ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

XVA Analysis for Bilateral Derivatives in Continuous Time

Master Thesis

Vito Gallo

September 12, 2020

Supervised by Prof. Dr. Cheridito

Department of Mathematics, ETH Zürich

Rämistrasse 101, 8092 Zürich

Abstract

XVAs are add-ons that a bank dealing bilateral derivatives charges to its clients to account for counterparty risk and its capital and funding implications. In this thesis we reformulate the continuous-time analysis of XVAs of [AC18] adding important theoretical results from the theory of invariance times of [CS17] that help us set rigorous assumptions for the well-posedness of the XVA equations and improved definition of the capital value adjustment (KVA) problem. We also generalise two important assumptions: we separate the margin value adjustment (MVA) from the funding value adjustment (FVA), and we allow the liquidation period of a trade due to default of the client to be positive. These generalisations permits us to obtain a more realistic XVA model, in which we distinguish the variation and initial margin. We also obtain a generalised counterparty exposure cash-flow, which is used in the formula for the credit value adjustment (CVA) and debt value adjustment (DVA). At the end of the thesis present a simple case study portfolio of interest rate swaps that could be used in an implementation of the XVA problem.

As in [AC18], we take a balance sheet perspective on the pricing and risk management of the bilateral derivatives portfolio of the bank; not only studying the pricing, but also the relative collateralisation, accounting, and dividend policy of the bank. Since the bank cannot hedge against default exposure cash-flows (of clients and of the bank itself), the bank's shareholders have to set aside a capital at risk and a wealth transfer from shareholders to bondholders occurs at the default of the bank. By consequence, the bank charges to the clients on top of the fair valuation of counterparty risk the so-called contra-liabilities and a cost-of-capital at inception of each new trade. This results in an all-inclusive XVA formula given by CVA + FVA + MVA + KVA.

Contents

Acronyms

BSDE Backward stochastic differential equation
CA Contra-assets valuation
CDS Credit default Swap
CET1 Core equity tier I capital
CL Contra-liabilities valuation
CM Clean margin
CVA Credit value adjustment
CR Capital at risk
CSA Credit support annex
DVA Debt value adjustment
EC Economic capital
ES Expected shortfall
FDA Funding debt adjustment
FTP Funds transfer price
FV Fair valuation of counterparty risk
FVA Funding value adjustment
IM Initial margin
IRS Interest rate swap
KVA Capital value adjustment
MDA Margin debt adjustment
MtM Mark-to-market
MVA Margin value adjustment
PIM Posted initial margin
RC Reserve capital
RIM Received initial margin
RM Risk margin
SCR Shareholders' capital at risk
SHC Shareholders' wealth
UC Uninvested capital

VaR Value at risk

VM Variation margin

XVA Generic “X” value adjustment

1 Introduction

1.1 Context

As illustrated in the historical introduction in [Gre15, Chapter 2], after the great financial crisis of 2008-09, the banking industry realised that counterparty credit risk was not negligible anymore, even when dealing with highly creditworthy counterparties (such as triple-A entities and global investment banks). Thus, the so-called *credit value adjustment* (CVA) on the price of OTC (over the counter; see [Gre15, Chapter 3] for an introduction on the topic) derivatives became more and more diffuse, also due to the fast developments in the regulatory requirements during the years after the crisis. Besides the pricing, also the management of counterparty risk became of great importance, with increasing capital requirements and collateralisation rules on OTC derivatives trading. This combined with the significant increment of the banks' spread (at which they could borrow unsecured funding), led to higher funding costs and the birth of *funding value adjustment* (FVA) and *margin value adjustment* (MVA); simultaneously, given the increasing capital requirements the *capital value adjustment* (KVA) was also introduced. We generally denote all these value adjustments by XVAs.

Another important member of the family of XVAs is the *debt value adjustment* (DVA), which accounts for the bank's own credit risk, and for this it was subject of great debates. In fact, subtracting the DVA from the CVA, yields a symmetrical formula for the fair valuation of bilateral counterparty risk (CVA − DVA), in line with the "law of one price" (see also our discussion in Section 4.2). However, this permits to banks to gain from a deterioration of their own creditworthiness, which goes against the Basel III capital rules (see [AA14, Section 3.1]). Moreover, the DVA (in a similar way as the *funding* and *margin debt adjustments* FDA and MDA) values the windfall benefit to the bank at its own default, due to the unused reserve capital; but, the shareholders cannot actually benefit from it, since they cannot perfectly hedge against the bank's own jump-to-default exposure. See [AC18, Section 1.1] for a more detailed discussion on the development of XVAs in the recent years.

