 0, 1$. Furthermore, we define $\hat{\boldsymbol{f}} = \mathcal{F}(\tilde{\boldsymbol{f}})$, $\hat{h} = \mathcal{F}(\tilde{h})$ with $\hat{\boldsymbol{G}}^l = \mathcal{F}(\tilde{\boldsymbol{G}}^l)$, $\hat{G}_2^l = \mathcal{F}(\tilde{G}_2^l)$ for $l = 0, 1$. By Lemma 4.8.1 and (4.8.2), $\hat{\boldsymbol{f}} \in H^{k-2}(I)^2$, $\hat{h} \in H^{k-1}(I)$ and for $l = 0, 1$ we have $\hat{\boldsymbol{G}}^l \in H^{k-l}(I)^2$ with $\hat{G}_2^l \in H^{k-l}(I)$. Moreover, (4.52)-(4.54) holds for all the functions above. By Theorem 4.7.3, for $k \geq 2$ system (4.49) exhibits a unique solution $(\hat{\boldsymbol{u}}, \hat{p}) \in H^k(I)^2 \times H^{k-1}(I)$ and (4.51) holds. By Lemma (4.8.3),

$$\begin{aligned}
&|||\hat{\boldsymbol{u}}|||^2_{H^k(I)} + |||\hat{p}|||^2_{H^{k-1}(I)} \leq C(|||\hat{\boldsymbol{f}}|||^2_{H^{k-2}(I)} + |||\hat{h}|||^2_{H^{k-1}(I)} + \\
&|V_D| \cdot |||\hat{\boldsymbol{G}}^0|||^2_{H^k(I)} + |V_N| \cdot |||\hat{\boldsymbol{G}}^1|||^2_{H^{k-1}(I)} + |V_G| \cdot (|||\hat{G}_2^0|||^2_{H^k(I)} + |||\hat{G}_2^1|||^2_{H^{k-1}(I)})).
\end{aligned}$$

As $\mathfrak{R}$ has no pole on the line $\mathfrak{L}_h = \{\lambda : \operatorname{Im}\lambda = h\}$ with $h = k-1-\beta$, we have $\frac{1}{\sqrt{2\pi}}\int_{-\infty+ih}^{+\infty+ih} e^{i\lambda t}(\hat{\boldsymbol{u}},\hat{p}) =: (\tilde{\boldsymbol{u}},\tilde{p}) \in H_h^k(D)^2 \times H_h^{k-1}(D)$, and by Lemma 4.8.2 we have

$$\begin{aligned}&|||\tilde{\boldsymbol{u}}|||^2_{H_h^k(D)} + |||\tilde{p}|||^2_{H_h^{k-1}(D)} \le C(|||\tilde{\boldsymbol{f}}|||^2_{H_h^{k-2}(D)} + |||\tilde{h}|||^2_{H_h^{k-1}(D)} + \\ &|V_D|\cdot|||\tilde{\boldsymbol{G}}^0|||^2_{H_h^k(D)} + |V_N|\cdot|||\tilde{\boldsymbol{G}}^1|||^2_{H_h^{k-1}(D)} + |V_G|\cdot(|||\tilde{G}_2^0|||^2_{H_h^k(D)} + |||\tilde{G}_2^1|||^2_{H_h^{k-1}(D)})).\end{aligned}$$

Finally, Lemma 4.8.1 implies that $(\tilde{\boldsymbol{u}},\tilde{p})(\log(\frac{1}{r}),\theta) =: (\overline{\boldsymbol{u}},p)(r,\theta) \in W_\beta^k(S_\infty)^2 \times W_\beta^{k-1}(S_\infty)$ and

$$\begin{aligned}\|\overline{\boldsymbol{u}}\|_{W_\beta^k(S_\infty)} + \|\overline{p}\|_{W_\beta^{k-1}(S_\infty)} \le & C(\|\overline{\boldsymbol{f}}\|_{W_\beta^{k-2}(S_\infty)} + \|h\|_{W_\beta^{k-1}(Q)} + \\ & \|\overline{\boldsymbol{g}}^0\|_{W_\beta^{k-1/2}(\Gamma_{D,S_\infty})} + \|\overline{\boldsymbol{g}}^1\|_{W_\beta^{k-3/2}(\Gamma_{N,S_\infty})} + \sum_{l=0,1}\|g_2^l\|_{W_\beta^{k-l-1/2}(\Gamma_{G,S_\infty})}).\end{aligned}$$

□

For the Stokes problem (4.1) we have:

Theorem 4.8.5. *Assume that $\beta \in (1-\kappa,1)\cap(0,1)$ where κ is the smallest positive imaginary part of the nonzero eigenvalues of $\mathfrak{R}(\lambda)$ with positive imaginary part and let $\boldsymbol{f} \in W_\beta^0(S_\infty)^2, h \in W_\beta^1(S_\infty), \boldsymbol{g}^0 \in W_\beta^{3/2}(\Gamma_{D,S_\infty})^2, \boldsymbol{g}^1 \in W_\beta^{1/2}(\Gamma_{N,S_\infty})^2$. For $l = 0,1$, let $g_2^l \in W_\beta^{2-l-1/2}(\Gamma_{G,S_\infty})$. Then the Stokes problem (4.1) has a solution $(\boldsymbol{u},p) \in W_\beta^2(S_\infty)^2 \times W_\beta^1(S_\infty)$ and there holds the a-priori estimate*

$$\begin{aligned}\|\overline{\boldsymbol{u}}\|_{W_\beta^2(S_\infty)} + \|p\|_{W_\beta^1(S_\infty)} \le & C(\|\overline{\boldsymbol{f}}\|_{W_\beta^0(S_\infty)} + \|h\|_{W_\beta^1(S_\infty)} + \\ & \|\overline{\boldsymbol{g}}^0\|_{W_\beta^{3/2}(\Gamma_{D,S_\infty})} + \|\overline{\boldsymbol{g}}^1\|_{W_\beta^{1/2}(\Gamma_{N,S_\infty})} + \sum_{l=0,1}\|g_2^l\|_{W_\beta^{2-l-1/2}(\Gamma_{G,S_\infty})}).\end{aligned} \tag{4.58}$$

Proof. We start from Theorem 4.8.4 with $k = 2$ and apply the transformation $\boldsymbol{u} = A^{-1}\overline{\boldsymbol{u}}$. Then this theorem can be validated using [16, Corollary 4.2]. □

Remark 4.8.6. *Note that since the operator $\mathfrak{U}$ has at most countably many points contained in its spectrum, the set of $\beta \in \mathbb{R}$ ensuring that the line $\mathfrak{L}_h = \{\lambda : \operatorname{Im}\lambda = 1-\beta\}$ has no pole of $\mathfrak{R}(\lambda)$ are dense in $\mathbb{R}$. Actually, $\mathfrak{R}(\lambda)$ has no poles in $\{\lambda : \operatorname{Im}\lambda \in (-\kappa,\kappa)\}$ if Dirichlet condition is prescribed on at least one edge of the sector and $\mathfrak{R}(\lambda)$ has the origin as the only pole in this strip otherwise. See Appendix A for more information on the eigenvectors of $\mathfrak{U}(\lambda)$ corresponding to the origin as an eigenvalue.*

Remark 4.8.7. *The Stokes equation we analyze in this chapter is $-\nabla\cdot\underline{\sigma}[\boldsymbol{u},p] = \boldsymbol{f}$. By Remark 4.0.1, above regularity results still hold for $-\nu\Delta\boldsymbol{u} + \nabla p = \boldsymbol{f}$ if $\boldsymbol{u}$ is divergence-free.*

Chapter 5

Analytic regularity of the incompressible stationary NSE

Recall that for a vector field $\boldsymbol{w} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, its polar component is

$$\overline{\boldsymbol{w}} = (w_r, w_\theta)^t := \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \boldsymbol{w}.$$

5.1 Auxiliary Stokes problem in a truncated sector

In this section, we temporarily drop Condition 2 in Remark 3.1.1 on the boundary condition. As a result, it is possible for a corner A_i to have Neumann boundary condition on both touching edges Γ_{i-1} and Γ_i or have Neumann boundary condition and Slip boundary condition on two edges. Without loss of generality we consider the vertex A_1 on $\partial\Omega$. Let (r, θ) be the polar coordinate system centered at A_1 such that $\{\theta = 0\}$ coincides with Γ_n and let $S_\delta := \{(r, \theta) : 0 < r < \delta < +\infty, 0 < \theta < \omega_1\} \subset \Omega$.

We define formally the following two Stokes operators

$$L_{st,1}(\boldsymbol{u}, p) := ((-\nu\Delta\boldsymbol{u} + \nabla p)^t, \nabla \cdot \boldsymbol{u})^t, \tag{5.1}$$

and

$$L_{st,2}(\boldsymbol{u}, p) := ((-\nabla \cdot \underline{\sigma}(\boldsymbol{u}, p))^t, \nabla \cdot \boldsymbol{u})^t. \tag{5.2}$$

Consider the following two Stokes problems:

$$\begin{aligned} L_{st,1}(\boldsymbol{u}, p) &= ((\boldsymbol{f})^t, 0)^t && \text{in } S_\delta \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } (\Gamma_1 \cup \Gamma_n) \cap \Gamma_D \cap \partial S_\delta =: \Gamma_{(1)}, \\ \begin{cases} \boldsymbol{u} \cdot \boldsymbol{n} = 0 \\ (\underline{\sigma}(\boldsymbol{u}, p)\boldsymbol{n}) \cdot \boldsymbol{t} = g_2^1 \end{cases} & && \text{on } (\Gamma_1 \cup \Gamma_n) \cap \Gamma_G \cap \partial S_\delta =: \Gamma_{(2)}, \\ \underline{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} &= \boldsymbol{g}^1 && \text{on } (\Gamma_1 \cup \Gamma_n) \cap \Gamma_N \cap \partial S_\delta =: \Gamma_{(3)}. \end{aligned} \tag{5.3}$$

$$\begin{aligned} L_{st,2}(\boldsymbol{u}, p) &= ((\boldsymbol{f})^t, 0)^t && \text{in } S_\delta \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma_{(1)}, \\ \begin{cases} \boldsymbol{u} \cdot \boldsymbol{n} = 0 \\ (\underline{\sigma}(\boldsymbol{u}, p)\boldsymbol{n}) \cdot \boldsymbol{t} = g_2^1 \end{cases} & && \text{on } \Gamma_{(2)}, \\ \underline{\sigma}(\boldsymbol{u}, p)\boldsymbol{n} &= \boldsymbol{g}^1 && \text{on } \Gamma_{(3)}. \end{aligned} \tag{5.4}$$

Here $\boldsymbol{f} \in L_\beta(S_\delta)^2$, $\overline{g}_2^1 \in W_\beta^{\frac{1}{2}}(\Gamma_{(2)})$ and $\overline{\boldsymbol{g}}^1 \in W_\beta^{\frac{1}{2}}(\Gamma_{(3)})^2$ with $\beta \in (0,1)$.

We assume that the function pair $(\boldsymbol{u}, p) \in H^1(S_\delta)^2 \times L^2(S_\delta)$ is the weak solution to (5.4). Here the weak solution means that for any $(\boldsymbol{v}, q) \in (H^1(S_\delta)^2 \cap \{\boldsymbol{v}|\boldsymbol{v} = \boldsymbol{0}$ on $\Gamma_{(1)} \cap (\{r = \delta\} \cap \partial S_\delta)$ and $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on $\Gamma_{(2)}\}) \times L^2(S_\delta)$ the following holds:

$$\begin{aligned} A(\boldsymbol{u}, \boldsymbol{v})_{S_\delta} + B(\boldsymbol{v}, p)_{S_\delta} &= \int_{S_\delta} \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x} + \int_{\Gamma_{(2)}} g_2^1(\boldsymbol{v} \cdot \boldsymbol{t})\, ds + \int_{\Gamma_{(3)}} \boldsymbol{g}^1 \cdot \boldsymbol{v}\, ds,, \\ B(\boldsymbol{u}, q)_{S_\delta} &= 0. \end{aligned} \tag{5.5}$$

Here $A(\cdot,\cdot)_{S_\delta}$ and $B(\cdot,\cdot)_{S_\delta}$ are those bilinear forms (3.2) and (3.3) but they are integrals over S_δ. It could be justified that this weak form also corresponds to the problem (5.3) due to the incompressible condition and thus $(\boldsymbol{u}, p)$ solves also (5.3) as a weak solution.

Recall that $S_\infty = \{(r, \theta) : 0 < r < +\infty, 0 < \theta < \omega_1\}$. For $l = 1, 2, 3$, set $\tilde{\Gamma}_{(l)}$ as the extension of $\Gamma_{(l)} \cap \partial S_\delta$ to the infinite sector S_∞ if $\Gamma_{(l)} \neq \emptyset$ and $\tilde{\Gamma}_{(l)} = \emptyset$ otherwise.

By $\phi_\delta(r)$ we denote a cut-off function $C^\infty(\mathbb{R})$ such that $\phi_\delta = 1$ for $0 < r < \frac{\delta}{2}$ and $\phi_\delta = 0$ for $r > \delta$. Set

$$(\tilde{\boldsymbol{u}}, \tilde{p}) := \phi_\delta(\boldsymbol{u}, p), \tag{5.6}$$

then clearly $(\tilde{\boldsymbol{u}}, \tilde{p}) = (\boldsymbol{u}, p)$ on $S_{\frac{\delta}{2}}$. By zero extension $(\tilde{\boldsymbol{u}}, \tilde{p}) \in H^1(S_\infty)^2 \times L^2(S_\infty)$ is well-defined in S_∞ and it solves in S_∞ the following equations as a weak solution:

$$\begin{aligned} L_{st,2}(\tilde{\boldsymbol{u}}, \tilde{p}) = \phi_\delta(\boldsymbol{f}, h) + L_1(\boldsymbol{u}, p, \phi_\delta) &=: ((\tilde{\boldsymbol{f}})^t, \tilde{h})^t && \text{in } S_\infty, \\ \tilde{\boldsymbol{u}} &= \boldsymbol{0} && \text{on } \tilde{\Gamma}_{(1)}, \\ \begin{cases} \tilde{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \\ (\underline{\sigma}(\tilde{\boldsymbol{u}}, \tilde{p})\boldsymbol{n}) \cdot \boldsymbol{t} = \phi_\delta g_2^1 + l_1(\boldsymbol{u}, p, \phi_\delta) \cdot \boldsymbol{t} \end{cases} && \text{on } \tilde{\Gamma}_{(2)}, \\ \underline{\sigma}(\tilde{\boldsymbol{u}}, \tilde{p})\boldsymbol{n} &= \phi_\delta \boldsymbol{g}^1 + l_1(\boldsymbol{u}, p, \phi_\delta) && \text{on } \tilde{\Gamma}_{(3)}. \end{aligned} \tag{5.7}$$

Here $L_1(\boldsymbol{u}, p, \phi_\delta) := L_{st,2}(\tilde{\boldsymbol{u}}, \tilde{p}) - \phi_\delta(\boldsymbol{u}, p)$ and $l_1(\boldsymbol{u}, p, \phi_\delta) := \underline{\sigma}(\tilde{\boldsymbol{u}}, \tilde{p}) \cdot \boldsymbol{n} - \phi_\delta \underline{\sigma}(\boldsymbol{u}, p) \cdot \boldsymbol{n}$ are lower order differential operators. The weak solution $(\tilde{\boldsymbol{u}}, \tilde{p})$ satisfies that for any $(\boldsymbol{v}, q) \in (H^1(S_\infty)^2 \cap \{\boldsymbol{v}|\boldsymbol{v} = \boldsymbol{0}$ on $\tilde{\Gamma}_{(1)})$ and $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on $\tilde{\Gamma}_{(2)}\}) \times L^2(S_\infty)$ the following holds:

$$\begin{aligned} A(\tilde{\boldsymbol{u}}, \boldsymbol{v})_{S_\infty} + B(\boldsymbol{v}, \tilde{p})_{S_\infty} &= \int_{S_\infty} \tilde{\boldsymbol{f}} \cdot \boldsymbol{v}\, d\boldsymbol{x} + \int_{\tilde{\Gamma}_{(2)}} g_2^1(\boldsymbol{v} \cdot \boldsymbol{t})\, ds + \int_{\tilde{\Gamma}_{(3)}} \boldsymbol{g}^1 \cdot \boldsymbol{v}\, ds, \\ B(\tilde{\boldsymbol{u}}, q)_{S_\infty} &= \int_{S_\infty} \tilde{h} q\, d\boldsymbol{x}. \end{aligned} \tag{5.8}$$

It is easy to justify that, by Lemma 2.2.6,

$$\begin{aligned} &\|\tilde{\boldsymbol{f}}\|_{L_\beta(S_\infty)} \le C(\|\boldsymbol{f}\|_{L_\beta(S_\delta)} + \|\boldsymbol{u}\|_{H^1(S_\delta \setminus S_{\delta/2})} + \|p\|_{L^2(S_\delta \setminus S_{\delta/2})}), \\ &\|\tilde{h}\|_{L_\beta(S_\infty)} \le C(\|\boldsymbol{u}\|_{H^1(S_\delta \setminus S_{\delta/2})} + \|p\|_{L^2(S_\delta \setminus S_{\delta/2})}), \\ &\|l_1(\boldsymbol{u}, p, \phi_\delta)\|_{W_\beta^{\frac{1}{2}}(\tilde{\Gamma}_{(2)} \cap \tilde{\Gamma}_{(3)})} \le C\|\boldsymbol{u}\|_{H^1(S_\delta \setminus S_{\delta/2})}, \\ &\|\phi_\delta g_2^1\|_{W_\beta^{\frac{1}{2}}(\tilde{\Gamma}_{(2)})} \le C\|g_2^1\|_{W_\beta^{\frac{1}{2}}(\tilde{\Gamma}_{(2)})}, \\ &\|\phi_\delta \boldsymbol{g}^1\|_{W_\beta^{\frac{1}{2}}(\tilde{\Gamma}_{(3)})} \le C\|\boldsymbol{g}^1\|_{W_\beta^{\frac{1}{2}}(\tilde{\Gamma}_{(3)})}. \end{aligned}$$

Here C is independent of $\boldsymbol{u}$ and p.

Lemma 5.1.1. *Let $(\boldsymbol{u}, p)$ be the weak solution to (5.3), $\tilde{\boldsymbol{f}}, \tilde{h}$ be defined as in (5.7) and $(\tilde{\boldsymbol{u}}, \tilde{p})$ be defined as in (5.6). Assume that $\beta \in (1-\kappa, 1) \cap (0,1)$ with κ defined as the smallest positive imaginary part of the nonzero eigenvalues of $\mathfrak{R}(\lambda)$ with positive imaginary part where $\mathfrak{R}(\lambda)$ is defined as in section 4.7.*

Then the problem (5.7) in S_∞ has at least one solution $(\boldsymbol{v}, q) \in W^2_\beta(S_\infty)^2 \times W^1_\beta(S_\infty)$. Moreover, there exists a constant $C_{SEC} = C_{SEC}(\beta, \delta)$ such that:

If $\tilde{\Gamma}_{(1)} \neq \emptyset$ or $\tilde{\Gamma}_{(1)} = \tilde{\Gamma}_{(3)} = \emptyset$, then the solution is unique, $(\boldsymbol{v}, q) = (\tilde{\boldsymbol{u}}, \tilde{p})$ and the solution $(\boldsymbol{u}, p)$ to (5.3) satisfies

$$\begin{aligned} &\|\overline{\boldsymbol{u}}\|_{W^2_\beta(S_{\delta/2})} + \|p\|_{W^1_\beta(S_{\delta/2})} \\ &\leq C_{SEC}(\|\bar{\boldsymbol{f}}\|_{L_\beta(S_\delta)} + \|\overline{\boldsymbol{u}}\|_{H^1(S_\delta \setminus S_{\delta/2})} + \|p\|_{L^2(S_\delta \setminus S_{\delta/2})} + \|g^1_2\|_{W^{\frac{1}{2}}_\beta(\Gamma_{(2)})} + \|\overline{\boldsymbol{g}}^1\|_{W^{\frac{1}{2}}_\beta(\Gamma_{(3)})}). \end{aligned} \tag{5.9}$$

In this case we have $\overline{\boldsymbol{u}} \in W^2_\beta(S_{\delta/2})^2 \subset H^{2,2}_\beta(S_{\delta/2})^2 \subset C^0(\overline{S_{\delta/2}})^2$ and $\boldsymbol{u}(A_1) = \boldsymbol{0}$.

Otherwise, the solution to (5.7) is unique modulo a linear subspace $V \subset \mathbb{R}^2$ and there exists a constant vector $\boldsymbol{e} \in V$ such that $(\boldsymbol{v}, q) = (\tilde{\boldsymbol{u}} - \boldsymbol{e}, \tilde{p})$. Moreover, the solution $(\boldsymbol{u}, p)$ to (5.3) satisfies

$$\begin{aligned} &\|\overline{\boldsymbol{u}} - \overline{\boldsymbol{e}}\|_{W^2_\beta(S_{\delta/2})} + \|p\|_{W^1_\beta(S_{\delta/2})} \\ &\leq C_{SEC}(\|\bar{\boldsymbol{f}}\|_{L_\beta(S_\delta)} + \|\overline{\boldsymbol{u}}\|_{H^1(S_\delta \setminus S_{\delta/2})} + \|p\|_{L^2(S_\delta \setminus S_{\delta/2})} + \|g^1_2\|_{W^{\frac{1}{2}}_\beta(\Gamma_{(2)})} + \|\overline{\boldsymbol{g}}^1\|_{W^{\frac{1}{2}}_\beta(\Gamma_{(3)})}). \end{aligned} \tag{5.10}$$

In this case we have $\boldsymbol{u}(A_1) = \boldsymbol{e}$ and $\overline{\boldsymbol{u}} - \overline{\boldsymbol{e}} \in W^2_\beta(S_{\delta/2})^2 \subset H^{2,2}_\beta(S_{\delta/2})^2 \subset C^0(\overline{S_{\delta/2}})^2$.

To prove this lemma we need the following lemma:

Lemma 5.1.2. *Let $(\boldsymbol{v}, q) \in W^2_\beta(S_\infty)^2 \times W^1_\beta(S_\infty)$ be a solution to (5.7) in the sector S_∞ and assume that if $\omega = \pi$ then different boundary conditions are prescribed on two edges of the sector. If $\tilde{\Gamma}_{(1)} \neq \emptyset$ or $\tilde{\Gamma}_{(1)} = \tilde{\Gamma}_{(3)} = \emptyset$, then*

$$\|\nabla \boldsymbol{v}\|^2_{L^2(S_\infty)} + \|r^{-1}\boldsymbol{v}\|^2_{L^2(S_\infty)} + \|q\|^2_{L^2(S_\infty)} < +\infty. \tag{5.11}$$

Otherwise, we have

$$\|\nabla \boldsymbol{v}\|^2_{L^2(S_\infty)} + \|q\|^2_{L^2(S_\infty)} < +\infty. \tag{5.12}$$

Here $|\nabla \boldsymbol{v}|^2 := |\partial_{x_1}\boldsymbol{v}|^2 + |\partial_{x_2}\boldsymbol{v}|^2$.

The proof follows the route of [16, Theorem 4.4, Corollary 4.3] and uses Lemma A.1.1.

Proof. For any vector field $\boldsymbol{v}$, define $|\mathscr{D}^1\boldsymbol{v}|^2 = \sum_{|\alpha|=1} r^{-2\alpha_2}|\mathscr{D}^\alpha \boldsymbol{v}|^2$. We list the following useful results here, which are checked in the proof of [16, Theorem 4.4]:

$$|\mathscr{D}^1\boldsymbol{v}|^2 \leq 2(|\mathscr{D}^1\overline{\boldsymbol{v}}|^2 + |r^{-1}\overline{\boldsymbol{v}}|^2), \tag{5.13}$$

$$|r^{-1}\boldsymbol{v}|^2 = |r^{-1}\overline{\boldsymbol{v}}|^2, \tag{5.14}$$

$$|\mathscr{D}^1\boldsymbol{v}|^2 = |\nabla \boldsymbol{v}|^2. \tag{5.15}$$

We first consider the two cases $\tilde{\Gamma}_{(1)} \neq \emptyset$ and $\tilde{\Gamma}_{(1)} = \tilde{\Gamma}_{(3)} = \emptyset$. By Lemma A.1.1 and Remark 4.8.6, we may select β, β' with $\beta < 1 < \beta'$ such that no pole of $\mathfrak{R}$ lies in $\{\lambda : \operatorname{Im}\lambda \in [1-\beta', 1-\beta]\}$. Note that since $\tilde{\boldsymbol{f}}$, $\tilde{h}$, $\phi_\delta g^1_2 + l_1(\boldsymbol{u}, p, \phi_\delta)\cdot \boldsymbol{t}$ and $\phi_\delta \boldsymbol{g}^1 + l_1(\boldsymbol{u}, p, \phi_\delta)$ have finite support, we have $\tilde{\boldsymbol{f}} \in L_{\beta'}(S_\infty)^2$, $\tilde{h} \in W^1_{\beta'}(S_\infty)$, $\phi_\delta g^1_2 + l_1(\boldsymbol{u}, p, \phi_\delta)\cdot \boldsymbol{t} \in W^{\frac{1}{2}}_{\beta'}(\tilde{\Gamma}_{(2)})$ and $\phi_\delta \boldsymbol{g}^1 + l_1(\boldsymbol{u}, p, \phi_\delta) \in W^{\frac{1}{2}}_{\beta'}(\tilde{\Gamma}_{(3)})^2$. Therefore, Theorem 4.8.4 implies that $(\overline{\boldsymbol{v}}, q) \in W^2_{\beta'}(S_\infty)^2 \times W^1_{\beta'}(S_\infty)$. Thus we have

$$\int_0^{\omega_1}\int_1^\infty (|\mathscr{D}^1\overline{\boldsymbol{v}}|^2 + |r^{-1}\overline{\boldsymbol{v}}|^2) r\, dr d\theta \leq \int_0^{\omega_1}\int_1^\infty (r^{2(\beta'-1)}|\mathscr{D}^1\overline{\boldsymbol{v}}|^2 + r^{2(\beta'-2)}|\overline{\boldsymbol{v}}|^2) r\, dr d\theta \leq \|\overline{\boldsymbol{v}}\|^2_{W^2_{\beta'}(S_\infty)}$$

and

$$\int_0^{\omega_1}\int_1^{\infty} q^2 r\ dr d\theta \le \int_0^{\omega_1}\int_1^{\infty} (r^{2(\beta'-1)}q^2) r\ dr d\theta \le \|q\|^2_{W^1_{\beta'}(S_\infty)}.$$

We also have

$$\int_0^{\omega_1}\int_0^{1} (|\mathscr{D}^1\overline{\boldsymbol{v}}|^2 + |r^{-1}\overline{\boldsymbol{v}}|^2) r\ dr d\theta \le \int_0^{\omega_1}\int_0^{1} (r^{2(\beta-1)}|\mathscr{D}^1\overline{\boldsymbol{v}}|^2 + r^{2(\beta-2)}|\overline{\boldsymbol{v}}|^2) r\ dr d\theta \le \|\overline{\boldsymbol{v}}\|^2_{W^2_\beta(S_\infty)}$$

and

$$\int_0^{\omega_1}\int_0^{1} q^2 r\ dr d\theta \le \int_0^{\omega_1}\int_0^{1} (r^{2(\beta-1)}q^2) r\ dr d\theta \le \|q\|^2_{W^1_\beta(S_\infty)}.$$

By combining above four inequalities and using (5.13)-(5.15) we have (5.11).

Now we consider the case that both $\tilde{\Gamma}_{(1)} \neq \emptyset$ and $\tilde{\Gamma}_{(1)} = \tilde{\Gamma}_{(3)} = \emptyset$ do not hold. By Lemma A.1.1, we could select $\beta < 1 < \beta'$ such that 0 is the only pole of $\mathfrak{R}$ that lies in $\{\lambda : \operatorname{Im}\lambda \in [1-\beta', 1-\beta]\}$. Then we consider the solution $(\overline{\boldsymbol{v}}', q')$ obtained by $(\overline{\boldsymbol{v}}', q') = (\tilde{\boldsymbol{v}}', \tilde{q}')(\log(\frac{1}{r}), \theta)$ with $(\tilde{\boldsymbol{v}}', q') = \frac{1}{\sqrt{2\pi}}\int_{-\infty+i(1-\beta')}^{\infty+i(1-\beta')} \mathfrak{R}(\lambda)[\hat{\boldsymbol{f}}, \hat{h}, \hat{\boldsymbol{g}}^l]e^{i\lambda t}\ d\lambda$. We now argue as above and obtain that $(\overline{\boldsymbol{v}}', q') \in W^2_{\beta'}(S_\infty)^2 \times W^1_{\beta'}(S_\infty)$. Then Lemma A.1.1 gives that $q = q'$ and

$$\overline{\boldsymbol{v}} = \overline{\boldsymbol{v}}' + c_1(\cos\theta, -\sin\theta)^t + c_2(\sin\theta, \cos\theta)^t,$$

where $c_1, c_2 \in \mathbb{R}$. Therefore

$$\boldsymbol{v} = \boldsymbol{v}' + (c_1, c_2)^t,$$

and $|\mathscr{D}^1\boldsymbol{v}|^2 = |\mathscr{D}^1\boldsymbol{v}'|^2$. Now, use (5.13):

$$\begin{aligned}&\int_0^{\omega_1}\int_1^{\infty} (|\mathscr{D}^1\boldsymbol{v}|^2 + q) r\ dr d\theta \le 2\int_0^{\omega_1}\int_1^{\infty} (r^{2(\beta'-1)}|\mathscr{D}^1\overline{\boldsymbol{v}}'|^2 + r^{2(\beta'-2)}|\boldsymbol{v}|^2 + r^{2(\beta'-1)}q'^2) r\ dr d\theta \\ &\le 2\|\overline{\boldsymbol{v}}'\|^2_{W^2_{\beta'}(S_\infty)} + 2\|q\|^2_{W^1_{\beta'}(S_\infty)},\end{aligned}$$

and we also have

$$\begin{aligned}&\int_0^{\omega_1}\int_0^{1} (|\mathscr{D}^1\boldsymbol{v}|^2 + q) r\ dr d\theta \le 2\int_0^{\omega_1}\int_0^{1} (r^{2(\beta-1)}|\mathscr{D}^1\overline{\boldsymbol{v}}|^2 + r^{2(\beta'-2)}|\boldsymbol{v}|^2 + r^{2(\beta-1)}q^2) r\ dr d\theta \\ &\le 2\|\overline{\boldsymbol{v}}\|^2_{W^2_\beta(S_\infty)} + 2\|q\|^2_{W^1_\beta(S_\infty)}.\end{aligned}$$

Now combine above inequalities and use (5.15) and we have (5.12). □

Proof of Lemma 5.1.1. The proof follows the route in the proof of [17, Theorem 5.2]. The existence of the solution $(\boldsymbol{v}, q)$ is an application of Theorem 4.8.5. By Lemma 2.2.6, $\boldsymbol{v} \in H^1_{loc}(S_\infty)^2$ and $q \in L^2(S_\infty)$.

