How curvature shapes space

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Hopf Lecture, ETH, Zürich

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The lecture will have three parts:

Part 1: Heinz Hopf and Riemannian geometry

Part 2: The Riemann curvature tensor

Part 3: Curvature pinching and the sphere theorem

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Credit: Some of the information about Hopf's role in Riemannian geometry comes from the book "A Panoramic View of Riemannian Geometry" by Marcel Berger.

Heinz Hopf

Part 1: Heinz Hopf and Riemannian geometry

1932 quote from Heinz Hopf:
"The problem of determining the global structure of a space form from its local metric properties and the connected one of metrizing–in the sense of differential geometry–a given topological space, may be worthy of interest for physical reasons."

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"The problem of determining the global structure of a space form from its local metric properties and the connected one of metrizing–in the sense of differential geometry–a given topological space, may be worthy of interest for physical reasons."

It suggest two questions:
1. What does the local geometry tell us about the global structure of a space?

2. Given a topological space (smooth manifold), find the best metric (geometry) it can support.

Hopf conjecture: curvature and Euler characteristic

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True for dimensions 2 and 4. For dimension 6 it is the inequality $b_3 < 2(b_2 + 1)$. There has been no progress to my knowledge.

Hopf conjecture: $S^2 \times S^2$

The two sphere S^2 has its constant curvature metric, and this induces the product metric on $S^2 \times S^2$ which then has non-negative curvature, but the curvature is not positive since a two plane spanned by unit vectors v_1 from the first factor and v_2 from the second has zero curvature.

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Hopf Question: Does $S^2 \times S^2$ have a Riemannian metric with positive curvature?

Despite being one of the best known problems in Riemannian geometry this question is still open. I believe that no such metric should exist, but the resolution will require a new idea. Under somewhat stronger positivity conditions such metrics can be shown not to exist.

Hopf conjecture: curvature and symmetric spaces

There is a more general question posed by Hopf concerning compact symmetric spaces. These are compact manifolds which have canonical metrics with large symmetry groups. These metrics always have non-negative curvature. There is a positive integer called the rank which is the largest dimension of a flat torus which can be embedded totally geodesically. For example $S^2 \times S^2$ has rank 2 since the product of equators from the factors is a 2-torus. The rank 1 symmetric spaces have strictly positive curvature.

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Hopf Question: Does a compact symmetric space of rank greater than 1 have a metric of positive curvature?

Rank one symmetric spaces

The compact rank 1 symmetric spaces are the sphere, and the projective spaces over the reals, complex numbers, and quaternions. There is also a projective plane over the octonians.

The rank 1 symmetric spaces have positive curvature, but only the sphere and the real projective space have constant curvature. The others have curvatures which lie between 1 and 4. This can be explained in terms of the Hopf fibrations.

Curvatures of projective spaces

In the complex case the Hopf fibration maps the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$ to $\mathbb{CP}^n$, the projective space of complex lines through the origin in $\mathbb{C}^{n+1}$. The fiber of a point $p \in \mathbb{CP}^n$ is the unit circle in the complex line p. The metric on $\mathbb{CP}^n$ is that induced from the orthogonal complement of the fiber in S^{2n+1}.

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A geodesic orthogonal to the fiber remains orthogonal to the fibers and meets each fiber at antipodal points, so it covers a geodesic of $\mathbb{CP}^n$ two times and so the closed geodesics of $\mathbb{CP}^n$ have length π. A two plane at a point of $\mathbb{CP}^n$ which is complex is tangent to a $\mathbb{CP}^1$ or S^2 which has diameter $\pi/2$ and hence curvature 4 (a sphere of radius $1/2$).

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A two plane which is real is tangent to $\mathbb{RP}^2$ of diameter $\pi/2$. This is double covered by an S^2 of diameter π (radius 1) and so has curvature 1.

Curvature pinching

Pinching Problem: Does a rank 1 symmetric space have a metric with curvature *strictly* between 1 and 4? More generally is a manifold with curvatures between 1 and 4 diffeomorphic to a quotient of a rank 1 symmetric space?

