How curvature shapes space

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Plan of Lecture

The lecture will have four parts:

Part 1: Review of the Einstein equations

Part 2: Local structure of the constraint manifold

Part 3: Localization of solutions

Part 4: An optimal extension problem and quasi-local mass

Part 1: Review of the Einstein equations

On a spacetime $\mathcal{S}^{n+1}$, the Einstein equations couple the gravitational field g (a Lorentz metric on $\mathcal{S}$) with the matter fields via their stress-energy tensor T

$$Ric(g) - \frac{1}{2} R \, g = T$$

where Ric denotes the Ricci curvature and $R = Tr_g(Ric(g))$ is the scalar curvature.

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where Ric denotes the Ricci curvature and $R = Tr_g(Ric(g))$ is the scalar curvature.

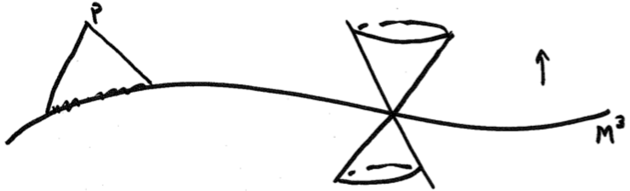
When there are no matter fields present the right hand side T is zero, and the equation reduces to

$$Ric(g) = 0.$$

These equations are called the **vacuum Einstein equations.**

Initial Data

The solution is determined by initial data given on a spacelike hypersurface M^n in $\mathcal{S}$.



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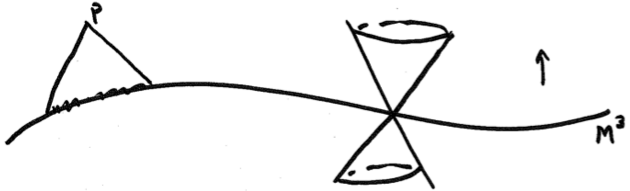

The fields at p are determined by initial data in the part of M which lies in the past of p.

The initial data for g are the induced (Riemannian) metric, also denoted g, and the second fundamental form p. These play the role of the initial position and velocity for the gravitational field. An initial data set is a triple (M, g, p).

The constraint equations

It turns out that $n + 1$ of the $(n + 1)(n + 2)/2$ Einstein equations can be expressed entirely in terms of the initial data and so are not dynamical. These come from the Gauss and Codazzi equations of differential geometry.

In case there is no matter present, the vacuum constraint equations become

$$R_M + Tr_g(p)^2 - \|p\|^2 = 0$$
$$\sum_{j=1}^{n} \nabla^j \pi_{ij} = 0$$

for $i = 1, 2, \ldots, n$ where R_M is the scalar curvature of M and $\pi_{ij} = p_{ij} - Tr_g(p) g_{ij}$.

The initial value problem

Given an initial data set (M, g, p) satisfying the vacuum constraint equations, there is a unique maximal globally hyperbolic spacetime which evolves from that data. This result involves the local solvability of a system of nonlinear wave equations.

Boundary conditions: Compact Cauchy surface

One case of interest for the Einstein equations is when the spacetime contains a compact Cauchy surface. This is often called the cosmological case. In this case the initial value problem can be formulated on a compact n-manifold and no boundary or asymptotic conditions are required.

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The compactness often makes the analysis easier, so this is a positive feature. On the other hand it is harder to interpret quantities such as gravitational energy and momentum in this setting.

Asymptotically flat manifolds

An important case for us is the asymptotically flat case. The requirement is that the initial manifold M outside a compact set be diffeomorphic to the exterior of a ball in R^n and that there be coordinates x in which g and p have appropriate falloff.