We firstly show (5.9). For any $(\boldsymbol{w}, \sigma) \in \widetilde{\boldsymbol{H}}^1_0(S_\infty) \times L^2(S_\infty)$ where $\widetilde{\boldsymbol{H}}^1_0(S_\infty) := \{\boldsymbol{u} \in H^1_{loc}(S_\infty)^2 | \|\nabla\boldsymbol{u}\|_{L^2(S_\infty)} < +\infty,\ \boldsymbol{u}|_{\tilde{\Gamma}_{(1)}} = \boldsymbol{0},\ \boldsymbol{u}\cdot\boldsymbol{n}|_{\tilde{\Gamma}_{(2)}} = 0\}$, we have

$$\begin{aligned}&A(\tilde{\boldsymbol{u}}, \boldsymbol{w})_{S_\infty} + B(\boldsymbol{w}, \tilde{p})_{S_\infty} = \int_{S_\infty} \tilde{f}\cdot\boldsymbol{w} + \int_{\tilde{\Gamma}_{(2)}} g^1_2(\boldsymbol{w}\cdot\boldsymbol{t}) + \int_{\tilde{\Gamma}_{(3)}} \boldsymbol{g}^1\cdot\boldsymbol{w}, \\ &B(\tilde{\boldsymbol{u}}, \sigma)_{S_\infty} = \int_{S_\infty} \tilde{h}\sigma.\end{aligned}$$

Here $A(\cdot,\cdot)_{S_\infty}$ and $B(\cdot,\cdot)_{S_\infty}$ are those bilinear forms (3.2) and (3.3) but they are integral over S_∞. Also, for any $(\boldsymbol{w}, \sigma) \in \widehat{\boldsymbol{H}}^1_0(S_\infty) \times L^2(S_\infty)$ where $\widehat{\boldsymbol{H}}^1_0(S_\infty) := \{\boldsymbol{u} \in \widetilde{\boldsymbol{H}}^1_0(S_\infty) | \boldsymbol{u}$ has bounded support in $S_\infty\}$ we

have

$$A(\boldsymbol{v},\boldsymbol{w})_{S_\infty}+B(\boldsymbol{w},q)_{S_\infty}=\int_{S_\infty}\tilde{f}\cdot\boldsymbol{w}+\int_{\tilde{\Gamma}_{(2)}}g_2^1(\boldsymbol{w}\cdot\boldsymbol{t})+\int_{\tilde{\Gamma}_{(3)}}\boldsymbol{g}^1\cdot\boldsymbol{w},$$
$$B(\boldsymbol{v},\sigma)_{S_\infty}=\int_{S_\infty}\tilde{h}\sigma.$$

Therefore, for any $(\boldsymbol{w},\sigma)\in\widehat{\boldsymbol{H}}_0^1(S_\infty)\times L^2(S_\infty)$,

$$A(\boldsymbol{v}-\tilde{\boldsymbol{u}},\boldsymbol{w})_{S_\infty}+B(\boldsymbol{w},q-\tilde{p})_{S_\infty}=0,$$
$$B(\boldsymbol{v}-\tilde{\boldsymbol{u}},\sigma)_{S_\infty}=0.$$

We define $\nabla\widehat{\boldsymbol{H}}_0^1(S_\infty):=\{\underline{\tau}|\exists\boldsymbol{v}\in\widehat{\boldsymbol{H}}_0^1(S_\infty),\underline{\tau}=\nabla\boldsymbol{v}\}$ and $\nabla\widetilde{\boldsymbol{H}}_0^1(S_\infty):=\{\underline{\tau}|\exists\boldsymbol{v}\in\widetilde{\boldsymbol{H}}_0^1(S_\infty),\underline{\tau}=\nabla\boldsymbol{v}\}$. By Lemma 5.1.2 and the density of $\nabla\hat{\boldsymbol{H}}_0^1(S_\infty)$ in $\nabla\tilde{\boldsymbol{H}}_0^1(S_\infty)$ with respect to $L^2(S_\infty)$-norm (this will be shown later in Lemma 5.1.4).

$$A(\boldsymbol{v}-\tilde{\boldsymbol{u}},\boldsymbol{v}-\tilde{\boldsymbol{u}})_{S_\infty}=0.$$

Therefore $\boldsymbol{v}-\tilde{\boldsymbol{u}}\in\mathbf{R}$. If $\tilde{\Gamma}_{(1)}\neq\emptyset$, then $\boldsymbol{v}-\tilde{\boldsymbol{u}}=\boldsymbol{0}$ on $\tilde{\Gamma}_{(1)}$ and $\boldsymbol{v}=\tilde{\boldsymbol{u}}$ on S_∞. Otherwise, both edges are equipped with slip boundary condition and thus $(\boldsymbol{v}-\tilde{\boldsymbol{u}})\cdot\boldsymbol{n}=0$ on ∂S_∞. Since in this case $\omega_1\neq\pi$, we must have $\boldsymbol{v}=\tilde{\boldsymbol{u}}$ on S_∞. And thus,

$$B(\boldsymbol{w},q-\tilde{p})_{S_\infty}=0,\qquad\forall\boldsymbol{w}\in\tilde{\boldsymbol{u}}_0^1(S_\infty).$$

By [14, Corollary 2.4] and [32, Lemma 2.3], for any $R>0$ there exists $\boldsymbol{w}\in\tilde{\boldsymbol{H}}_0^1(S_R)$ such that $\nabla\cdot\boldsymbol{w}=q-\tilde{p}$ in S_R and $\|\nabla\cdot\boldsymbol{w}\|_{L^2(S_R)}\le c\|q-\tilde{p}\|_{L^2(S_\infty)}$. We extend $\boldsymbol{w}$ to S_∞ such that the $H^1(S_\infty)$ norm is still bounded and denote the extension still by $\boldsymbol{w}$. Set $\tilde{\boldsymbol{w}}=\psi_\delta\boldsymbol{w}$ where ψ_δ is a cut-off function such that $\psi_\delta=1$ on S_R and $\psi_\delta=0$ on $S_\infty\setminus S_{R+\delta}$. We have

$$0=B(\tilde{\boldsymbol{w}},q-\tilde{p})_{S_\infty}=\int_{S_R}(q-\tilde{p})^2+\int_{S_{R+\delta}\setminus S_R}(q-\tilde{p})\nabla\cdot\tilde{\boldsymbol{w}},$$

and

$$|\int_{S_{R+\delta}\setminus S_R}(q-\tilde{p})\nabla\cdot\tilde{\boldsymbol{w}}|\le C\|q-\tilde{p}\|_{L^2(S_{R+\delta}\setminus S_R)}\|\boldsymbol{w}\|_{H^1(S_\infty)}.$$

Since $q-\tilde{p}\in L^2(S_\infty)$, $\|q-\tilde{p}\|_{L^2(S_{R+\delta}\setminus S_R)}\to 0$ as $\delta\to 0$. Therefore, $\|q-\tilde{p}\|_{L^2(S_R)}=0$. As R is arbitrary, we have $q=\tilde{p}$ on S_∞. Therefore, by Theorem 4.8.5, $(\tilde{\boldsymbol{u}},p)=(\boldsymbol{v},q)\in W_\beta^2(S_\infty)^2\times W_\beta^1(S_\infty)$ and

$$\begin{aligned}&\|\overline{\boldsymbol{u}}\|_{W_\beta^2(S_{\delta/2})}+\|p\|_{W_\beta^1(S_{\delta/2})}\\&\le C_{SEC,1}(\|\overline{\boldsymbol{f}}\|_{L_\beta(S_\delta)}+\|\overline{\boldsymbol{u}}\|_{H^1(S_\delta\setminus S_{\delta/2})}+\|p\|_{L^2(S_\delta\setminus S_{\delta/2})}+\|g_2^1\|_{W_\beta^{\frac{1}{2}}(\tilde{\Gamma}_{(2)})}+\|\overline{\boldsymbol{g}}^1\|_{W_\beta^{\frac{1}{2}}(\tilde{\Gamma}_{(3)})}).\end{aligned}$$

By Lemma 2.2.6 we have $\overline{\boldsymbol{u}}\in W_\beta^2(S_{\delta/2})^2\subset H_\beta^{2,2}(S_{\delta/2})^2\subset C^0(\overline{S_{\delta/2}})^2$ and we must have $\boldsymbol{u}(A_1)=\overline{\boldsymbol{u}}(A_1)=\boldsymbol{0}$ to ensure that $\boldsymbol{u}\in W_\beta^2(S_{\frac{\delta}{2}})^2$.

We now consider the case that $\tilde{\Gamma}_{(1)}=\emptyset$ and $\tilde{\Gamma}_{(3)}\neq\emptyset$. For any $(\boldsymbol{w},\sigma)\in\tilde{\boldsymbol{H}}^1(S_\infty)\times L^2(S_\infty)$ where $\widetilde{\boldsymbol{H}}^1(S_\infty):=\{\boldsymbol{u}\in H_{loc}^1(S_\infty)^2|\|\nabla\boldsymbol{u}\|_{L^2(S_\infty)}<+\infty,\ \boldsymbol{u}\cdot\boldsymbol{n}|_{\tilde{\Gamma}_{(2)}}=0\}$, we have

$$A(\tilde{\boldsymbol{u}},\boldsymbol{w})_{S_\infty}+B(\boldsymbol{w},\tilde{p})_{S_\infty}=\int_{S_\infty}\tilde{f}\cdot\boldsymbol{w}+\int_{\tilde{\Gamma}_{(2)}}g_2^1(\boldsymbol{w}\cdot\boldsymbol{t})+\int_{\tilde{\Gamma}_{(3)}}\boldsymbol{g}^1\cdot\boldsymbol{w},$$
$$B(\tilde{\boldsymbol{u}},\sigma)_{S_\infty}=\int_{S_\infty}\tilde{h}\sigma.$$

Also, for any $(\boldsymbol{w},\sigma)\in\widehat{\boldsymbol{H}}^1(S_\infty)\times L^2(S_\infty)$ where $\widehat{\boldsymbol{H}}^1(S_\infty):=\{\boldsymbol{u}\in\widetilde{\boldsymbol{H}}^1(S_\infty)|\boldsymbol{u}$ has bounded support in $S_\infty\}$ we have

$$A(\boldsymbol{v},\boldsymbol{w})_{S_\infty}+B(\boldsymbol{w},q)_{S_\infty}=\int_{S_\infty}\tilde{f}\cdot\boldsymbol{w}+\int_{\tilde{\Gamma}_{(2)}}g_2^1(\boldsymbol{w}\cdot\boldsymbol{t})+\int_{\tilde{\Gamma}_{(3)}}\boldsymbol{g}^1\cdot\boldsymbol{w},$$
$$B(\boldsymbol{v},\sigma)_{S_\infty}=\int_{S_\infty}\tilde{h}\sigma.$$

Therefore, for any $(\boldsymbol{w},\sigma)\in\widehat{\boldsymbol{H}}^1(S_\infty)\times L^2(S_\infty)$,

$$\begin{aligned}&A(\boldsymbol{v}-\tilde{\boldsymbol{u}},\boldsymbol{w})_{S_\infty}+B(\boldsymbol{w},q-\tilde{p})_{S_\infty}=0,\\&B(\boldsymbol{v}-\tilde{\boldsymbol{u}},\sigma)_{S_\infty}=0.\end{aligned}$$

We define $\nabla\widehat{\boldsymbol{H}}^1(S_\infty):=\{\underline{\tau}|\exists\boldsymbol{v}\in\widehat{\boldsymbol{H}}^1(S_\infty),\underline{\tau}=\nabla\boldsymbol{v}\}$ and $\nabla\widetilde{\boldsymbol{H}}^1(S_\infty):=\{\underline{\tau}|\exists\boldsymbol{v}\in\widetilde{\boldsymbol{H}}^1(S_\infty),\underline{\tau}=\nabla\boldsymbol{v}\}$. Use the density of $\nabla\widehat{\boldsymbol{H}}^1(S_\infty)$ in $\nabla\widetilde{\boldsymbol{H}}^1(S_\infty)$ with respect to $L^2(S_\infty)$-norm(this is shown in Lemma 5.1.4) and we have

$$A(\boldsymbol{v}-\tilde{\boldsymbol{u}},\boldsymbol{v}-\tilde{\boldsymbol{u}})_{S_\infty}=0.$$

Therefore,

$$\boldsymbol{v}-\tilde{\boldsymbol{u}}\in\mathbf{R}.$$

By Lemma 5.1.2, $\|D(\boldsymbol{v}-\tilde{\boldsymbol{u}})\|_{L^2(S_\infty)}<+\infty$. So $\boldsymbol{v}-\tilde{\boldsymbol{u}}\in span\{(1,0)^t,(0,1)^t\}$ and therefore $A(\boldsymbol{v}-\tilde{\boldsymbol{u}},\boldsymbol{w})_{S_\infty}=0$ for any $\boldsymbol{w}\in\tilde{\boldsymbol{H}}^1(S_\infty)$. Based on this we have $B(\boldsymbol{w},q-\tilde{p})_{S_\infty}=0$ for any $\boldsymbol{w}\in\tilde{\boldsymbol{H}}^1(S_\infty)$. Argue as before and we have $q=\tilde{p}$ in $L^2(S_\infty)$. By Theorem 4.8.5, there exists a constant vector $\boldsymbol{e}$ such that

$$\begin{aligned}&\|\overline{\boldsymbol{u}}-\overline{\boldsymbol{e}}\|_{W^2_\beta(S_{\delta/2})}+\|p\|_{W^1_\beta(S_{\delta/2})}\\&\le C_{SEC,2}(\|\overline{\boldsymbol{f}}\|_{L_\beta(S_\delta)}+\|\overline{\boldsymbol{u}}\|_{H^1(S_\delta\setminus S_{\delta/2})}+\|p\|_{L^2(S_\delta\setminus S_{\delta/2})}+\|g_2^1\|_{W^{\frac{1}{2}}_\beta(\tilde{\Gamma}_{(2)})}+\|\overline{\boldsymbol{g}}^1\|_{W^{\frac{1}{2}}_\beta(\tilde{\Gamma}_{(3)})}).\end{aligned}$$

Lemma 2.2.6 implies that $\tilde{\boldsymbol{u}}\in C^0(\overline{S}_\delta)^2$ and we must have $\overline{\boldsymbol{u}}(A_1)-\overline{\boldsymbol{e}}=\boldsymbol{0}$ to ensure that $\overline{\boldsymbol{u}}-\overline{\boldsymbol{e}}\in W^2_\beta(S_\delta)^2$. Therefore $\boldsymbol{u}(A_1)=\boldsymbol{e}$ and $\overline{\boldsymbol{u}}-\overline{\boldsymbol{e}}\in W^2_\beta(S_{\delta/2})^2\subset H^{2,2}_\beta(S_{\delta/2})^2\subset C^0(\overline{S_{\delta/2}})^2$. The proof is completed by setting $C_{SEC}=\max(C_{SEC,1},C_{SEC,2})$. □

Remark 5.1.3. *The proof of Lemma 5.1.1 shows that any solution $(\boldsymbol{u},p)\in W^2_\beta(S_\infty)^2\times W^1_\beta(S_\infty)$ to the Stokes problem (4.1) with $0<\beta<1$, $\boldsymbol{f}=\boldsymbol{0}$ and zero boundary condition must satisfy that $\boldsymbol{u}\in span\{(0,1)^t,(1,0)^t\}$ and that $p=0$. Due to the fact that $span\{(0,1)^t,(1,0)^t\}\cap W^2_\beta(S_\infty)^2=(0,0)^t$, the homogeneous Stokes problem in a sector can only have zero solution in $W^2_\beta(S_\infty)^2\times W^1_\beta(S_\infty)$ and thus the solution $(\boldsymbol{u},p)\in W^2_\beta(S_\infty)^2\times W^1_\beta(S_\infty)$ to the Stokes problem (4.1) is unique in the scenario of Theorem 4.8.5 if $0<\beta<1$.*

We need to prove the following lemma used before.

Lemma 5.1.4. *If $\tilde{\Gamma}_{(1)}\neq\emptyset$ or $\tilde{\Gamma}_{(1)}=\tilde{\Gamma}_{(3)}=\emptyset$, then $\nabla\widehat{\boldsymbol{H}}^1_0(S_\infty)$ is dense in $\nabla\widetilde{\boldsymbol{H}}^1_0(S_\infty)$ with respect to $L^2(S_\infty)$-norm. Otherwise, $\nabla\widehat{\boldsymbol{H}}^1(S_\infty)$ is dense in $\nabla\widetilde{\boldsymbol{H}}^1(S_\infty)$ with respect to $L^2(S_\infty)$-norm.*

Proof. Without loss of generality, let A_1 be located at the origin. We assume firstly that the edge $\{\theta=0\}=\{(x_1,x_2)|x_1>0,x_2=0\}$. Define the following spaces:

$$\begin{aligned}&\widetilde{H}^1_0(S_\infty):=\{u\in H^1_{loc}(S_\infty)|\|\nabla u\|_{L^2(S_\infty)}<+\infty,u=0\text{ on }\{\theta=0\}\},\\&\widehat{H}^1_0(S_\infty):=\{u\in\tilde{H}^1_0(S_\infty)|u\text{ has bounded support}\},\\&\widetilde{H}^1(S_\infty):=\{u\in H^1_{loc}(S_\infty)|\|\nabla u\|_{L^2(S_\infty)}<+\infty\},\\&\widehat{H}^1(S_\infty):=\{u\in\tilde{H}^1(S_\infty)|u\text{ has bounded support}\}.\end{aligned}$$

For any two constants $0 \le R_1 < R_2 \le +\infty$, define $S_{[R_1,R_2]} := S_\infty \cap \{(r,\theta) : R_1 < r < R_2\}$. Then $S_\infty = S_{[0,1]} \cup (\cup_{n=0}^{+\infty} S_{[2^n,2^{n+1}]})$. Furthermore, we fix a cut-off function $\phi(r) \in C^\infty(\mathbb{R})$ such that $\phi(r) = 1$ for $r < 0$ and $\phi(r) = 0$ for $r > 1$. Then it is straightforward to show that $\phi_n(x_1,x_2) := \phi(\frac{\sqrt{x_1^2+x_2^2}-2^n}{2^n})$ satisfies $\|\nabla\phi_n\|_{L^\infty(S_\infty)} \le \frac{K}{2^n}$ where K is a constant depending only on ϕ.

We prove two claims before verifying the statements in the lemma.

Claim 1: Given any $u \in \widetilde{H}_0^1(S_\infty)$ and any $\epsilon > 0$, there exists $v \in \widehat{H}_0^1(S_\infty)$ such that $\|\nabla u - \nabla v\|_{L^2(S_\infty)} \le \epsilon$. Moreover, if $u = 0$ on ∂S_∞, then $v = 0$ on ∂S_∞.

Proof of Claim 1:

Fix a $u \in \widetilde{H}_0^1(S_\infty)$. The first Poincaré inequality implies that for any function $w \in H^1(S_{[1,2]})$ which vanishes on $\{(r,\theta) : \theta = 0, r \in (1,2)\}$ there exists a constant $C_{poin,1} > 0$ such that the following holds:

$$\|w\|_{L^2(S_{[1,2]})} \le C_{poin,1}\|\nabla w\|_{L^2(S_{[1,2]})}.$$

By applying the homothetic scaling $\Phi_n : S_{[2^n,2^{n+1}]} \to S_{[1,2]} : (x_1,x_2) \mapsto (\frac{x_1}{2^n}, \frac{x_2}{2^n})$ it is easy to verify that for any $n \in \mathbb{N}$ and any $w \in H^1(S_{[2^n,2^{n+1}]})$ which vanishes on $\{(r,\theta) : \theta = 0, r \in (2^n, 2^{n+1})\}$ the following Poincaré inequality holds:

$$\|w\|_{L^2(S_{[2^n,2^{n+1}]})} \le 2^n C_{poin,1}\|\nabla w\|_{L^2(S_{[2^n,2^{n+1}]})}.$$

For any $\epsilon > 0$ we could choose $\tilde{n}$ sufficiently large such that $\|\nabla u\|_{L^2(S_{[2^{\tilde{n}},+\infty]})} < \frac{\epsilon}{3+KC_{poin,1}}$. We show now that $v := \phi_{\tilde{n}} u$ is the function we desire. Clearly $v = 0$ on any edge on which $u = 0$ and v has bounded support in S_∞. Moreover, we have

$$\begin{aligned}
&\|\nabla u - \nabla v\|_{L^2(S_\infty)} \\
&\le \|\nabla u - \nabla(\phi_{\tilde{n}} u)\|_{L^2(S_{[2^{\tilde{n}},2^{\tilde{n}+1}]})} + \|\nabla u\|_{L^2(S_{[2^{\tilde{n}+1},+\infty]})} \\
&\le \|(1-\phi_{\tilde{n}})\nabla u\|_{L^2(S_{[2^{\tilde{n}},2^{\tilde{n}+1}]})} + \|\nabla\phi_{\tilde{n}} u\|_{L^2(S_{[2^{\tilde{n}},2^{\tilde{n}+1}]})} + \|\nabla u\|_{L^2(S_{[2^{\tilde{n}},+\infty]})} \\
&\le 3\|\nabla u\|_{L^2(S_{[2^{\tilde{n}},+\infty]})} + \|\nabla\phi_{\tilde{n}}\|_{L^\infty(S_\infty)}\|u\|_{L^2(S_{[2^{\tilde{n}},2^{\tilde{n}+1}]})} \\
&\le \frac{3\epsilon}{3+KC_{poin,1}} + \frac{K}{2^{\tilde{n}}}2^{\tilde{n}}C_{poin,1}\|\nabla u\|_{L^2(S_{[2^{\tilde{n}},2^{\tilde{n}+1}]})} \qquad \text{by the Poincaré inequality in } S_{[2^{\tilde{n}},2^{\tilde{n}+1}]} \text{ stated above} \\
&\le \frac{3\epsilon}{3+KC_{poin,1}} + KC_{poin,1}\frac{\epsilon}{3+KC_{poin,1}} = \epsilon.
\end{aligned}$$

Therefore v satisfies the requirement in the claim.

Obviously Claim 1 also works for functions taking zero value on $\{\theta = \omega_1\}$.

Claim 2: Given any $u \in \widetilde{H}^1(S_\infty)$ and any $\epsilon > 0$, there exists $v \in \widehat{H}^1(S_\infty)$ such that $\|\nabla u - \nabla v\|_{L^2(S_\infty)} \le \epsilon$.

Proof of Claim 2:

By the second Poincaré inequality, for any function $w \in H^1(S_{[1,2]})$ there exists a constant $C_{poin,2} > 0$ such that the following holds:

$$\inf_{c\in\mathbb{R}} \|w - c\|_{L^2(S_{[1,2]})} \le C_{poin,2}\|\nabla w\|_{L^2(S_{[1,2]})}.$$

Similar to the proof of Claim 1, by using the homothetic scaling Φ_n it is easy to verify that for any $n \in \mathbb{N}$ and any $w \in H^1(S_{[2^n,2^{n+1}]})$ the following Poincaré inequality in $S_{[2^n,2^{n+1}]}$ holds:

$$\inf_{c\in\mathbb{R}} \|w - c\|_{L^2(S_{[2^n,2^{n+1}]})} \le 2^n C_{poin,2}\|\nabla w\|_{L^2(S_{[2^n,2^{n+1}]})}.$$

For any $\epsilon > 0$ we could choose $\bar{n}$ sufficiently large such that $\|\nabla u\|_{L^2(S_{[2^{\bar{n}},+\infty]})} < \frac{\epsilon}{3+KC_{poin,2}}$. With aid of the Poincaré inequality in $S_{[2^{\bar{n}},2^{\bar{n}+1}]}$ there exists a constant $c_{\bar{n}} \in \mathbb{R}$ such that

$$\|u - c_{\bar{n}}\|_{L^2(S_{[2^{\bar{n}},2^{\bar{n}+1}]})} \leq 2^{\bar{n}} C_{poin,2} \|\nabla u\|_{L^2(S_{[2^{\bar{n}},2^{\bar{n}+1}]})} \leq \frac{2^{\bar{n}} C_{poin,2}\epsilon}{3 + KC_{poin,2}}.$$

We show now that $v := \phi_{\bar{n}}(u - c_{\bar{n}})$ is the function we desire. Obviously v has bounded support in S_∞. Moreover, we have

$$\begin{aligned}
&\|\nabla u - \nabla v\|_{L^2(S_\infty)} \\
&\leq \|\nabla u - \nabla(\phi_{\bar{n}}(u - c_{\bar{n}}))\|_{L^2(S_{[2^{\bar{n}},2^{\bar{n}+1}]})} + \|\nabla u\|_{L^2(S_{[2^{\bar{n}+1},+\infty]})} \\
&\leq \|(1 - \phi_{\bar{n}})\nabla u\|_{L^2(S_{[2^{\bar{n}},2^{\bar{n}+1}]})} + \|\nabla \phi_{\bar{n}}(u - c_{\bar{n}})\|_{L^2(S_{[2^{\bar{n}},2^{\bar{n}+1}]})} + \|\nabla u\|_{L^2(S_{[2^{\bar{n}+1},+\infty]})} \\
&\leq 3\|\nabla u\|_{L^2(S_{[2^{\bar{n}},+\infty]})} + \|\nabla \phi_{\bar{n}}\|_{L^\infty(S_\infty)} \|u - c_{\bar{n}}\|_{L^2(S_{[2^{\bar{n}},2^{\bar{n}+1}]})} \\
&\leq \frac{3\epsilon}{3 + KC_{poin,2}} + \frac{K}{2^{\bar{n}}} \frac{2^{\bar{n}} C_{poin,2}\epsilon}{3 + KC_{poin,2}} = \epsilon.
\end{aligned}$$

Therefore v satisfies the requirement in the claim.

We now turn to the proof of the lemma. It is easy to verify that if $\omega_1 \neq \pi$, then there exists an invertible matrix $B := [\boldsymbol{n}|_{\theta=0}, \boldsymbol{n}|_{\theta=\omega_1}]^t$ such that $(u_a, u_b)^t := B\boldsymbol{u} = (\boldsymbol{u} \cdot \boldsymbol{n}|_{\theta=0}, \boldsymbol{u} \cdot \boldsymbol{n}|_{\theta=\omega_1})^t$.

Firstly we justify the density of $\nabla \widehat{\boldsymbol{H}}_0^1(S_\infty)$ in $\nabla \widetilde{\boldsymbol{H}}_0^1(S_\infty)$ given that $\tilde{\Gamma}_{(1)} \neq \emptyset$ or $\tilde{\Gamma}_{(1)} = \tilde{\Gamma}_{(3)} = \emptyset$. Fix $\boldsymbol{u} \in \widetilde{\boldsymbol{H}}_0^1(S_\infty)$. We have the following cases to deal with:

Case 1: Slip boundary condition is not prescribed on both edges(which means that only Dirichlet and Neumann boundary conditions are on the boundary, note that we could not have only Neumann boundary condition here). Then by Claim 1 there exists $\boldsymbol{v} \in \widehat{\boldsymbol{H}}_0^1(S_\infty)$ such that $\|\nabla(\boldsymbol{u} - \boldsymbol{v})\|_{L^2(S_\infty)} \leq \epsilon$.

Case 2: Dirichlet and Slip boundary conditions are on the boundary. We may assume that Slip boundary condition is on $\{\theta = \omega_1\}$ and Dirichlet boundary condition is on $\{\theta = 0\}$. If $\omega_1 \neq \pi$, then $(u_a, u_b)^t = \boldsymbol{0}$ on $\{\theta = 0\}$ and $u_b = 0$ on $\{\theta = \omega_1\}$. By Claim 1, there exist $v_a \in \widehat{H}_0^1(S_\infty)$ and $v_b \in \widehat{H}_0^1(S_\infty) \cap \{w | w = 0 \text{ on } \{\theta = \omega_1\}\}$ such that $\|\nabla((u_a, u_b)^t - (v_a, v_b)^t)\|_{L^2(S_\infty)} \leq \frac{\epsilon}{2\|B^{-1}\|_\infty}$(Here $\|\cdot\|_\infty$ is the *maximum row sum matrix* norm, see [19, Chapter 5.6]). Now $\boldsymbol{v} := B^{-1}(v_a, v_b)^t$ satisfies the boundary condition required in $\widehat{\boldsymbol{H}}_0^1(S_\infty)$ and

$$\|\nabla(\boldsymbol{u} - \boldsymbol{v})\|_{L^2(S_\infty)} \leq 2\|B^{-1}\|_\infty \|\nabla(B(\boldsymbol{u} - \boldsymbol{v}))\|_{L^2(S_\infty)} \leq 2\|B^{-1}\|_\infty \|\nabla((u_a, u_b)^t - (v_a, v_b)^t)\|_{L^2(S_\infty)} \leq \epsilon.$$

If $\omega_1 = \pi$, then boundary conditions imply that $u_1 = 0$ on $\{\theta = 0\}$ and $u_2 = 0$ on ∂S_∞(since we assume before that $\{\theta = 0\} = \{(x_1, x_2) | x_1 > 0, x_2 = 0\}$, $\boldsymbol{n}|_{\theta=0,\pi} = (0, -1)^t$). Now we could find $\boldsymbol{v} = (v_1, v_2)^t \in \widehat{\boldsymbol{H}}_0^1(S_\infty)$ such that $\|\nabla(\boldsymbol{u} - \boldsymbol{v})\|_{L^2(S_\infty)} \leq \epsilon$ by obtaining v_i from u_i with Claim 1 for $i = 1, 2$.

Case 3: Only Slip boundary condition is on ∂S_∞. In this case $\omega_1 \neq \pi$. Now u_a vanishes on $\{\theta = 0\}$ and u_b vanishes on $\{\theta = \omega_1\}$ and thus by Claim 1 there exists $(v_a, v_b)^t$ with v_a vanishing on $\{\theta = 0\}$ and v_b vanishing on $\{\theta = \omega_1\}$ such that $\|\nabla((u_a, u_b)^t - (v_a, v_b)^t)\|_{L^2(S_\infty)} \leq \frac{\epsilon}{2\|B^{-1}\|_\infty}$. We now use the same argument as in Case 2 before and it is clear that $\boldsymbol{v} := B^{-1}(v_a, v_b)^t \in \widehat{\boldsymbol{H}}_0^1(S_\infty)$ and $\|\nabla(\boldsymbol{u} - \boldsymbol{v})\|_{L^2(S_\infty)} \leq \epsilon$.

Therefore $\nabla \widehat{\boldsymbol{H}}_0^1(S_\infty)$ is dense in $\nabla \widetilde{\boldsymbol{H}}_0^1(S_\infty)$ with respect to $L^2(S_\infty)$-norm. We now show that $\nabla \widehat{\boldsymbol{H}}^1(S_\infty)$ is dense in $\nabla \widetilde{\boldsymbol{H}}^1(S_\infty)$ with respect to $L^2(S_\infty)$-norm given that $\tilde{\Gamma}_{(1)} = \emptyset$ and $\tilde{\Gamma}_{(3)} \neq \emptyset$. Fix $\boldsymbol{u} \in \tilde{\boldsymbol{H}}^1(S_\infty)$. Two cases shall be considered:

Case 1: Neumann and Slip boundary conditions are on the boundary. We may assume that Slip boundary condition is on $\{\theta = 0\}$ and Neumann boundary condition is on $\{\theta = \omega_1\}$. If $\omega_1 \neq \pi$, then $u_a = 0$ on $\{\theta = 0\}$. By Claim 1 and Claim 2, there exist $v_a \in \widehat{H}_0^1(S_\infty)$ and $v_b \in \widehat{H}^1(S_\infty)$ such that $\|\nabla((u_a, u_b)^t - (v_a, v_b)^t)\|_{L^2(S_\infty)} \leq \frac{\epsilon}{2\|B^{-1}\|_\infty}$. Now $\boldsymbol{v} := B^{-1}(v_a, v_b)^t$ satisfies the boundary condition required in $\widehat{\boldsymbol{H}}^1(S_\infty)$ and

$$\|\nabla(\boldsymbol{u} - \boldsymbol{v})\|_{L^2(S_\infty)} \leq 2\|B^{-1}\|_\infty \|\nabla(B(\boldsymbol{u} - \boldsymbol{v}))\|_{L^2(S_\infty)} \leq 2\|B^{-1}\|_\infty \|\nabla((u_a, u_b)^t - (v_a, v_b)^t)\|_{L^2(S_\infty)} \leq \epsilon.$$

If $\omega = \pi$, then boundary conditions imply that $u_2 = 0$ on $\{\theta = 0\}$. Now we could find $\boldsymbol{v} := (v_1, v_2)^t \in \widehat{\boldsymbol{H}}^1(S_\infty)$ such that $\|\nabla(\boldsymbol{u} - \boldsymbol{v})\|_{L^2(S_\infty)} \leq \epsilon$ by obtaining v_1 from u_1 with Claim 2 and obtaining v_2 from u_2 with Claim 1.

Case 2: Only Neumann boundary condition is on ∂S_∞. Then by Claim 2 there exists $\boldsymbol{v} \in \widehat{\boldsymbol{H}}^1(S_\infty)$ such that

$$\|\nabla(\boldsymbol{u} - \boldsymbol{v})\|_{L^2(S_\infty)} \leq \epsilon.$$

Above analysis ensures the density of $\nabla\widehat{\boldsymbol{H}}^1(S_\infty)$ in $\nabla\widetilde{\boldsymbol{H}}^1(S_\infty)$.

We finally show that the density results still hold if our assumption about $\{\theta = 0\}$ at the beginning is dropped. To see this we introduce a new coordinate system $\widehat{\boldsymbol{x}} := (\hat{x}_1, \hat{x}_2)^t$ induced by using a rotation matrix R such that the relation between $\widehat{\boldsymbol{x}}$ and the old coordinate $\boldsymbol{x}$ is $\widehat{\boldsymbol{x}} = R\boldsymbol{x}$ and that $\{\theta = 0\} = \{(\hat{x}_1, \hat{x}_2) | \hat{x}_1 > 0, \hat{x}_2 = 0\}$. Then the normal vector will be presented using the new coordinate system as $\widehat{\boldsymbol{n}} = R\boldsymbol{n}$. For any $\boldsymbol{u} \in H^1_{loc}(S_\infty)^2 \cap \{\boldsymbol{v} | \|\nabla_{\boldsymbol{x}}\boldsymbol{u}\|_{L^2(S_\infty)} < +\infty\}$, define $\widehat{\boldsymbol{u}} = R\boldsymbol{u}$. Then it is easy to show that $\widehat{\boldsymbol{u}} \in H^1_{loc}(S_\infty)^2 \cap \{\boldsymbol{v} | \|\nabla_{\hat{\boldsymbol{x}}}\widehat{\boldsymbol{v}}\|_{L^2(S_\infty)} < +\infty\}$ under the new coordinate system and $\widehat{\boldsymbol{u}} = \boldsymbol{0}$ on any edge if and only if $\boldsymbol{u} = \boldsymbol{0}$. Moreover, we have

$$\widehat{\boldsymbol{u}} \cdot \widehat{\boldsymbol{n}} = (R\boldsymbol{u}) \cdot (R\boldsymbol{n}) = \boldsymbol{u} \cdot \boldsymbol{n},$$

since the rotation matrix R is orthogonal. Therefore $\widehat{\boldsymbol{u}} \cdot \widehat{\boldsymbol{n}} = 0 \Longleftrightarrow \boldsymbol{u} \cdot \boldsymbol{n} = 0$. Now we could apply above approximation results on $\widehat{\boldsymbol{u}}$ and transform those results back to $\boldsymbol{u}$ to support the density results in the general case. □

5.2 Analytic regularity over Ω

Recall that $\mathbf{W} := \{\boldsymbol{u} \in H^1(\Omega)^2 : \boldsymbol{u} = \boldsymbol{0} \text{ on } \Gamma_D \text{ and } \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_G\}$, $\mathbf{M}_1 := \{\boldsymbol{v} \in \mathbf{W}, \|\boldsymbol{v}\|_{H^1(\Omega)} \leq (\frac{1}{2} - \sqrt{\frac{1}{4} - \frac{C_{conv}}{C_{coer}^2}\|\boldsymbol{f}\|_{\mathbf{W}^*}})\frac{C_{coer}}{C_{conv}}\}$ and $\mathbf{M}_2 := \{\boldsymbol{v} \in \mathbf{W}, \|\boldsymbol{v}\|_{H^1(\Omega)} \leq \frac{\|\boldsymbol{f}\|_{\mathbf{W}^*}}{C_{coer}}\}$ (see the statement of Theorem 3.1.3). The following will be the main regularity result.

