MOTIVES AND COMPLEX MULTIPLICATION

Minicourses

FABRIZIO ANDREATTA – *Special cycles on Shimura varieties*

The goal of these lectures is to outline the strategy and provide the main ingredients to prove a conjecture of Bruinier-Kudla-Yang expressing the arithmetic intersection between Heegner divisors and suitable CM points on orthogonal Shimura varieties.

The first lecture will be devoted to introducing two of the main actors appearing in the conjecture, namely orthogonal Shimura varieties and their Heegner divisors. In low dimension one recovers more classical settings such as modular curves and Hilbert modular surfaces.

In the second and third lecture I will focus on the definition of the so called big CM points and of the realizations of the motive associated to them: étale, crystalline, de Rham etc. Examples in low dimension will be provided. I will then use the comparison between crystalline and de Rham realizations to compute explicitly the contribution at finite places to the arithmetic intersection appearing in the BKY conjecture. This is the key step in proving (many cases of) the conjecture together with the work of Bruinier, Kudla, Yang dealing with the contribution at the archimedean places.

KAI-WEN LAN – *An example-based introduction to Shimura varieties*

I will start by explaining why Shimura varieties are natural generalizations of modular curves and Siegel modular varieties, and give many examples of groups and symmetric domains with complex coordinates. If time permits, I will also introduce their boundary components, and discuss about their models over rational numbers or integers. The lectures will be for people who are not already familiar with these topics—for most of them, some willingness to see matrices larger than 2x2 ones should suffice. (I hope to allow simple factors of all possible types A, B, C, D, and E to show up if time permits, but it is not necessary to know beforehand what this means.)

FRANS OORT – *Jacobians with complex multiplication*

In 1987 Robert Coleman made the conjecture that for $g > 3$ the number of CM Jacobians (algebraic curves for which the Jacobian has sufficiently many complex multiplications) should be finite. This approach has created new developments. The conjecture is still wide open for $g > 7$. For $3 < g < 8$ there are counter examples (Shimura, De Jong - Noot, and many others). We will discuss some of these examples showing such counter examples. These examples are via families of Galois covers of curves. We feel we are still far from a general understanding of possibilities of families of curves giving a special subvariety.

We ask how to prove existence / to construct CM Jacobians and we study "linearity" or "curvature" of the Torelli locus.

We show that for a variant (Weyl CM Jacobians) the equivalent of the Coleman conjecture indeed holds (Chai and FO, 2012).

One can try a geometric approach. We study the question whether for a given value of g there exists a special subvariety generically contained in the open Torelli locus. Assuming the André-Oort conjecture, now a theorem for the moduli space of polarized abelian varieties, this is equivalent to the Coleman conjecture for g. We expect that for $g \gg 0$ such special subvarieties do not exist (FO, 1995, 1997). Many special cases have been studied; the general situation seems unclear. We discuss algebraic curves with many automorphisms; we show there are infinitely many curves with many automorphisms that are not CM curves.

We discuss Jacobians in mixed characteristic, and we show that the Serre-Tate canonical lift of an ordinary Jacobian in general is not a Jacobian. One of these methods, as developed by Dwork-Ogus (1986), has been applied in order to show certain families of curves do not give a special subvariety (De Jong - Noot, Moonen and others).

The lecture will discuss results, examples, and many open problems. We think we are still far from a true understanding of this topic.

Talks

GIUSEPPE ANCONA – *On the standard conjecture of Hodge type for abelian varieties*

In the sixties, Grothendieck formulated four conjectures on algebraic cycles that he called "standard". They are still widely open in general, but for abelian varieties several results are available : they are all known in characteristic zero and two of them (Künneth and Lefschetz) are also known in positive characteristic (Liebermann, 1968). Over finite fields any abelian variety has a CM structure (Tate, 1966), in particular its motive can be decomposed into simple factors of dimension one. This is the starting point for Clozel (1999) to show that numerical and homological equivalence coincide in some cases, another standard conjecture. Following his method, we will give some results on the forth standard conjecture (of Hodge type) about the signature of the pairing induced by the intersection product on cycles modulo numerical equivalence.

MASANORI ASAKURA – *Regulator of hypergeometric fibrations*

We call a fibration $f\colon X \to \mathbb{P}^1$ a hypergeometric fibration if it satisfies some conditions, for example, f is smooth outside $t = 0, 1, \infty$, etc. Taking the base change $t \to a - t^n$, one gets fibrations $f_a\colon X_a \to \mathbb{P}^1$, which are parametrized by

a. We discuss the Beilinson regulator of certain elements of the motivic cohomology groups $H^3_{\mathcal{M}}(X_a, \mathbb{Q}(2))$. Regarding it as a function of a, the main result is that the regulator is a generalized hypergeometric function ${}_3F_2$. As a remarkable application, we can get a numerical suffucient condition for that ${}_3F_2$ is a logarithmic function (this gives a new formula of hypergeometric functions, as far as we know). This is a joint work with Noriyuki Otsubo.