1.2 Main reference

In this thesis we follow the XVA analysis developed in [AC18] regarding a dealer bank trading bilateral derivatives with its clients. The key assumptions in the latter paper are that the bank cannot hedge against its own jump-to-default exposure, and it also cannot perfectly hedge the counteprarty default losses. Under this *market incompleteness assumption*, the paper specifies what needs to be priced and what does not in the context of XVAs, and it also establishes "the corresponding collateralisation, accounting, and dividend policy of the bank." Specifically, under their continuous-time "cost-of-capital" XVA approach, at inception of each contract the clients pay to the bank the add-on (note that, in [AC18], MVA is considered part of the FVA)

$$\mathrm{CVA} + \mathrm{FVA} + \mathrm{KVA} = \mathrm{CA} + \mathrm{KVA} = \mathrm{FV} + \mathrm{CL} + \mathrm{KVA}, \tag{1.1}$$

where $\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA}$ denotes the *contra-assets*, which are the sum of the fair valuation of counterparty risk $\mathrm{FV} = \mathrm{CVA} - \mathrm{DVA}$ and the *contra-liabilities* $\mathrm{CL} = \mathrm{DVA} + \mathrm{FDA}$ (again, note that MDA is considered part of FDA here). In a certain sense, these add-ons are computed unilaterally (as we discuss also in Section 6.2), even though the default of the bank is included in the model, which ensures that the capital of the shareholders does not increase as an effect of the sole deterioration of the bank's own creditworthiness, as mentioned above. The KVA in [AC18] is a risk premium for the capital at risk of shareholders, remunerating them at a hurdle rate h, making the shareholders' equity a submartingale with drift corresponding to h. All together, these add-ons (together with the collateralisation, accounting, and dividend strategy illustrated in the paper) guarantee to the shareholders the so-called "soft-landing option", for which the bank could go into run-off (that is, stop making new trades) at any time in the future, always ensuring the hurdle rate payments to shareholders. In other words, one obtains a "sustainable strategy of profits retention" (as it is called in [AC18]).

Another important contribution given by [AC18] is the "balance sheet optimisation" perspective they take in the pricing and risk management of bilateral derivatives, as opposed to the "hedging paradigm" used in the past. As detailed also in this thesis, the resulting XVA problem in the model of [AC18] is "self-contained" and "self-consistent", as CVA (and also MVA, in this thesis) is the input of the FVA problem, and both CVA and FVA (as well as MVA in this thesis) are inputs for the KVA problem.

As shown in [Alb+19] and [ACC17], an important achievement of the XVA model of [AC18] is that it is actually implementable at the scale of real bilateral derivatives portfolio of a dealer bank.

1.3 Contributions of this thesis

In this thesis, we provide an analysis of the XVA metrics under the cost-of-capital XVA setup of [AC18]. We only focus on the continuous time model, without considering the discrete case. We also refer to other papers of the same authors using a similar approach to XVAs, that are [AC20], [CSS20], [ACC17], and [Alb+19].

A fundamental tool for all these papers in the proof of the well-posedness of the XVA equations is the theory of invariance times of [CS17] and [CS18], which permits us to deal with a model with two pricing frameworks: one (larger) taking into account the risk of default of the bank itself, and the other (smaller) free of the credit risk of the bank, which we sometimes refer to as "clean". Another important tool to solve the FVA and KVA problems is the theory of BSDEs (backward stochastic differential equations), which we took from [KP15] to set the sufficient assumptions for the well-posedness and comparison principle for our BSDEs.

In this thesis, we carefully summarise all the necessary basics of semimartingales theory (our main references are [HWY92] and [JS03]), including stopping times, optional and predictable projections of processes, semimartingales, stochastic integrals with respect to local martingales and compensated random measures, and a martingale representation theory useful in the study of BSDEs. This help us (and hopefully the reader) clearly understand what are the important assumptions of the model. For instance, in comparison to [AC18] we additionally assume the quasi-left-continuity of the filtrations; this is necessary to apply the BSDEs theory of [KP15], but it is also useful to ensure the existence of predictable projection of progressive measurable processes and justify the assumptions of [AC18] for their incremental XVA approach (which describes the XVA prices and accounting strategy at each new trade).