Theorem 5.2.1. *Let $0 < \beta_f = (\beta_1, \beta_2, \cdots, \beta_n) < 1$ and $\boldsymbol{f} \in B^0_{\beta_f}(\Omega)^2 \cap \mathbf{W}^*$ such that (3.9) holds true. Assume that for each $i \in \{1, 2, \cdots, n\}$, $\beta_i \in (1 - \kappa_i, 1) \cap (0, 1)$ with κ_i defined as the smallest positive imaginary part of the nonzero eigenvalues of $\mathfrak{R}_i(\lambda)$ with positive imaginary part where $\mathfrak{R}_i(\lambda)$ is defined regarding the corner A_i as in section 4.7.*

Then, to (3.1) there exists

- *a weak solution $(\boldsymbol{u}, p) \in \mathbf{W} \times L^2(\Omega)$ such that $\boldsymbol{u}$ is uniquely determined in $\mathbf{M}_1$ and p associated with that $\boldsymbol{u}$ is uniquely determined in $L^2(\Omega)$ in the case that $|\Gamma_N| > 0$.*
- *a unique weak solution $(\boldsymbol{u}, p) \in \mathbf{W} \times L_0(\Omega)$ such that $\boldsymbol{u} \in \mathbf{M}_2$ in the case that $|\Gamma_N| = 0$.*

Moreover, any weak solution pair $(\boldsymbol{u}, p) \in B^2_{\beta_f}(\Omega)^2 \times B^1_{\beta_f}(\Omega)$.

The existence and uniqueness of the solution $(\boldsymbol{u}, p)$ are a consequence of Theorem 3.1.3. Before we complete the proof of this theorem, we give a corollary which is more applicable:

Corollary 5.2.2. *Let $0 < \beta_f = (\beta_1, \beta_2, \cdots, \beta_n) < 1$ and $\boldsymbol{f} \in B^0_{\beta_f}(\Omega)^2 \cap \mathbf{W}^*$ such that (3.9) holds true. Then there exists $\tilde{\beta} \in (0,1)^n$ satisfying $\tilde{\beta} > \beta_f$ such that all the existence and uniqueness results in Theorem 5.2.1 hold and $(\boldsymbol{u}, p) \in B^2_{\tilde{\beta}}(\Omega)^2 \times B^1_{\tilde{\beta}}(\Omega)$.*

Proof. We only need to show the regularity of the solution $(\boldsymbol{u}, p)$. There always exists $(\tilde{\beta}_1, \cdots, \tilde{\beta}_n) =: \tilde{\beta} \in (0,1)^n$ such that $\tilde{\beta} > \beta_f$ and that for each $i \in \{1, 2, \cdots, n\}$, $\tilde{\beta}_i > 1 - \kappa_i$ where κ_i is defined as in the statement of Theorem 5.2.1. Also, it is clear that $\boldsymbol{f} \in B^0_{\tilde{\beta}}(\Omega)^2$. Now we apply Theorem 5.2.1 with $\tilde{\beta}$ and we have the result. □

We will prove a series of lemmas in the following to justify the weighted analytical regularity of the solution. For any $0 < \delta \leq \frac{1}{4}\min_{i,j} d(A_i, A_j)$, define $S^i_\delta = B(A_i, \delta) \cap \Omega$ the truncated sector at A_i where $B(A_i, \delta)$ is the ball centered at A_i with radius δ. For $0 \leq a < b \leq \frac{1}{4}\min_{i,j} d(A_i, A_j)$ we also set $S^i_b \supset S^i_{[a,b]} := S^i_b \setminus \overline{S^i_a}$. Define further $\Omega_\delta := \Omega \setminus \cup_{i=1}^n S^i_\delta$. Note that Ω_δ is a Lipschitz domain.

Lemma 5.2.3. *Given $\boldsymbol{f} \in B^0_{\beta_f}(\Omega)^2$ with $0 < \beta_f = (\beta_1, \beta_2, \cdots, \beta_n) < 1$, then $\boldsymbol{f} \in L^t(\Omega)^2$ for $t \in (1, \frac{2}{1+\max_i \beta_i})$.*

Proof. [28, Lemma 2.4] yields $H^{2,0}_{\beta_f}(\Omega) \subset L^t(\Omega)$ for any $t \in (1, \frac{2}{1+\max_i \beta_i})$. Since $\boldsymbol{f} \in B^0_{\beta_f}(\Omega)^2 \subset H^{2,0}_{\beta_f}(\Omega)^2$ by the definition of $B^0_\beta(\Omega)$, $\boldsymbol{f} \in L^t(\Omega)^2$ with $t \in (1, \frac{2}{1+\max_i \beta_i})$. □

Lemma 5.2.4. *For any $\boldsymbol{v} \in H^1(\Omega)^2$, $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} \in L^s(\Omega)^2$ for $1 < s < 2$.*

Proof. This is an application of the Sobolev embedding $H^1(\Omega) \hookrightarrow L^q(\Omega)$ valid for any $q \in (1, +\infty)$ and of the Hölder inequality. □

Lemma 5.2.5. *Let $0 < \beta_f < 1$, $\boldsymbol{f} \in B^0_{\beta_f}(\Omega)^2$ and let $(\boldsymbol{u}, p) \in \mathbf{M} \times L^2(\Omega)$ be a weak solution to (3.1) with right-hand side $\boldsymbol{f}$. In particular, $(\boldsymbol{u}, p)$ solves (3.5).*

Then $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in L^2(\Omega)^2$. Moreover, given any $\delta \in (0, \frac{1}{4}\min_{i,j} d(A_i, A_j)]$, it holds that $(\boldsymbol{u}, p)|_{\Omega_\delta} \in H^k(\Omega_\delta)^2 \times H^{k-1}(\Omega_\delta)$ for any $k \in \mathbb{N}_{\geq 1}$. Furthermore, $(\boldsymbol{u}, p) \in C^{k,\mu}_{loc}(\Omega_\delta)^2 \times C^{k,\mu}_{loc}(\Omega_\delta)$ for any $k \in \mathbb{N}$ and $\mu \in [0, 1]$.

Proof. We move the nonlinear term to the right-hand side and consider the corresponding Stokes problem (3.11). Choose a positive number $\gamma < 1 - \beta_f$ such that the operator pencil $\mathcal{A}(\alpha)$, which was generated by introducing polar coordinates and applying the Mellin transform with respect to the parameter α on the Stokes operator in each truncated sector (see [32, Section 3.2]), has no eigenvalue or has $\alpha = 0$ as the unique eigenvalue in the strip $\{\alpha : -\epsilon < \operatorname{Re}\alpha \leq \gamma\}$, where ϵ is a small positive real number.

By Lemma 5.2.3 and Lemma 5.2.4, $\boldsymbol{f} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in L^{\frac{2}{2-\gamma}}(\Omega)^2$. Therefore, by [33, Corollary 4.2], $(\boldsymbol{u}, p) \in W^{2,\frac{2}{2-\gamma}}(\Omega)^2 \times W^{1,\frac{2}{2-\gamma}}(\Omega)$. The Sobolev embedding theorem implies $\boldsymbol{u} \in C^0(\overline{\Omega})^2$ and then $\|\boldsymbol{u}\|_{L^\infty(\Omega)} < +\infty$. With $\boldsymbol{u} \in H^1(\Omega)^2$ we obtain $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} \in L^2(\Omega)^2$.

As $(\boldsymbol{u}, p) \in H^1(\Omega)^2 \times L^2(\Omega)$, $(\boldsymbol{u}, p)|_{\Omega_\delta} \in H^1(\Omega_\delta)^2 \times L^2(\Omega_\delta)$ for any $\delta \in (0, \frac{1}{4}\min_{i,j} d(A_i, A_j)]$. To show that $(\boldsymbol{u}, p)|_{\Omega_\delta} \in H^2(\Omega_\delta)^2 \times H^1(\Omega_\delta)$, we apply the standard elliptic regularity for Stokes problem:

$$\|\boldsymbol{u}\|_{H^2(\Omega_\delta)} + \|p\|_{H^1(\Omega_\delta)} \leq C\|\boldsymbol{f} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\|_{L^2(\Omega_{\delta/2})} < +\infty.$$

We prove that $(\boldsymbol{u}, p)|_{\Omega_\delta} \in H^k(\Omega_\delta)^2 \times H^{k-1}(\Omega_\delta)$ for $k \in \mathbb{N}_{\geq 2}$ and for any $\delta \in (0, \frac{1}{4}\min_{i,j} d(A_i, A_j)]$ by induction.

Assume that this holds for $k = m - 1$ where $m \in \mathbb{N}_{\geq 3}$. By using a Sobolev extension operator (see, for example, the operator introduced by Stein [42, Chapter VI, Theorem 5]) on $\boldsymbol{u}$ we see that there exists $\tilde{\boldsymbol{u}} \in H^{m-1}(\mathbb{R}^2)^2$ which is identical to $\boldsymbol{u}$ on Ω_δ. Now, [6, Theorem C.10] implies that $(\tilde{\boldsymbol{u}} \cdot \nabla)\tilde{\boldsymbol{u}} \in H^{m-2}(\mathbb{R}^2)^2$. The restriction of $(\tilde{\boldsymbol{u}} \cdot \nabla)\tilde{\boldsymbol{u}}$ to Ω_δ, which is $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}$, clearly belongs to $H^{m-2}(\Omega_\delta)^2$. The elliptic regularity for the Stokes problem implies (note that $\boldsymbol{f} \in H^{m-2}(\Omega_\delta)^2$):

$$\|\boldsymbol{u}\|_{H^m(\Omega_\delta)} + \|p\|_{H^{m-1}(\Omega_\delta)} \leq C\|\boldsymbol{f} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\|_{H^{m-2}(\Omega_{\delta/2})} < +\infty.$$

Therefore $(\boldsymbol{u}, p)|_{\Omega_\delta} \in H^m(\Omega_\delta)^2 \times H^{m-1}(\Omega_\delta)$.

The local Hölder continuity of the solution can be justified by using the Sobolev embedding $H^k(\Omega_\delta) \hookrightarrow C^{k-2}(\overline{\Omega_\delta})$ for any $k \in \mathbb{N}_{\geq 2}$. □

Now we could prove the interior analyticity of the solution using the local Hölder continuity of $(\boldsymbol{u}, p)$ in Ω_δ from Lemma 5.2.5 with [31, Theorem 6.7.6] or directly using [20, Theorem 1.2]. The analyticity in regular parts of boundary can be derived using local Hölder continuity of the solution with [31, Theorem 6.7.6'].

Lemma 5.2.6. *For any $0 < \delta \leq \frac{1}{4}\min_{i,j\in\{1,2,\cdots,n\},i\neq j} d(A_i, A_j)$, the solution $(\boldsymbol{u}, p)$ to (3.1) with $\boldsymbol{f}$ satisfying (3.9) is analytic in $\overline{\Omega_{\delta/2}}$.*

The task now is to investigate the regularity of the solution pair in each truncated sector $S^i_{\delta/2}, i \in \{1, 2, \cdots, n\}$.

In Lemma 5.2.5 we have shown that $(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} \in L^2(\Omega)^2 \subset L_{\beta_f}(\Omega)^2$. By moving the nonlinear term to the right-hand side and using Lemma 5.1.1 on each sector S^i_δ, we have $(\overline{\boldsymbol{u}}, p) \in W^2_{\beta_i}(S^i_\delta)^2 \times W^1_{\beta_i}(S^i_\delta)$ (recall that by Condition 2 in Remark 3.1.1, only the first case in Lemma 5.1.1 happens here).

From now on we fix a small $\delta \in (0, 1)$ such that for any i, $\|\overline{\boldsymbol{u}}\|_{W^2_{\beta_i}(S^i_\delta)} \leq 1$ and $\|p\|_{W^1_{\beta_i}(S^i_\delta)} \leq 1$. This is possible since for each i, $\lim_{\delta\to 0}\|\overline{\boldsymbol{u}}\|_{W^2_{\beta_i}(S^i_\delta)} = 0$ and $\lim_{\delta\to 0}\|p\|_{W^1_{\beta_i}(S^i_\delta)} = 0$, by the Dominated Convergence Theorem.

Without loss of generality we focus on the sector S^1_δ. We will use the coordinate system (r, θ) centered at A_1 such that $S^1_\delta = \{(r, \theta) : 0 < r < \delta, 0 < \theta < \omega_1\}$. We claim that:

Lemma 5.2.7. *Let all assumptions in Theorem 5.2.1 hold true and let $(\boldsymbol{u}, p)$ be a weak solution to (3.1). Then there exists two constants $D_u, E_u \geq 1$ such that for all $\alpha \in \mathbb{N}^2$ satisfying $|\alpha| \geq 2$:*

$$\|r^{\beta_1+\alpha_1-2}\mathscr{D}^\alpha u_r\|_{L^2(S^1_{\delta/2})} \leq D_u^{|\alpha|-2}E_u^{\max(\alpha_2-\frac{4}{3},0)}(|\alpha|-2)!, \tag{5.16}$$

$$\|r^{\beta_1+\alpha_1-2}\mathscr{D}^\alpha u_\theta\|_{L^2(S^1_{\delta/2})} \leq D_u^{|\alpha|-2}E_u^{\max(\alpha_2-\frac{4}{3},0)}(|\alpha|-2)!, \tag{5.17}$$

and for any $|\alpha| \geq 1$:

$$\|r^{\beta_1+\alpha_1-1}\mathscr{D}^\alpha p\|_{L^2(S^1_{\delta/2})} \leq D_u^{|\alpha|-1}E_u^{\alpha_2}(|\alpha|-1)!. \tag{5.18}$$

To show this we need the following lemmas:

Lemma 5.2.8.

$$\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}} = \begin{pmatrix} -\frac{u_\theta^2}{r} + u_r\partial_r u_r + \frac{u_\theta\partial_\theta u_r}{r} \\ \frac{u_r u_\theta}{r} + u_r\partial_r u_\theta + \frac{u_\theta\partial_\theta u_\theta}{r} \end{pmatrix}$$

Proof. Elementary calculus yields:

$$\partial_{x_1} = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta, \qquad \partial_{x_2} = \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta.$$

Then:

$$[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]_1 = (u_1\partial_{x_1} + u_2\partial_{x_2})u_1 =$$
$$(\cos\theta u_r - \sin\theta u_\theta)(\cos^2\theta\partial_r u_r - \cos\theta\sin\theta\partial_r u_\theta + \frac{\sin^2\theta}{r}u_r - \frac{\cos\theta\sin\theta}{r}\partial_\theta u_r + \frac{\cos\theta\sin\theta}{r}u_\theta + \frac{\sin^2\theta}{r}\partial_\theta u_\theta) +$$
$$(\sin\theta u_r + \cos\theta u_\theta)(\cos\theta\sin\theta\partial_r u_r - \sin^2\theta\partial_r u_\theta - \frac{\cos\theta\sin\theta}{r}u_r + \frac{\cos^2\theta}{r}\partial_\theta u_r - \frac{\cos^2\theta}{r}u_\theta - \frac{\cos\theta\sin\theta}{r}\partial_\theta u_\theta)$$
$$= -\frac{\cos\theta}{r}u_\theta^2 - \frac{\sin\theta}{r}u_r u_\theta + \cos\theta u_r\partial_r u_r - \sin\theta u_r\partial_r u_\theta + \frac{\cos\theta}{r}u_\theta\partial_\theta u_r - \frac{\sin\theta}{r}u_\theta\partial_\theta u_\theta.$$

Similarly:

$$
\begin{aligned}
&[(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}]_2 = (u_1\partial_{x_1} + u_2\partial_{x_2})u_2 = \\
&= -\frac{\sin\theta}{r}u_\theta^2 + \frac{\cos\theta}{r}u_r u_\theta + \sin\theta u_r\partial_r u_r + \cos\theta u_r\partial_r u_\theta + \frac{\sin\theta}{r}u_\theta\partial_\theta u_r + \frac{\cos\theta}{r}u_\theta\partial_\theta u_\theta.
\end{aligned}
$$

Therefore:

$$
\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}} = A((\boldsymbol{u}\cdot\nabla)\boldsymbol{u}) = \begin{pmatrix} -\frac{u_\theta^2}{r} + u_r\partial_r u_r + \frac{u_\theta\partial_\theta u_r}{r} \\ \frac{u_r u_\theta}{r} + u_r\partial_r u_\theta + \frac{u_\theta\partial_\theta u_\theta}{r} \end{pmatrix}.
$$

□

The next lemma is similar to [28, Lemma 1.10] but it is written in polar coordinates.

Lemma 5.2.9. *Let $\eta, \hat{\eta}, a, b \in \mathbb{R}$ such that $\eta > \hat{\eta} + \frac{1}{2}$ and $0 \le a < b \le \delta < 1$. Then there exists a constant $C_{INT} = C_{INT}(\eta, \hat{\eta}, a, b) > 0$ such that for any multi-index $\alpha \in \mathbb{N}^2$ and any function ϕ satisfying $\|r^{\hat{\eta}+\alpha_1+\gamma_1}\mathscr{D}^{\alpha+\gamma}\phi\|_{L^2(S^1_{[a,b]})} < +\infty$ for all $\gamma \in \mathbb{N}^2$ with $|\gamma| \le 1$, there holds*

$$
\begin{aligned}
&\|r^{\eta+\alpha_1}\mathscr{D}^\alpha\phi\|_{L^4(S^1_{[a,b]})} \\
&\le C_{INT}\|r^{\hat{\eta}+\alpha_1}\mathscr{D}^\alpha\phi\|^{\frac{1}{2}}_{L^2(S^1_{[a,b]})}\Big(\sum_{|\gamma|\le 1}\|r^{\hat{\eta}+\alpha_1+\gamma_1}\mathscr{D}^{\alpha+\gamma}\phi\|^{\frac{1}{2}}_{L^2(S^1_{[a,b]})} + \alpha_1^{\frac{1}{2}}\|r^{\hat{\eta}+\alpha_1}\mathscr{D}^\alpha\phi\|^{\frac{1}{2}}_{L^2(S^1_{[a,b]})}\Big).
\end{aligned} \tag{5.19}
$$

Proof. Without loss of generality we set $b = 1$.

Given $j \in \mathbb{N}_0$, we introduce $S^j := \{(r,\theta) : 2^{-j-1}<r<2^{-j}, 0<\theta<\omega_1\} \subset S^1_{[0,1]}$ and the homothetic scaling $\Psi_j : S^j \to S^0 : (r,\theta) \mapsto (2^j r, \theta)$ and set $\hat{\phi}_j = \phi \circ \Psi_j^{-1}$. Then we have, for any $q \in [1, +\infty)$:

$$
\|r^{\eta+\alpha_1}\mathscr{D}^\alpha\phi\|_{L^q(S^j)} = 2^{-j(\eta+\frac{2}{q})}\|r^{\eta+\alpha_1}\hat{\mathscr{D}}^\alpha\hat{\phi}_j\|_{L^q(S^0)}. \tag{5.20}
$$

Let now $c \in (0,1)$. As $S^1_{[c,1]}$ satisfies the cone condition (see [1]), by [1, Theorem 3] there exists a constant $C_0 = C_0(c)$ such that for any $\psi \in W^{1,2}(S^1_{[c,1]})$:

$$
\|\psi\|_{L^4(S^1_{[c,1]})} \le C_0\|\psi\|^{\frac{1}{2}}_{H^1(S^1_{[c,1]})} \cdot \|\psi\|^{\frac{1}{2}}_{L^2(S^1_{[c,1]})}.
$$

Note that:

$$
\partial_{x_1} = \cos\theta\partial_r - \frac{\sin\theta}{r}\partial_\theta, \qquad \partial_{x_2} = \sin\theta\partial_r + \frac{\cos\theta}{r}\partial_\theta.
$$

Since $S^1_{[c,1]}$ is a bounded set and is bounded away from the vertex, $r^{\alpha_1}\mathscr{D}^\alpha\phi \in H^1(S^1_{[c,1]})$. Also, for all $\psi \in H^1(S^1_{[c,1]})$,

$$
\|\psi\|^2_{H^1(S^1_{[c,1]})} \le C_1(\|\psi\|^2_{L^2(S^1_{[c,1]})} + \|\partial_r\psi\|^2_{L^2(S^1_{[c,1]})} + \|\partial_\theta\psi\|^2_{L^2(S^1_{[c,1]})}).
$$

The constant C_1 here depends on c but is independent of ψ. Therefore:

$$
\|\psi\|^2_{L^4(S^1_{[c,1]})} \le C_0^2 C_1\|\psi\|_{L^2(S^1_{[c,1]})} \cdot \Big(\sum_{|\gamma|\le 1}\|\mathscr{D}^\gamma(r^{\alpha_1}\mathscr{D}^\alpha\psi)\|^2_{L^2(S^1_{[c,1]})}\Big)^{\frac{1}{2}}.
$$

Set $\psi = r^{\eta+\alpha_1}\mathscr{D}^\alpha\phi$ in this inequality:

$$
\begin{aligned}
&\|r^{\eta+\alpha_1}\mathscr{D}^\alpha\phi\|_{L^4(S^1_{[c,1]})} \le \max(1,c^\eta)\|r^{\alpha_1}\mathscr{D}^\alpha\phi\|_{L^4(S^1_{[c,1]})}\\
&\le C_0C_1^{\frac12}\max(1,c^\eta)\|r^{\alpha_1}\mathscr{D}^\alpha\phi\|^{\frac12}_{L^2(S^1_{[c,1]})}(\sum_{|\gamma|\le1}\|\mathscr{D}^\gamma(r^{\alpha_1}\mathscr{D}^\alpha\phi)\|^2_{L^2(S^1_{[c,1]})})^{\frac14}\\
&\le C_0C_1^{\frac12}\max(1,c^\eta)\|r^{\alpha_1}\mathscr{D}^\alpha\phi\|^{\frac12}_{L^2(S^1_{[c,1]})}(\sum_{|\gamma|\le1}\|r^{\gamma_1+\alpha_1}\mathscr{D}^{\alpha+\gamma}\phi\|^{\frac12}_{L^2(S^1_{[c,1]})}+|\alpha_1|^{\frac12}\cdot\|r^{\alpha_1-1}\mathscr{D}^\alpha\phi\|^{\frac12}_{L^2(S^1_{[c,1]})})\\
&\le C_0C_1^{\frac12}\max(1,c^\eta)\max(1,c^{-\hat\eta},c^{-\hat\eta-1})\|r^{\hat\eta+\alpha_1}\mathscr{D}^\alpha\phi\|^{\frac12}_{L^2(S^1_{[c,1]})}\\
&\cdot(\sum_{|\gamma|\le1}\|r^{\hat\eta+\gamma_1+\alpha_1}\mathscr{D}^{\alpha+\gamma}\phi\|^{\frac12}_{L^2(S^1_{[c,1]})}+|\alpha_1|^{\frac12}\cdot\|r^{\hat\eta+\alpha_1}\mathscr{D}^\alpha\phi\|^{\frac12}_{L^2(S^1_{[c,1]})}).
\end{aligned}
$$

If $a\neq0$, then set $c=a$ in the above inequality and (5.19) holds if $C_{INT}\ge C_0C_1^{\frac12}\max(1,a^\eta)\max(1,a^{-\hat\eta},a^{-\hat\eta-1})$. Otherwise, set $c=\frac12$ in the inequality and use (5.20):

$$
\begin{aligned}
&\|r^{\eta+\alpha_1}\mathscr{D}^\alpha\phi\|_{L^4(S^j)} = 2^{-j(\eta+\frac12)}\|r^{\eta+\alpha_1}\hat{\mathscr{D}}^\alpha\hat\phi_j\|_{L^4(S^0)}\\
&\le C_0C_1^{\frac12}\max(1,(\frac12)^\eta)\max(1,(\frac12)^{-\hat\eta},(\frac12)^{-\hat\eta-1})2^{-j(\eta+\frac12)}\|r^{\hat\eta+\alpha_1}\hat{\mathscr{D}}^\alpha\hat\phi_j\|^{\frac12}_{L^2(S^1_{[c,1]})}\\
&\cdot(\sum_{|\gamma|\le1}\|r^{\hat\eta+\gamma_1+\alpha_1}\hat{\mathscr{D}}^{\alpha+\gamma}\hat\phi_j\|^{\frac12}_{L^2(S^1_{[c,1]})}+|\alpha_1|^{\frac12}\cdot\|r^{\hat\eta+\alpha_1}\hat{\mathscr{D}}^\alpha\hat\phi_j\|^{\frac12}_{L^2(S^1_{[c,1]})})\\
&\le C_0C_1^{\frac12}\max(1,(\frac12)^\eta)\max(1,(\frac12)^{-\hat\eta},(\frac12)^{-\hat\eta-1})2^{-j(\eta+\frac12)+j(\hat\eta+1)}\|r^{\eta+\alpha_1}\mathscr{D}^\alpha\phi\|^{\frac12}_{L^2(S^j)}\\
&\cdot(\sum_{|\gamma|\le1}\|r^{\hat\eta+\gamma_1+\alpha_1}\mathscr{D}^{\alpha+\gamma}\phi\|^{\frac12}_{L^2(S^j)}+|\alpha_1|^{\frac12}\cdot\|r^{\hat\eta+\alpha_1}\mathscr{D}^\alpha\phi\|^{\frac12}_{L^2(S^j)}).
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\|r^{\eta+\alpha_1}\mathscr{D}^\alpha\phi\|_{L^4(S^1_{[0,1]})} \le \sum_{j\in\mathbb{N}}\|r^{\eta+\alpha_1}\mathscr{D}^\alpha\phi\|_{L^4(S^j)}\\
&\le C_0C_1^{\frac12}\max(1,(\frac12)^\eta)\max(1,(\frac12)^{-\hat\eta},(\frac12)^{-\hat\eta-1})\sum_{j\in\mathbb{N}}(2^{-j(\eta+\frac12)+j(\hat\eta+1)}\|r^{\eta+\alpha_1}\mathscr{D}^\alpha\phi\|^{\frac12}_{L^2(S^j)}\\
&\cdot(\sum_{|\gamma|\le1}\|r^{\hat\eta+\gamma_1+\alpha_1}\mathscr{D}^{\alpha+\gamma}\phi\|^{\frac12}_{L^2(S^j)}+|\alpha_1|^{\frac12}\cdot\|r^{\hat\eta+\alpha_1}\mathscr{D}^\alpha\phi\|^{\frac12}_{L^2(S^j)}))\\
&\le C_0C_1^{\frac12}\max(1,(\frac12)^\eta)\max(1,(\frac12)^{-\hat\eta},(\frac12)^{-\hat\eta-1})(\sum_{j\in\mathbb{N}}2^{-j(\eta+\frac12)+j(\hat\eta+1)})\\
&\|r^{\eta+\alpha_1}\mathscr{D}^\alpha\phi\|^{\frac12}_{L^2(S^1_{[0,1]})}\cdot(\sum_{|\gamma|\le1}\|r^{\hat\eta+\gamma_1+\alpha_1}\mathscr{D}^{\alpha+\gamma}\phi\|^{\frac12}_{L^2(S^1_{[0,1]})}+|\alpha_1|^{\frac12}\cdot\|r^{\hat\eta+\alpha_1}\mathscr{D}^\alpha\phi\|^{\frac12}_{L^2(S^1_{[0,1]})}).
\end{aligned}
$$

Conclude the two cases above and it is clear now that we could set

$$
\begin{aligned}
&C_{INT} := \max(C_0C_1^{\frac12}\max(1,a^\eta)\max(1,a^{-\hat\eta},a^{-\hat\eta-1}),\\
&C_0C_1^{\frac12}\max(1,(\frac12)^\eta)\max(1,(\frac12)^{-\hat\eta},(\frac12)^{-\hat\eta-1})(\sum_{j\in\mathbb{N}}2^{-j(\eta+\frac12)+j(\hat\eta+1)}))
\end{aligned}
$$

(since $\eta>\hat\eta+\frac12$ the sum here is finite) to validate (5.19). This concludes the proof. □

Lemma 5.2.10. *Let $\beta \in (0,1)$, $k \in \mathbb{N}_{\geq 1}, 0 \leq a < b \leq \delta \leq 1$ and let u be a function such that $\|u\|_{W^2_\beta(S^1_\delta)} \leq 1$. Assume that there exists two constants $D_u, E_u \geq 1$ independent of k, a, b such that, for any $2 \leq |\alpha| \leq k+1$:*

$$\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha u\|_{L^2(S^1_{[a,b]})} \leq D_u^{|\alpha|-2} E_u^{\alpha_2}(|\alpha|-2)!$$

Then there is a constant $C_{ECN}>0$ independent of D_u, E_u, k, a, b such that:

(1): for any $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq k$:

$$\|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^\alpha u\|_{L^4(S^1_{[a,b]})} \leq C_{ECN}(|\alpha|+1)^{\frac{1}{2}} D_u^{\max(|\alpha|-\frac{3}{2},0)} E_u^{\alpha_2+\frac{1}{2}}(\max(|\alpha|-2,0))!. \tag{5.21}$$

(2): for any $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq k$:

$$\|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^\alpha (ru)\|_{L^4(S^1_{[a,b]})} \leq C_{ECN}(|\alpha|+1)^{\frac{1}{2}} D_u^{\max(|\alpha|-\frac{3}{2},0)} E_u^{\alpha_2+\frac{1}{2}}(\max(|\alpha|-2,0))!. \tag{5.22}$$

(3): for any $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq k-1$:

$$\|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^\alpha (r\partial_r u)\|_{L^4(S^1_{[a,b]})} \leq C_{ECN}(|\alpha|+1)^{\frac{1}{2}} D_u^{\max(|\alpha|-\frac{1}{2},0)} E_u^{\alpha_2+\frac{1}{2}}(\max(|\alpha|-1,0))!. \tag{5.23}$$

(4): for any $\alpha \in \mathbb{N}^2$ with $|\alpha| \leq k-1$:

$$\|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^\alpha (\partial_\theta u)\|_{L^4(S^1_{[a,b]})} \leq C_{ECN}(|\alpha|+1)^{\frac{1}{2}} D_u^{\max(|\alpha|-\frac{1}{2},0)} E_u^{\alpha_2+\frac{3}{2}}(\max(|\alpha|-1,0))!. \tag{5.24}$$

Proof. We start by showing (5.21). By Lemma 5.2.9 (note that since $\beta < 1$, $\frac{\beta}{2} - 1 > \beta - 2 + \frac{1}{2}$ and thus Lemma 5.2.9 is applicable here), there holds, for all $\alpha \in \mathbb{N}^2$,

$$\begin{aligned}
&\|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^\alpha u\|_{L^4(S^1_{[a,b]})} \\
&\leq C_{INT}\|r^{\beta-2+\alpha_1}\mathscr{D}^\alpha u\|^{\frac{1}{2}}_{L^2(S^1_{[a,b]})} \cdot (\sum_{|\gamma|\leq 1}\|r^{\beta-2+\alpha_1+\gamma_1}\mathscr{D}^{\alpha+\gamma} u\|^{\frac{1}{2}}_{L^2(S^1_{[a,b]})} + \alpha_1^{\frac{1}{2}}\|r^{\beta-2+\alpha_1}\mathscr{D}^\alpha u\|^{\frac{1}{2}}_{L^2(S^1_{[a,b]})}).
\end{aligned}$$

Therefore, when $|\alpha| \geq 2$:

$$\begin{aligned}
&\|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^\alpha u\|_{L^4(S^1_{[a,b]})} \\
&\leq 2C_{INT}(D_u^{\max(|\alpha|-2,0)} E_u^{\alpha_2}(|\alpha|-2)!)^{\frac{1}{2}}(D_u^{\max(|\alpha|-1,0)} E_u^{\alpha_2+1}(|\alpha|-1)!)^{\frac{1}{2}} \\
&+ (1+\alpha_1^{\frac{1}{2}})C_{INT} D_u^{\max(|\alpha|-2,0)} E_u^{\alpha_2}(|\alpha|-2)! \\
&\leq 4C_{INT}(|\alpha|+1)^{\frac{1}{2}} D_u^{\max(|\alpha|-\frac{3}{2},0)} E_u^{\alpha_2+\frac{1}{2}}(|\alpha|-2)!
\end{aligned}$$

and in the case that $|\alpha| \leq 1$, recall that $\|u\|_{W^2_\beta(S^1_\delta)} \leq 1$:

$$\|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^\alpha u\|_{L^4(S^1_{[a,b]})} \leq 4C_{INT}.$$

It follows that

$$\|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^\alpha u\|_{L^4(S^1_{[a,b]})} \leq 4C_{INT}(|\alpha|+1)^{\frac{1}{2}} D_u^{\max(|\alpha|-\frac{3}{2},0)} E_u^{\alpha_2+\frac{1}{2}}(\max(|\alpha|-2,0))!.$$

In order to prove (5.22), remark that for all $\alpha \in \mathbb{N}^2$,

$$\begin{aligned}
&\|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^\alpha(ru)\|_{L^4(S^1_{[a,b]})} \leq \|r^{\frac{\beta}{2}+\alpha_1}\mathscr{D}^\alpha u\|_{L^4(S^1_{[a,b]})} + \alpha_1\|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^{\alpha-(1,0)}u\|_{L^4(S^1_{[a,b]})} \\
&\leq \delta(\|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^\alpha u\|_{L^4(S^1_{[a,b]})} + \alpha_1\|r^{\frac{\beta}{2}-1+\alpha_1-1}\mathscr{D}^{\alpha-(1,0)}u\|_{L^4(S^1_{[a,b]})}).
\end{aligned}$$