BRUNO KLINGLER – *Hodge theory and atypical intersections*

Given a smooth family X of quasi-projective varieties over a smooth quasi-projective base S, or more generally a variation V of mixed Hodge structures on S, one would like to understand the Hodge locus of (S, V): the locus of points $s \in S$ for which the fiber V_s admits more Hodge tensors than the general fiber. A deep theorem of Cattani-Deligne-Kaplan states that this Hodge locus is a countable union of irreducible algebraic subvarieties of S: the special subvarieties of (S, V). In this talk I will define a refinement of the Hodge locus: the atypical locus of (S, V), and state a general conjecture about the geometry of this atypical locus. For Shimura varieties and their subvarieties one recovers the Zilber-Pink conjecture.

DAMIAN RÖSSLER – *Logarithmic derivatives of Dirichlet L-functions and Arakelov theory*

We shall revisit some conjectures made by V. Maillot and the speaker a few years ago. These conjectures relate the logarithmic derivatives of the Artin L-functions of a CM field K to some arithmetic Chern classes living on Shimura varieties parameterising abelian varieties with subcomplex multiplication by K. They can be proven up to some log(p) factors when the Galois group of K over Q is abelian, as an application of the arithmetic Lefschetz formula proven by K. Köhler, S. Tang and the speaker. We shall outline a new more direct approach, which involves more accessible analytic tools than those that are needed to prove the arithmetic Lefschetz formula. More precisely, we shall show that a proof of the conjectures in the abelian case can be given, which depends only on Bismut's equivariant curvature formula.

LENNY TAELMAN – *Complex multiplication and K3 surfaces over finite fields*

K3 surfaces form a widely studied class of algebraic surfaces. Having Kodaira dimension 0 (just like abelian surfaces), they exhibit more interesting geometry than say rational surfaces, yet at the same time they are more tractable than the relatively unstructured surfaces of general type.

Over the complex numbers K3 surfaces are classified in terms of Hodge structures and quadratic forms, by what are commonly called the 'Torelli theorems for K3 surfaces'. This is similar to the classification of complex elliptic curves or abelian varieties in terms of lattices. Although this classification is a purely transcendental affair, one can show that complex K3 surfaces whose Hodge structures have Complex Multiplication are defined over number fields. Reducing such CM K3 surfaces at primes of good reduction gives a powerful method of producing K3 surfaces over finite fields.

In this lecture, I'll explain how this method can be used to attack questions about K3 surfaces over finite fields. In particular, I'll show how one can (almost) classify the zeta functions of K3 surfaces over a given finite field $\mathbf{F}_q$.

No prior knowledge of K3 surfaces will be assumed.

JACOB TSIMERMAN – *Jacobians isogenous to abelian varieties in finite characteristic*

(Joint with Ananth Shankar) Oort asked whether, for every dimension $g > 3$, over an algebraically closed field k, there exists an abelian variety not isogenous to a Jacobian. In characteristic 0, it is now a theorem (Chai-Oort-T.) that this is true. We present a heuristic probabilistic argument that suggests this is false for k the algebraic closure of a finite field. Strangely enough, our heuristics suggests that such abelian varieties do not exist for $g \leq 9$, but exist 'generically' for $g > 9$. To support our heuristic, we use additive combinatorics to prove that there is a hypersurface H in $X(1)^{27}$ such that every k-point in $X(1)^{27}$ is coordinate-wise isogenous to a point in H.

MARYNA VIAZOVSKA – *CM Values of Green's Functions*

Higher Green functions are real-valued functions of two variables on the upper half-plane which are bi-invariant under the action of a congruence subgroup, have logarithmic singularity along the diagonal, and satisfy the equation $\Delta f = k(1-k)f$, where Δ is a hyperbolic Laplace operator and k is a positive integer. The significant arithmetic properties of these functions were noticed B. Gross and D. Zagier "Heegner points and derivatives of L-series"(1986). In particular, it was conjectured that the values of higher Green's functions at CM points are equal to the logarithms of algebraic numbers. In this talk we will present a proof of the conjecture for any k and any pair of CM points lying in the same imaginary quadratic field.