With the same goal in mind (to understand clearly what needs to be assumed in the model and what are the consequences), we also give a summary of the invariance time theory from [CS17] and [CS18] adapting the results to our particular setup, also adding proofs, where we deemed it useful or in case they are omitted in the original papers.

Using the theoretical knowledge of semimartingales, BSDEs, and invariance times, we are able to clearly state what are the sufficient assumptions (in particular, $\mathbb{L}^2$ and measurability assumptions) on the various cash-flows processes in the model of [AC18]. Specifically, we give a more rigorous definition of the KVA problem, and we enhance the discussion on the spaces of semimartingales important for the statement of the XVA problems, showing in details the relations between them. A fundamental result we obtain in this discussion is the Corollary 2.2, which justifies the invariance valuation principle in [AC18, Assumption 4.2] in the continuous time setup.

Two generalisations of the setup in [AC18] are the fact that in this thesis we separate MVA from FVA, resulting in an additional cash-flow process describing the costs due to unsecured funding of the posted initial margin, and that we allow the liquidation (or close-out) period of a trade (due to default of a client) to be "non-instantaneous". However, the liquidation period due to default of the bank is assumed instantaneous; since for the computation of the XVAs we only observe the cash-flows up to just before the default of the bank, maintaining an instantaneous liquidation assumption for the bank is not a limitation. These generalisations result in an improvement of the credit exposure cash-flows (see (3.12)), a different DVA formula, and an enhancement of the final example in [AC18, Section 7.4]. In particular, in the formula for the credit exposure cash-flow we allow the default of the bank to occur during the liquidation period of a trade (due to default of the client) and we separate variation and initial margin when dealing with the collateral (which is a generalisation of the approach in [AC20]). When the default of the bank happens during the liquidation of a trade, the bank does not only have a windfall at its default due to the unpaid promised cash-flows (this is the assumption in [AC18]), but it may also have a shortfall due to counterparty default expenses. By consequence, the DVA is changed as well and it is not necessarily non-negative. By these generalisation and our clear statement of the cash-flows assumptions, we also get slightly generalised well-posedness results for the CVA, FVA, and MVA problem; see Remark 5.17. We apply these generalisations to the final concrete example (see Section 6.1), also allowing contractual promised cash-flows to occur during the liquidation period of a trade. This result in a Theorem 6.10 that is a generalisation of [AC18, Proposition 7.2] (which is equivalent to [AC20, Proposition 5.1]).

Finally, inspired by [Alb+19], we also give a case study portfolio of interest rate swaps that could be used in future research to implement a solution of the XVA problem. The portfolio we designed is a very basic example of how this theory could be applied to a real bilateral derivatives portfolio of a bank. For future research, it would be also interesting to improve the complexity of this example to the level of a real banking portfolio.

1.4 Outline

The thesis is organised as follows. In Chapter 2, we present the notation and basic setup, and we introduce and study the spaces of semimartingales in which the XVA problems will be defined. Chapter 3 introduces the financial setup of the model bank, subdivided in different interconnected trading desks and different stakeholders. We also introduce the various cash-flow processes relative to each trading desk, and give a picture of the balance sheet of the bank in which we model the XVAs. In Chapter 4, we state the XVA problems defining the XVA processes; first we precisely state all the needed assumptions for the well-posedness of the problems in the invariance valuation setup, and then we state the equations (under the well-posedness assumption). Note that, during the whole thesis we assume the run-off assumption; in Section 4.4 we deal with the incremental XVA approach and show that stating the XVA equations under the run-off assumption is not a loss of generality. The well-posedness of all the XVA equations it then proved in Chapter 5 after a few minor simplifications of the assumptions. In Chapter 6, we first give a more concrete example of the cash-flow processes, which permits us to obtain more concrete formulas for the XVA problem (seen as a unique problem as a whole). By looking at this example, we also add a brief discussion on the unilateral form of the obtained XVA formulas, as opposed to bilateral XVAs, even though we do include the default of the bank in the modelling. Lastly, we apply the XVA problem to a simple case study portfolio of interest rate swaps.