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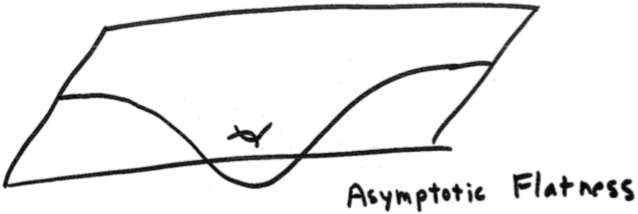

Minkowski and Schwarzschild Solutions

The following are two basic examples of asymptotically flat spacetimes:

1) The Minkowski spacetime is R^{n+1} with the flat metric $g = -dx_0^2 + \sum_{i=1}^{n} dx_i^2$. It is the spacetime of special relativity.

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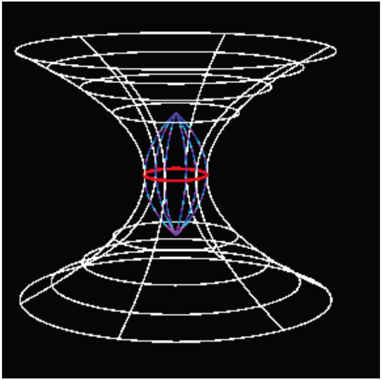
2) The Schwarzschild spacetime is determined by initial data with $p = 0$ and

$$g_{ij} = (1 + \frac{E}{2|x|^{n-2}})^{\frac{4}{n-2}} \delta_{ij}$$

for $|x| > 0$. It is a vacuum solution describing a static black hole with mass E. It is the analogue of the exterior field in Newtonian gravity induced by a point mass.

The Schwarzschild spacetime

Here is a picture of the extended Schwarzschild initial manifold. Its features lead to important notions for general asymptotically flat solutions such as the ADM energy-momentum and the notions of black holes and trapped surfaces.



Part 2: Local structure of the constraint manifold

Notice that the initial value problem allows us to parametrize solutions of the Einstein equations by solutions of the constraint equations. On the other hand we may think of the constraint 'manifold' as the set $\Phi(g,p)=0$ where (g,p) consist of a metric and a symmetric $(0,2)$ tensor on a given manifold M. Notice that the domain of Φ is an open subset of a vector space. The map Φ is the constraint map

$$\Phi(g,p) = (R(g) + Tr_g(p)^2 - \|p\|^2, \sum_{j=1}^{n} \nabla^j \pi_{ij})$$

where $\pi = p - Tr_g(p)g$.

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where $\pi = p - Tr_g(p)g$.

A solution of the Einstein equations is said to be *linearization stable* if every infinitesimal deformation is tangent to a family of deformations.

A simple example

Let $\Phi(x, y) = x^2 - y^2$ defined on $\mathbb{R}^2$ and consider the set $\Sigma = \{\Phi = 0\}$. At a point $(x, y) \in \Sigma$ the space of infinitesimal deformations consists of the kernel of $d\Phi$ at the point (x, y). For points $(x, y) \neq (0, 0)$ this defines the tangent line to Σ and each such vector is tangent to a curve in Σ.

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At $(0, 0)$ we have $d\Phi \equiv 0$, and every vector at this point is an infinitesimal deformation. The only vectors which are tangent to curves in Σ are those which make a 45^0 angle with the coordinate axes.

Linearization stability

As in the example, one expects the constraint manifold to be smooth at a solution (g, p) if (g, p) is linearization stable. In fact linearization stability is the necessary and sufficient condition for smoothness of the constraint manifold near a given point. In order to formulate this it is necessary to be precise about topologies on the space of tensors (g, p).

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The question of integrating infinitesimal deformations comes up in any problem involving a moduli space of solutions. Linearization stability is the condition that this can be done for all infinitesimal deformations.

Linearization stability and symmetry

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It was shown by Corvino and S. that linearization stability does hold in suitable decay spaces for the asymptotically flat case.

Extensions of the idea; localization of supports

A natural question to ask is whether it is possible to integrate infinitesimal deformations at a point (g, p) which are supported in some open set Ω of M in such a way that the resulting path of solutions is equal to (g, p) outside Ω.