Now we apply (5.21) to obtain that

$$\begin{aligned}
&\|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^\alpha(ru)\|_{L^4(S^1_{[a,b]})} \\
&\leq 4\delta C_{INT}(|\alpha|+1)^{\frac{1}{2}}D_u^{\max(|\alpha|-\frac{3}{2},0)}E_u^{\alpha_2+\frac{1}{2}}(\max(|\alpha|-2,0))! \\
&+ 4\delta\alpha_1 C_{INT}|\alpha|^{\frac{1}{2}}D_u^{\max(|\alpha|-\frac{5}{2},0)}E_u^{\alpha_2+\frac{1}{2}}(\max(|\alpha|-3,0))! \\
&\leq (4\delta + 4\delta\max_{j\in\mathbb{N}}(\frac{j^{\frac{3}{2}}}{(j+1)^{\frac{1}{2}}\max(j-2,1)}))C_{INT}(|\alpha|+1)^{\frac{1}{2}}D_u^{\max(|\alpha|-\frac{3}{2},0)}E_u^{\alpha_2+\frac{1}{2}}(\max(|\alpha|-2,0))! \\
&= (4+6\sqrt{3})\delta C_{INT}(|\alpha|+1)^{\frac{1}{2}}D_u^{\max(|\alpha|-\frac{3}{2},0)}E_u^{\alpha_2+\frac{1}{2}}(\max(|\alpha|-2,0))!.
\end{aligned}$$

We now prove (5.23). We have, by applying (5.21) again, that

$$\begin{aligned}
&\|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^\alpha(r\partial_r u)\|_{L^4(S^1_{[a,b]})} \\
&\leq \|r^{\frac{\beta}{2}-1+\alpha_1+1}\mathscr{D}^{\alpha+(1,0)}u\|_{L^4(S^1_{[a,b]})} + \alpha_1\|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^\alpha u\|_{L^4(S^1_{[a,b]})} \\
&\leq 4C_{INT}(|\alpha|+2)^{\frac{1}{2}}D_u^{\max(|\alpha|-\frac{1}{2},0)}E_u^{\alpha_2+\frac{1}{2}}(\max(|\alpha|-1,0))! \\
&+ 4\alpha_1 C_{INT}(|\alpha|+1)^{\frac{1}{2}}D_u^{\max(|\alpha|-\frac{3}{2},0)}E_u^{\alpha_2+\frac{1}{2}}(\max(|\alpha|-2,0))! \\
&\leq (4C_{INT}\max_{j\in\mathbb{N}}((\frac{j+2}{j+1})^{\frac{1}{2}} + \frac{j}{\max(j-1,1)}))(|\alpha|+1)^{\frac{1}{2}}D_u^{\max(|\alpha|-\frac{1}{2},0)}E_u^{\alpha_2+\frac{1}{2}}(\max(|\alpha|-1,0))! \\
&\leq (8+\frac{8}{3}\sqrt{3})(|\alpha|+1)^{\frac{1}{2}}D_u^{\max(|\alpha|-\frac{1}{2},0)}E_u^{\alpha_2+\frac{1}{2}}(\max(|\alpha|-1,0))!.
\end{aligned}$$

Finally, for (5.24), we still use (5.21):

$$\begin{aligned}
&\|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^\alpha(\partial_\theta u)\|_{L^4(S^1_{[a,b]})} = \|r^{\frac{\beta}{2}-1+\alpha_1}\mathscr{D}^{\alpha+(0,1)}u\|_{L^4(S^1_{[a,b]})} \\
&\leq 4C_{INT}(|\alpha|+1)^{\frac{1}{2}}D_u^{\max(|\alpha|-\frac{1}{2},0)}E_u^{\alpha_2+\frac{3}{2}}(\max(|\alpha|-1,0))!.
\end{aligned}$$

We set $C_{ECN} = (4+6\sqrt{3})C_{INT}$ and this concludes the proof. □

Lemma 5.2.11 (Weighted analytic regularity of the quadratic nonlinearity)**.** *Let $0 \leq a < b \leq \delta \leq 1$, $\beta \in (0,1)$ and $k \in \mathbb{N}_{\geq 1}$. Furthermore let $\boldsymbol{u} : S^1_\delta \subset \Omega \to \mathbb{R}^2$ be a vector field such that $\|\overline{\boldsymbol{u}}\|_{W^2_\beta(S^1_\delta)} \leq 1$. Assume that there exists two constants $D_u, E_u \geq 1$ such that, for any $\alpha \in \mathbb{N}^2$ with $2 \leq |\alpha| \leq k+1$:*

$$\begin{aligned}
\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha u_r\|_{L^2(S^1_{[a,b]})} &\leq D_u^{|\alpha|-2}E_u^{\alpha_2}(|\alpha|-2)!, \\
\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha u_\theta\|_{L^2(S^1_{[a,b]})} &\leq D_u^{|\alpha|-2}E_u^{\alpha_2}(|\alpha|-2)!.
\end{aligned}$$

Then, there exists a constant $C_{ANT} > 0$ independent of D_u, E_u, k, a, b such that for any $1 \leq |\alpha| \leq k$,

$$\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(r^2\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})\|_{L^2(S^1_{[a,b]})} \leq C_{ANT}D_u^{|\alpha|-1}E_u^{\alpha_2+2}|\alpha|!. \tag{5.25}$$

Proof. We estimate $\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(ru_\theta^2)\|_{L^2(S^1_{[a,b]})}$. By the Leibniz rule, the Hölder inequality and Lemma 5.2.10, for all $\alpha \in \mathbb{N}^2$ with $1 \le |\alpha| \le k$,

$$
\begin{aligned}
&\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(ru_\theta^2)\|_{L^2(S^1_{[a,b]})}\\
&\le \sum_{j=0}^{|\alpha|}\sum_{|\gamma|=j,\gamma\le\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\|r^{\frac{\beta}{2}+\gamma_1-1}\mathscr{D}^\gamma(ru_\theta)\|_{L^4(S^1_{[a,b]})}\|r^{\frac{\beta}{2}+\alpha_1-\gamma_1-1}\mathscr{D}^{\alpha-\gamma}(u_\theta)\|_{L^4(S^1_{[a,b]})}\\
&\le \sum_{j=0}^{|\alpha|}\sum_{|\gamma|=j,\gamma\le\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}(C_{ECN}(|\gamma|+1)^{\frac12}D_u^{\max(|\gamma|-\frac32,0)}E_u^{\gamma_2+\frac12}\max(|\gamma|-2,0)!)\\
&\cdot(C_{ECN}(|\alpha-\gamma|+1)^{\frac12}D_u^{\max(|\alpha-\gamma|-\frac32,0)}E_u^{\alpha_2-\gamma_2+\frac12}\max(|\alpha-\gamma|-2,0)!)\\
&\le \max_{j\ge0}\Big(\frac{j+1}{\max(j-1,1)}\Big)^3C_{ECN}^2\sum_{j=0}^{|\alpha|}\sum_{|\gamma|=j,\gamma\le\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}D_u^{\max(|\alpha|-\frac32,0)}E_u^{\alpha_2+1}j!(|\alpha|-j)!\frac{1}{(\max(j,1)\max(|\alpha|-j,1))^{\frac32}}\\
&\le 27C_{ECN}^2\sum_{j=0}^{|\alpha|}\sum_{|\gamma|=j,\gamma\le\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}D_u^{\max(|\alpha|-\frac32,0)}E_u^{\alpha_2+1}j!(|\alpha|-j)!\frac{1}{(\max(j,1)\max(|\alpha|-j,1))^{\frac32}}.
\end{aligned}
$$

In [21, Proposition 2.1] it was shown that

$$
\sum_{|\gamma|=j,\gamma\le\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}=\begin{pmatrix}|\alpha|\\ j\end{pmatrix}.
$$

Then,

$$
\begin{aligned}
&\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(ru_\theta^2)\|_{L^2(S^1_{[a,b]})}\\
&\le 27C_{ECN}^2D_u^{\max(|\alpha|-\frac32,0)}E_u^{\alpha_2+1}|\alpha|!\sum_{j=0}^{|\alpha|}\frac{1}{(\max(j,1)\max(|\alpha|-j,1))^{\frac32}}\\
&\le 27C_{ECN}^2D_u^{\max(|\alpha|-\frac32,0)}E_u^{\alpha_2+1}|\alpha|!\sum_{j=0}^{|\alpha|}\frac{1}{\max(j,1)^{\frac32}}\\
&\le 27C_{ECN}^2\cdot(2+\int_1^\infty\frac{1}{x^{\frac32}}dx)D_u^{\max(|\alpha|-\frac32,0)}E_u^{\alpha_2+1}|\alpha|!\\
&\le 108C_{ECN}^2D_u^{\max(|\alpha|-\frac32,0)}E_u^{\alpha_2+1}|\alpha|!.
\end{aligned}
$$

We could estimate $\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(ru_ru_\theta)\|_{L^2(S^1_{[a,b]})}$ by following the same steps. For all $\alpha \in \mathbb{N}^2$ with $1 \le |\alpha| \le k$,

$$
\begin{aligned}
&\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(ru_ru_\theta)\|_{L^2(S^1_{[a,b]})}\\
&\le \sum_{j=0}^{|\alpha|}\sum_{|\gamma|=j,\gamma\le\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\|r^{\frac{\beta}{2}+\gamma_1-1}\mathscr{D}^\gamma(ru_r)\|_{L^4(S^1_{[a,b]})}\|r^{\frac{\beta}{2}+\alpha_1-\gamma_1-1}\mathscr{D}^{\alpha-\gamma}(u_\theta)\|_{L^4(S^1_{[a,b]})}\\
&\le \sum_{j=0}^{|\alpha|}\sum_{|\gamma|=j,\gamma\le\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}(C_{ECN}(|\gamma|+1)^{\frac12}D_u^{\max(|\gamma|-\frac32,0)}E_u^{\gamma_2+\frac12}\max(|\gamma|-2,0)!)\\
&\cdot(C_{ECN}(|\alpha-\gamma|+1)^{\frac12}D_u^{\max(|\alpha-\gamma|-\frac32,0)}E_u^{\alpha_2-\gamma_2+\frac12}\max(|\alpha-\gamma|-2,0)!)\\
&\le \max_{j\ge0}\Big(\frac{j+1}{\max(j-1,1)}\Big)^3C_{ECN}^2\sum_{j=0}^{|\alpha|}\sum_{|\gamma|=j,\gamma\le\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}D_u^{\max(|\alpha|-\frac32,0)}E_u^{\alpha_2+1}j!(|\alpha|-j)!\frac{1}{(\max(j,1)\max(|\alpha|-j,1))^{\frac32}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq 27C_{ECN}^2 \sum_{j=0}^{|\alpha|} \sum_{|\gamma|=j,\gamma\leq\alpha} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} D_u^{\max(|\alpha|-\frac{3}{2},0)} E_u^{\alpha_2+1} j!(|\alpha|-j)! \frac{1}{(\max(j,1)\max(|\alpha|-j,1))^{\frac{3}{2}}}. \\
&\leq 27C_{ECN}^2 D_u^{\max(|\alpha|-\frac{3}{2},0)} E_u^{\alpha_2+1} |\alpha|! \sum_{j=0}^{|\alpha|} \frac{1}{(\max(j,1)\max(|\alpha|-j,1))^{\frac{3}{2}}} \\
&\leq 27C_{ECN}^2 D_u^{\max(|\alpha|-\frac{3}{2},0)} E_u^{\alpha_2+1} |\alpha|! \sum_{j=0}^{|\alpha|} \frac{1}{\max(j,1)^{\frac{3}{2}}} \\
&\leq 27C_{ECN}^2 \cdot (2 + \int_1^\infty \frac{1}{x^{\frac{3}{2}}} dx) D_u^{\max(|\alpha|-\frac{3}{2},0)} E_u^{\alpha_2+1} |\alpha|! \\
&\leq 108C_{ECN}^2 D_u^{\max(|\alpha|-\frac{3}{2},0)} E_u^{\alpha_2+1} |\alpha|!.
\end{aligned}
$$

Next, we consider bounding the term $\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(r^2 u_r \partial_r u_r)\|_{L^2(S^1_{[a,b]})}$. There holds

$$
\begin{aligned}
&\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(r^2 u_r \partial_r u_r)\|_{L^2(S^1_{[a,b]})} \\
&\leq \sum_{j=1}^{|\alpha|} \sum_{|\gamma|=j,\gamma\leq\alpha} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \|r^{\frac{\beta}{2}+\gamma_1-1}\mathscr{D}^\gamma(ru_r)\|_{L^4(S^1_{[a,b]})} \|r^{\frac{\beta}{2}+\alpha_1-\gamma_1-1}\mathscr{D}^{\alpha-\gamma}(r\partial_r u_r)\|_{L^4(S^1_{[a,b]})} \\
&+ \|r^{\beta_1+\alpha_1-2} u_r \mathscr{D}^\alpha(r\partial_r u_r)\|_{L^2(S^1_{[a,b]})}.
\end{aligned} \tag{5.26}
$$

The first term above can be estimated as before:

$$
\begin{aligned}
&\sum_{j=1}^{|\alpha|} \sum_{|\gamma|=j,\gamma\leq\alpha} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \|r^{\frac{\beta}{2}+\gamma_1-1}\mathscr{D}^\gamma(ru_r)\|_{L^4(S^1_{[a,b]})} \|r^{\frac{\beta}{2}+\alpha_1-\gamma_1-1}\mathscr{D}^{\alpha-\gamma}(r\partial_r u_r)\|_{L^4(S^1_{[a,b]})} \\
&\leq \sum_{j=1}^{|\alpha|} \sum_{|\gamma|=j,\gamma\leq\alpha} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} (C_{ECN}(|\gamma|+1)^{\frac{1}{2}} D_u^{\max(|\gamma|-\frac{3}{2},0)} E_u^{\gamma_2+\frac{1}{2}} \max(|\gamma|-2,0)!) \\
&\cdot (C_{ECN}(|\alpha-\gamma|+1)^{\frac{1}{2}} D_u^{\max(|\alpha-\gamma|-\frac{1}{2},0)} E_u^{\alpha+2-\gamma_2+\frac{1}{2}} \max(|\alpha-\gamma|-1,0)!) \\
&\leq \max_{j\in\mathbb{N}} (\frac{j+1}{\max(j-1,1)})^2 C_{ECN}^2 \sum_{j=0}^{|\alpha|} \sum_{|\gamma|=j,\gamma\leq\alpha} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} D_u^{|\alpha|-1} E_u^{\alpha_2+1} j!(|\alpha|-j)! \frac{1}{(j)^{\frac{3}{2}}(\max(|\alpha|-j,1))^{\frac{1}{2}}} \\
&\leq 9C_{ECN}^2 D_u^{|\alpha|-1} E_u^{\alpha_2+1} |\alpha|! \sum_{j=1}^{|\alpha|} \frac{1}{(j)^{\frac{3}{2}}(\max(|\alpha|-j,1))^{\frac{1}{2}}} \\
&\leq 9C_{ECN}^2 D_u^{|\alpha|-1} E_u^{\alpha_2+1} |\alpha|! \sum_{j=1}^{|\alpha|} \frac{1}{j^{\frac{3}{2}}} \\
&\leq 9C_{ECN}^2 (1 + \int_1^\infty \frac{1}{x^{\frac{3}{2}}} dx) D_u^{|\alpha|-1} E_u^{\alpha_2+1} |\alpha|! \\
&\leq 27C_{ECN}^2 D_u^{|\alpha|-1} E_u^{\alpha_2+1} |\alpha|!.
\end{aligned}
$$

For the second term in (5.26), since $\overline{\boldsymbol{u}} \in W^2_\beta(S^1_\delta)^2 \subset C^0(\overline{S^1_\delta})^2$ and $\|\overline{\boldsymbol{u}}\|_{W^2_\beta(S^1_\delta)} \leq 1$, if $|\alpha| = 1$,

$$
\begin{aligned}
&\|r^{\beta+\alpha_1-2} u_r \mathscr{D}^\alpha(r\partial_r u_r)\|_{L^2(S^1_{[a,b]})} \\
&\leq \|u_r\|_{L^\infty(S^1_\delta)} \cdot (\|r^{\beta-2+\alpha_1+1}\mathscr{D}^\alpha(\partial_r u_r)\|_{L^2(S^1_{[a,b]})} + \alpha_1 \|r^{\beta-2+\alpha_1}\mathscr{D}^\alpha u_r\|_{L^2(S^1_{[a,b]})}) \\
&\leq 2\|u_r\|_{L^\infty(S^1_\delta)},
\end{aligned}
$$

and if $|\alpha| \geq 2$,

$$\begin{aligned}
&\|r^{\beta+\alpha_1-2}u_r\mathscr{D}^\alpha(r\partial_r u_r)\|_{L^2(S^1_{[a,b]})}\\
&\leq \|u_r\|_{L^\infty(S^1_\delta)}\cdot(\|r^{\beta-2+\alpha_1+1}\mathscr{D}^\alpha(\partial_r u_r)\|_{L^2(S^1_{[a,b]})}+\alpha_1\|r^{\beta-2+\alpha_1}\mathscr{D}^\alpha u_r\|_{L^2(S^1_{[a,b]})})\\
&\leq 2\|u_r\|_{L^\infty(S^1_\delta)}D_u^{|\alpha|-1}E_u^{\alpha_2}(|\alpha|-1)!.
\end{aligned}$$

In conclusion:

$$\begin{aligned}
&\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(r^2u_r\partial_r u_r)\|_{L^2(S^1_{[a,b]})}\\
&\leq 27C_{ECN}^2D_u^{|\alpha|-1}E_u^{\alpha_2+1}|\alpha|!+\max(2\|u_r\|_{L^\infty(S^1_\delta)},2\|u_r\|_{L^\infty(S^1_\delta)}D_u^{|\alpha|-1}E_u^{\alpha_2}(|\alpha|-1)!)\\
&\leq (27C_{ECN}^2+2\|u_r\|_{L^\infty(S^1_\delta)})D_u^{|\alpha|-1}E_u^{\alpha_2+1}|\alpha|!.
\end{aligned}$$

We bound $\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(r^2u_r\partial_r u_\theta)\|_{L^2(S^1_{[a,b]})}$ in the same way. There holds

$$\begin{aligned}
&\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(r^2u_r\partial_r u_r)\|_{L^2(S^1_{[a,b]})}\\
&\leq\sum_{j=1}^{|\alpha|}\sum_{|\gamma|=j,\gamma\leq\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\|r^{\frac{\beta}{2}+\gamma_1-1}\mathscr{D}^\gamma(ru_r)\|_{L^4(S^1_{[a,b]})}\|r^{\frac{\beta}{2}+\alpha_1-\gamma_1-1}\mathscr{D}^{\alpha-\gamma}(r\partial_r u_\theta)\|_{L^4(S^1_{[a,b]})}\\
&+\|r^{\beta+\alpha_1-2}u_r\mathscr{D}^\alpha(r\partial_r u_\theta)\|_{L^2(S^1_{[a,b]})}.
\end{aligned}\tag{5.27}$$

For the first term we have,

$$\begin{aligned}
&\sum_{j=1}^{|\alpha|}\sum_{|\gamma|=j,\gamma\leq\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\|r^{\frac{\beta}{2}+\gamma_1-1}\mathscr{D}^\gamma(ru_r)\|_{L^4(S^1_{[a,b]})}\|r^{\frac{\beta}{2}+\alpha_1-\gamma_1-1}\mathscr{D}^{\alpha-\gamma}(r\partial_r u_\theta)\|_{L^4(S^1_{[a,b]})}\\
&\leq\sum_{j=1}^{|\alpha|}\sum_{|\gamma|=j,\gamma\leq\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}(C_{ECN}(|\gamma|+1)^{\frac12}D_u^{\max(|\gamma|-\frac32,0)}E_u^{\gamma_2+\frac12}\max(|\gamma|-2,0)!)\\
&\cdot(C_{ECN}(|\alpha-\gamma|+1)^{\frac12}D_u^{\max(|\alpha-\gamma|-\frac12,0)}E_u^{\alpha_2-\gamma_2+\frac12}\max(|\alpha-\gamma|-1,0)!)\\
&\leq\max_{j\in\mathbb{N}}(\frac{j+1}{\max(j-1,1)})^2C_{ECN}^2\sum_{j=0}^{|\alpha|}\sum_{|\gamma|=j,\gamma\leq\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}D_u^{|\alpha|-1}E_u^{\alpha_2+1}j!(|\alpha|-j)!\frac{1}{(j)^{\frac32}(\max(|\alpha|-j,1))^{\frac12}}\\
&\leq 9C_{ECN}^2D_u^{|\alpha|-1}E_u^{\alpha_2+1}|\alpha|!\sum_{j=1}^{|\alpha|}\frac{1}{(j)^{\frac32}(\max(|\alpha|-j,1))^{\frac12}}\\
&\leq 9C_{ECN}^2D_u^{|\alpha|-1}E_u^{\alpha_2+1}|\alpha|!\sum_{j=1}^{|\alpha|}\frac{1}{j^{\frac32}}\\
&\leq 9C_{ECN}^2(1+\int_1^\infty\frac{1}{x^{\frac32}}dx)D_u^{|\alpha|-1}E_u^{\alpha_2+1}|\alpha|!\\
&\leq 27C_{ECN}^2D_u^{|\alpha|-1}E_u^{\alpha_2+1}|\alpha|!.
\end{aligned}$$

For the second term in (5.27), if $|\alpha|=1$,

$$\begin{aligned}
&\|r^{\beta+\alpha_1-2}u_r\mathscr{D}^\alpha(r\partial_r u_\theta)\|_{L^2(S^1_{[a,b]})}\\
&\leq \|u_r\|_{L^\infty(S^1_\delta)}\cdot(\|r^{\beta-2+\alpha_1+1}\mathscr{D}^\alpha(\partial_r u_\theta)\|_{L^2(S^1_{[a,b]})}+\alpha_1\|r^{\beta-2+\alpha_1}\mathscr{D}^\alpha u_\theta\|_{L^2(S^1_{[a,b]})})\\
&\leq 2\|u_\theta\|_{L^\infty(S^1_\delta)},
\end{aligned}$$

and if $|\alpha| \geq 2$,

$$\begin{aligned}
&\|r^{\beta+\alpha_1-2}u_r\mathscr{D}^\alpha(r\partial_r u_\theta)\|_{L^2(S^1_{[a,b]})}\\
&\leq \|u_r\|_{L^\infty(S^1_\delta)}\cdot(\|r^{\beta-2+\alpha_1+1}\mathscr{D}^\alpha(\partial_r u_\theta)\|_{L^2(S^1_{[a,b]})}+\alpha_1\|r^{\beta-2+\alpha_1}\mathscr{D}^\alpha u_\theta\|_{L^2(S^1_{[a,b]})})\\
&\leq 2\|u_\theta\|_{L^\infty(S^1_\delta)}D_u^{|\alpha|-1}E_u^{\alpha_2}(|\alpha|-1)!.
\end{aligned}$$

In conclusion,

$$\begin{aligned}
&\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(r^2u_r\partial_r u_r)\|_{L^2(S^1_{[a,b]})}\\
&\leq 27C_{ECN}^2D_u^{|\alpha|-1}E_u^{\alpha_2+1}|\alpha|!+\max(2\|u_\theta\|_{L^\infty(S^1_\delta)},2\|u_\theta\|_{L^\infty(S^1_\delta)}D_u^{|\alpha|-1}E_u^{\alpha_2}(|\alpha|-1)!)\\
&\leq (27C_{ECN}^2+2\|u_\theta\|_{L^\infty(S^1_\delta)})D_u^{|\alpha|-1}E_u^{\alpha_2+1}|\alpha|!.
\end{aligned}$$

We now evaluate $\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(ru_\theta\partial_\theta u_r)\|_{L^2(S^1_{[a,b]})}$:

$$\begin{aligned}
&\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(ru_\theta\partial_\theta u_r)\|_{L^2(S^1_{[a,b]})}\\
&\leq\sum_{j=1}^{|\alpha|}\sum_{|\gamma|=j,\gamma\leq\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\|r^{\frac{\beta}{2}+\gamma_1-1}\mathscr{D}^\gamma(ru_\theta)\|_{L^4(S^1_{\frac{\delta}{2}})}\|r^{\frac{\beta}{2}+\alpha_1-\gamma_1-1}\mathscr{D}^{\alpha-\gamma}(\partial_\theta u_r)\|_{L^4(S^1_{\frac{\delta}{2}})}\\
&+\|r^{\beta+\alpha_1-2}ru_\theta\mathscr{D}^\alpha(\partial_\theta u_r)\|_{L^2(S^1_{[a,b]})}.
\end{aligned}\tag{5.28}$$

The first term here can be estimated as before and we obtain

$$\begin{aligned}
&\sum_{j=1}^{|\alpha|}\sum_{|\gamma|=j,\gamma\leq\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\|r^{\frac{\beta}{2}+\gamma_1-1}\mathscr{D}^\gamma(ru_\theta)\|_{L^4(S^1_{\frac{\delta}{2}})}\|r^{\frac{\beta}{2}+\alpha_1-\gamma_1-1}\mathscr{D}^{\alpha-\gamma}(\partial_\theta u_r)\|_{L^4(S^1_{\frac{\delta}{2}})}\\
&\leq 9C_{ECN}^2D_u^{\max(|\alpha|-\frac{3}{2},0)}E_u^{\alpha_2+2}|\alpha|!\sum_{j=1}^{|\alpha|}\frac{1}{(j)^{\frac{3}{2}}(\max(|\alpha|-j,1))^{\frac{1}{2}}}\\
&\leq 9C_{ECN}^2D_u^{\max(|\alpha|-\frac{3}{2},0)}E_u^{\alpha_2+2}|\alpha|!\sum_{j=1}^{|\alpha|}\frac{1}{j^{\frac{3}{2}}}\\
&\leq 9C_{ECN}^2(1+\int_1^\infty\frac{1}{x^{\frac{3}{2}}}dx)D_u^{\max(|\alpha|-\frac{3}{2},0)}E_u^{\alpha_2+2}|\alpha|!\\
&\leq 27C_{ECN}^2D_u^{\max(|\alpha|-\frac{3}{2},0)}E_u^{\alpha_2+2}|\alpha|!.
\end{aligned}$$

For the second term in (5.28), we get that

$$\begin{aligned}
&\|r^{\beta+\alpha_1-2}ru_\theta\mathscr{D}^\alpha(\partial_\theta u_r)\|_{L^2(S^1_{[a,b]})}\leq\|ru_\theta\|_{L^\infty(S^1_\delta)}\cdot\|r^{\beta-2+\alpha_1}\mathscr{D}^{\alpha+(0,1)}u_r\|_{L^2(S^1_{[a,b]})}\\
&\leq\delta\|u_\theta\|_{L^\infty(S^1_\delta)}D_u^{|\alpha|-1}E_u^{\alpha_2+1}(|\alpha|-1)!.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(ru_\theta\partial_\theta u_r)\|_{L^2(S^1_{[a,b]})}\leq 27C_{ECN}^2D_u^{\max(|\alpha|-\frac{3}{2},0)}E_u^{\alpha_2+2}|\alpha|!+\delta\|u_\theta\|_{L^\infty(S^1_\delta)}D_u^{|\alpha|-1}E_u^{\alpha_2+1}(|\alpha|-1)!\\
&\leq(27C_{ECN}^2+\delta\|u_\theta\|_{L^\infty(S^1_\delta)})D_u^{|\alpha|-1}E_u^{\alpha_2+2}|\alpha|!.
\end{aligned}$$

Similar arguments can be applied on $\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(ru_\theta\partial_\theta u_\theta)\|_{L^2(S^1_{[a,b]})}$:

$$\begin{aligned}
&\|r^{\beta+\alpha_1-2}\mathscr{D}^\alpha(ru_\theta\partial_\theta u_\theta)\|_{L^2(S^1_{[a,b]})}\\
&\leq\sum_{j=1}^{|\alpha|}\sum_{|\gamma|=j,\gamma\leq\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\|r^{\frac{\beta}{2}+\gamma_1-1}\mathscr{D}^\gamma(ru_\theta)\|_{L^4(S^1_{\frac{\delta}{2}})}\|r^{\frac{\beta}{2}+\alpha_1-\gamma_1-1}\mathscr{D}^{\alpha-\gamma}(\partial_\theta u_\theta)\|_{L^4(S^1_{\frac{\delta}{2}})}\\
&+\|r^{\beta+\alpha_1-2}ru_\theta\mathscr{D}^\alpha(\partial_\theta u_\theta)\|_{L^2(S^1_{[a,b]})}.
\end{aligned}\tag{5.29}$$

The first term here can be estimated as before and we obtain

$$\begin{aligned}
&\sum_{j=1}^{|\alpha|}\sum_{|\gamma|=j,\gamma\le\alpha}\begin{pmatrix}\alpha\\ \gamma\end{pmatrix}\|r^{\frac{\beta}{2}+\gamma_1-1}\mathscr{D}^{\gamma}(ru_\theta)\|_{L^4(S^1_{\frac{\delta}{2}})}\|r^{\frac{\beta}{2}+\alpha_1-\gamma_1-1}\mathscr{D}^{\alpha-\gamma}(\partial_\theta u_\theta)\|_{L^4(S^1_{\frac{\delta}{2}})}\\
&\le 9C_{ECN}^2D_u^{\max(|\alpha|-\frac{3}{2},0)}E_u^{\alpha_2+2}|\alpha|!\sum_{j=1}^{|\alpha|}\frac{1}{(j)^{\frac{3}{2}}(\max(|\alpha|-j,1))^{\frac{1}{2}}}\\
&\le 9C_{ECN}^2D_u^{\max(|\alpha|-\frac{3}{2},0)}E_u^{\alpha_2+2}|\alpha|!\sum_{j=1}^{|\alpha|}\frac{1}{j^{\frac{3}{2}}}\\
&\le 9C_{ECN}^2(1+\int_1^\infty\frac{1}{x^{\frac{3}{2}}}dx)D_u^{\max(|\alpha|-\frac{3}{2},0)}E_u^{\alpha_2+2}|\alpha|!\\
&\le 27C_{ECN}^2D_u^{\max(|\alpha|-\frac{3}{2},0)}E_u^{\alpha_2+2}|\alpha|!.
\end{aligned}$$

For the second term in (5.29), we get that

$$\begin{aligned}
&\|r^{\beta+\alpha_1-2}ru_\theta\mathscr{D}^{\alpha}(\partial_\theta u_\theta)\|_{L^2(S^1_{[a,b]})}\le\|ru_\theta\|_{L^\infty(S^1_\delta)}\cdot\|r^{\beta-2+\alpha_1}\mathscr{D}^{\alpha+(0,1)}u_\theta\|_{L^2(S^1_{[a,b]})}\\
&\le\delta\|u_\theta\|_{L^\infty(S^1_\delta)}D_u^{|\alpha|-1}E_u^{\alpha_2+1}(|\alpha|-1)!.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|r^{\beta+\alpha_1-2}\mathscr{D}^{\alpha}(ru_\theta\partial_\theta u_\theta)\|_{L^2(S^1_{[a,b]})}\le 27C_{ECN}^2D_u^{\max(|\alpha|-\frac{3}{2},0)}E_u^{\alpha_2+2}|\alpha|!+\delta\|u_\theta\|_{L^\infty(S^1_\delta)}D_u^{|\alpha|-1}E_u^{\alpha_2+1}(|\alpha|-1)!\\
&\le(27C_{ECN}^2+\delta\|u_\theta\|_{L^\infty(S^1_\delta)})D_u^{|\alpha|-1}E_u^{\alpha_2+2}|\alpha|!.
\end{aligned}$$

It is clear with all estimates above and Lemma 5.2.8 that if we set $C_{ANT}:=2^{\frac{1}{2}}(162C_{ECN}^2+(2+\delta)\|\overline{\boldsymbol{u}}\|_{L^\infty(S^1_\delta)})$, then (5.25) holds.