The important basic notions and important results on semimartingales theory and BSDEs are the material of Appendix A, while Appendix B focuses on the relevant theory of invariance times from [CS17] and [CS18]. Appendix C summarises the notions of value at risk and expected shortfall.

1.5 Acknowledgements

I would like to thank my supervisor Prof. Dr. Patrick Cheritido for the support and the important feedbacks on my work. I would also like to thank Dr. Sebastian Becker for the discussions on the possible implementation of the results of my thesis.

2 Preliminaries

In this chapter we give the notation and basic setup of the thesis in Section 2.1, we introduce some important spaces of semimartingales, in which the XVAs are defined, in Section 2.2, and we define the notion of valuation with respect to our two pricing frameworks in Section 2.3. The basic knowledge of semimartingale theory that is necessary for the understanding of this thesis and the useful theory of invariance times of [CS17] are summarised in Appendix A and Appendix B, respectively.

Unlike [AC18], this thesis directly focuses on a continuous time setup, without considering the discrete one.

2.1 Probabilistic setup

We assume that $(\Omega, \mathscr{A}, \mathbb{Q})$ is a sufficiently rich probability space, so that all the introduced random objects are well defined. We let $\mathbb{G} = (\mathscr{G}_t)_{t\geq 0}$ be a filtration of sub-σ-fields of $\mathscr{A}$.

In general, when we introduce a stochastic process X, we only assume that it is *measurable*, in the sense that $X\colon \Omega \times [0,\infty[\to \mathbb{R}$ is $\mathscr{A} \otimes \mathscr{B}([0,\infty[)$-measurable, where $\mathscr{B}([0,\infty[)$ denotes the Borel σ-field on the positive real line. See Appendix A.1.1 for the basics on stochastic processes. In particular, note that all equalities and order relationships between stochastic processes are intended in the indistinguishable sense, and when we speak of uniqueness of a process, we mean up to indistinguishability. Also, when we speak of the continuity (right-, left-continuity or being càdlàg) of a process, we mean up to a $\mathbb{Q}$-nullset. Analogously, the equalities between random variables are intended $\mathbb{Q}$-a.s..

When we introduce a function without specifying the codomain, we mean real-valued function. For a function $f\colon \Omega \times E \to F$, $(\omega, x) \mapsto f(\omega, x)$, for some spaces E and F, we usually omit the dependence on ω and simply write $f(x)$, or f_x (when E is a time interval, and f is a stochastic process).

See Appendix A.1.1 for the basic notation and definitions on stopping times and stochastic intervals, which are denoted with the double square brackets $[\![\cdot,\cdot]\!]$. For two stopping times $\theta \leq \eta$, we use the notation $[\theta, \eta]$ to indicate a random subinterval of $[0,\infty]$ (so, not the stochastic interval $[\![\theta,\eta]\!] \subseteq \Omega \times [0,\infty[$). This can be used when referring to a property of the trajectories of a process: saying that a process X is right-continuous on $[0,\theta]$ means that, for all ω in a measurable set of probability one, $X_.(\omega)$ is right-continuous on the interval $[0,\theta(\omega)]$.

See Appendix A.1.2 and Appendix A.1.3 for the definition and useful properties of optional and predictable projections of stochastic processes (as well as dual predictable projections, also called compensators) and for a summary on the notation and conventions around local martingales, semimartingales, and stochastic integrals.

We fix a finite time $T > 0$ as upper bound on the maturity of all derivative contracts in the portfolio of the bank, including any additional liquidation period, which can last two weeks or more (see Remark 3.25). Even if we are only interested in the time interval $[0,T]$, we sometimes need to define processes on the whole positive real line $[0,\infty[$; in general, we assume that all the cumulative cash-flows processes are stopped at T and all the XVAs and price processes vanish on $]T,\infty[$. Additionally, we assume that no payment is due at time 0, that is, all the cumulative cash-flow processes start at 0 at time $t = 0$.

The time of default of the bank—in Section 3.3 we explain in more details what we mean by default—is represented by a totally inaccessible $\mathbb{G}$-stopping time τ with an intensity γ, which is a non-negative $\mathbb{G}$-predictable process vanishing on $]\tau, \infty[$ (see Definition A.5 and Definition A.20).