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Berger refers to the problem has Hopf's pinching problem, and says that H. Rauch, who was the first to prove a partial result around 1950, spent the year 1948-49 at ETH and learned the problem from Hopf. It was a central problem that led to the development of techniques in global Riemannian geometry after 1950.

Part 2: The Riemann curvature tensor

Let M^n be a smooth n-manifold (space which is locally diffeomorphic to $\mathbb{R}^n$)

A **Riemannian metric** g on M is an assignment of inner product to each tangent space which varies smoothly from point to point.

If X, Y are smooth vector fields then $g(X, Y)$ is a smooth function which is bilinear, symmetric, and positive definite at each point.

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Surfaces in $\mathbb{R}^3$:

$g(X, Y) = X \cdot Y$ for tangent vector fields X and Y.

Sign of curvature

The metric g enables us to measure lengths, angles, and volumes. Under reasonable assumptions (Hopf-Rinow Theorem) we can find shortest curves or **geodesics** and we can consider geodesic triangles.

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The sign of the curvature is determined by the sum of interior angles of geodesic triangles.

We say that M has **positive curvature** if the sum of interior angles of nondegenerate geodesic triangles in M is greater than π.

Riemann curvature tensor

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R is a 4-linear function on tangent vectors which satisfies the symmetries

$$R(X,Y,Z,W) = -R(Y,X,Z,W) = R(Z,W,X,Y)$$
$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$$

for all tangent vectors $X, Y, Z, W \in T_pM$.

If Π is a 2-plane in T_pM and X, Y an orthonormal basis for Π, then $K(\Pi) = R(X, Y, X, Y)$ is basis independent, and is called the **sectional curvature** of Π.

The notion of positive curvature described above is equivalent to the condition that all sectional curvatures are positive.

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The sectional curvature $K(\Pi)$ is equal to the Gauss curvature at the point p of the surface swept out by geodesics in M which are tangent to Π. Because of the symmetries of R, the sectional curvatures determine all components of the curvature tensor.

Ricci and scalar curvature

The curvature tensor has two distinct traces called the **Ricci and scalar curvature**. They are given by

$$Ric(X, Y) = \sum_{k=1}^{n} R(X, e_k, Y, e_k)$$

and

$$scal = \sum_{k=1}^{n} Ric(e_k, e_k)$$

where X, Y are arbitrary tangent vectors and the e_k form an orthonormal basis at p.

The Ricci and scalar curvature play a central role in the theory of general relativity.

Manifolds of constant positive curvature

Spherical space forms: These are the compact Riemannian manifolds with all sectional curvatures equal to 1. They have universal covers which are isometric to S^n, and so may be viewed as the quotient of S^n by a finite subgroup Γ of $O(n+1)$ which acts freely on S^n (an early theorem of Hopf).

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For even n these are just S^n and $\mathbb{RP}^n$, but for each odd $n \geq 3$ there are infinitely many such manifolds which have been completely classified by group theoretic methods.

CROSS: compact rank one symmetric spaces

The CROSS manifolds which are not of constant curvature are the following:

$$\mathbb{CP}^n, \quad \mathbb{HP}^n, \quad \mathbb{OP}^2$$

where $n \geq 2$ and $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ denote respectively the complex numbers, the quaternions, and the octonians. The manifolds indicated have real dimension $2n$, $4n$, and 16 respectively.

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The CROSS manifolds have natural Riemannian metrics with sectional curvatures in the interval $[1, 4]$.

Part 3: Curvature pinching and the sphere theorem

For $\delta \in (0, 1]$, a manifold of positive curvature is said to be δ-**pinched** if the sectional curvature of (M, g) satisfies $\delta \le K_2/K_1$ for any two sectional curvatures K_1 and K_2. If the strict inequality holds, we say that (M, g) is **strictly** δ-pinched.

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We say that (M, g) is **pointwise** δ**-pinched** if $\delta \leq K_2/K_1$ for all points $p \in M$ and sectional curvatures K_1 and K_2 **at the point** p. If the strict inequality holds, we say that (M, g) is **strictly** pointwise δ-pinched.