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A natural question to ask is whether it is possible to integrate infinitesimal deformations at a point (g, p) which are supported in some open set Ω of M in such a way that the resulting path of solutions is equal to (g, p) outside Ω.

It turns out that if (g, p) is locally linearization stable in the sense that there are no spacetime Killing vector fields in Ω, there are methods to achieve this.

Enlarging the class of deformations

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For example if we have a submanifold Σ given by $\Phi = 0$ in $\mathbb{R}^n$ and a point $p \in \Sigma$, we could determine a path in Σ by taking a path $p + tv$ in $\mathbb{R}^n$, and projecting it to Σ using a local retraction from a neighborhood of Σ to Σ.

Enlarging the class of deformations

Combining these ideas, we can consider taking a point (g, p) in the constraint manifold and perturbing it in the space of tensors to a nearby $(\tilde{g}, \tilde{p})$ which may not be in the constraint manifold. We can then attempt to project into the constraint manifold to obtain $(\hat{g}, \hat{p})$ which is near (g, p) and satisfies the constraints.

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A natural application of this idea would be to simplify the geometry of (g, p) inside an open set Ω without changing it outside Ω. We will see some applications of this in the next part of the talk.

Understanding the obstructions

The fact that it is not generally possible to do this is illustrated by the following example. Let (g, p) be the euclidean metric on $\mathbb{R}^3$ and $p = 0$, so that it is initial data for Minkowski space. Now take symmetric $(0, 2)$ tensors h and k to have compact support. Let $\tilde{g} = g + \epsilon h$ and $\tilde{p} = \epsilon k$ for ϵ small.

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It is not possible to perturb to $(\hat{g}, \hat{p})$ satisfying the constraint equations and having compact support because this would lead to a compactly supported solution of the constraint equations violating the positive energy theorem.

It turns out that one can account for the obstruction by allowing flexibility in the exterior solution one uses, so that by allowing an exterior Schwarzschild or Kerr solution the construction can be made.

Part 3: Localization of solutions

As far as we know, the first instance of this general technique was used by the speaker and Yau in the early 1980s to simplify the asymptotics of general asymptotically flat metrics with $p = 0$. The idea is, given g a general asymptotically flat metric with $R = 0$, we let

$$\tilde{g} = \chi g + (1 - \chi)\delta$$

where χ is a cutoff function which is 1 in a large ball and 0 outside a ball of twice the radius.

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where χ is a cutoff function which is 1 in a large ball and 0 outside a ball of twice the radius.

We then use conformal deformation to construct a metric $\hat{g} = u^4 \tilde{g}$ to impose the constraint equations $\hat{R} = 0$. In this way we can approximate g by a solution which is conformally flat outside a compact set. This can be done for virtually any metric g for which the ADM energy exists.

The spacetime case

There is a spacetime ($p \neq 0$) analogue of this result. Notice that the metric $u^4\delta$ has zero scalar curvature if and only if u is harmonic. Since harmonic functions have nice asymptotic behavior, it is natural to look for solutions of the general constraint equations which are given by harmonic functions near infinity.

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If we require the conditions

$$g = u^4\delta, \text{ and } \pi = u^2[L_X g - div_g(X)g],$$

then to leading order the constraint equations imply that u and the components of the vector field X are harmonic.

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It was shown by Corvino and the speaker that solutions with this asymptotic form are dense in the constraint manifold of asymptotically flat solutions. With this asymptotic behavior the ADM conserved quantities appear as terms in the asymptotic expansion.

Specifying precise asymptotic behavior

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It was shown by J. Corvino ($p = 0$) and by Corvino and S. (also Chruściel and Delay) that the set of initial data which are identical to a boosted slice of the Kerr (generalization of Schwarzschild) spacetime are dense in a natural topology in the space of all data with reasonable decay.