In the invariance time setup introduced in Appendix B, we also have a second filtration $\mathbb{F} = (\mathscr{F}_t)_{t\geq 0}$ on $(\Omega, \mathscr{A})$, which is a subfiltration of $\mathbb{G}$, that is, $\mathscr{F}_t \subseteq \mathscr{G}_t$ for all $t \geq 0$, and such that τ is not an $\mathbb{F}$-stopping time. We assume that both $\mathbb{G}$ and $\mathbb{F}$ satisfy the usual condition and are quasi-left-continuous, so that all the results of Appendix A and Appendix B apply to both filtrations. In particular, the quasi-left-continuity is necessary in Chapter 5 for the well-posedness of the BSDEs (see also Appendix A.2), and it is useful to simplify the requirements on a stochastic process X so that a predictable projection ${}^pX$ exists (see Remark A.13). We also assume that the σ-filed $\mathscr{G}_0$ is *trivial*, in the sense that all the elements of $\mathscr{G}_0$ have $\mathbb{Q}$-measure either 0 or 1;[1] this is no loss of generality, since we assume that all the (cumulative) cash-flows start at 0 at time 0. By this assumption it follows that the conditional expectation of a random variable with respect to $\mathscr{G}_0$ is ($\mathbb{Q}$-a.s.) equal to its expected value.

In this setup with two filtrations and a stopping time τ, we assume that the Condition(C) of Appendix B.3 holds. This means that the Condition(B) holds (that is, all $\mathbb{G}$-predictable processes have an $\mathbb{F}$-predictable reduction; see Appendix B.1), $S_T > 0$, where S denotes the Azéma supermartingale of Definition B.3, and $e^{\Gamma_{\tau\wedge T}}$ is $\mathbb{Q}$-integrable, where $\Gamma := \gamma \centerdot \lambda$ and λ is the Lebesgue measure on $[0, \infty[$. Recall that we assume that the intensity γ of τ is $\mathbb{F}$-predictable and vanishing after T; so, Γ is also $\mathbb{F}$-predictable and stopped at T. Thank to the Condition(C) we can benefit from all the important results form the theory of invariance times summarised in Appendix B. In particular, we have the existence of an invariance measure $\mathbb{P}$ as in Condition(A) in Appendix B.2, for which any $(\mathbb{F}, \mathbb{P})$-local martingale on $[0, T]$ stopped before τ defines a unique (up to indistinguishability) $(\mathbb{G}, \mathbb{Q})$-local martingale on $[0, \tau \wedge T]$ with no jump at τ, and the $\mathbb{F}$-optional reduction of any $(\mathbb{G}, \mathbb{Q})$-local martingale on $[0, \tau\wedge T]$ with no jump at τ defines a unique $(\mathbb{F}, \mathbb{P})$-local martingale on $[0, T]$.[2]

Given our setup with multiple filtrations and probability measures, it becomes important to always distinguish to which stochastic basis we are applying a particular result: $(\Omega, \mathscr{A}, \mathbb{G}, \mathbb{Q})$, $(\Omega, \mathscr{A}, \mathbb{F}, \mathbb{Q})$, or $(\Omega, \mathscr{A}, \mathbb{F}, \mathbb{P})$. The expectations with respect to $\mathbb{Q}$ and $\mathbb{P}$ are respectively denoted by $\mathbb{E}$ and $\mathbb{E}'$, while the conditional expectation with respect to $(\mathscr{G}_t, \mathbb{Q})$ and $(\mathscr{F}_t, \mathbb{P})$, for $t > 0$, are denoted by $\mathbb{E}_t$ $\mathbb{E}'_t$, respectively. When dealing with the predictable projection of some process, we need to have in mind which filtration and which probability is used. This is why, from now on, we alway specify it by adding a reference to the filtration and probability used, when needed; for example, we may say that ${}^pX$ denotes the $(\mathbb{G}, \mathbb{Q})$-predictable projection of the process X. We also assume, without loss of generality, that any optional or predictable reductions vanish on $]T, \infty[$, since we are only interested in the cash-flows on the interval $[0, T]$.