Curvature pinching (II)

To clarify the pinching conditions we have defined, note that a spherical space form is 1-pinched while a non-constant curvature CROSS is $1/4$-pinched, but not strictly $1/4$-pinched.

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For $n = 2$ there is only one sectional curvature at each point, so any positively curved surface is pointwise 1-pinched, even though its global pinching constant may be arbitrarily small.

Sphere theorems

Theorems in Riemannian geometry which characterize the sphere or spherical space forms are called **sphere theorems**. In 1951, H. Rauch showed that a compact, simply connected Riemannian manifold which is δ-pinched is homeomorphic to S^n ($\delta \approx 0.75$). He also posed the problem of determining the optimal δ.

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This question was settled around 1960 by the celebrated theorem of M. Berger and W. Klingenberg.

Topological sphere theorem: Let (M, g) be a compact, simply connected Riemannian manifold which is strictly 1/4-pinched. Then M is homeomorphic to S^n.

Proof involves refined use of comparison and variational techniques for geodesics.

Two unresolved issues

1) **Diffeomorphism:** For simply connected manifolds, the topological sphere theorem left open the question of whether a strictly 1/4-pinched manifold is *diffeomorphic* rather than just homeomorphic to a sphere; in other words, the question of whether an exotic sphere might admit such a metric.

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1) **Diffeomorphism:** For simply connected manifolds, the topological sphere theorem left open the question of whether a strictly $1/4$-pinched manifold is *diffeomorphic* rather than just homeomorphic to a sphere; in other words, the question of whether an exotic sphere might admit such a metric.

2) **Fundamental groups:** For non-simply connected manifolds, the sphere theorem says only that the universal cover is homeomorphic to a sphere. There are known examples of exotic free actions of finite groups on spheres (even $\mathbb{Z}_2$ actions on S^4). The curvature properties of such exotic space forms are unknown.

One can also formulate the sphere theorem equivariantly and ask if a strictly $1/4$-pinched metric with a given symmetry group is equivariantly diffeomorphic to a sphere.

Some answers

1) 1960's, D. Gromoll and E. Calabi, $\delta(n)$-pinching

1970's, M. Sugimoto, K. Shiohama, and H. Karcher, $\delta = 0.87$

K. Grove, H. Karcher, and E. Ruh, $\delta = 0.76$

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2) 1970's, K. Grove, H. Karcher, and E. Ruh, $\delta = 0.97$ and for a decreasing sequence $\delta(n)$ converging to .68 as $n \to \infty$.

1980's, E. Ruh, *pointwise* $\delta(n)$-pinched manifolds, new analytic method

M. Micallef and D. Moore, topological sphere theorem with pointwise $1/4$-pinching, variational theory for minimal two spheres in M

More answers

1980's, G. Huisken, C. Margerin, S. Nishikawa, pointwise $\delta(n)$-pinching, used Ricci flow

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In higher dimensions, C. Böhm and B. Wilking developed new convergence methods for the Ricci flow and were able to show that a manifold with two-positive curvature operator is a spherical space form.

Optimal sphere theorems

The following results from 2007, which are joint with Simon Brendle, give an optimal differentiable sphere theorem under pointwise pinching assumptions for non-simply connected manifolds.

The method also preserves any symmetry which exists for the pinched metric and therefore provides a sharp equivariant sphere theorem.

Theorem A. (Brendle, Schoen) Let (M, g) be a compact Riemannian manifold which is strictly pointwise 1/4-pinched. Then M is diffeomorphic to a spherical space form. In particular, no exotic sphere admits a metric with strictly pointwise 1/4-pinched sectional curvature.

For $1/4$-pinched manifolds we have the following result.

Theorem B. (Brendle, Schoen) Let (M, g) be a compact Riemannian manifold which is pointwise $1/4$-pinched. Then either M is diffeomorphic to a spherical space form or M is isometric to a locally CROSS manifold.