□

Proof of Lemma 5.2.7. We rewrite the Navier-Stokes equation in the sector S^1_δ using differential and boundary operators $\overline{L}_{st,1}(\cdot,\cdot)$ and $\overline{B}(\cdot,\cdot)$ introduced in Remark 4.1.5:

$$\begin{aligned}
\overline{L}_{st,1}(\boldsymbol{u},p)&=((\overline{\boldsymbol{f}}-\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})^t,0)^t,\\
\overline{B}(\boldsymbol{u},p)&=\boldsymbol{0}\qquad\qquad\text{on }(\Gamma_1\cup\Gamma_n)\cap\partial S^1_\delta.
\end{aligned}\tag{5.30}$$

This set of equations has the following specific form:

$$-\nu(\partial_r^2u_r+\frac{1}{r}\partial_ru_r+\frac{1}{r^2}\partial_\theta^2u_r-\frac{u_r}{r^2}-\frac{2}{r^2}\partial_\theta u_\theta)+\partial_rp=f_r-(\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})_r,\tag{5.31}$$

$$-\nu(\partial_r^2u_\theta+\frac{1}{r}\partial_ru_\theta+\frac{1}{r^2}\partial_\theta^2u_\theta-\frac{u_\theta}{r^2}+\frac{2}{r^2}\partial_\theta u_r)+\frac{1}{r}\partial_\theta p=f_\theta-(\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})_\theta,\tag{5.32}$$

$$\partial_ru_r+r^{-1}u_r+r^{-1}\partial_\theta u_\theta=0.\tag{5.33}$$

$$\overline{\boldsymbol{u}}=\boldsymbol{0}\qquad\text{on }(\Gamma_1\cup\Gamma_n)\cap\Gamma_D\cap\partial S^1_\delta,\tag{5.34}$$

$$\begin{pmatrix}\boldsymbol{u}\cdot\boldsymbol{n}\\(\underline{\sigma}(\boldsymbol{u},p)\boldsymbol{n})\cdot\boldsymbol{t}\end{pmatrix}=\begin{pmatrix}u_\theta\\ \nu(\partial_ru_\theta+\frac{1}{r}\partial_\theta u_r-\frac{1}{r}u_\theta)\end{pmatrix}=\boldsymbol{0}\qquad\text{on }(\Gamma_1\cup\Gamma_n)\cap\Gamma_G\cap\partial S^1_\delta,\tag{5.35}$$

$$\overline{\underline{\sigma}(\boldsymbol{u},p)\boldsymbol{n}}=\begin{pmatrix}\nu(r^{-1}\partial_\theta u_r+\partial_ru_\theta-r^{-1}u_\theta)\\ -p+2\nu r^{-1}(\partial_\theta u_\theta+u_r)\end{pmatrix}=\boldsymbol{0}\qquad\text{on }(\Gamma_1\cup\Gamma_n)\cap\Gamma_N\cap\partial S^1_\delta.\tag{5.36}$$

By Lemma 4.1.2, Lemma 5.2.6, Lemma 5.2.11 and the fact that $\boldsymbol{f}\in B^0_\beta(\Omega)^2$, there exists constants $K_f,K_{nonli},K_u,K_p\ge1$ such that for any multi-index $|\alpha|\ge1$:

$$\|r^{\beta_1+\alpha_1-2}\mathscr{D}^\alpha(r^2\overline{\boldsymbol{f}})\|_{L^2(S^1_\delta)} \leq K_f^{|\alpha|}|\alpha|!, \tag{5.37}$$

$$\|r^{\beta_1+\alpha_1-2}\mathscr{D}^\alpha(r^2\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})\|_{L^2(S^1_{[\frac{\delta}{2},\delta]})} \leq K_{nonli}^{|\alpha|}|\alpha|!, \tag{5.38}$$

and

$$\|r^{\beta_1-1+\alpha_1}\mathscr{D}^\alpha p\|_{L^2(S^1_{[\frac{\delta}{2},\delta]})} \leq K_p^{|\alpha|-1}(|\alpha|-1)!. \tag{5.39}$$

Also, for any $k \in \mathbb{N}_{\geq 1}$,

$$\|r^k\partial_r^k\overline{\boldsymbol{u}}\|_{H^1(S^1_{[\frac{\delta}{2},\delta]})} \leq K_u^k k!. \tag{5.40}$$

Recall that we have fixed $\delta \in (0,1)$. From (5.39) and the fact that $\beta_1<1$ we have, for any $k \in \mathbb{N}_{\geq 1}$,

$$\|r^k\partial_r^k p\|_{L^2(S^1_{[\frac{\delta}{2},\delta]})} \leq K_p^k k!. \tag{5.41}$$

Set $K = \max(K_f, K_{nonli}, K_u, K_p)$, then all inequalities above still hold true if these constants are replaced by K. We will always do that replacement when we use these inequalities later. We claim that all bounds in Lemma 5.2.7 hold if we set

$$E_u = \max([33(\nu + C_{ANT} + 1)]^3, [42(\frac{1}{\nu} + C_{ANT} + 1)]^{\frac{3}{2}}) \geq 1 \tag{5.42}$$

and

$$\begin{aligned} D_u = \max(&22(C_{SEC}+1)(1+\frac{1}{\nu})K, 2(C_{SEC}+1)(C_{ANT}+7)E_u^2, \\ &33(\nu + C_{ANT}+1)E_u^2, 42(\frac{1}{\nu}+\frac{C_{ANT}}{\nu}+1)E_u^{\frac{7}{3}}) \geq 1. \end{aligned} \tag{5.43}$$

Before we prove this by induction, we present the following elementary results about D_u and E_u, which will be useful later. For $k, N \in \mathbb{N}_{\geq 1}$, there holds, under (5.42) and (5.43),

$$(C_{SEC}+1)(11K^k k! + (C_{ANT}+7)D_u^{k-1}E_u^2 k!) \leq D_u^k k!, \tag{5.44}$$

since

$$\begin{aligned} &(C_{SEC}+1)(11K^k k! + (C_{ANT}+7)D_u^{k-1}E_u^2 k!) \\ &= 11(C_{SEC}+1)K^k k! + (C_{SEC}+1)(C_{ANT}+7)D_u^{k-1}E_u^2 k! \\ &\leq \frac{1}{2}D_u^k k! + \frac{1}{2}D_u^k k! \leq D_u^k k!. \end{aligned}$$

Furthermore,

$$3D_u^k E_u^{N-\frac{4}{3}} \leq D_u^k E_u^{N-\frac{1}{3}}, \tag{5.45}$$

since $E_u \geq [33(\nu + C_{ANT} + 1)]^3 \geq 3$. Also,

$$11(\nu + C_{ANT} + 1)(D_u^{k-1}E_u^{N+2}k! + D_u^k E_u^{N-\frac{1}{3}}k!) + K^k k! \leq D_u^k E_u^N k!, \tag{5.46}$$

since

$$\begin{aligned}
&11(\nu + C_{ANT} + 1)(D_u^{k-1}E_u^{N+2}k! + D_u^k E_u^{N-\frac{1}{3}}k!) + K^k k! \\
&= 11(\nu + C_{ANT} + 1)D_u^{k-1}E_u^{N+2}k! + 11(\nu + C_{ANT} + 1)D_u^k E_u^{N-\frac{1}{3}}k! + K^k k! \\
&\leq \frac{1}{3}D_u^k E_u^N k! + \frac{1}{3}D_u^k E_u^N k! + \frac{1}{3}D_u^k E_u^N k! \leq D_u^k E_u^N k!.
\end{aligned}$$

Finally,

$$14(\frac{1}{\nu} + \frac{C_{ANT}}{\nu} + 1)(D_u^{k-1}E_u^{N+2}k! + D_u^k E_u^{N-1}k!) + \frac{1}{\nu}K^k k! \leq D_u^k E_u^{N-\frac{1}{3}}k!, \tag{5.47}$$

since

$$\begin{aligned}
&14(\frac{1}{\nu} + \frac{C_{ANT}}{\nu} + 1)(D_u^{k-1}E_u^{N+2}k! + D_u^k E_u^{N-1}k!) + \frac{1}{\nu}K^k k! \\
&= 14(\frac{1}{\nu} + \frac{C_{ANT}}{\nu} + 1)D_u^{k-1}E_u^{N+2}k! + 14(\frac{1}{\nu} + \frac{C_{ANT}}{\nu} + 1)D_u^k E_u^{N-1}k! + \frac{1}{\nu}K^k k! \\
&\leq \frac{1}{3}D_u^k E_u^{N-\frac{1}{3}}k! + \frac{1}{3}D_u^k E_u^{N-\frac{1}{3}}k! + \frac{1}{3}D_u^k E_u^{N-\frac{1}{3}}k! \leq D_u^k E_u^{N-\frac{1}{3}}k!.
\end{aligned}$$

To prove the weighted analytic regularity we introduce the following induction hypothesis.

Induction Hypothesis H_k

For any $k \in \mathbb{N}_{\geq 1}$, we say that the hypothesis H_k holds if inequalities (5.16) and (5.17), i.e.

$$\begin{aligned}
\|r^{\beta_1+\alpha_1-2}\mathscr{D}^\alpha u_r\|_{L^2(S^1_{\delta/2})} &\leq D_u^{|\alpha|-2}E_u^{\max(\alpha_2-\frac{4}{3},0)}(|\alpha|-2)!, \\
\|r^{\beta_1+\alpha_1-2}\mathscr{D}^\alpha u_\theta\|_{L^2(S^1_{\delta/2})} &\leq D_u^{|\alpha|-2}E_u^{\max(\alpha_2-\frac{4}{3},0)}(|\alpha|-2)!,
\end{aligned}$$

are satisfied for $2 \leq |\alpha| \leq k+1$ and (5.18), which is

$$\|r^{\beta_1+\alpha_1-1}\mathscr{D}^\alpha p\|_{L^2(S^1_{\delta/2})} \leq D_u^{|\alpha|-1}E_u^{\alpha_2}(|\alpha|-1)!,$$

is satisfied for $1 \leq |\alpha| \leq k$ with D_u and E_u defined in (5.42) and (5.43). Clearly these constants are independent of k.

By our setting on δ, $\|\overline{\boldsymbol{u}}\|_{W^2_{\beta_1}(S^1_\delta)} \leq 1$ and $\|p\|_{W^1_{\beta_1}(S^1_\delta)} \leq 1$. Therefore (5.16) and (5.17) hold for $|\alpha| = 2$ and (5.18) holds for $|\alpha| = 1$ and H_1 is correct. Now we assume that H_k holds for some $k \in \mathbb{N}_{\geq 1}$. To show that H_{k+1} is true we analyze in the following two steps: The first step is dedicated to prove that (5.16) and (5.17) hold for $|\alpha| = k+2$ with $\alpha_2 \leq 2$ and (5.18) holds for $|\alpha| = k+1$ with $\alpha_2 \leq 1$. This is done by applying Lemma 5.1.1 on the function pair $\overline{\boldsymbol{v}} := r^k\partial_r^k(\overline{\boldsymbol{u}})$ and $q := r^k\partial_r^k p$, which is the solution of an auxiliary Stokes problem (this problem is constructed below). And in the second step, we finish justifying H_{k+1} by establishing relations between derivatives of $(\boldsymbol{u}, p)$ with lower and higher α_2 using a specific order of differentiation of (5.31), (5.32) and (5.33), this will enable us to bound derivatives with higher α_2 using derivatives with lower α_2.

Step 1: By Lemma 5.2.11, there exists a constant C_{ANT} independent of D_u, E_u, k such that for any $\alpha \in \mathbb{N}^2$ with $1 \le |\alpha| \le k$

$$\|r^{\beta_1+\alpha_1-2}\mathscr{D}^\alpha(r^2\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})\|_{L^2(S^1_{\delta/2})} \le C_{ANT}D_u^{|\alpha|-1}E_u^{\alpha_2+2}|\alpha|!. \tag{5.48}$$

The following lemma is crucial for Step 1.

Lemma 5.2.12. *$(\boldsymbol{v}, q)$ defined by $\overline{\boldsymbol{v}} := r^k\partial_r^k(\overline{\boldsymbol{u}})$ and $q := r^k\partial_r^k p$ solves formally in S^1_δ:*

$$\overline{L}_{st,1}(\boldsymbol{v},q)_{12} = r^{k-2}\partial_r^k(r^2(\overline{\boldsymbol{f}}-\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})) - kr^{k-2}(r\partial_r^k p + (k-1)\partial_r^{k-1}p, \partial_r^{k-1}\partial_\theta p)^t, \tag{5.49}$$

$$\overline{L}(\boldsymbol{v},q)_3 = 0, \tag{5.50}$$

$$\overline{\boldsymbol{v}} = \boldsymbol{0} \qquad \text{on } (\Gamma_1\cup\Gamma_n)\cap\Gamma_D\cap\partial S^1_\delta, \tag{5.51}$$

$$\begin{pmatrix} \boldsymbol{v}\cdot\boldsymbol{n} \\ (\underline{\sigma}(\boldsymbol{v},p)\boldsymbol{n})\cdot\boldsymbol{t} \end{pmatrix} = \boldsymbol{0} \qquad \text{on } (\Gamma_1\cup\Gamma_n)\cap\Gamma_G\cap\partial S^1_\delta, \tag{5.52}$$

$$\overline{\underline{\sigma}(\boldsymbol{v},q)\boldsymbol{n}} = \begin{pmatrix} 0 \\ kr^{k-1}\partial_r^{k-1}p \end{pmatrix} \qquad \text{on } (\Gamma_1\cup\Gamma_n)\cap\Gamma_N\cap\partial S^1_\delta. \tag{5.53}$$

Proof. We verify all equations above in order.

Verification of (5.49): We firstly multiply both sides of $\overline{L}_{st,1}(\boldsymbol{u},p)_{12} = \overline{\boldsymbol{f}} - \overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}}$ by r^2, then differentiate both sides by ∂_r^k and finally multiply both sides by r^{k-2}. We obtain:

$$\begin{pmatrix} -\nu((r^k\partial_r^{k+2} + kr^{k-1}\partial_r^{k+1} + k(k-1)r^{k-2}\partial_r^k)u_r + (r^{k-1}\partial_r^{k+1} + kr^{k-2}\partial_r^k)u_r \\ +r^{k-2}\partial_r^k\partial_\theta^2 u_r - r^{k-2}\partial_r^k u_r - 2r^{k-2}\partial_r^k\partial_\theta u_\theta) \\ +(r^k\partial_r^{k+1} + 2kr^{k-1}\partial_r^k + k(k-1)r^{k-2}\partial_r^{k-1})p \\ -\nu((r^k\partial_r^{k+2} + kr^{k-1}\partial_r^{k+1} + k(k-1)r^{k-2}\partial_r^k)u_\theta + (r^{k-1}\partial_r^{k+1} + kr^{k-2}\partial_r^k)u_\theta \\ +r^{k-2}\partial_r^k\partial_\theta^2 u_\theta - r^{k-2}\partial_r^k u_\theta + 2r^{k-2}\partial_r^k\partial_\theta u_r) \\ +(r^{k-1}\partial_r^k\partial_\theta + kr^{k-2}\partial_r^{k-1}\partial_\theta)p \end{pmatrix} = r^{k-2}\partial_r^k(r^2(\overline{\boldsymbol{f}}-\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})).$$

Moreover, we have, by (4.13),

$$\begin{aligned}\overline{L}_{st,1}(\boldsymbol{v},q)_{12} &:= \begin{pmatrix} -\nu(\partial_r^2 v_r + \frac{1}{r}\partial_r v_r + \frac{1}{r^2}\partial_\theta^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\partial_\theta v_\theta) + \partial_r q \\ -\nu(\partial_r^2 v_\theta + \frac{1}{r}\partial_r v_\theta + \frac{1}{r^2}\partial_\theta^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\partial_\theta v_r) + \frac{1}{r}\partial_\theta q \end{pmatrix} \\ &= \begin{pmatrix} -\nu((r^k\partial_r^{k+2} + kr^{k-1}\partial_r^{k+1} + k(k-1)r^{k-2}\partial_r^k)u_r + (r^{k-1}\partial_r^{k+1} + kr^{k-2}\partial_r^k)u_r \\ +r^{k-2}\partial_r^k\partial_\theta^2 u_r - r^{k-2}\partial_r^k u_r - 2r^{k-2}\partial_r^k\partial_\theta u_\theta) \\ +(r^k\partial_r^{k+1} + kr^{k-1}\partial_r^k)p \\ -\nu((r^k\partial_r^{k+2} + kr^{k-1}\partial_r^{k+1} + k(k-1)r^{k-2}\partial_r^k)u_\theta + (r^{k-1}\partial_r^{k+1} + kr^{k-2}\partial_r^k)u_\theta \\ +r^{k-2}\partial_r^k\partial_\theta^2 u_\theta - r^{k-2}\partial_r^k u_\theta + 2r^{k-2}\partial_r^k\partial_\theta u_r) \\ +r^{k-1}\partial_r^k\partial_\theta p \end{pmatrix}.\end{aligned}$$

Therefore,

$$\overline{L}_{st,1}(\boldsymbol{v},q)_{12} + \begin{pmatrix} kr^{k-1}\partial_r^k p + k(k-1)r^{k-2}\partial_r^{k-1}p \\ kr^{k-2}\partial_r^{k-1}\partial_\theta p \end{pmatrix} = r^{k-2}\partial_r^k(r^2(\overline{\boldsymbol{f}}-\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})).$$

This leads to (5.49).

Verification of (5.50): Note that (5.50) is equivalent to

$$r^k\partial_r^{k+1}u_r + (k+1)r^{k-1}\partial_r^k u_r + r^{k-1}\partial_r^k\partial_\theta u_\theta = 0. \tag{5.54}$$

The incompressibility of $\boldsymbol{u}$ (see (5.33)) implies

$$r\partial_r u_r + u_r + \partial_\theta u_\theta = 0.$$

Differentiate it by ∂_r^k and we have

$$r\partial_r^{k+1}u_r + (k+1)\partial_r^k u_r + \partial_r^k \partial_\theta u_\theta = 0.$$

By multiplying this equality by r^{k-1} we obtain (5.54) and thus (5.50) is justified.

Verification of (5.51): The equation (5.51) can be shown by differentiating (5.34) with ∂_r^k and multiplying it by r^k.

Verification of (5.52): We could show similarly as in the verification of (5.51) that the first component of the left-hand side of (5.52) vanishes. Now, (5.35) implies

$$\nu(r\partial_r u_\theta + \partial_\theta u_r - u_\theta) = 0.$$

Differentiate it by ∂_r^k and multiply the resulting equality by r^{k-1}, we have

$$\nu(kr^{k-1}\partial_r^k u_\theta + r^k\partial_r^{k+1}u_\theta + r^{k-1}\partial_r^k\partial_\theta u_r - r^{k-1}\partial_r^k u_\theta) = 0.$$

It is straightforward to show that this is equivalent to

$$\nu(\partial_r v_\theta + \frac{1}{r}\partial_\theta v_r - \frac{1}{r}v_\theta) = 0.$$

Due to (5.35), the left-hand side is exactly the second component of (5.52). By concluding above derivations we verify (5.52).

Verification of (5.53): We firstly look at the first component of it. (5.36) indicates

$$\nu(\partial_\theta u_r + r\partial_r u_\theta - u_\theta) = 0.$$

Differentiate it by ∂_r^k and multiply the resulting equality by r^{k-1}, we have

$$\nu(r^{k-1}\partial_r^k\partial_\theta u_r + kr^{k-1}\partial_r^k u_\theta + r^k\partial_r^{k+1}u_\theta - r^{k-1}\partial_r^k u_\theta) = 0.$$

This implies

$$\nu(r^{-1}\partial_\theta v_r + \partial_r v_\theta - r^{-1}v_\theta) = 0, \tag{5.55}$$

which implies, due to the polar-component form of the Neumann boundary condition (5.36), the first component of (5.53) vanishes. For the second component of (5.53), we note that (5.36) implies

$$-rp + 2\nu(\partial_\theta u_\theta + u_r) = 0$$

Differentiate it by ∂_r^k and multiply the resulting equality by r^{k-1}, we have

$$-kr^{k-1}\partial_r^{k-1}p - r^k\partial_r^k p + 2\nu(r^{k-1}\partial_r^k\partial_\theta u_\theta + r^{k-1}\partial_r^k u_r) = 0.$$

Therefore,

$$-q + 2\nu r^{-1}(\partial_\theta v_\theta + v_r) = -r^k\partial_r^k p + 2\nu(r^{k-1}\partial_r^k\partial_\theta u_\theta + r^{k-1}\partial_r^k u_r) = kr^{k-1}\partial_r^{k-1}p.$$

Note that the left-hand side is the polar-component form of the second component of (5.53)(see (5.36)), this together with (5.55) validates (5.53). □

We remark here that all computations in the preceding proof are formal. (5.49)-(5.53) will then be justified by taking suitable weighted Sobolev norms on those formal relations.

By H_k, (5.37), (5.38) and (5.39), we have

$$\begin{aligned}
&\|\overline{L}_{st,1}(\boldsymbol{v},q)_{12}\|_{L_{\beta_1}(S^1_\delta)}\\
&\le \|r^{\beta_1+k-2}\partial_r^k(r^2\overline{\boldsymbol{f}}-r^2\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})\|_{L^2(S^1_\delta)}+k(k-1)\|r^{\beta_1+k-2}\partial_r^{k-1}p\|_{L^2(S^1_\delta)}+k\|r^{\beta_1+k-1}\partial_r^k p\|_{L^2(S^1_\delta)}\\
&+k\|r^{\beta_1+k-2}\partial_r^{k-1}\partial_\theta p\|_{L^2(S^1_\delta)}\\
&\le (2K^k k!+C_{ANT}D_u^{k-1}E_u^2k!)+(D_u^{k-2}k!+K^{k-2}k!)+(D_u^{k-1}k!+K^{k-1}k!)+(D_u^{k-1}E_uk!+K^{k-1}k!)\\
&\le 5K^kk!+(C_{ANT}+3)D_u^{k-1}E_u^2k!,
\end{aligned}$$

and

$$\begin{aligned}
&\|\overline{\underline{\sigma}(\boldsymbol{v},q)\boldsymbol{n}}\|_{W^{\frac12}_{\beta_1}((\Gamma_1\cup\Gamma_n)\cap\Gamma_N\cap\partial S^1_\delta)}\\
&\le \|kr^{k-1}\partial_r^{k-1}p\|_{W^1_{\beta_1}(S^1_\delta)}\\
&\le \|kr^{\beta_1+k-2}\partial_r^{k-1}p\|_{L^2(S^1_\delta)}+\|kr^{\beta_1+k-2}\partial_r^{k-1}\partial_\theta p\|_{L^2(S^1_\delta)}+\|kr^{\beta_1+k-1}\partial_r^k p\|_{L^2(S^1_\delta)}\\
&+\|k(k-1)r^{\beta_1+k-2}\partial_r^{k-1}p\|_{L^2(S^1_\delta)}\\
&\le (kD_u^{k-2}(k-2)!+kK^{k-2}(k-2)!)+(kD_u^{k-1}E_u(k-1)!+kK^{k-1}(k-1)!)\\
&+(kD_u^{k-1}(k-1)!+kK^{k-1}(k-1)!)+(k(k-1)D_u^{k-2}(k-2)!+k(k-1)K^{k-2}(k-2)!)\\
&\le 4K^{k-1}k!+4D_u^{k-1}E_uk!.
\end{aligned}$$

By Lemma 5.1.1, the above two inequalities, (5.40) and (5.41),

$$\begin{aligned}
&\|\overline{\boldsymbol{v}}\|_{W^2_\beta(S^1_{\delta/2})}+\|q\|_{W^1_\beta(S^1_{\delta/2})}\\
&\le C_{SEC}(\|\overline{L}_{st,1}(\overline{\boldsymbol{v}},q)_{12}\|_{L_{\beta_1}(S^1_\delta)}+\|\overline{\boldsymbol{v}}\|_{W^{1,2}(S^1_{[\frac{\delta}{2},\delta]})}+\|q\|_{L^2(S^1_{[\frac{\delta}{2},\delta]})}+\|\overline{\underline{\sigma}(\boldsymbol{v},q)\boldsymbol{n}}\|_{W^{\frac12}_{\beta_1}((\Gamma_1\cup\Gamma_n)\cap\Gamma_N\cap\partial S^1_\delta)})\\
&\le C_{SEC}(11K^kk!+(C_{ANT}+7)D_u^{k-1}E_u^2k!).
\end{aligned}$$

Based on this and H_k we have:

$$\begin{aligned}
&\sum_{|\gamma|=2}\|r^{\beta_1-2+k+\gamma_1}\mathscr{D}^{(k,0)+\gamma}\overline{\boldsymbol{u}}\|_{L^2(S^1_{\frac{\delta}{2}})}\\
&\le \|r^{\beta_1-2+2}\partial_r^2(\overline{\boldsymbol{v}})-2kr^{\beta_1+k-1}\partial_r^{k+1}(\overline{\boldsymbol{u}})-k(k-1)r^{\beta_1+k-2}\partial_r^k(\overline{\boldsymbol{u}})\|_{L^2(S^1_{\frac{\delta}{2}})}\\
&+\|r^{\beta_1-2+2}\mathscr{D}^{(1,1)}(\overline{\boldsymbol{v}})-kr^{\beta_1+k-1}\mathscr{D}^{(k,1)}(\overline{\boldsymbol{u}})\|_{L^2(S^1_{\frac{\delta}{2}})}+\|r^{\beta_1-2+2}\mathscr{D}^{(0,2)}(\overline{\boldsymbol{v}})\|_{L^2(S^1_{\frac{\delta}{2}})}\\
&\le \|\overline{\boldsymbol{v}}\|_{W^2_\beta(S^1_\delta)}+3D_u^{k-1}k!+D_u^{k-2}k!\\
&\le (C_{SEC}+1)(11K^kk!+(C_{ANT}+7)D_u^{k-1}E_u^2k!).
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{|\gamma|=2}\|r^{\beta_1-2+k+\gamma_1}\mathscr{D}^{(k,0)+\gamma}\overline{\boldsymbol{u}}\|_{L^2(S^1_{\frac{\delta}{2}})}\\
&\le \|r^{\beta_1-1+1}\partial_r q-kr^{\beta_1-1+k}\partial_r^k p\|_{L^2(S^1_{\frac{\delta}{2}})}+\|r^{\beta_1-1}\partial_\theta q\|_{L^2(S^1_{\frac{\delta}{2}})}\\
&\le \|\overline{\boldsymbol{v}}\|_{W^2_\beta(S^1_\delta)}+D_u^{k-1}k!+D_u^{k-2}k!\\
&\le (C_{SEC}+1)(11K^kk!+(C_{ANT}+7)D_u^{k-1}E_u^2k!).
\end{aligned}$$

Now, we obtain by using (5.44),

$$\sum_{|\gamma|=2} \|r^{\beta_1-2+k+\gamma_1}\mathscr{D}^{(k,0)^t+\gamma}u_r\|_{L^2(S^1_{\frac{\delta}{2}})} \leq D_u^k k!,$$
$$\sum_{|\gamma|=2} \|r^{\beta_1-2+k+\gamma_1}\mathscr{D}^{(k,0)^t+\gamma}u_\theta\|_{L^2(S^1_{\frac{\delta}{2}})} \leq D_u^k k!,$$

and

$$\sum_{|\gamma|=1} \|r^{\beta_1-1+k+\gamma_1}\mathscr{D}^{(k,0)^t+\gamma}p\|_{L^2(S^1_{\frac{\delta}{2}})} \leq D_u^k k!.$$

Hence, (5.16) and (5.17) hold for $|\alpha| = k+2$ with $\alpha_2 \leq 2$ and (5.18) holds for $|\alpha| = k+1$ with $\alpha_2 \leq 1$.

Step 2: In this step we prove that (5.16) and (5.17) hold for $|\alpha| = k+2$ and (5.18) holds for $|\alpha| = k+1$. Part of this statement has already been proven in the previous step and we need to prove that (5.16) and (5.17) hold for $|\alpha| = k+2$ with $2 < \alpha_2 \leq k+2$ and (5.18) holds for $|\alpha| = k+1$ with $1 < \alpha_2 \leq k+1$.

We proceed by induction with respect to α_2. Let $N \in \mathbb{N}$, $2 \leq N \leq k+1$ and assume that (5.16) and (5.17) hold true for $|\alpha| = k+2$ with $\alpha_2 \leq N$ and (5.18) holds true for $|\alpha| = k+1$ with $\alpha_2 \leq N-1$ (The case $N = 2$ has been shown in step 1). We now show that (5.16) and (5.17) hold true for $|\alpha| = k+2$ with $\alpha_2 = N+1$ and (5.18) holds true for $|\alpha| = k+1$ with $\alpha_2 = N$.

Now, (5.33) implies that formally

$$r\partial_r u_r + u_r + \partial_\theta u_\theta = 0.$$

Differentiate this equality with $\mathscr{D}^{(k+1-N,N)}$ and multiply by $r^{(\beta_1-2)+(k+1-N)}$ on both sides:

$$r^{(\beta_1-2)+(k+2-N)}\mathscr{D}^{(k+2-N,N)}u_r + (k+2)r^{(\beta_1-2)+(k+1-N)}\mathscr{D}^{(k+1-N,N)}u_r +$$
$$r^{(\beta_1-2)+(k+1-N)}\mathscr{D}^{(k+1-N,N+1)}u_\theta = 0.$$

This implies that, by our assumption:

$$\begin{aligned}
&\|r^{(\beta_1-2)+(k+1-N)}\mathscr{D}^{(k+1-N,N+1)}u_\theta\|_{L^2(S^1_{\frac{\delta}{2}})} \\
&\leq \|r^{(\beta_1-2)+(k+2-N)}\mathscr{D}^{(k+2-N,N)}u_r\|_{L^2(S^1_{\frac{\delta}{2}})} + (k+2)\|r^{(\beta_1-2)+(k+1-N)}\mathscr{D}^{(k+1-N,N)}u_r\|_{L^2(S^1_{\frac{\delta}{2}})} \\
&\leq 3D_u^k E_u^{N-\frac{4}{3}} k!.
\end{aligned}$$

Therefore, from (5.45), it follows that

$$\|r^{(\beta_1-2)+(k+1-N)}\mathscr{D}^{(k+1-N,N+1)}u_\theta\|_{L^2(S^1_{\frac{\delta}{2}})} \leq D_u^k E_u^{(N+1)-\frac{4}{3}} k! \tag{5.56}$$

and (5.17) holds for $|\alpha| = k+2$ with $\alpha_2 = N+1$. Now multiply both sides of (5.32) by r^2, differentiate by $\mathscr{D}^{(k+1-N,N-1)}$ and then multiply by $r^{(\beta_1-2)+(k+1-N)}$. We obtain the following formal relation:

$$\begin{aligned}
&r^{(\beta_1-1)+(k-N+1)}\mathscr{D}^{(k-N+1,N)}p = r^{(\beta_1-2)+(k-N+1)}\mathscr{D}^{(k+1-N,N-1)}(r^2(f_\theta - (\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})_\theta)) \\
&- (k+1-N)r^{(\beta_1-1)+(k-N)}\mathscr{D}^{(k-N,N)}p \\
&- \begin{pmatrix} \nu r^{(\beta_1-2)+k+3-N}\mathscr{D}^{(k+3-N,N-1)} + \nu(2(k+1-N)+1)r^{(\beta_1-2)+k+2-N}\mathscr{D}^{(k+2-N,N-1)} \\ +\nu((k+1-N)(k-N)+(k+1-N)-1)r^{(\beta_1-2)+k+1-N}\mathscr{D}^{(k+1-N,N-1)} \\ +\nu r^{(\beta_1-2)+k+1-N}\mathscr{D}^{(k+1-N,N+1)} \end{pmatrix} u_\theta \\
&+ 2\nu r^{(\beta_1-2)+(k+1-N)}\mathscr{D}^{(k+1-N,N)}u_r.
\end{aligned}$$

Hence, by the fact that H_k holds and by (5.37), (5.48) and (5.56),

$$\begin{aligned}
&\|r^{(\beta_1-1)+(k-N+1)}\mathscr{D}^{(k-N+1,N)}p\|_{L^2(S^1_{\frac{\delta}{2}})}\\
&\le K^k k! + C_{ANT}D_u^{k-1}E_u^{N+2}k! + D_u^{k-1}E_u^N k! + \nu D_u^k E_u^{\max(N-\frac{7}{3},0)}k! + 3\nu D_u^{k-1}E_u^{\max(N-\frac{7}{3},0)}k!\\
&+ 3\nu D_u^{k-2}E_u^{\max(N-\frac{7}{3},0)}k! + \nu D_u^k E_u^{N+1-\frac{4}{3}}k! + 2\nu D_u^{k-1}E_u^{N-\frac{4}{3}}k!\\
&\le 11(\nu + C_{ANT}+1)(D_u^{k-1}E_u^{N+2}k! + D_u^k E_u^{N-\frac{1}{3}}k!) + K^k k!.
\end{aligned}$$

It follows from (5.46) that

$$\|r^{(\beta_1-1)+(k-N+1)}\mathscr{D}^{(k-N+1,N)}p\|_{L^2(S^1_{\frac{\delta}{2}})} \le D_u^k E_u^N k!. \tag{5.57}$$

Therefore (5.18) holds for $|\alpha| = k+1$ with $\alpha_2 = N$.

Finally, multiply both sides of (5.31) by r^2, differentiate by $\mathscr{D}^{(k+1-N,N-1)}$ and then multiply by $r^{(\beta_1-2)+(k+1-N)}$. We obtain formally:

$$\begin{aligned}
&\nu r^{(\beta_1-2)+k+1-N}\mathscr{D}^{(k+1-N,N+1)}u_r = -r^{(\beta_1-2)+k+1-N}\mathscr{D}^{(k+1-N,N-1)}(r^2(f_r - (\overline{(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}})_r))\\
&+\begin{pmatrix} r^{(\beta_1-1)+k+2-N}\mathscr{D}^{(k+2-N,N-1)}\\ +2(k+1-N)r^{(\beta_1-1)+k+1-N}\mathscr{D}^{(k+1-N,N-1)}\\ +(k+1-N)(k-N)r^{(\beta_1-1)+k-N}\mathscr{D}^{(k-N,N-1)}\end{pmatrix}p\\
&+\begin{pmatrix} -\nu(r^{(\beta_1-2)+k+3-N}\mathscr{D}^{(k+3-N,N-1)}\\ +(2(k+1-N)+1)r^{(\beta_1-2)+k+2-N}\mathscr{D}^{(k+2-N,N-1)}\\ +((k+1-N)(k-N)+(k+1-N)-1)r^{(\beta_2-2)+k+1-N}\mathscr{D}^{(k+1-N,N-1)})\end{pmatrix}u_r\\
&+2\nu r^{(\beta_1-2)+k+1-N}\mathscr{D}^{(k+1-N,N)}u_\theta.
\end{aligned}$$

Therefore, by the validity of H_k, (5.37), (5.48), (5.56) and (5.57),

$$\begin{aligned}
&\|r^{(\beta_1-2)+k+1-N}\mathscr{D}^{(k+1-N,N+1)}u_r\|_{L^2(S^1_{\frac{\delta}{2}})}\\
&\le \frac{1}{\nu}(C_{ANT}D_u^{k-1}E_u^{N+2}k! + K^k k! + D_u^N E_u^{N-1}k! + 2D_u^{k-1}E_u^{N-1}k! + 2D_u^{k-2}E_u^{N-1}k! + \nu D_u^k E_u^{N-\frac{7}{3}}k!\\
&+ 3\nu D_u^{k-1}E_u^{\max(N-\frac{7}{3},0)}k! + 3\nu D_u^{k-2}E_u^{\max(N-\frac{7}{3},0)}k! + 2\nu D_u^{k-1}E_u^{\max(N-\frac{7}{3},0)}k!)\\
&\le 14(\frac{1}{\nu} + \frac{C_{ANT}}{\nu} + 1)(D_u^{k-1}E_u^{N+2}k! + D_u^k E_u^{N-1}k!) + \frac{1}{\nu}K^k k!.
\end{aligned}$$

Then by (5.47), we obtain

$$\|r^{(\beta_1-2)+(k-N+1)}\mathscr{D}^{(k-N+1,N+1)}u_r\|_{L^2(S^1_{\frac{\delta}{2}})} \le D_u^k E_u^{N-\frac{1}{3}}k!. \tag{5.58}$$

Therefore (5.16) holds for $|\alpha| = k+2$ with $\alpha_2 = N+1$.