Localization of initial data

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A good linear analogue is data (E, B) for the Maxwell equations on $\mathbb{R}^3$ with $B = 0$ and E satisfying the constraint equation $divE = 4\pi q$ where q is a compactly supported charge density. There is then a total charge and so the solution cannot be approximated by solutions of compact support. There is considerable flexibility in the asymptotic form of E which can be achieved.

Asymptotic behavior

The energy and linear momentum can be shown to exist under very weak asymptotic decay

$$g_{ij} = \delta_{ij} + O_2(|x|^{-q}),\ \ p_{ij} = O_1(|x|^{-q-1})$$

for any $q > (n-2)/2$.

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In order to understand the global properties of the Einstein evolution it is important to understand what asymptotic form is reasonable to assume. The positive energy theorem implies that there are no solutions of the constraint equations with compact support.

A further consequence of positive energy

If we let U denote the open subset of M consisting of those points at which the Ricci curvature of g is nonzero, then we have the following. It shows that under reasonable decay conditions the set U must include a positive 'angle' at infinity.

Proposition Assume that (M, g, p) satisfies the decay conditions

$$g_{ij} = \delta_{ij} + O_2(|x|^{2-n}),\ p_{ij} = O_1(|x|^{1-n}).$$

Unless the initial data is trivial, we have

$$\liminf_{\sigma \to \infty} \sigma^{1-n} Vol(U \cap \partial B_\sigma) > 0.$$

Proof of proposition

The energy can be written in terms of the Ricci curvature

$$E = -c_n \lim_{\sigma \to \infty} \sigma \int_{S_\sigma} Ric(\nu, \nu)\ da$$

for a positive constant c_n. If our initial data is nontrivial, then we have $E > 0$, and so for any σ sufficiently large we have

$$E/2 < c_n \sigma \int_{S_\sigma} |Ric(\nu, \nu)|\ da \leq c\sigma^{1-n} Vol(U \cap \partial B_\sigma)$$

where the second inequality follows from the decay assumption.

□

Localizing solutions in a cone

Let us consider an asymptotically flat manifold (M, g) with $R_g = 0$ and with decay

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In joint work with A. Carlotto we have shown that there is a metric $\hat{g}$ which satisfies $R_{\hat{g}} = 0$ with $\hat{g} = g$ inside a cone based at a point far out in the asymptotic region while $\hat{g} = \delta$ outside a cone with slightly larger angle. Moreover $\hat{g}$ is close to g in a topology in which the energy is continuous, so $\hat{E}$ is arbitrarily close to E. The metric $\hat{g}$ satisfies

$$\hat{g}_{ij} = \delta_{ij} + O(|x|^{-q})$$

provided $q < n-2$.

Where is the energy?

Since there is very little contribution to the energy inside the region where $\bar{g} = g$ and none in the euclidean region, most of the energy resides on the transition region. This shows that one cannot impose too much decay on this region and makes the weakened decay plausible.

Part 4: An optimal extension problem and quasi-local mass

Given a compact region (Ω, g, p) one can look for extensions to asymptotically flat data $(M, \hat{g}, \hat{p})$ which contain Ω as a subdomain. If $\Sigma = \partial\Omega$, then it turns out that the extension problem depends on the metric g restricted to Σ and the mean curvature H of Σ in Ω.

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The infimum of the ADM mass among all extensions in which Σ satisfies an outer minimizing condition is called the *Bartnik* quasi-local mass of the region Ω. denoted $m_B(\Omega)$. It is very difficult to compute and the question of existence of an optimal extension is open.

A lower bound

The inverse mean curvature flow which was proposed by R. Geroch and developed by G. Huisken and T. Ilmanen implies the lower bound

$$m_H(\Omega) \leq m_B(\Omega)$$

where $m_H(\Omega)$ is the Hawking mass given by

$$m_H(\Omega) = \sqrt{\frac{A}{16\pi}}\big(1 - \frac{1}{16\pi}\int_\Sigma H^2 \; da\big).$$

The black hole case

In the case when $H = 0$ the surface Σ is an apparent horizon it was shown by C. Mantoulidis and S. that one can construct extensions in which Σ is outer minimizing and where the ADM mass is arbitrarily close to the Hawking mass. This shows that in this case we have the equality

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The method was extended recently by Cabrera, Cederbaum, McCormick, and Miao to give an effective upper bound in the case of small constant H.