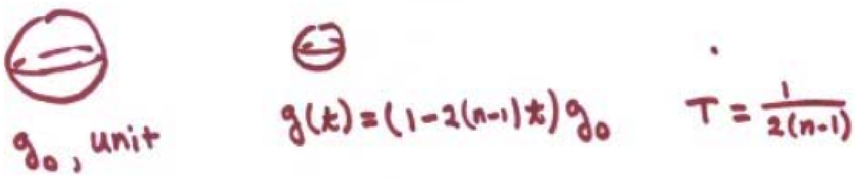
Ricci flow

The fundamental idea is to start with a given Riemannian manifold (M, g_0), and evolve the metric by the evolution equation

$$\frac{\partial}{\partial t} g(t) = -2\, Ric_{g(t)}, \qquad g(0) = g_0.$$

Here, $Ric_{g(t)}$ denotes the Ricci tensor of the time-dependent metric $g(t)$. The Ricci flow, in suitable coordinates, is a nonlinear heat equation for the Riemannian metric, and therefore, there is a short time existence theorem for any smooth initial metric.

An important example:



This solution is defined for all $t \in [0, \frac{1}{2(n-1)})$, and collapses to a point as $t \to \frac{1}{2(n-1)}$.

Ricci flow and curvature (I)

Preservation of curvature condition: One must study the evolution of the curvature tensor R which is determined by the equation

$$\frac{\partial}{\partial t} R = \Delta R + Q(R),$$

where $Q(R)$ is a quadratic polynomial in the components of R. There is a corresponding ODE on the vector space of curvature-type tensors on $\mathbb{R}^n$ which has the form $dR/dt = Q(R)$.

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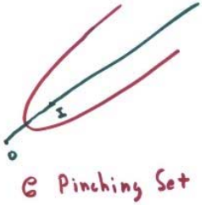
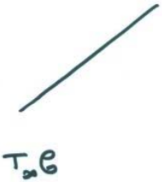
Theorem. (Hamilton) If the curvatures of g_0 lie in an invariant set $\mathcal{C}$, and the flow is smooth up to time t, then the curvatures of $g(t)$ also lie in $\mathcal{C}$.

Proof uses the maximum principle.

Ricci flow and curvature (II)

Convergence criterion: A closed set $\mathcal{C}$ is called a **pinching set** if $\mathcal{C}$ is an invariant set, and if for all $\epsilon > 0$, there exists Λ such that for all $R \in \mathcal{C}$ with $\|R\| > \Lambda$ we have $\|\frac{R}{\|R\|} - I\| < \epsilon$. (Here I denotes the curvature tensor of S^n.)





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Theorem. (Hamilton) If the curvatures of g_0 lie in a pinching set $\mathcal{C}$, then the flow exists up to a finite time T and the volume of the metric $g(t)$ converges to 0 as $t \to T$. Moreover, the renormalized unit volume metrics $\hat{g}(t) = \epsilon(t)^{-2} g(t)$ converge to a constant curvature metric as $t \to T$. In particular, M is diffeomorphic to a spherical space form.

General strategy

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In 1988, M. Micallef and J. D. Moore introduced a new curvature condition, called PIC, which was motivated from the variational study of two dimensional minimal surfaces in M; in particular, from the second variation of area.

In 1997 Hamilton showed that PIC is preserved by Ricci flow in four dimensions. The first step in the proof of Theorem A is the extension of this to higher dimensions. This was done by Brendle and the speaker, and independently by H. Nguyen.

Theorem. If g_0 is (strictly) PIC and the flow is smooth up to time t, then $g(t)$ is also (strictly) PIC.

Conditions related to PIC

We define a set of curvature tensors $\tilde{\mathcal{C}}$ by the requirement that the trivial extension of $R \in \tilde{\mathcal{C}}$ to $\mathbb{R}^{n+1}$ be PIC. This corresponds to the condition that $M \times \mathbb{R}$ be PIC.

Similarly, let $\hat{\mathcal{C}}$ be the set of curvature tensors such that the trivial extension to $\mathbb{R}^{n+2}$ is PIC. This corresponds to the condition that $M \times \mathbb{R}^2$ be PIC.