By combining (5.56), (5.57) and (5.58) we have that (5.16) and (5.17) are true for $|\alpha| = k+2$ and (5.18) is true for $|\alpha| = k+1$. Hence H_{k+1} holds based on the validity of H_k. By induction we prove that H_k holds for any $k \in N_{\ge 1}$ and the proof is finished.

□

Proof of Theorem 5.2.1. We have stated that the existence and uniqueness of the solution follow from Theorem 3.1.3.

Lemma 5.2.7 indicates that there exists a constant $D_1 > 1$ depending on ω_1, δ and β_1 such that for all $\alpha \in \mathbb{N}^2$ satisfying $|\alpha| \geq 2$:

$$\|r_1^{\beta_1+\alpha_1-2}\mathscr{D}^\alpha u_r\|_{L^2(S^1_{\delta/2})} \leq D_1^{|\alpha|-2}(|\alpha|-2)!,$$
$$\|r_1^{\beta_1+\alpha_1-2}\mathscr{D}^\alpha u_\theta\|_{L^2(S^1_{\delta/2})} \leq D_1^{|\alpha|-2}(|\alpha|-2)!,$$

and for any $|\alpha| \geq 1$:

$$\|r_1^{\beta_1+\alpha_1-1}\mathscr{D}^\alpha p\|_{L^2(S^1_{\delta/2})} \leq D_1^{|\alpha|-1}(|\alpha|-1)!.$$

This result on the truncated sector $S^1_{\frac{\delta}{2}}$ in the proof of Lemma 5.2.7 can be conducted analogously on other sectors $S^i_{\frac{\delta}{2}}, i = 2, \cdots, n$ and similar regularity results will be obtained: For each corner A_i, there exists a constant $D_i > 1$ depending on ω_i, δ and β_i such that for all $\alpha \in \mathbb{N}^2$ satisfying $|\alpha| \geq 2$:

$$\|r_i^{\beta_i+\alpha_1-2}\mathscr{D}^\alpha u_r\|_{L^2(S^i_{\delta/2})} \leq D_i^{|\alpha|-2}(|\alpha|-2)!, \tag{5.59}$$
$$\|r_i^{\beta_i+\alpha_1-2}\mathscr{D}^\alpha u_\theta\|_{L^2(S^i_{\delta/2})} \leq D_i^{|\alpha|-2}(|\alpha|-2)!, \tag{5.60}$$

and for any $|\alpha| \geq 1$:

$$\|r_i^{\beta_i+\alpha_1-1}\mathscr{D}^\alpha p\|_{L^2(S^i_{\delta/2})} \leq D_i^{|\alpha|-1}(|\alpha|-1)!. \tag{5.61}$$

To see this, we re-index the corner by clockwise rotation of the indices: for $i \in \{1, \cdots, n\}$, we re-index in the following way: $A_{i+1} \to A_i$, $\omega_{i+1} \to \omega_i$, $\beta_{i+1} \to \beta_i$ and $r_i \to r_{i-1}$. In this scenario, the proof of Lemma 5.2.7 is actually made on the truncated sector $S^2_{\frac{\delta}{2}}$ and then (5.59)-(5.61) are proved with $i = 2$. The validity of (5.59)-(5.61) with $i > 2$ will be shown by doing multiple rotations until $A_i, \omega_i, \beta_i, r_i$ are re-indexed as $A_1, \omega_1, \beta_1, r_1$.

Now Lemma 2.2.5 and Lemma 4.1.3 give that for each corner A_i, there exists a constant $F_i > 1$ depending on ω_i, δ and β_i such that for all $\alpha \in \mathbb{N}^2$ satisfying $|\alpha| \geq 2$:

$$\|r_i^{\beta_i+|\alpha|-2}D^\alpha u_1\|_{L^2(S^i_{\delta/2})} \leq \hat{D}_i^{|\alpha|-2}(|\alpha|-2)!, \tag{5.62}$$
$$\|r_i^{\beta_i+|\alpha|-2}D^\alpha u_2\|_{L^2(S^i_{\delta/2})} \leq \hat{D}_i^{|\alpha|-2}(|\alpha|-2)!, \tag{5.63}$$

and for any $|\alpha| \geq 1$:

$$\|r_i^{\beta_i+|\alpha|-1}D^\alpha p\|_{L^2(S^i_{\delta/2})} \leq \hat{D}_i^{|\alpha|-1}(|\alpha|-1)!. \tag{5.64}$$

By the setting of δ, it is easy to check that there exists a constant C_b depending on β_f, Ω and δ such that for any $i \in \{1, \cdots, n\}$, any $j \in \{1, 2\}$ and any $|\alpha| \geq j$.

$$\Phi_{\beta_f+|\alpha|-j} < r_i^{\beta_i+|\alpha|-j}C_b^{|\alpha|-j}.$$

Therefore, for all i and all $\alpha \in \mathbb{N}^2$ satisfying $|\alpha| \geq 2$:

$$\|\Phi_{\beta_f+|\alpha|-2}D^\alpha u_1\|_{L^2(S^i_{\delta/2})} \leq (C_b\hat{D}_i)^{|\alpha|-2}(|\alpha|-2)!,$$
$$\|\Phi_{\beta_f+|\alpha|-2}D^\alpha u_2\|_{L^2(S^i_{\delta/2})} \leq (C_b\hat{D}_i)^{|\alpha|-2}(|\alpha|-2)!,$$

and for any $|\alpha| \geq 1$:

$$\|\Phi_{\beta_f+|\alpha|-1} D^\alpha p\|_{L^2(S^i_{\delta/2})} \leq (C_b \hat{D}_i)^{|\alpha|-1}(|\alpha|-1)!.$$

Lemma 5.2.6 implies that there exists $C_0 > nC_b\hat{D}_i$ for any i such that for all $\alpha \in \mathbb{N}^2$ satisfying $|\alpha| \geq 2$:

$$\|\Phi_{\beta_f+|\alpha|-2} D^\alpha u_1\|_{L^2(\Omega\setminus\cup_i S^i_{\delta/2})} \leq C_0^{|\alpha|-2}(|\alpha|-2)!,$$
$$\|\Phi_{\beta_f+|\alpha|-2} D^\alpha u_2\|_{L^2(\Omega\setminus\cup_i S^i_{\delta/2})} \leq C_0^{|\alpha|-2}(|\alpha|-2)!,$$

and for any $|\alpha| \geq 1$:

$$\|\Phi_{\beta_f+|\alpha|-1} D^\alpha p\|_{L^2(\Omega\setminus\cup_i S^i_{\delta/2})} \leq C_0^{|\alpha|-1}(|\alpha|-1)!.$$

Summarize all the inequalities above and we have that for all $\alpha \in \mathbb{N}^2$ satisfying $|\alpha| \geq 2$:

$$\|\Phi_{\beta_f+|\alpha|-2} D^\alpha u_1\|_{L^2(\Omega)} \leq 2C_0^{|\alpha|-2}(|\alpha|-2)!,$$
$$\|\Phi_{\beta_f+|\alpha|-2} D^\alpha u_2\|_{L^2(\Omega)} \leq 2C_0^{|\alpha|-2}(|\alpha|-2)!,$$

and for any $|\alpha| \geq 1$:

$$\|\Phi_{\beta_f+|\alpha|-1} D^\alpha p\|_{L^2(\Omega)} \leq 2C_0^{|\alpha|-1}(|\alpha|-1)!.$$

These results together with the fact that $(\boldsymbol{u}, p)|_{S^i_\delta} \in W^2_{\beta_i}(S^i_\delta)^2 \times W^1_{\beta_i}(S^i_\delta) \subset H^{2,2}_{\beta_i}(S^i_\delta)^2 \times H^{1,1}_{\beta_i}(S^i_\delta)$ holds in each corner S^i_δ imply that $(\boldsymbol{u}, p) \in B^2_{\beta_f}(\Omega)^2 \times B^1_{\beta_f}(\Omega)$. □

Part II

hp-DGFEM Discretization

Chapter 6

hp-DGFEM Discretization of the stationary incompressible NSE

From this chapter on we analyze the mixed *hp*-DGFEM for the stationary incompressible NSE. The analysis is mainly based on [40] and [36]. We assume from now on that $\partial\Omega = \Gamma_D$. Then (3.1) reduces to the following problem:

$$\begin{aligned} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p &= \boldsymbol{f} && \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 && \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \partial\Omega. \end{aligned} \tag{6.1}$$

We introduce here a weak formulation of (6.1) which is different from (3.5). Define the following bilinear form

$$A_{noslip}(\boldsymbol{u},\boldsymbol{v}) = \nu\int_\Omega \nabla\boldsymbol{u} : \nabla\boldsymbol{v}\, d\boldsymbol{x}, \tag{6.2}$$

The variational problem now reads: Find $(\boldsymbol{u}, p) \in \mathbf{W}\times L_0$ such that for all $\boldsymbol{v}\in\mathbf{W}$ and $q\in L_0$,

$$\begin{aligned} A_{noslip}(\boldsymbol{u},\boldsymbol{v}) + O(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) + B(\boldsymbol{v},p) &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\, d\boldsymbol{x}, \\ B(\boldsymbol{u},q) &= 0. \end{aligned} \tag{6.3}$$

It has been shown in [14, Chapter IV, Theorem 2.1] that there exists a solution $(\boldsymbol{u},p)\in\mathbf{W}\times L_0$ to (3.5) such that $\boldsymbol{u}$ belongs to the kernel $\mathbf{Z} := \{\boldsymbol{v}\in\mathbf{W} | \nabla\cdot\boldsymbol{v} = 0 \text{ in } L^2(\Omega)\}$. Moreover, by setting $\boldsymbol{v}=\boldsymbol{u}$ in (6.3) and using the Cauchy-Schwarz and Poincaré inequalities we get the stability bound

$$\|\nabla\boldsymbol{u}\|_{L^2(\Omega)} \le \frac{C_P\|\boldsymbol{f}\|_{L^2(\Omega)}}{\nu}, \tag{6.4}$$

where

$$C_P := \Big(\inf_{\boldsymbol{v}\in H_0^1(\Omega)^2} \frac{(\int_\Omega \|\nabla\boldsymbol{v}\|_F^2)^{\frac{1}{2}}}{\|\boldsymbol{v}\|_{L^2(\Omega)}}\Big)^{-1}$$

is the Poincaré constant in Ω. Here $\|\cdot\|_F$ is the so-called *Frobenius* norm on matrix(see [19, Chapter 5.2]).

It has also been shown in [14, Chapter IV, Theorem 2.2] that the uniqueness of the solution for (3.5) can be ensured under the *small data assumption*

$$\frac{C_O C_P \|\boldsymbol{f}\|_{L^2(\Omega)}}{\nu^2} < 1, \tag{6.5}$$

where the norm of the convective form is

$$C_O := \sup_{\boldsymbol{v},\boldsymbol{u},\boldsymbol{w}\in\mathbf{V}} \frac{O(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})}{|\boldsymbol{u}|_{H^1(\Omega)}|\boldsymbol{v}|_{H^1(\Omega)}|\boldsymbol{w}|_{H^1(\Omega)}} < +\infty.$$

We introduce a mixed *hp*-DGFEM discretization for (6.1) in this chapter. This discretization, which combines the numerical scheme for the stationary Stokes system in [40] and a discrete trilinear form for the convection term[9, Chapter 6], is proposed in [36].

6.1 Meshes and finite element space

Let $\mathfrak{T}$ be a collection of meshes $\mathcal{T}$ on Ω. We assume that each element is the affine mapping of the reference triangle $\hat{T} = \{(x,y) \in \mathbb{R}^2 : x > 0, y > 0, x + y < 1\}$ or the reference square $\hat{Q} = (0,1)^2$. We allow irregular meshes but we require that the intersection of the closures of two neighbouring elements is a common vertex or a complete edge of at least one of those two elements. This setting makes it possible to construct a geometric mesh towards a corner on $\partial\Omega$, which is perfect in handling singularity of the solution.

We denote by h_K the diameter of the element $K \in \mathcal{T}$ and $\underline{h} = \{h_K\}_{K\in\mathcal{T}}$ the diameter vector. To build the finite element space we also assign to each element K a polynomial degree $k_K \geq 1$ and store those polynomial degrees in a vector $\underline{k} = \{k_K\}_{K\in\mathcal{T}}$. We set the meshwidth of $\mathcal{T}$ as $h_\mathcal{T} = \max_{K\in\mathcal{T}} h_K$ and the maximum polynomial degree of $\mathcal{T}$ as $k_\mathcal{T} = \max_{K\in\mathcal{T}} k_K$. Furthermore, we have the following restrictions on those parameters mentioned above:

- (Shape-regularity) There exist two positive constants κ_1 and κ_2 which are uniform with respect to $\mathfrak{T}$ such that
$$\forall K \in \mathcal{T} : \|J_F\|_{L^\infty(\hat{K})} \leq \kappa_1 h_K^2 \qquad \|J_{F^{-1}}\|_{L^\infty(K)} \leq \kappa_1 h_{\hat{K}}^{-2}$$
where $J_F(J_{F^{-1}})$ is the Jacobian of the mapping $F(F^{-1})$ and $\hat{K}$ is $\hat{T}$ or $\hat{Q}$.

- (Bounded local variation of mesh sizes and elemental polynomial degrees) There exist two positive constants κ_3 and κ_4 which are uniform with respect to $\mathfrak{T}$ such that for any two elements K and K' sharing an interior edge
$$\kappa_3 h_K \leq h_{K'} \leq \kappa_3^{-1} h_K \qquad \kappa_4 k_K \leq k_{K'} \leq \kappa_4^{-1} k_K.$$

An interior edge E_i is the non-empty interior of $\partial K \cap \partial K'$, where K and K' are two adjacent elements. Here we assume that E is the entire edge of at least one of those two elements. A boundary edge E_b is the non-empty interior of $\partial K \cap \partial\Omega$ such that E_b is the entire edge of K. We denote by $\mathcal{E}_\mathcal{I}(\mathcal{T})$ the set of all interior edges and $\mathcal{E}_\mathcal{D}(\mathcal{T})$ the set of all boundary edges and set $\mathcal{E}(\mathcal{T}) := \mathcal{E}_\mathcal{I} \cup \mathcal{E}_\mathcal{D}(\mathcal{T})$.

Given a mesh $\mathcal{T}$ and a vector $\underline{k}$ listing the elemental polynomial degrees, we introduce the following local polynomial space $\mathbb{S}_k(K) := \{q = \hat{q} \circ F_K^{-1} : \hat{q} \in \hat{\mathbb{S}}_{k_K}(\hat{K})\}$ where, for an positive integer k, $\hat{\mathbb{S}}_k(\hat{K}) = \mathbb{P}_k(\hat{K})$ if K is a triangle and $\hat{\mathbb{S}}_k(\hat{K}) = \mathbb{Q}_k(\hat{K})$ if K is a quadrilateral. For definition of $\mathbb{P}_k$ and $\mathbb{Q}_k$, refer to [10, Chapter 1.2].

We now define the discontinuous Galerkin space

$$S^{\underline{k}}(\mathcal{T}) := \{v \in L^2(\Omega) : v|_K \in \mathbb{S}_{k_K}(K), K \in \mathcal{T}\}$$

and introduce the following finite element spaces approximating $\mathbf{W}$ and L_0:

$$\mathbf{V}_{DG} := [S^{\underline{k}}(\mathcal{T})]^2, \qquad \tilde{Q}_{DG} := S^{\underline{k}-1}(\mathcal{T}), \qquad Q_{DG} := L_0 \cap \tilde{Q}_{DG}.$$

We also define $\underline{\Sigma}_{DG} := [S^{\underline{k}}(\mathcal{T})]^{2\times 2}$, which will be used in the analysis of DG-FEM method.

6.2 Trace operators

The trace operators, which describe the property of the (numerical) solutions near the interior or boundary edges, play a central role in the design of the discontinuous Galerkin methods. We introduce here some trace operators which will be used later.

Given a mesh $\mathcal{T} \in \mathfrak{T}$ and functions $\boldsymbol{v} \in H^1(\mathcal{T})^2$, $q \in H^1(\mathcal{T})$ and $\underline{\tau} \in H^1(\mathcal{T})^{2\times 2}$, consider an interior edge $E \in \mathcal{E}_{\mathcal{I}}$ shared by two elements $K^{\pm}$. Denote by $\boldsymbol{n}^{\pm}$ the unit outward normals on $\partial K^{\pm}$ and by $(\boldsymbol{v}^{\pm}, q^{\pm}, \underline{\tau}^{\pm})$ the traces of $(\boldsymbol{v}, q, \underline{\tau})$ on E from $K^{\pm}$. We define the mean value operator $\{\{\cdot\}\}$ as

$$\{\{\boldsymbol{v}\}\} := (\boldsymbol{v}^+ + \boldsymbol{v}^-)/2, \qquad \{\{q\}\} := (q^+ + q^-)/2, \qquad \{\{\underline{\tau}\}\} := (\underline{\tau}^+ + \underline{\tau}^-)/2.$$

We also introduce jump operator $[\![\cdot]\!]$ as

$$[\![q]\!] := q^+\boldsymbol{n}^+ + q^-\boldsymbol{n}^-, \qquad [\![\boldsymbol{v}]\!] := \boldsymbol{v}^+\cdot\boldsymbol{n}^+ + \boldsymbol{v}^-\cdot\boldsymbol{n}^-, \qquad \underline{[\![\boldsymbol{v}]\!]} := \boldsymbol{v}^+\otimes\boldsymbol{n}^+ + \boldsymbol{v}^-\otimes\boldsymbol{n}^-, \qquad [\![\underline{\tau}]\!] := \underline{\tau}^+\boldsymbol{n}^+ + \underline{\tau}^-\boldsymbol{n}^-.$$

For a boundary edge $E \in \mathcal{E}_{\mathcal{D}}$, we define the mean value operators as $\{\{\boldsymbol{v}\}\} := \boldsymbol{v}$, $\{\{q\}\} := q$, $\{\{\underline{\tau}\}\} := \underline{\tau}$ and $\{\{\underline{\tau}\}\} := \underline{\tau}$, we also set jump operator as $[[q]] := q\boldsymbol{n}^+$,$[[\boldsymbol{v}]] := \boldsymbol{v}\cdot\boldsymbol{n}^+$ and $\underline{[\![\boldsymbol{v}]\!]} := \boldsymbol{v}\otimes\boldsymbol{n}^+$, where $\boldsymbol{n}^+$ is the unit outward normal on the edge E. Due to the trace theorem for H^1 functions in two-dimensional case, all operators defined above are well-defined. Moreover, the following lemma holds.

Lemma 6.2.1. On an edge E, we have for $\boldsymbol{u} : \Omega \to \mathbb{R}^2$ and $\underline{\tau} : \Omega \to \mathbb{R}^{2\times 2}$

$$[\![\underline{\tau}\boldsymbol{u}]\!] = [\![\underline{\tau}]\!] \cdot \{\{\boldsymbol{u}\}\} + \{\{\underline{\tau}\}\} : \underline{[\![\boldsymbol{u}]\!]}. \tag{6.6}$$

Here we assume that all trace operators are well-defined with respect to $\boldsymbol{u}$ and $\underline{\tau}$.

Proof. Elementary matrix manipulations. □

6.3 Corner elements

Define:

$$\mathcal{T}_{vert} := \{K \in \mathcal{T} : \overline{K} \cap \{A_1, \cdots, A_n\} = \emptyset\}, \qquad \mathcal{T}_{int} := \mathcal{T} \setminus \mathcal{T}_{vert}$$

We assume that each element each element in $\mathcal{T}_{vert}$ touches at most one corner.

Given an element $K \in \mathcal{T}_{vert}$ and assume that $K \cap \partial\Omega = \{A_i\}$. We introduce the auxiliary space $H^{k,l}_{\beta}(K)$ which is defined as the space $H^{k,l}_{\beta}(S_\delta)$ introduced in Chapter 2 but all integrals in the norm will be taken on K instead of S_δ. We simply write $f \in H^{k,l}_{\beta_i}(K)$ if $f|_K \in H^{k,l}_{\beta_i}(K)$.

We state the following auxiliary results.

Lemma 6.3.1 ([40], Lemma 3.1)**.** *Let $K \in \mathcal{T}_{vert}$, then*
(1): $H^{0,0}_{\beta}(K) \subset L^1(K)$ and for any $\phi \in H^{0,0}_{\beta_i}(K)$ we have

$$\|\phi\|_{L^1(K)} \le C h_K^{1-\beta_i} \|\phi\|_{H^{0,0}_{\beta_i}(K)}. \tag{6.7}$$

(2): Let $\phi \in H^{0,0}_{\beta_i}(K)$ and $v \in L^\infty(K)$, then

$$|\int_K \phi v \, d\boldsymbol{x}| \le C h_K^{1-\beta_i} \|v\|_{L^\infty} \|\phi\|_{H^{0,0}_{\beta_i}(K)}. \tag{6.8}$$

(3): Let $\phi \in H^{1,1}_{\beta_i}(K)$, then the trace $\phi|_{\partial K} \in L^1(\partial K)$ and

$$\|\phi\|_{L^1(\partial K)} \leq C(\|\phi\|_{L^2(K)} + h_K^{1-\beta_i}\|\phi\|_{H^{1,1}_{\beta_i}(K)}). \tag{6.9}$$

All the constants $C > 0$ are independent of the discretization parameter $\underline{h}, \underline{k}$.

Lemma 6.3.2 ([40], Lemma 3.2)**.** *Let $K \in \mathcal{T}_{vert}, \underline{\tau} \in H^{1,1}_{\beta_i}(K)^{2\times 2}$ and $\boldsymbol{v} \in C^1(\overline{K})^2$. Then the following integration-by-part formula holds*

$$\int_K \underline{\tau} : (\nabla \otimes \boldsymbol{v})\, d\boldsymbol{x} = -\int_K \underline{\tau} : \nabla \boldsymbol{v}\, d\boldsymbol{x} + \int_{\partial K} \underline{\tau} : (\boldsymbol{v} \otimes \boldsymbol{n})\, ds, \tag{6.10}$$

here $\boldsymbol{n}$ is the unit outward normal vector.

6.4 Lifting operators

In this section we introduce some lifting operators, which map functions defined on edges to functions defined on elements. As we will see later, they play a role in the design of the discrete forms.

The (global) lifting operator $\underline{\mathcal{L}} : H^1(\mathcal{T})^2 \to \underline{\Sigma}_{DG}$ is defined by

$$\int_\Omega \underline{\mathcal{L}}(\boldsymbol{v}) : \underline{\tau}\, d\boldsymbol{x} = \int_{\mathcal{E}(\mathcal{T})} \underline{[\![\boldsymbol{v}]\!]} : \{\{\underline{\tau}\}\}\, ds \qquad \forall \underline{\tau} \in \underline{\Sigma}_{DG}.$$

Also, the lifting operator $\mathcal{M} : H^1(\mathcal{T})^2 \to Q_{DG}$ is given by

$$\int_\Omega \mathcal{M}(\boldsymbol{v}) q\, dx = \int_{\mathcal{E}(\mathcal{T})} [\![\boldsymbol{v}]\!]\{\{q\}\}\, ds \qquad \forall q \in Q_{DG}.$$

These operators are introduced and thoroughly studied in [38].

6.5 Discretization on the variational problem

We consider the following mixed method: find $(\boldsymbol{u}_{DG}, p_{DG}) \in \mathbf{V}_{DG} \times Q_{DG}$ such that for any $\boldsymbol{v} \in \mathbf{V}_{DG}$ and $q \in Q_{DG}$:

$$\begin{aligned} A_{DG}(\boldsymbol{u}_{DG}, \boldsymbol{v}) + O_{DG}(\boldsymbol{u}_{DG}; \boldsymbol{u}_{DG}, \boldsymbol{v}) + B_{DG}(\boldsymbol{v}, p_{DG}) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, d\boldsymbol{x}, \\ B_{DG}(\boldsymbol{u}_{DG}, q) &= 0. \end{aligned} \tag{6.11}$$

A_{DG}, B_{DG} and O_{DG} discretize $A(\cdot,\cdot)$, $B(\cdot,\cdot)$ and $O(\cdot;\cdot,\cdot)$. Given $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbf{V}_{DG}$, those forms are defined as:

$$\begin{aligned} A_{DG}(\boldsymbol{u}, \boldsymbol{v}) &:= \nu \int_\Omega \nabla_h \boldsymbol{u} : \nabla_h \boldsymbol{v}\, d\,\boldsymbol{x} - \nu\Big(\int_{\mathcal{E}(\mathcal{T})} \underline{[\![\boldsymbol{u}]\!]} : \{\{\nabla_h \boldsymbol{v}\}\} + \underline{[\![\boldsymbol{v}]\!]} : \{\{\nabla_h \boldsymbol{u}\}\}\, d\boldsymbol{s}\Big) + \nu \int_{\mathcal{E}(\mathcal{T})} j\underline{[\![\boldsymbol{u}]\!]} : \underline{[\![\boldsymbol{v}]\!]}\, d\boldsymbol{s}, \\ B_{DG}(\boldsymbol{v}, q) &:= -\int_\Omega q \nabla_h \cdot \boldsymbol{v}\, d\boldsymbol{x} + \int_{\mathcal{E}(T)} \{\{q\}\}[\![\boldsymbol{v}]\!]\, d\boldsymbol{s}, \\ O_{DG}(\boldsymbol{w}; \boldsymbol{u}, \boldsymbol{v}) &:= \int_\Omega ((\boldsymbol{w} \cdot \nabla_h)\boldsymbol{u}) \cdot \boldsymbol{v}\, d\boldsymbol{x} + \frac{1}{2}\int_\Omega (\nabla_h \cdot \boldsymbol{w})\boldsymbol{u} \cdot \boldsymbol{v}\, d\boldsymbol{x} - \int_{\mathcal{E}_{\mathcal{I}}(\mathcal{T})} \underline{[\![\boldsymbol{u}]\!]} : (\{\{\boldsymbol{v}\}\} \otimes \{\{\boldsymbol{w}\}\}\,) d\boldsymbol{s} \\ &- \frac{1}{2}\int_{\mathcal{E}(\mathcal{T})} [\![\boldsymbol{w}]\!]\{\{\boldsymbol{u} \cdot \boldsymbol{v}\}\}\, d\boldsymbol{s}. \end{aligned} \tag{6.12}$$

Here ∇_h and $\nabla_h\cdot$ denote the broken gradient and broken divergence operators, see [10, Chapter 1.2]. Also, the function j is the interior penalty stabilization function which is defined, for all edge $E \in \mathcal{E}(\mathcal{T})$ as $j|_E := j_0 k_E^2 h_E^{-1}$ with $j_0>0$ as a sufficient large constant independent of $\underline{h}$, $\underline{k}$ and ν and with k_E and h_E defined as:

$$k_E := \begin{cases} \max\{k_K, k_{K'}\} & \text{if} \quad E = \partial K \cap \partial K' \in \mathcal{E}_{\mathcal{I}}(\mathcal{T}), \\ k_K & \text{if} \quad E = \partial K \cap \partial\Omega \in \mathcal{E}_{\mathcal{D}}(\mathcal{T}), \end{cases} \tag{6.13}$$

and

$$h_E := \begin{cases} \min\{h_K, h_{k'}\} & \text{if} \quad E = \partial K \cap \partial K' \in \mathcal{E}_{\mathcal{I}}(\mathcal{T}), \\ h_K & \text{if} \quad E = \partial K \cap \partial\Omega \in \mathcal{E}_{\mathcal{D}}(\mathcal{T}). \end{cases} \tag{6.14}$$

With lifting operators given in section 6.4 we rewrite A_{DG} and B_{DG} as:

$$\begin{aligned} A_{DG}(\boldsymbol{u}, \boldsymbol{v}) &:= \nu \int_\Omega (\nabla_h \boldsymbol{u} : \nabla_h \boldsymbol{v} - \underline{\mathcal{L}}(\boldsymbol{u}) : \nabla_h \boldsymbol{v} - \underline{\mathcal{L}}(\boldsymbol{v}) : \nabla_h \boldsymbol{u})\, d\boldsymbol{x} + \nu \int_{\mathcal{E}(\mathcal{T})} j \underline{[\![\boldsymbol{u}]\!]} : \underline{[\![\boldsymbol{v}]\!]}\, d\boldsymbol{s}, \\ B_{DG}(\boldsymbol{v}, q) &:= -\int_\Omega q[\nabla_h \cdot \boldsymbol{v} - \mathcal{M}(\boldsymbol{v})]\, d\boldsymbol{x}. \end{aligned} \tag{6.15}$$

Chapter 7

Existence and uniqueness of the discrete solution

In this chapter, we show the existence and uniqueness of the solution to the numerical scheme (6.11).

For a function $\boldsymbol{v} \in H^1(\mathcal{T})^2$ we introduce the auxiliary norm $\|\boldsymbol{v}\|_{DG}$ by

$$\|\boldsymbol{v}\|_{DG}^2 := \|\nabla_h \boldsymbol{v}\|_{L^2(\Omega)}^2 + \int_{e\in\mathcal{E}(\mathcal{T})} j|\underline{[\![\boldsymbol{v}]\!]}|^2 \, d\boldsymbol{s}.$$

We have the following result:

Lemma 7.0.1. *There exists a constant $j_{\min}$ such that for $j_0 \geq j_{\min}$ and for any $p \in [1, +\infty)$, there exists a constant $C_{emb} = C_{emb}(\Omega, p, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ such that for any $\boldsymbol{v} \in H^1(\mathcal{T})^2$*

$$\|\boldsymbol{v}\|_{L^p(\Omega)} \leq C_{emb}\|\boldsymbol{v}\|_{DG}.$$

Proof. The following broken norm

$$\|\boldsymbol{v}\|_{1,\mathcal{T}}^2 := \|\nabla_h \boldsymbol{v}\|_{L^2(\Omega)}^2 + \int_{\mathcal{E}(\mathcal{T})} h^{-1}|[\![\boldsymbol{v}]\!]|^2 \, ds$$

is introduced in [36]. [36, Lemma 4.1] indicates that there exists $C_0 := C_0(\Omega, p, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ such that $\|\boldsymbol{v}\|_{L^p(\Omega)} \leq C_0\|\boldsymbol{v}\|_{1,\mathcal{T}}$.

Set $j_{\min} := \frac{1}{(\min \underline{k})^2}$, then it is clear that $\|\boldsymbol{v}\|_{1,\mathcal{T}} \leq \|\boldsymbol{v}\|_{DG}$. Combine all above claims and the proof is finished. □

From now on we assume that $j_0 \geq j_{\min}$.

7.1 Properties of discrete forms

In this section we list some properties for forms A_{DG}, B_{DG} and O_{DG}. The following lemma shows that A_{DG} is continuous in $H^1(\mathcal{T})^2$ and coercive in $\mathbf{V}_{DG}$.

Lemma 7.1.1. *There exists a constant $C_{A_{DG}} = C_{A_{DG}}(j_0, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ such that for any $\boldsymbol{v}, \boldsymbol{w} \in H^1(\mathcal{T})^2$*

$$|A_{DG}(\boldsymbol{v}, \boldsymbol{w})| \leq C_{A_{DG}}\nu\|\boldsymbol{v}\|_{DG}\|\boldsymbol{w}\|_{DG}. \tag{7.1}$$

Moreover, there is a constant $C_{coer} = C_{coer}(j_{\min}, \kappa_1, \kappa_2, \kappa_3, \kappa_4)$ with

$$A_{DG}(\boldsymbol{v}, \boldsymbol{v}) \geq C_{coer}\nu\|\boldsymbol{v}\|_{DG}^2, \qquad \forall \boldsymbol{v} \in \mathbf{V}_{DG}. \tag{7.2}$$

The proof of this lemma is based on stability estimates on the lifting operator $\underline{\mathcal{L}}$, see [38, Lemma 7.1-Lemma 7.6] for technical details.

The following inf-sup condition for B_{DG} is proved in [36, Lemma 4.3]

Lemma 7.1.2. *Suppose that* $\min_{K\in\mathcal{T}} k_K \geq 2$ *and let*

$$\alpha := \begin{cases} 1 & \text{if the mesh } \mathcal{T} \text{ contains at least one quadrilaterial} \\ 0 & \text{otherwise} \end{cases}$$

Then there exists a constant C_{is} independent of $\underline{h}$,$\underline{k}$ and ν such that the following inf-sup condition for $\mathbf{V}_{DG}$ and Q_{DG} holds true:

$$\inf_{0q\in Q_{DG}} \sup_{\mathbf{0}\boldsymbol{v}\in\mathbf{V}_{DG}} \frac{B_{DG}(\boldsymbol{v},q)}{\|\boldsymbol{v}\|_{DG}\|q\|_{L^2(\Omega)}} \geq C_{is}|\underline{k}|^{-\alpha} > 0 \tag{7.3}$$

The following result is proved in [36, Proposition 4.1].

Lemma 7.1.3. *For any $\boldsymbol{w},\boldsymbol{u}\in\mathbf{V}_{DG}$, we have $O_{DG}(\boldsymbol{w};\boldsymbol{u},\boldsymbol{u}) = 0$. Moreover, there is a constant $C_{O_{DG}}$ independent of $\underline{h}$,$\underline{k}$ and ν such that for any $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in H^1(\mathcal{T})^2$*

$$|O_{DG}(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})| \leq C_{O_{DG}}\|\boldsymbol{w}\|_{DG}\|\boldsymbol{u}\|_{DG}\|\boldsymbol{v}\|_{DG}. \tag{7.4}$$

7.2 Existence and uniqueness of discrete solutions

We introduce the discrete kernel $\mathbf{Z}_{DG} := \{\boldsymbol{v}\in\mathbf{V}_{DG}|B_{DG}(\boldsymbol{v},q) = 0, \forall q\in Q_{DG}\}$.