Geometry of black holes

The work with Mantoulidis also characterizes the metrics on S^2 which can arise on apparent horizons (stable minimal surfaces). They are the metrics g such that the operator $-\Delta + K$ has positive first eigenvalue. In particular any metric of positive Gauss curvature can be achieved.

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On the other hand we also show that there are metrics on apparent horizons with $\int_\Sigma K_-$ arbitrarily large.

Brown-York quasi-local mass

We consider the special case $p = 0$ so that M^3 is an asymptotically flat manifold with $R \geq 0$. Consider a compact region $\Omega \subset M$ with $\Sigma = \partial\Omega$.

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Under the assumptions that Σ has positive Gauss curvature K and positive mean curvature H, Brown and York defined a quasi-local mass quantity by using the Weyl Embedding Theorem to embed Σ isometrically into $\mathbb{R}^3$. If H_0 denotes the mean curvature of the embedded surface in $\mathbb{R}^3$, the mass is defined by

$$m_{BY}(\Sigma) = \frac{1}{8\pi} \int_\Sigma (H_0 - H)\, da.$$

Two dimensional motivation

If Ω is simply connected surface with boundary curve Γ with $K \geq 0$, the Gauss-Bonnet theorem says

$$\int_{\Omega} K \; da = 2\pi - \int_{\Gamma} k \; ds$$

where k is the geodesic curvature of Γ. Notice that we could write

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Thus if $K \geq 0$ we have $\int_{\Gamma}(k_0 - k) \; ds \geq 0$ with equality only if $K \equiv 0$.

Positivity of the Brown-York mass

In 2002 Y. Shi and L. F. Tam proved the following positivity theorem for the Brown-York mass.

Theorem: Assuming that Ω has non-negative scalar curvature and both K and H are positive, then $m_{BY}(\Sigma) \geq 0$ with equality only if (Ω, g) is flat.

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Theorem: Assuming that Ω has non-negative scalar curvature and both K and H are positive, then $m_{BY}(\Sigma) \geq 0$ with equality only if (Ω, g) is flat.

The proof involves the construction of an extension of Ω to an asymptotically flat manifold whose ADM mass is less than or equal to $m_{BY}(\Sigma)$ and applying the positive mass theorem.

Comparison of three quasi-local masses

Assuming we take asymptotically flat extensions in which Σ is outer minimizing, we have the inequalities

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The first inequality follows from the work of Huisken and Ilmanen using the inverse mean curvature flow. The second inequality follows from the Shi-Tam proof which involved the construction of an extension whose ADM mass was at most $m_{BY}(\Sigma)$, while $m_B(\Sigma)$ is defined to be the infimum of the ADM masses over all such extensions.

Examples with strict inequalities

If we let Σ be the long thin ellipsoid below with its mean curvature H_0 being that from $\mathbb{R}^3$, we see that both $m_B(\Sigma)$ and $m_{BY}(\Sigma)$ are zero. On the other hand we recall that

$$m_H(\Omega) = \sqrt{\frac{A}{16\pi}}\left(1 - \frac{1}{16\pi}\int_\Sigma H_0^2 \, da\right),$$

so $m_H(\Sigma)$ is very **negative**.

Making m_B much smaller than m_{BY}

If we take the same ellipsoid, but now take $H = 0$, then this is allowable by the work described above, and we have $m_H(\Sigma) = m_B(\Sigma)$ while

$$m_{BY}(\Sigma) = \frac{1}{8\pi} \int_\Sigma H_0 \, da$$

which can be made arbitrarily large.