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Micallef and Moore showed that pointwise $1/4$-pinched metrics are PIC. We showed that they also satisfy the stronger condition $\hat{\mathcal{C}}$.

Convergence in $\hat{\mathcal{C}}$

Theorem. Given any compact set K contained in the interior of $\hat{\mathcal{C}}$, there exists a pinching set $\mathcal{P}_K$ such that $K \subset \mathcal{P}_K \subset \hat{\mathcal{C}}$.

Convergence Result: If g_0 has curvature interior to $\hat{\mathcal{C}}$, then the Ricci flow has a finite time singularity and the renormalized metrics converge to a constant curvature metric. In particular, M is diffeomorphic to a spherical space form.

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The construction of the pinching set $\mathcal{P}_K$ is a direct adaptation of the work of Böhm and Wilking, which involves deforming invariant sets to construct *pinching families* and to fashion pinching sets from these families.

Comparison of the conditions

To summarize the inclusions of our curvature conditions we have

$$
\begin{array}{ccccccccc}
C_{1/4} & \subseteq & \hat{C} & \subseteq & \tilde{C} & \subseteq & PIC & \subseteq & C_{Scal\geq 0} \\
 & & \cap\!| & & \cap\!| & & & & \\
 & & C_{K\geq 0} & & C_{Ric\geq 0} & & & &
\end{array}
$$

In three dimensions we have $\mathcal{C}_{Ric\geq 0} = \tilde{\mathcal{C}}$. In that case we recover Hamilton's original condition of positive Ricci curvature.

Proof of Theorems A and B

The discussion on the previous slides finishes the proof of Theorem A, for if M is strictly $1/4$-pinched, it follows that its curvatures lie interior to $\hat{\mathcal{C}}$. From the convergence theorem we find that M is diffeomorphic to a spherical space form.

Proof of Theorems A and B

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The proof of Theorem B relies on a variant of the strong maximum principle for degenerate elliptic equations developed by J. M. Bony to show the holonomy is reduced. The proof then uses M. Berger's classification of holonomy groups.

Below a quarter

Where does it leave the classification of positively curved manifolds? Any metric g of positive curvature has a pinching constant $\delta(g) \in (0, 1]$. We have a complete classification of those manifolds having metrics with $\delta(g) \in [1/4, 1]$. It turns out that $1/4$ is a hard barrier though.

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In 1996, U. Abresch and W. Meyer showed that there exists $\delta_0 < 1/4$ ($\delta_0 = \frac{1}{4(1+10^{-6})^2}$) such that any odd dimensional, simply connected, δ_0-pinched manifold is homeomorphic to a sphere. (Note that δ_0 does not depend on n.)

Below a quarter

Where does it leave the classification of positively curved manifolds? Any metric g of positive curvature has a pinching constant $\delta(g) \in (0, 1]$. We have a complete classification of those manifolds having metrics with $\delta(g) \in [1/4, 1]$. It turns out that $1/4$ is a hard barrier though.

In 1996, U. Abresch and W. Meyer showed that there exists $\delta_0 < 1/4$ ($\delta_0 = \frac{1}{4(1+10^{-6})^2}$) such that any odd dimensional, simply connected, δ_0-pinched manifold is homeomorphic to a sphere. (Note that δ_0 does not depend on n.)

P. Petersen and T. Tao used a compactness argument, together with our work, to show that there exists a $\delta(n) \in (0, 1/4)$ with the property: if (M, g) is a compact, simply connected Riemannian manifold of dimension n which is $\delta(n)$-pinched, then M is diffeomorphic to a sphere or a CROSS. (Note that the constant $\delta(n)$ is likely to tend to $1/4$ as n tends to infinity.)

Further results

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B. Andrews and H. Nguyen formulated the notion of 1/4-pinched flag curvature which requires comparison of sectional curvatures only for planes with a nontrivial intersection. In dimension 4, they were able to prove the sphere theorem under this weaker pinching condition. L. Ni and B. Wilking elucidated this condition and extended their work to all dimensions.