All above properties of the discrete form A_{DG}, B_{DG}, O_{DG} can be used to derive the following existence and uniqueness result for the discrete solution of (6.11)[36, Proposition 4.2]

Lemma 7.2.1. *There exists a discrete solution* $(\boldsymbol{u}_{DG},p_{DG})\in\mathbf{V}_{DG}\times Q_{DG}$ such that $\boldsymbol{u}_{DG}\in\mathbf{Z}_{DG}$ *and*

$$\|\boldsymbol{u}_{DG}\|_{DG} \leq \frac{C_{emb}\|\boldsymbol{f}\|_{L^2(\Omega)}}{\nu C_{coer}} \tag{7.5}$$

Moreover, under the small data assumption

$$\frac{C_{O_{DG}}C_{emb}\|\boldsymbol{f}\|_{L^2(\Omega)}}{C_{coer}^2\nu^2} < 1 \tag{7.6}$$

the discrete problem (6.11) has a unique solution.

By the continuous and discrete small data assumptions (6.5) and (7.6) we know that if we define $C_{sm} := \frac{\max\{C_O,C_{O_{DG}}\}\max\{C_P,C_{emb}\}}{\min\{1,C_{coer}^2\}}$, then both (3.5) and (6.11) will have unique solutions if the following condition

$$C_{sm}\nu^{-2}\|\boldsymbol{f}\|_{L^2(\Omega)} \leq \frac{1}{2} \tag{7.7}$$

holds.

Chapter 8

Error analysis

8.1 Weak residual

We firstly introduce a weak residual which measures the non-conformity of the scheme (6.11). Given the solution $(\boldsymbol{u}, p) \in \mathbf{W} \times L_0$, define

$$R_{DG}(\boldsymbol{u}, p; \boldsymbol{v}) := A_{DG}(\boldsymbol{u}, \boldsymbol{v}) + O_{DG}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{v}) + B_{DG}(\boldsymbol{v}, p) - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, d\boldsymbol{x} \tag{8.1}$$

for any $\boldsymbol{v} \in \mathbf{V}_{DG}$.

The error estimate is then defined as the weak residual

$$\mathcal{R}_{DG}(\boldsymbol{u}, p) := \sup_{0 \neq \boldsymbol{v} \in \mathbf{V}_{DG}} \frac{|R_{DG}(\boldsymbol{u}, p; v)|}{\nu^{1/2} \|\boldsymbol{v}\|_{DG}}. \tag{8.2}$$

We also introduce the following norm

$$|||(\boldsymbol{u}, p)|||^2 := \nu \|\boldsymbol{u}\|_{DG}^2 + \nu^{-1} \|p\|_{L^2(\Omega)}^2 \tag{8.3}$$

for any function pair $(\boldsymbol{u}, p) \in \mathbf{W} \times L_0$.

The following theorem holds.

Theorem 8.1.1. *Assume that there exists a positive constant C_{sm}* such that for $0 < \nu \leq 1$ *(7.7) holds. Let $(\boldsymbol{u}, p) \in \mathbf{W} \times L_0$ be the continuous solution to (3.5) and let $(\boldsymbol{u}_{DG}, p_{DG}) \in \mathbf{V}_{DG} \times Q_{DG}$ be the discrete solution to (6.11). Then we have the following error estimates:*

$$\nu^{1/2} \|\boldsymbol{u} - \boldsymbol{u}_{DG}\|_{DG} \leq C |\underline{k}|^{\alpha} [\inf_{(v,q) \in \mathbf{V}_{DG} \times Q_{DG}} |||(\boldsymbol{u} - \boldsymbol{v}, p - q)||| + \mathcal{R}_{DG}(\boldsymbol{u}, p)] \tag{8.4}$$

$$\nu^{-1/2} \|p - p_{DG}\|_{L^2(\Omega)} \leq C |\underline{k}|^{2\alpha} [\inf_{(v,q) \in \mathbf{V}_{DG} \times Q_{DG}} |||(\boldsymbol{u} - \boldsymbol{v}, p - q)||| + \mathcal{R}_{DG}(\boldsymbol{u}, p)] \tag{8.5}$$

where α is defined in Theorem 7.1.2 and where the constant C is independent of $\underline{h}$,$\underline{k}$ and *ν.*

For proof see [36, Theorem 6.1].

With this theorem we are able to bound the error term $\|\boldsymbol{u} - \boldsymbol{u}_{DG}\|_{DG}$ and $\|p - p_{DG}\|_{L^2(\Omega)}$ by estimating $\inf_{(v,q) \in \mathbf{V} \times Q} |||(\boldsymbol{u} - \boldsymbol{v}, p - q)|||$ and $\mathcal{R}_{DG}(\boldsymbol{u}, p)$. The following lemmas, which are similar to [40, Lemma 3-Lemma 6, Theorem 2] will focus on this task.

Lemma 8.1.2. *Let $\boldsymbol{f} \in L^2(\Omega)^2$ and $(\boldsymbol{u}, p) \in \mathbf{W} \times L_0$ be a solution to (6.1). Then we have for any $\boldsymbol{v} \in \mathbf{V}_{DG}$,*

$$R_{DG}(\boldsymbol{u}, p; \boldsymbol{v}) = \int_\Omega (\nu \nabla \boldsymbol{u} - p \underline{I}) : \nabla_h \boldsymbol{v} \, d\boldsymbol{x} - \int_\Omega ((\boldsymbol{u} \cdot \nabla) \boldsymbol{u} - \boldsymbol{f}) \cdot \boldsymbol{v} \, d\boldsymbol{x} - \int_\Omega \nu \nabla \boldsymbol{u} : \underline{\mathcal{L}}(\boldsymbol{v}) \, d\boldsymbol{x} + \int_\Omega p \mathcal{M}(\boldsymbol{v}) \, d\boldsymbol{x}. \tag{8.6}$$

This lemma can be derived directly from the definitions of all discrete forms and (8.1).

Lemma 8.1.3. *If all assumptions in Corollary 5.2.2 and (6.5) hold true, then for any interior edge $E \in \mathcal{E}_{\mathcal{I}}(\mathcal{T})$, we have that $[[\nu\nabla \boldsymbol{u} - p\underline{I}]] = \boldsymbol{0}$ in $L^2(E)^2$ on E.*

Proof. By Corollary 5.2.2, there exists $\beta \in (0,1)^n$ such that $(\boldsymbol{u}, p) \in B^2_\beta(\Omega)^2 \times B^1_\beta(\Omega)$. The proof then follows the same way as in the proof of [40, Lemma 3.3] □

Lemma 8.1.4. *Let $\underline{P} : L^2(\Omega)^{2\times 2} \to \underline{\Sigma}_{DG}$ and $P : L^2_0(\Omega) \to Q_{DG}$ denote the L^2-projection onto $\underline{\Sigma}_{DG}$ and Q_{DG}. If all assumptions in Corollary 5.2.2 and (6.5) hold true, then we have, for all $\boldsymbol{v} \in \mathbf{V}_{DG}$,*

$$R_{DG}(\boldsymbol{u}, p; \boldsymbol{v}) = \nu \int_{\mathcal{E}} \underline{[\![\boldsymbol{v}]\!]} : \{\{\nabla \boldsymbol{u} - \underline{P}\nabla \boldsymbol{u}\}\} - \int_{\mathcal{E}} [\![\boldsymbol{v}]\!]\{\{p - P(p)\}\}. \tag{8.7}$$

Proof. By partial integration (this can be justified by using Lemma 6.3.2), (6.6) and Lemma 8.1.3 we deduce

$$\begin{aligned}
&\int_\Omega (\nu\nabla \boldsymbol{u} - p\underline{I}) : \nabla_h \boldsymbol{v} \\
&= \sum_{K\in\mathcal{T}} \Big(\int_{\partial K} (\nu\nabla \boldsymbol{u} - p\underline{I}) : (\boldsymbol{v} \otimes \underline{n}_K) - \int_K (\nu\Delta \boldsymbol{u} - \nabla p)\cdot \boldsymbol{v}\Big) \\
&= -\int_\Omega (\nu\Delta \boldsymbol{u} - \nabla p)\cdot \boldsymbol{v} + \sum_{E\in\mathcal{E}(\mathcal{T})} \int_E [\![(\nu\nabla \boldsymbol{u} - p\underline{I})\cdot \boldsymbol{v}]\!] \\
&= -\int_\Omega (\nu\Delta \boldsymbol{u} - \nabla p)\cdot \boldsymbol{v} + \int_{\mathcal{E}_{\mathcal{I}}(\mathcal{T})} [\![\nu\nabla \boldsymbol{u} - p\underline{I}]\!]\cdot\{\{\boldsymbol{v}\}\} + \int_{\mathcal{E}} \{\{\nu\nabla \boldsymbol{u} - p\underline{I}\}\} : \underline{[\![\boldsymbol{v}]\!]} \\
&= -\int_\Omega (\nu\Delta \boldsymbol{u} - \nabla p)\cdot \boldsymbol{v} + \int_{\mathcal{E}} \{\{\nu\nabla \boldsymbol{u} - p\underline{I}\}\} : \underline{[\![\boldsymbol{v}]\!]}.
\end{aligned}$$

So we have by using (8.6)

$$\begin{aligned}
R_{DG}(\boldsymbol{u}, p; \boldsymbol{v}) &= -\int_\Omega (\nu\Delta \boldsymbol{u} - \nabla p)\cdot \boldsymbol{v} + \int_{\mathcal{E}} \{\{\nu\nabla \boldsymbol{u} - p\underline{I}\}\} : \underline{[\![\boldsymbol{v}]\!]} - \int_\Omega ((\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{f})\cdot \boldsymbol{v} \\
&\quad - \int_\Omega \nu\nabla \boldsymbol{u} : \underline{\mathcal{L}}(\boldsymbol{v}) + \int_\Omega p\mathcal{M}(\boldsymbol{v}) \\
&= \int_{\mathcal{E}} \{\{\nu\nabla \boldsymbol{u} - p\underline{I}\}\} : \underline{[\![\boldsymbol{v}]\!]} - \int_\Omega \nu\nabla \boldsymbol{u} : \underline{\mathcal{L}}(\boldsymbol{v}) + \int_\Omega p\mathcal{M}(\boldsymbol{v}).
\end{aligned} \tag{8.8}$$

Also, due to the properties of projection operators and the definition of lifting operators $\underline{\mathcal{L}}$ and $\mathcal{M}$ we have

$$\int_\Omega \nu\nabla \boldsymbol{u} : \underline{\mathcal{L}}(\boldsymbol{v}) = \nu \int_\Omega \underline{P}(\nabla \boldsymbol{u}) : \underline{\mathcal{L}}(\boldsymbol{v}) = \nu \int_{\mathcal{E}(\mathcal{T})} \underline{[[\boldsymbol{v}]]} : \{\{\underline{P}(\nabla \boldsymbol{u})\}\}, \tag{8.9}$$

$$\int_\Omega p\mathcal{M}(\boldsymbol{v}) = \int_\Omega P(p)\mathcal{M}(\boldsymbol{v}) = \int_{\mathcal{E}(\mathcal{T})} [[\boldsymbol{v}]]\{\{P(p)\}\}. \tag{8.10}$$

Finally, insert (8.9) and (8.10) into (8.8):

$$\begin{aligned}
R_{DG}(\boldsymbol{u}, p; \boldsymbol{v}) &= \int_{\mathcal{E}} \{\{\nu\nabla \boldsymbol{u} - p\underline{I}\}\} : \underline{[\![\boldsymbol{v}]\!]} - \nu \int_{\mathcal{E}(\mathcal{T})} \underline{[[\boldsymbol{v}]]} : \{\{\underline{P}(\nabla \boldsymbol{u})\}\} + \int_{\mathcal{E}(\mathcal{T})} [[\boldsymbol{v}]]\{\{P(p)\}\} \\
&= \nu \int_{\mathcal{E}} \underline{[\![\boldsymbol{v}]\!]} : \{\{\nabla \boldsymbol{u} - \underline{P}\nabla \boldsymbol{u}\}\} - \int_{\mathcal{E}} [\![\boldsymbol{v}]\!]\{\{p - P(p)\}\}.
\end{aligned}$$

□

Lemma 8.1.5. *Assume that all assumptions in Corollary 5.2.2 and (6.5) hold true.*

Then there exists C depending only on κ_1, κ_2, κ_3, κ_4 and ν such that for any $\boldsymbol{v}, \boldsymbol{w} \in \mathbf{V}_{DG}$ and $q \in Q_{DG}$ we have

$$|R_{DG}(\boldsymbol{u}, p; \boldsymbol{w})| \leq C(|||(\boldsymbol{u}-\boldsymbol{v}, p-q)|||\|\boldsymbol{w}\|_{DG}) + |\nu \int_{\mathcal{E}} \{\{\nabla \boldsymbol{u} - \nabla \boldsymbol{v}\}\} : \underline{[\![w]\!]} - \int_{\mathcal{E}} \{\{p-q\}\}[\![\boldsymbol{w}]\!]|. \tag{8.11}$$

Proof. For fixed but arbitrary $\boldsymbol{v}, \boldsymbol{w} \in \mathbf{V}_{DG}$ and $q \in Q_{DG}$, with (8.7) and the fact that L^2-projection reproduces polynomials in $\underline{\Sigma}_{DG}$ and Q_{DG} we have

$$\begin{aligned} R_{DG}(\boldsymbol{u}, p; \boldsymbol{w}) &= \nu \int_{\mathcal{E}} \underline{[\![\boldsymbol{w}]\!]} : \{\{\nabla \boldsymbol{u} - \nabla_h \boldsymbol{v} - \underline{P}(\nabla \boldsymbol{u}) - \underline{P}(\nabla_h \boldsymbol{v})\}\} - \int_{\mathcal{E}} [\![\boldsymbol{w}]\!]\{\{p - q - P(p) + P(q)\}\} \\ &= \underbrace{(\nu \int_{\mathcal{E}} \underline{[\![\boldsymbol{w}]\!]} : \{\{\underline{P}(\nabla(\boldsymbol{v}-\boldsymbol{u}))\}\} + \int_{\mathcal{E}} [\![\boldsymbol{w}]\!]\{\{P(p-q)\}\})}_{A} + \underbrace{(\nu \int_{\mathcal{E}} \{\{\nabla \boldsymbol{u} - \nabla \boldsymbol{v}\}\} : \underline{[\![w]\!]} - \int_{\mathcal{E}} \{\{p-q\}\}[\![\boldsymbol{w}]\!])}_{B}. \end{aligned}$$

We consider bounding term A:

$$\begin{aligned} &|\nu \int_{\mathcal{E}} \underline{[\![\boldsymbol{w}]\!]} : \{\{\underline{P}(\nabla(\boldsymbol{v}-\boldsymbol{u}))\}\} + \int_{\mathcal{E}} [\![\boldsymbol{w}]\!]\{\{P(p-q)\}\}| \\ &\leq \nu |\int_{\mathcal{E}} j^{\frac{1}{2}} \underline{[\![\boldsymbol{w}]\!]} : j^{-\frac{1}{2}} \{\{\underline{P}(\nabla(\boldsymbol{v}-\boldsymbol{u}))\}\}| + |\int_{\mathcal{E}} j^{\frac{1}{2}} [\![\boldsymbol{w}]\!] j^{-\frac{1}{2}} \{\{P(p-q)\}\}| \\ &\leq \nu \int_{\mathcal{E}} j |\underline{[\![\boldsymbol{w}]\!]}|^2 \int_{\mathcal{E}} j^{-1} |\{\{\underline{P}(\nabla(\boldsymbol{v}-\boldsymbol{u}))\}\}|^2 + \int_{\mathcal{E}} j |[\![\boldsymbol{w}]\!]|^2 \int_{\mathcal{E}} j^{-1} |\{\{P(p-q)\}\}|^2 \\ &\leq C\|\boldsymbol{w}\|_{DG} \sum_{K \in \mathcal{T}} [\frac{h_K}{k_K^2} \nu \|\underline{P}(\nabla(\boldsymbol{v}-\boldsymbol{u}))\|^2_{L^2(\partial K)} + \frac{h_K}{k_K^2} \|P(p-q)\|^2_{L^2(\partial K)}]^{\frac{1}{2}} \\ &\leq C\|\boldsymbol{w}\|_{DG} [\nu \|\underline{P}(\nabla(\boldsymbol{v}-\boldsymbol{u}))\|_{L^2(\Omega)} + \|P(p-q)\|_{L^2(\Omega)}] \\ &\leq C\|\boldsymbol{w}\|_{DG} \nu^{\frac{1}{2}} [\nu^{\frac{1}{2}} \|\nabla(\boldsymbol{v}-\boldsymbol{u})\|_{L^2(\Omega)} + \nu^{-\frac{1}{2}} \|p-q\|_{L^2(\Omega)}] \\ &\leq C\|\boldsymbol{w}\|_{DG} \nu^{\frac{1}{2}} [\nu^{\frac{1}{2}} \|\boldsymbol{u}-\boldsymbol{v}\|_{DG} + \nu^{-\frac{1}{2}} \|p-q\|_{L^2(\Omega)}] \\ &\leq C\|\boldsymbol{w}\|_{DG} |||(\boldsymbol{u}-\boldsymbol{v}, p-q)|||. \end{aligned}$$

The constant C here only depends on κ_1, κ_2, κ_3, κ_4 and ν. In above derivation we use the Cauchy-Schwarz inequality, the definition of j, h_K, k_K, the inequality $|[\![\boldsymbol{w}]\!]|^2 \leq |\underline{[\![\boldsymbol{w}]\!]}|^2$ and the following trace inequality

$$\|\phi\|^2_{L^2(\partial K)} \leq C k_K^2 h_K^{-1} \|\phi\|^2_{L^2(\Omega)}$$

which holds for any polynomial $\phi \in \mathbb{Q}_{k_K}(K)$, see [41, Theorem 4.76]. Combine all results above and we have the result.

□

8.2 Quasioptimality of the discontinuous Galerkin method

Lemma 8.2.1. *Assume that all assumptions in Corrollary 5.2.2 and the small data assumption (7.7) hold true. Denote by $(\boldsymbol{u}, p) \in \mathbf{W} \times L_0$ the unique solution to (3.1) and $(\boldsymbol{u}_{DG}, p_{DG}) \in \mathbf{V}_{DG} \times Q_{DG}$ the unique solution to (6.11) with $\underline{k} \geq 2$. Then we have*

$$|||(\boldsymbol{u}-\boldsymbol{u}_{DG}, p-p_{DG})||| \leq C|\underline{k}|^{2\alpha+1} \inf_{(\boldsymbol{v},\tilde{q}) \in \mathbf{V}_{DG} \times \tilde{Q}_{DG}} (E_1 + E_2 + E_3) \tag{8.12}$$

where E_1,E_2 and E_3 are defined as

$$
\begin{aligned}
E_1^2 &= \sum_{K\in\mathcal{T}} (|\boldsymbol{u}-\boldsymbol{v}|^2_{H^1(K)} + h_K^{-2}\|\boldsymbol{u}-\boldsymbol{v}\|^2_{L^2(K)} + \|p-\tilde{q}\|^2_{L^2(K)}),\\
E_2^2 &= \sum_{K\in\mathcal{T}_{int}} h_K^2(|\boldsymbol{u}-\boldsymbol{v}|^2_{H^2(K)} + |p-\tilde{q}|^2_{H^1(K)}),\\
E_3^2 &= \sum_{K\in\mathcal{T}_{vert}} h_K^{2(1-\beta_K)}(|\boldsymbol{u}-\boldsymbol{v}|^2_{H^{2,2}_{\beta_K}(K)} + |p-\tilde{q}|^2_{H^{1,1}_{\beta_K}(K)}).
\end{aligned}
$$

Here the constant C is independent of the discretization and we write $\beta_K = \beta_i$ if K touches A_i.

Proof. By Corollary 5.2.2, $\boldsymbol{u} \in B^2_\beta(\Omega)^2$ and $p \in B^1_\beta(\Omega)$ for some $\beta \in (0,1)^n$. In view of Lemma 8.1.5 and Theorem 8.1.1 the crucial task is to bound $|||(\boldsymbol{u}-\boldsymbol{v}, p-q)|||$ and

$$
\sup_{\boldsymbol{0}\neq \boldsymbol{w}\in\mathbf{V}_{DG}} \inf_{(\boldsymbol{v},q)\in\mathbf{V}_{DG}\times Q_{DG}} \Big(\frac{1}{\|\boldsymbol{w}\|_{DG}}|\nu\int_{\mathcal{E}}\{\{\nabla\boldsymbol{u}-\nabla\boldsymbol{v}\}\} : \underline{[\![w]\!]} - \int_{\mathcal{E}}\{\{p-q\}\}[\![\boldsymbol{w}]\!]|\Big).
$$

For any $\tilde{q} \in \tilde{Q}_{DG}$, set $q = \tilde{q} - \frac{1}{|\Omega|}\int_\Omega \tilde{q} \in Q_{DG}$.

We firstly consider bounding $|||(\boldsymbol{u}-\boldsymbol{v}, p-q)|||$. In the following steps we will use the trace inequality

$$
\|\phi\|^2_{L^2(\partial K)} \le C[h_K^{-1}\|\phi\|^2_{L^2(K)} + h_K|\phi|^2_{H^1(K)}], \qquad \forall\phi\in H^1(K). \tag{8.13}
$$

We have

$$
\begin{aligned}
&\|\boldsymbol{u}-\boldsymbol{v}\|^2_{DG} = \sum_{K\in\mathcal{T}}\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2_{L^2(K)} + \int_{\mathcal{E}} j|\underline{[\![\boldsymbol{u}-\boldsymbol{v}]\!]}|^2 d\boldsymbol{s}\\
&\le \sum_{K\in\mathcal{T}}\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2_{L^2(K)} + C\sum_{K\in\mathcal{T}}\int_{E\in\partial K} k_E^2 h_E^{-1}|\boldsymbol{u}-\boldsymbol{v}|^2 d\boldsymbol{s}\\
&\le \sum_{K\in\mathcal{T}}\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2_{L^2(K)} + C\sum_{K\in\mathcal{T}} k_K^2 h_K^{-1}\|\boldsymbol{u}-\boldsymbol{v}\|^2_{L^2(\partial K)}\\
&\le C|\underline{k}|^2\sum_{K\in\mathcal{T}}(\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2_{L^2(K)} + h_K^{-1}\|\boldsymbol{u}-\boldsymbol{v}\|^2_{L^2(\partial K)})\\
&\le C|\underline{k}|^2\sum_{K\in\mathcal{T}}(|\boldsymbol{u}-\boldsymbol{v}|^2_{H^1(K)} + h_K^{-2}\|\boldsymbol{u}-\boldsymbol{v}\|^2_{L^2(K)} + |\boldsymbol{u}-\boldsymbol{v}|^2_{H^1(K)})\\
&\le C|\underline{k}|^2 E_1^2.
\end{aligned}
$$

Here the constant C depends only on κ_3, κ_4 and j_0. Moreover, for any $q \in Q_{DG}$,

$$
\begin{aligned}
\|p-q\|_{L^2(\Omega)} &= \|p-\tilde{q} - |\Omega|^{-1}\int_\Omega (p-\tilde{q})\|_{L^2()}\\
&\le \|p-\tilde{q}\|_{L^2(\Omega)} + |\Omega|^{-\frac{1}{2}}\int_\Omega |p-\tilde{q}|\\
&\le 2\|p-\tilde{q}\|_{L^2(\Omega)} \le 2E_1.
\end{aligned} \tag{8.14}
$$

So

$$
|||(\boldsymbol{u}-\boldsymbol{v}, p-q)|||^2 \le \nu C|\underline{k}|^2E_1^2 + \nu^{-1}E_1^2 \le C|\underline{k}|^2E_1^2.
$$

The constant C here depends only on κ_3, κ_4, j_0 and ν. Take the infimum among $(\boldsymbol{v}, q) \in \mathbf{V}_{DG}\times Q_{DG}$ and we have

$$
\inf_{(\boldsymbol{v},q)\in\mathbf{V}_{DG}\times Q_{DG}} |||(\boldsymbol{u}-\boldsymbol{v}, p-q)|||^2 \le C|\underline{k}|^2E_1^2. \tag{8.15}
$$

To bound $\sup_{\boldsymbol{0}\neq\boldsymbol{w}\in\mathbf{V}_{DG}}\inf_{(\boldsymbol{v},q)\in\mathbf{V}_{DG}\times Q_{DG}}(\frac{1}{\|\boldsymbol{w}\|_{DG}}|\nu\int_{\mathcal{E}}\{\{\nabla\boldsymbol{u}-\nabla\boldsymbol{v}\}\}:\underline{[\![w]\!]}-\int_{\mathcal{E}}\{\{p-q\}\}[\![\boldsymbol{w}]\!]|)$, we fix $\boldsymbol{v},\boldsymbol{w}\in\mathbf{V}_{DG}$ and $q\in Q_{DG}$. Then we have

$$
\begin{aligned}
&|\nu\int_{\mathcal{E}}\{\{\nabla\boldsymbol{u}-\nabla\boldsymbol{v}\}\}:\underline{[\![w]\!]}-\int_{\mathcal{E}}\{\{p-q\}\}[\![\boldsymbol{w}]\!]|\\
&\leq\sum_{E\in\mathcal{E}}\int_E\nu|\{\{\nabla\boldsymbol{u}-\nabla\boldsymbol{v}\}\}:\underline{[\![\boldsymbol{w}]\!]}|+|\{\{p-q\}\}[\![\boldsymbol{w}]\!]|\ d\boldsymbol{s}\\
&\leq\sum_{E\in\mathcal{E}}\int_E(\nu|\{\{\nabla\boldsymbol{u}-\nabla\boldsymbol{v}\}\}|+|\{\{p-q\}\}|)\cdot|\underline{[\![\boldsymbol{w}]\!]}|\ d\boldsymbol{s}\\
&\leq\sum_{E\in\mathcal{E}}\|[\![\boldsymbol{w}]\!]\|_{L^\infty(E)}\int_E\nu|\{\{\nabla\boldsymbol{u}-\nabla\boldsymbol{v}\}\}|+|\{\{p-q\}\}|\ d\boldsymbol{s}.
\end{aligned}
$$

Since $\boldsymbol{w}\in\mathbf{V}_{DG}$, each component of $[\![\boldsymbol{w}]\!]$ is still a polynomial. Apply the trace inequality for polynomials (see [34, Lemma 1]):

$$
\|[\![\boldsymbol{w}]\!]\|_{L^\infty(E)}=\||[\![\boldsymbol{w}]\!]|^2\|^{\frac{1}{2}}_{L^\infty(E)}\leq C\frac{k_E}{\sqrt{h_E}}\||[\![\boldsymbol{w}]\!]|^2\|^{\frac{1}{2}}_{L^1(E)}=C\frac{k_E}{\sqrt{h_E}}\|[\![\boldsymbol{w}]\!]\|_{L^2(E)}.
$$

Notice the inequality $|[\![\boldsymbol{w}]\!]|^2\leq|\underline{[\![\boldsymbol{w}]\!]}|^2$, we have

$$
\begin{aligned}
&|\nu\int_{\mathcal{E}}\{\{\nabla\boldsymbol{u}-\nabla\boldsymbol{v}\}\}:\underline{[\![w]\!]}-\int_{\mathcal{E}}\{\{p-q\}\}[\![\boldsymbol{w}]\!]|\\
&\leq C\sum_{E\in\mathcal{E}}(\frac{k_E}{\sqrt{h_E}}\|\underline{[\![\boldsymbol{w}]\!]}\|_{L^2(E)}(\int_E\nu|\{\{\nabla\boldsymbol{u}-\nabla\boldsymbol{v}\}\}|+|\{\{p-q\}\}|\ d\boldsymbol{s}))\\
&\leq C(\sum_{E\in\mathcal{E}}\|\frac{k_E}{\sqrt{h_E}}\underline{[\![\boldsymbol{w}]\!]}\|^2_{L^2(E)})^{\frac{1}{2}}(\sum_{E\in\mathcal{E}}(\int_E\nu|\{\{\nabla\boldsymbol{u}-\nabla\boldsymbol{v}\}\}|+|\{\{p-q\}\}|\ d\boldsymbol{s})^2)^{\frac{1}{2}}\\
&\leq C(\int_{\mathcal{E}}k_E^2h_E^{-1}|\underline{[\![\boldsymbol{w}]\!]}|^2\ d\boldsymbol{s})^{\frac{1}{2}}((\int_{\mathcal{E}}\nu|\{\{\nabla\boldsymbol{u}-\nabla\boldsymbol{v}\}\}|)^2+(\int_{\mathcal{E}}|\{\{p-q\}\}|)^2)^{\frac{1}{2}}\\
&\leq C\|\boldsymbol{w}\|_{DG}\cdot[\sum_{K\in\mathcal{T}}(\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2_{L^1(\partial K)}+\|p-q\|^2_{L^1(\partial K)})]^{\frac{1}{2}}.
\end{aligned}
$$

In the case that $K\in\mathcal{T}_{int}$, since $\boldsymbol{u}\in B^2_\beta(\Omega)^2$ and $p\in B^1_\beta(\Omega)$, $\nabla\boldsymbol{u}|_K\in H^1(K)^{2\times2}$ and $p|_K\in H^1(K)$. By applying the Cauchy-Schwarz inequality and the trace inequality (8.13):

$$
\begin{aligned}
\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2_{L^1(\partial K)}&\leq Ch_K\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2_{L^2(\partial K)}\leq C\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2_{L^2(K)}+Ch_K^2|\nabla(\boldsymbol{u}-\boldsymbol{v})|^2_{H^1(K)}\\
&\leq C(|\boldsymbol{u}-\boldsymbol{v}|^2_{H^1(K)}+h_K^2|\boldsymbol{u}-\boldsymbol{v}|^2_{H^2(K)}).
\end{aligned}
$$

Similarly $\|p-q\|^2_{L^1(\partial K)}\leq C(\|p-q\|^2_{L^2(K)}+h_K^2|p-q|^2_{H^1(K)})$. Here the constant C depends only on κ_1 and κ_2.

In the case that $K\in\mathcal{T}_{vert}$, we apply the third assertions of Lemma 6.3.1 on $\nabla\boldsymbol{u}\in H^{1,1}_{\beta_K}(K)^{2\times2}$ and $p\in H^{1,1}_{\beta_K}(K)$. We have

$$
\begin{aligned}
\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2_{L^1(\partial K)}&\leq C(\|\nabla(\boldsymbol{u}-\boldsymbol{v})\|^2_{L^2(K)}+h_K^{2(1-\beta_K)}|\boldsymbol{u}-\boldsymbol{v}|^2_{H^{2,2}_{\beta_K}(K)}),\\
\|p-q\|^2_{L^1(\partial K)}&\leq C(\|p-q\|^2_{L^2(K)}+h_K^{2(1-\beta_K)}|p-q|^2_{H^{1,1}_{\beta_K}(K)}).
\end{aligned}
$$

By all claims above and noticing that $\nabla(q-\tilde{q}) \equiv \boldsymbol{0}$ we have

$$
\begin{aligned}
&\frac{1}{\|\boldsymbol{w}\|_{DG}}|\nu\int_{\mathcal{E}}\{\{\nabla\boldsymbol{u}-\nabla\boldsymbol{v}\}\}:\underline{[\![w]\!]}-\int_{\mathcal{E}}\{\{p-q\}\}[\![\boldsymbol{w}]\!]| \\
&\leq C[\sum_{K\in\mathcal{T}}(|\boldsymbol{u}-\boldsymbol{v}|^2_{H^1(K)}+\|p-q\|^2_{L^2(K)})+\sum_{K\in\mathcal{T}_{int}}h_K^2(|\boldsymbol{u}-\boldsymbol{v}|^2_{H^1(K)}+|p-q|^2_{H^1(K)})+ \\
&\sum_{K\in\mathcal{T}_{vert}}h_K^{2(1-\beta_K)}(|\boldsymbol{u}-\boldsymbol{v}|^2_{H^{2,2}_{\beta_K}(K)}+|p-q|^2_{\mathcal{K}^1_{H^{1,1}_{\beta_K}(K)}})]^{\frac{1}{2}} \\
&\leq C[\sum_{K\in\mathcal{T}}(|\boldsymbol{u}-\boldsymbol{v}|^2_{H^1(K)}+\|p-\tilde{q}\|^2_{L^2(K)})+\sum_{K\in\mathcal{T}_{int}}h_K^2(|\boldsymbol{u}-\boldsymbol{v}|^2_{H^1(K)}+|p-\tilde{q}|^2_{H^1(K)})+ \\
&\sum_{K\in\mathcal{T}_{vert}}h_K^{2(1-\beta_K)}(|\boldsymbol{u}-\boldsymbol{v}|^2_{H^{2,2}_{\beta_K}(K)}+|p-\tilde{q}|^2_{\mathcal{K}^1_{H^{1,1}_{\beta_K}(K)}})]^{\frac{1}{2}} \\
&\leq C(E_1^2+E_2^2+E_3^2)^{\frac{1}{2}}.
\end{aligned}
$$

Here the constant C is independent of $|\underline{k}|$. As the choices of $\boldsymbol{v},\boldsymbol{w}\in\mathbf{V}_{DG}$ and $q\in Q_{DG}$ are arbitrary,

$$
\sup_{\boldsymbol{0}\neq\boldsymbol{w}\in\mathbf{V}_{DG}}\inf_{(\boldsymbol{v},q)\in\mathbf{V}_{DG}\times Q_{DG}}(\frac{1}{\|\boldsymbol{w}\|_{DG}}|\nu\int_{\mathcal{E}}\{\{\nabla\boldsymbol{u}-\nabla\boldsymbol{v}\}\}:\underline{[\![w]\!]}-\int_{\mathcal{E}}\{\{p-q\}\}[\![\boldsymbol{w}]\!]|)\leq C(E_1^2+E_2^2+E_3^2)^{\frac{1}{2}}.
$$

Now, by Lemma 8.1.5, (8.15) and the definition of $\mathcal{R}_{DG}(\boldsymbol{u},p)$:

$$
\mathcal{R}_{DG}(\boldsymbol{u},p)^2\leq C(|\underline{k}|^2E_1^2+E_1^2+E_2^2+E_3^2)\leq C|\underline{k}|^2(E_1^2+E_2^2+E_3^2). \tag{8.16}
$$

Finally, use Theorem 8.1.1, (8.15) and (8.16):

$$
\begin{aligned}
&|||\boldsymbol{u}-\boldsymbol{u}_{DG},p-p_{DG}|||^2 \\
&=\nu\|\boldsymbol{u}-\boldsymbol{u}_{DG}\|^2_{DG}+\nu^{-1}\|p-p_{DG}\|_{L^2(\Omega)} \\
&\leq C(|\underline{k}|^{2\alpha}+|\underline{k}|^{4\alpha})[|\underline{k}|^2E_1^2+|\underline{k}|^2(E_1^2+E_2^2+E_3^2)] \\
&\leq C|\underline{k}|^{4\alpha+2}(E_1^2+E_2^2+E_3^2).
\end{aligned}
$$

Taking the square root of each side and then taking the infimum among all $(\boldsymbol{v},q)\in\mathbf{V}_{DG}\times Q_{DG}$ concludes the proof. □

Chapter 9

Exponential rate of convergence

In this chapter, we show that the error estimate in Lemma 8.2.1 is exponentially convergent on geometrical meshes. Theorem 9.2.1 will be the main result of the *hp*-DGFEM's performance on the stationary incompressible NSE with zero Dirichlet boundary condition.

9.1 Geometrical meshes

We follow the steps in [36, Section 6.3] which introduce the *hp*-DG discretization that uses geometrically-refined mesh towards the corners with linear polynomial slope. We fix a refinement ratio $\sigma \in (0, \frac{1}{2})$, a polynomial slope $s>0$ and a minimum polynomial degree k_{min}. Let $R := \min_{i,j\in 1,\cdot,n} \frac{d(A_i,A_j)}{2}$. We also fix a constant $n_L \in \mathbb{N}$.

We firstly consider a corner $A \in \{A_1, \cdots, A_n\}$ and the corner mesh $\mathcal{T}_A^l := \{K \in \mathcal{T} : d(K, A) < R\}$. Suppose that $\mathcal{T}_A^l$ can be divided into mesh layers $\mathcal{T}_A^l := \mathfrak{L}_0^l \cup \cdots \cup \mathfrak{L}_l^l$ such that

- (1) $\mathfrak{L}_l^l$ contains all elements touching $\mathfrak{c}$.
- (2) $\min_{K\in\mathfrak{L}_j^l} d(K, A) \simeq \max_{K\in\mathfrak{L}_j^l} d(K, A) \simeq R\sigma^j$.
- (3) For all $K \in \mathfrak{L}_j^l$, $h_K \simeq R\sigma^j$.
- (4) $|\mathfrak{L}_j^l| \simeq n_L$.

Here all relations are uniform with respect to l and j. Furthermore, if $T \in \mathcal{T}_A^l$ is a triangle, we assume that there exists an affine map F_T such that $T = F_T(\hat{T})$ and that for $Q_T := F_T(\hat{Q})$, we have

- (5) $Q_T \subset \Omega$.
- (6) $d(Q_T, A) \geq Cd(T, A)$ uniformly with respect to T and to the refinement level l.

Remark 9.1.1. *It has been shown in [11, Lemma 1] that (5) holds for meshes that are sufficiently refined using the Newest-vertex Bisection technique. Moreover, it can be shown that for any T, there exists an affine mapping such that $d(Q_T, A) = d(T, A)$. The key is to construct an affine mapping that maps the segment $y = -x + 1, x \in (0, 1)$ to the edge of T which is the furthest edge away from A.*

The refinement of the mesh $\mathcal{T}_A^l$ is $\mathcal{T}_A^{l+1} := \mathfrak{L}_0^{l+1} \cup \cdots \cup \mathfrak{L}_{l+1}^{l+1}$ with $\mathfrak{L}_j^{l+1} = \mathfrak{L}_j^l$ for $j = 0, 1, \cdots, l-1$.

Now, the mesh $\mathcal{T}^l$ on Ω is obtained by using $\mathcal{T}_{\mathfrak{c}_i}^l$ for $i = 1, \cdots, n$ and by using a fixed quasi-uniform partition $\mathcal{T}_{fixed}$ in the rest part of Ω. Moreover, for each element $K \in \mathfrak{L}_j^l \subset \mathfrak{T}_{A_i}^l$, the polynomial degree corresponding to K is set as $k_K = k_{min} + \lfloor s(i-j) \rfloor$ and for each element $K \in \mathcal{T}_{fixed}$, $k_K = k_{min} + \lfloor sl \rfloor$.

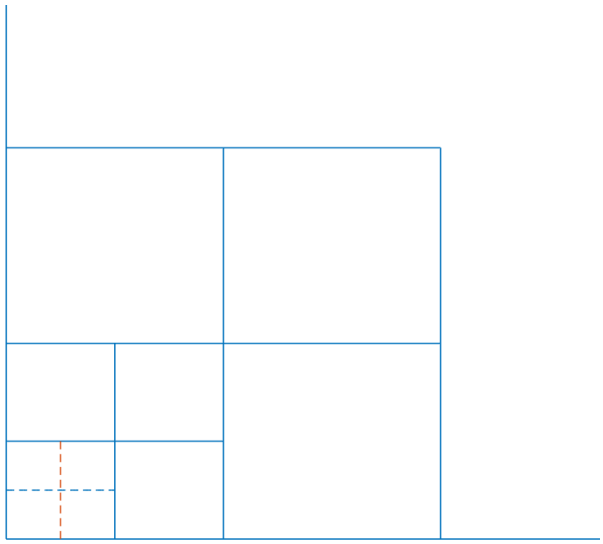

Figure 9.1: An example of the corner mesh near a right-angle corner. The dashed lines show how to refine the corner mesh at the next refinement step. In this case $n_L = 3$.

9.2 Exponential convergence

We now prove the exponential convergence of the discretization stated in Chapter 6 with the mesh described in Section 9.1.

Theorem 9.2.1. *Assume that (7.7) holds and let $(\boldsymbol{u}, p)$ be the solution to (3.1) with $\boldsymbol{f} \in B^0_{\beta_f}(\Omega)^2$ for a vector $\beta_f \in (0,1)^n$. Let $\mathbf{V}_{DG}$ and Q_{DG} be the spaces defined in (6.1) with respect to the mesh $\mathcal{T}^l$ and the polynomial setting shown in Section 9.1 and let $(\boldsymbol{u}_{DG}, p_{DG}) \in \mathbf{V}_{DG} \times Q_{DG}$ denote the numerical solution corresponding to (6.11). Then there exist two constants b and C independent of l such that for $N = dim(\mathbf{V}_{DG}) \simeq dim(Q_{DG})$ we have*

$$|||(\boldsymbol{u}_{DG} - \boldsymbol{u}, p_{DG} - p)||| \leq C \exp(-bN^{\frac{1}{3}}).$$

The proof follows the lines of [40, Theorem 6.4] and [36, Theorem 6.3].

Proof. By Corollary 5.2.2, there exists $\beta = (\beta_1, \cdots, \beta_n) \in (0,1)^n$ such that $\boldsymbol{u} \in B^2_\beta(\Omega)^2$ and $p \in B^1_\beta(\Omega)$. For any element K touching a corner A_j we write $\beta_K = \beta_j$ and $r_K = r_i$.

Now we consider bounding $E_1^2 + E_2^2 + E_3^2$ in Lemma 8.2.1. We start by considering elements in $\mathcal{T}_{vert}$. Split E_1^2 as:

$$E_1^2 = \int_{K \in \mathcal{T}_{int}} (|\boldsymbol{u} - \boldsymbol{v}|^2_{H^1(K)} + h_K^{-2} \|\boldsymbol{u} - \boldsymbol{v}\|^2_{L^2(K)} + \|p - \tilde{q}\|^2_{L^2(K)})$$
$$+ \int_{K \in \mathcal{T}_{vert}} (|\boldsymbol{u} - \boldsymbol{v}|^2_{H^1(K)} + h_K^{-2} \|\boldsymbol{u} - \boldsymbol{v}\|^2_{L^2(K)} + \|p - \tilde{q}\|^2_{L^2(K)}) =: E^2_{1,int} + E^2_{1,vert}.$$

By [37, Proposition 5.1, Proposition 5.4] and the assumption (3) on the corner mesh in Section 9.1, for

any $K \in \mathcal{K}_{vert}$ there exist $\boldsymbol{v} \in \mathbb{Q}_1(K)^2$ and $\tilde{q} \in \mathbb{Q}_0(K)$ such that

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_{vert}} (|\boldsymbol{u}-\boldsymbol{v}|^2_{H^1(K)} + h_K^{-2}\|\boldsymbol{u}-\boldsymbol{v}\|^2_{L^2(K)} + \|p-\tilde{q}\|^2_{L^2(K)}) \\
&\le C \sum_{K\in\mathcal{T}_{vert}} h_K^{2(1-\beta_K)}(|\boldsymbol{u}|_{H^{2,2}_\beta(K)} + |p|_{H^{1,1}_\beta(K)}) \\
&\le C \sum_{K\in\mathcal{T}_{vert}} \sigma^{2(1-\beta_K)l}(|\boldsymbol{u}|_{H^{2,2}_\beta(K)} + |p|_{H^{1,1}_\beta(K)}).
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_{vert}} h_K^{2(1-\beta_K)}(|\boldsymbol{u}-\boldsymbol{v}|^2_{H^{2,2}_\beta(K)} + \|p-\tilde{q}\|^2_{H^{1,1}_\beta(K)}) \le C \sum_{K\in\mathcal{T}_{vert}} h_K^{2(1-\beta_K)}(|\boldsymbol{u}|_{H^{2,2}_\beta(K)} + |p|_{H^{1,1}_\beta(K)}) \\
&\le C \sum_{K\in\mathcal{T}_{vert}} \sigma^{2(1-\beta_K)l}(|\boldsymbol{u}|_{H^{2,2}_\beta(K)} + |p|_{H^{1,1}_\beta(K)}).
\end{aligned} \tag{9.1}$$

So there exists two constants C_{vert}, b_{vert} independent of l such that

$$E^2_{1,vert} + E^2_3 \le C_{vert}\exp(-b_{vert}l). \tag{9.2}$$

Now it remains to estimate

$$E^2_{1,int} + E^2_2 = \sum_{K\in\mathcal{T}_{int}} (\sum_{i=0,1,2} h_K^{2(i-1)}|\boldsymbol{u}-\boldsymbol{v}|^2_{H^i(K)} + \sum_{i=0,1} h_K^{2i}|p-q|^2_{H^i(K)}).$$

We fix an element $K \in \mathcal{T}_{int}$. We firstly consider the case that K is a quadrilateral. For any $s \le k_K$ such that $s \le k_K$, let $\gamma \in \mathbb{N}_0^2$ be a multi index such that $|\gamma| = s > 0$. It has been shown in [41, Corollary 4.47] that there exists a polynomial $\boldsymbol{v} \in \mathbb{Q}_{k_K}(K)^2$ such that

$$\sum_{i=0,1,2} h_K^{2(i-1)}|\boldsymbol{u}-\boldsymbol{v}|^2_{H^i(K)} \le Ch_K^{2s} \sum_{i=0,1,2} (\frac{(k_K-s)!}{(k_K+s+2-2i)!} \sum_{|\alpha|=1,2,3} h_K^{2|\alpha|-2}\|D^{\alpha+\gamma}\boldsymbol{u}\|^2_{L^2(K)}). \tag{9.3}$$

Set $\beta_{max} := \max \beta_j$. It can be inferred from conditions (2), (3) in Section 9.1 that $r_K^\beta \simeq h_K^{\beta_{max}}$ uniformly in j and l on K. we have

$$\sum_{i=0,1,2} h_K^{2(i-1)}|\boldsymbol{u}-\boldsymbol{v}|^2_{H^i(K)} \le C^{s+1}h_K^{2(1-\beta_{max})} \sum_{i=0,1,2} (\frac{(k_K-s)!}{(k_K+s+2-2i)!} \sum_{|\alpha|=1,2,3} \|r^{|\alpha|+|\gamma|+\beta-2}D^{\alpha+\gamma}\boldsymbol{u}\|^2_{L^2(K)}). \tag{9.4}$$

In the case that $K = T$ is a triangle, we consider Q_T defined in Section 9.1 and note that for $k \in \mathbb{N}$, $\hat{\mathbb{Q}}_{\lfloor k_T/2\rfloor} \subset \hat{\mathbb{P}}_k$. So all functions in $\mathbb{Q}_{\lfloor k_T/2\rfloor}(Q_T)$ can be restricted to functions in $\mathbb{P}_{k_T}(T)$, i.e., for $i = 0, 1, 2$ there exists a polynomial $\boldsymbol{v} \in \mathbb{Q}_{\lfloor \frac{k_T}{2}\rfloor}(Q_T)^2$ such that $\boldsymbol{v}|_T \in \mathbb{P}_{k_T}(T)^2$ and

$$|\boldsymbol{u}-\boldsymbol{v}|_{H^i(T)} \le C|\boldsymbol{u}-\boldsymbol{v}|_{H^i(Q_T)}.$$

Here the constant is independent of k_T and h_T. It can also be derived from conditions (2), (3) and (6) in Section 9.1 that on Q_T we have $r^\beta \simeq h_{Q_T}^{\beta_{max}} \simeq h_T^{\beta_{max}}$. Use the above argument we obtain that there exists a polynomial $\boldsymbol{v} \in \mathbb{P}_{k_T}(K)^2$ such that for $s \le \lfloor \frac{k_T}{2}\rfloor$,

$$\sum_{i=0,1,2} h_K^{2(i-1)}|\boldsymbol{u}-\boldsymbol{v}|^2_{H^i(K)} \le C^{s+1}h_K^{2(1-\beta_{max})} \sum_{i=0,1,2} (\frac{(\lfloor \frac{k_T}{2}\rfloor-s)!}{(\lfloor \frac{k_T}{2}\rfloor+s+2-2i)!} \sum_{|\alpha|=1,2,3} \|r^{|\alpha|+|\gamma|+\beta-2}D^{\alpha+\gamma}\boldsymbol{u}\|^2_{L^2(K)}). \tag{9.5}$$

This estimate is weaker than (9.4), so it still holds for quadrilateral elements and it will be used in the following. We still denote $\mathbb{S}_{k_K}(K) = \mathbb{Q}_{k_K}(K)$ if K is a quadrilateral and $\mathbb{S}_{k_K}(K) = \mathbb{P}_{k_K}(K)$ if it is a triangle.

As $\boldsymbol{u} \in B^2_\beta(\Omega)^2$, there exists two constant $A_u, C_u > 1$ such that for any multi-index $\alpha \in \mathbb{N}_0^2$ with $|\alpha| > 0$ we have $\|r^{|\alpha|+\beta-2} D^\alpha \boldsymbol{u}\|_{L^2(\Omega)} \le C_u A_u^{|\alpha|-2}(|\alpha|-2)!$. Therefore, selecting C_1 and A_1 which are independent of s (they depend only on $\boldsymbol{u}$) and satisfying $9C^{s+1}C_u^2 A_u^{2(s+1)}(s+1)^2 \le C_1 A_1^s$, we obtain that for any $K \in \mathcal{T}_{int}$ there exists $\boldsymbol{v} \in \mathbb{S}_{k_K}(K)^2$ such that

$$\sum_{i=0,1,2} h_K^{2(i-1)} |\boldsymbol{u} - \boldsymbol{v}|^2_{H^i(K)} \le C^{s+1} C_u^2 h_K^{2(1-\beta_{max})} \sum_{i=0,1,2} \Big(\frac{(\lfloor \frac{k_K}{2} \rfloor - s)!}{(\lfloor \frac{k_K}{2} \rfloor + s + 2 - 2i)!} \sum_{|\alpha|=1,2,3} A_u^{2(s+|\alpha|-2)} (s + |\alpha| - 2)!^2 \Big) \tag{9.6}$$

$$\le C_1 h_K^{2(1-\beta_{max})} \frac{(\lfloor \frac{k_K}{2} \rfloor - s)!}{(\lfloor \frac{k_K}{2} \rfloor + s - 2)!} A_1^s (s!)^2.$$

Now, notice that $p \in B^1_\beta(\Omega)$, use the same arguments as in (9.3)-(9.6) and we obtain that there exists a polynomial $\tilde{q} \in \mathbb{S}_{k_K-1}(K)$ and two constants A_2, C_2 (they only depend on the pressure p) such that

$$\sum_{i=0,1} h_K^{2i} |p - \tilde{q}|^2_{H^i(K)} \le C_2 h_K^{2(1-\beta_{max})} \frac{(\lfloor \frac{k_K}{2} \rfloor - s)!}{(\lfloor \frac{k_K}{2} \rfloor + s - 2)!} A_2^s (s!)^2. \tag{9.7}$$

Now we use (9.6) and (9.7) to bound $E^2_{1,int} + E^2_2$. We note that if $K \in \mathfrak{L}^l_j \subset \mathcal{T}^l_{\mathfrak{c}}$, then $h_K \simeq \sigma^j$ and $k_K = k_j := k_{min} + s\lfloor l - j \rfloor$, otherwise ($K \in \mathcal{T}_{fixed}$) $h_K \le diam(\Omega)$ is fixed and $k_K = k_0$. Set $A = \max(A_1, A_2)$ and we have

$$\begin{aligned}
E^2_{1,int} + E^2_2 = & \sum_{K \in \mathcal{T}_{int} \cup (\cup_{i=1,2\cdots,n} \mathcal{T}^l_{\mathfrak{c}_i})} \Big(\sum_{i=0,1,2} h_K^{2(i-1)} |\boldsymbol{u} - \boldsymbol{v}|^2_{H^i(K)} + \sum_{i=0,1} h_K^{2i} |p - q|^2_{H^i(K)} \Big) \\
& + \sum_{K \in \mathcal{T}_{fixed}} \Big(\sum_{i=0,1,2} h_K^{2(i-1)} |\boldsymbol{u} - \boldsymbol{v}|^2_{H^i(K)} + \sum_{i=0,1} h_K^{2i} |p - q|^2_{H^i(K)} \Big) \\
& \le C \Big[\sum_{j=0}^{l} \sigma^{2(1-\beta_{max})j} \min_{s=1,2,\cdots,\lfloor k_j/2 \rfloor} \frac{(\lfloor \frac{k_j}{2} \rfloor - s)!}{(\lfloor \frac{k_j}{2} \rfloor + s - 2)!} A^s (s!)^2 \Big] \\
& \le C_{int} \exp(-b_{int} l)
\end{aligned} \tag{9.8}$$

for some positive constants C_{int} and b_{int}. Here the final inequality can be derived from [39, Lemma 5.9 and Lemma 5.12]. The proof is finished by combining (9.2), (9.8) and the fact that $N \simeq l^3$ as $l \to +\infty$. $\square$

Chapter 10

Discussion on main results

In this chapter, we conclude our work and give some discussion.

The two major results in this thesis are Theorem 5.2.1, which show the analytic regularity of the solution to the NSE with specific boundary condition(see Chapter 3) using the weighted, analytic function spaces $B^l_\beta(\Omega)$ with $0 < \beta < 1$ and Theorem 9.2.1, which establishes exponential convergence of a suitable hp-DGFEM applied to the NSE with an analytic solution in a polygon. We give some further discussion on these theorems.

10.1 Discussion on Theorem 5.2.1

Theorem 5.2.1 justifies the analytic regularity of the solution in polygon given analytic data and homogeneous boundary conditions. To the best of our knowledge, there is no previous result about the analytic regularity of the stationary NSE in a polygon with mixed boundary conditions (A result which considers only Dirichlet boundary condition was presented recently in [28]).

The core method we use here follows from [16] and [17], which studies the polar-component form of the equation using polar coordinate: in Step 1 of the proof of Lemma 5.2.7, we evaluate $W^2_{\beta_1}(S^1_{\frac{\delta}{2}})^2 \times W^1_{\beta_1}(S^1_{\frac{\delta}{2}})$ regularity of higher order derivatives $(r^k\partial_r^k\overline{\boldsymbol{u}}, r^k\partial_r^k p)$ by considering an auxiliary Stokes problem with $(r^k\partial_r^k\overline{\boldsymbol{u}}, r^k\partial_r^k p)$ as the solution. Here, the usage of polar coordinate in the proof ensures that the boundary conditions of this auxiliary problem can be easily determined by derivatives of the boundary value of $(\boldsymbol{u}, p)$ and lower-order derivatives of p. This method was not possible to use with Cartesian coordinates.

What is the flaw in this method? One might notice that our restriction on the boundary condition appears strange, particularly with regard to Condition 2 in Remark 3.1.1:

Condition 2: Each corner A_i must have at least one touching edge with Dirichlet boundary condition or have both touching edges with slip boundary condition.

This condition rules out the possibility that we have only Neumann boundary condition or Neumann boundary condition combined with slip boundary condition near a corner. From a practical point of view, we may find it kind of acceptable as these cases are uncommon in physical application. However, they are still of mathematical interest. Why is Condition 2 imposed here?

We revisit Theorem 5.1.1, which studies the local property of the solution to the Stokes problem near a corner. Theorem 5.1.1 says that if the boundary condition near a corner (say, A_1) follows Condition 2, then the solution $(\boldsymbol{u}, p)$ satisfies that $\boldsymbol{u} \in W^2_{\beta_1}(S^1_\delta)^2 \subset C^0(\overline{S_{\delta/2}})^2$ and $\boldsymbol{u}(A_1) = \boldsymbol{0}$. However, if Condition 2 is violated, then $\boldsymbol{u}$ is still continuous but not necessarily vanish at A_1 and thus it is possible that $\boldsymbol{u}, \overline{\boldsymbol{u}} \notin W^2_{\beta_1}(S^1_\delta)^2$. Similarly, the auxiliary Stokes problem's solution $(r^k\partial_r^k\overline{\boldsymbol{u}}, r^k\partial_r^k p)$ might not exhibit $W^2_{\beta_1}(S^1_{\frac{\delta}{2}})^2 \times W^1_{\beta_1}(S^1_{\frac{\delta}{2}})$ regularity. Thus we could not evaluate higher order derivatives of $\boldsymbol{u}$ by calculating $W^2_{\beta_1}(S^1_{\frac{\delta}{2}})$-norm.

Therefore, if we follow the route in this thesis, [16] and [17], we must enforce Condition 2 on the problem. On the other hand, for the study of Stokes or stationary incompressible NSE in a polygon with general boundary conditions, the Kondrat'ev space $W^2_\beta(S_\delta)$ with $0 < \beta < 1$ is not large enough to contain the solution(However, the space $B^l_\beta(\Omega)$ describing the analyticity might still be applicable). To study the analytic regularity to these problems, a larger weighted Sobolev space should be constructed.

The good news is that the aforementioned flaw only affects equations with vector solutions. For a scalar equation, we do not need to take the polar component of the solution and thus do not face the above problem. As an illustration, we can use the method described in this thesis to investigate the analytic regularity of the solution to the following equation with specified boundary conditions in a polygon

$$L(u) + g(u) = f,$$

where $L(\cdot)$ is an elliptic operator and $g(\cdot)$ is a function satisfying particular conditions. The analytic regularity result for the linearized version, which omits g, is studied using polar coordinates in [4].

10.2 Discussion on Theorem 9.2.1

A central task in the *hp*-DGFEM discretization for stationary incompressible NSE with zero Dirichlet boundary condition is to discretize the functional forms (6.2), (3.3) and (3.4).

We remark here that other possible discretization strategies exist. As for (6.2), this thesis uses the so-called symmetric interior penalty discretization of the Laplace operator. A list of other possible discretizations can be found in [3]. [38] justifies, for some of those discretizations in [3], the stability and consistency properties. With minor modifications to the proof of Theorem 9.2.1, those discretizations shall lead to exponential convergence as well.

A future generalization for this theorem (or the discretization) is to consider NSE with mixed boundary conditions and apply *hp*-DGFEM on (3.5) instead of (6.3). For this aim, we need to consider discretizing (3.2). Possible strategies for this could be found in [18, 45] treating the elastic problem, which is related to the Stokes problem/NSE. Related discrete functional analysis tools are already developed in these references. We expect that with the regularity result in Part 1 and with mesh design similar to that presented in this thesis, *hp*-DGFEM will achieve exponential convergence as well. Details for that shall be completed in a future paper.

Appendix A

Eigenvalues and eigenvectors of the operator pencil $[\hat{L}, \hat{B}]$

A.1 Determination of the eigenvalues and eigenvectors

Consider the following homogeneous problem:

$$\begin{aligned} \hat{L}(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}) &= (\boldsymbol{0},0) \qquad \text{on} \quad (0,\omega) \\ \hat{B}(D,\lambda)(\hat{\boldsymbol{u}},\hat{p}) &= (0,0,0) \qquad \text{on} \quad \{0,\omega\} \end{aligned} \tag{A.1}$$

We recall that $\partial_\theta = iD$ and the operators are defined as:

$$\hat{L}(D,\lambda) = \begin{pmatrix} \nu D^2 + 2\nu(1+\lambda^2) & \nu(3+i\lambda)iD & -(1+i\lambda) \\ -\nu(3-i\lambda)iD & 2\nu D^2 + \nu(1+\lambda^2) & iD \\ 1-i\lambda & iD & 0 \end{pmatrix}$$

and

$$\begin{aligned} \hat{B}(D,\lambda)|_{V_D} &= A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ \hat{B}(D,\lambda)|_{V_N} &= A_2 = \begin{pmatrix} \nu iD & -\nu(1+i\lambda) & 0 \\ 2\nu & 2\nu iD & -1 \end{pmatrix} \\ \hat{B}(D,\lambda)|_{V_G} &= A_3 = \begin{pmatrix} 0 & 1 & 0 \\ iD & -(1+i\lambda) & 0 \end{pmatrix} \end{aligned}$$

We firstly study the fundamental solutions. Since $[\hat{L}, \hat{B}]$ has only constant coefficients, all fundamental solutions can be written in the form $\exp(b\theta)E$ where b is a constant and E is a vector, b and E satisfy $\hat{L}(-ib,\lambda)E = 0$. As $\det(\hat{L}(-ib,\lambda)) = \nu(\lambda^2 + (1-ib)^2)(\lambda^2 + (-1-ib)^2)$, we have $b_1 = -i-\lambda$, $b_2 = -i+\lambda$, $b_3 = i-\lambda$, $b_4 = i+\lambda$. If $\lambda \neq 0, \pm i$, then for b_1 we have

$$\hat{L}(-ib_1,\lambda) = \begin{pmatrix} \nu(\lambda+i)(\lambda-3i) & -i\nu(\lambda-3i)(\lambda+i) & -i(\lambda-i) \\ -i\nu(\lambda+3i)(\lambda+i) & -\nu(\lambda+i)(\lambda+3i) & -(\lambda+i) \\ -i(\lambda+i) & -(\lambda+i) & 0 \end{pmatrix}$$

Therefore, the fundamental solution corresponding to b_1 is $\exp(b_1\theta)E_1$ where $E_1 = (1,-i,0)^t$. Similarly, we have

$$\hat{L}(-ib_2,\lambda) = \begin{pmatrix} \nu(\lambda-i)(\lambda+3i) & i\nu(\lambda-3i)(\lambda-i) & -i(\lambda-i) \\ i\nu(\lambda+3i)(\lambda-i) & -\nu(\lambda-i)(\lambda-3i) & \lambda-i \\ -i(\lambda+i) & \lambda-i & 0 \end{pmatrix}$$

and we obtain for b_2 the fundamental solution $\exp(b_2\theta)E_2$ where $E_2 = (i(\lambda - i), -(\lambda + i), 4i\nu\lambda)^t$. Furthermore,

$$\hat{L}(-ib_3, \lambda) = \begin{pmatrix} \nu(\lambda - i)(\lambda + 3i) & -i\nu(\lambda - 3i)(\lambda - i) & -i(\lambda - i) \\ -i\nu(\lambda + 3i)(\lambda - i) & -\nu(\lambda - i)(\lambda - 3i) & -(\lambda - i) \\ -i(\lambda + i) & -(\lambda - i) & 0 \end{pmatrix}$$

and $\exp(b_3\theta)E_3$ with $E_3 = (i(\lambda - i), \lambda + i, 4i\nu\lambda)^t$ is a fundamental solution for b_3. Finally.

$$\hat{L}(-ib_4, \lambda) = \begin{pmatrix} \nu(\lambda + i)(\lambda - 3i) & i\nu(\lambda - 3i)(\lambda + i) & -i(\lambda - i) \\ i\nu(\lambda + 3i)(\lambda + i) & -\nu(\lambda + i)(\lambda + 3i) & \lambda + i \\ -i(\lambda + i) & \lambda + i & 0 \end{pmatrix}$$

and the fundamental solution for b_4 is $\exp(b_4\theta)E_4$ where $E_4 = (1, i, 0)^t$. Combine all above and we have that the solution of the homogeneous problem has the form

$$(\hat{\boldsymbol{u}}, \hat{p}) = \sum_{j=1}^{4} B_j \exp(b_j\theta)E_j. \tag{A.2}$$

If $\lambda = 0$, then $b_1 = b_2 = -i$ with the eigenvector $E_1^0 = (1, -i, 0)^t$ and $b_3 = b_4 = i$ with the eigenvector $E_3^0 = (1, i, 0)^t$. Therefore the solution can always be represented as

$$(\hat{\boldsymbol{u}}, \hat{p}) = \sum_{j=1}^{4} B_j \exp(b_j\theta)E_j^0. \tag{A.3}$$

Here $E_2^0 = (i\theta - 1, \theta, -2\nu)^t$ and $E_4^0 = (-i\theta - 1, \theta, -2\nu)^t$.

If $\lambda = i$, then $b_1 = -2i$ with the eigenvector $E_1^i = (1, -i, 0)^t$, $b_2 = b_3 = 0$ with the eigenvectors $E_2^i = (0, 2i, -4\nu)^t$ and $E_3^i = (0, -2i, -4\nu)^t$ and $b_4 = 2i$ with the eigenvector $E_4^i = (1, i, 0)^t$ and the solution has the following form:

$$(\hat{\boldsymbol{u}}, \hat{p}) = \sum_{j=1}^{4} B_j \exp(b_j\theta)E_j^i. \tag{A.4}$$

And if $\lambda = -i$, then $b_1 = b_4 = 0$ with the eigenvectors $E_1^{-i} = (1, i, 0)^t$, $E_4^{-i} = (1, -i, 0)^t$, $b_2 = -2i$ with the eigenvectors $E_2^{-i} = (2, 0, 4\nu)^t$ and $b_3 = 2i$ with the eigenvector $E_4^{-i} = (2, 0, 4\nu)^t$ and the solution has the following form:

$$(\hat{\boldsymbol{u}}, \hat{p}) = \sum_{j=1}^{4} B_j \exp(b_j\theta)E_j^{-i}. \tag{A.5}$$

The vector $B = (B_1, B_2, B_3, B_4)^t$ could be determined according to the boundary conditions.

Lemma A.1.1. *Set $\alpha = -i\lambda$. If $\{\{0\}, \{\omega\}\} \subset V_D$, then the solution λ to the equation*

$$\alpha^2 \sin^2(\omega) = \sin^2(\alpha\omega), \qquad \alpha \neq 0 \tag{A.6}$$

are the eigenvalues of the operator pencil $[\hat{L}, \hat{B}]$.

If $\{\{0\}\} \subset V_D, \{\{\omega\}\} \subset V_N$, then the solution λ to the equation

$$\alpha^2 \sin^2(\omega) = \cos^2(\alpha\omega) \tag{A.7}$$

are the eigenvalues of the operator pencil.

If $\{\{0\}\} \subset V_D, \{\{\omega\}\} \subset V_G$, then the solution λ to the equation

$$2\alpha^2 \sin\omega \cos\omega = \sin(2\alpha\omega), \qquad \alpha \neq 0 \tag{A.8}$$

are the eigenvalues of the operator pencil.

If $\{\{0\}, \{\omega\}\} \subset V_N$*, then the solution* λ *to the equation*

$$\alpha^2 \sin^2(\omega) = \sin^2(\alpha\omega) \tag{A.9}$$

are the eigenvalues of the operator pencil. Moreover, the eigenvectors corresponding to $\lambda = 0$ *are* $(\cos\theta, -\sin\theta, 0)^t, (\sin\theta, \cos\theta, 0)^t$.

If $\{\{0\}\} \subset V_G, \{\{\omega\}\} \subset V_N$*, then the solution* λ *to the equation*

$$2\alpha^2 \sin\omega \cos\omega = -\sin(2\alpha\omega) \tag{A.10}$$

are the eigenvalues of the operator pencil. Moreover, the eigenvector corresponding to $\lambda = 0$ *is* $(\cos\theta, -\sin\theta, 0)^t$.

If $\{\{0\}, \{\omega\}\} \subset V_G$*, then the solution* λ *to the equation*

$$\sin((1+\alpha)\omega)\sin((1-\alpha)\omega) = 0 \tag{A.11}$$

are the eigenvalues of the operator pencil (Therefore, 0 *is not an eigenvalue as by our assumption on the domain* $\omega \neq \pi$*).*

Proof. Let $\mathcal{A}(\alpha)$ be defined as in the proof of Lemma 5.2.5, then it is easy to check, from [32, Section 3.2], that $\mathcal{A}(-i\lambda) = [\hat{L}(D,\lambda), \hat{B}(D,\lambda)]$. The assertion follows now from [33] and [32, Example 3.2]. □

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Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

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