We denote the predictable (respectively optional) filtration with respect to $\mathbb{G}$ and $\mathbb{F}$ by $\mathscr{P}(\mathbb{G})$ and $\mathscr{P}(\mathbb{F})$ (respectively $\mathscr{O}(\mathbb{G})$ and $\mathscr{O}(\mathbb{F})$). The spaces of local martingales with respect to $(\mathbb{G}, \mathbb{Q})$ is denoted by $\mathcal{M}_{loc}(\mathbb{G}, \mathbb{Q})$; the spaces of $(\mathbb{F}, \mathbb{Q})$ and $(\mathbb{F}, \mathbb{P})$ local martingales are denoted similarly. See Appendix A.1 for more details.

2.2 Spaces of Semimartingales

From now on, we denote $\bar{\tau} := \tau \wedge T$. Since the stopping before τ will be used very often in the rest of the thesis, we simplify the notation in the following way (as in [AC18]): for a semimartingale

[1] We can see $\mathscr{G}_0$ as the smallest σ-field containing all the $\mathbb{Q}$-nullsets.

[2] This is exactly the result of Theorem B.18.

Y on $[0, \infty[$ we write

$$Y^\circ := Y^{\tau-} = Y \mathbb{1}_{[\![0,\tau[\![} + Y_{\tau-} \mathbb{1}_{[\![\tau,\infty[\![}, \text{ and } Y^\bullet := Y^\circ - Y.$$

Since we only focus on the interval $[0, \bar{\tau}]$, we can generally assume that any process Y vanishes on $]T, \infty[$,[3] so $Y^\bullet$ can be seen as a "bullet cash-flow"—when Y represents a cash-flow—at time τ given by

$$Y^\bullet = -(Y_\tau - Y_{\tau-}) \mathbb{1}_{[\![\tau]\!]} \mathbb{1}_{\{\tau<T\}} = -\Delta Y_\tau \mathbb{1}_{[\![\tau]\!]} \mathbb{1}_{\{\tau<T\}}.$$

Note that $\mathbb{Q}[\tau = T] = 0$ since τ is totally inaccessible; so it does not make any difference if we multiply times $\mathbb{1}_{\{\tau\leq T\}}$ instead of $\mathbb{1}_{\{\tau<T\}}$ above. We say that Y is without jump at τ if $\Delta Y_\tau = Y_\tau - Y_{\tau-} = 0$, that is, if $Y = Y^\circ$.

We can now define the following spaces of semimartingales, which are useful in the study of the well-posedness of the XVA equations in Chapter 5.

First, we define

$$\mathfrak{S}_2 := \left\{ Y \ \mathbb{G}\text{-optional process on } [0, \bar{\tau}] \colon \ \mathbb{E}\left[Y_0^2 + \int_0^T e^{\Gamma_s} \mathbb{1}_{\{s<\tau\}} d\left(Y_s^*\right)^2 \right] < \infty \right\}, \tag{2.1}$$

where $\left(Y_t^*\right)^2 := \sup_{s\leq t} |Y_s|^2$.

Note that $e^\Gamma \geq 1$, since $\gamma \geq 0$; hence, for any $Y \in \mathfrak{S}_2$:

$$\mathbb{E}\left[\sup_{\substack{0\leq t\leq T \\ t<\tau}} |Y_t|^2 \right] < \infty. \tag{2.2}$$

This means that Y° is bounded in L^2 with respect to $\mathbb{Q}$, and thus it is of class(D). We denote the set of all $\mathbb{G}$-optional processes on $[0, \bar{\tau}]$ bounded in L^2 with respect to $\mathbb{Q}$ by

$$\mathbb{S}^2(\mathbb{G}) := \left\{ X \ \mathbb{G}\text{-optional process on } [0, \bar{\tau}] \colon \ \mathbb{E}\left[\sup_{0\leq t\leq \bar{\tau}} |X_t|^2 \right] < \infty \right\}.$$

Remark 2.1. Any $Y \in \mathfrak{S}_2$ is $\mathbb{G}$-optional, so it has an $\mathbb{F}$-optional reduction Y', for which (as we show below) it holds

$$\mathbb{E}'\left[\sup_{t\in[0,T]} |Y_t'|^2 \right] = \mathbb{E}\left[Y_0^2 + \int_0^T e^{\Gamma_s} \mathbb{1}_{\{s<\tau\}} d\left(Y_s^*\right)^2 \right] < \infty. \tag{2.3}$$

In fact, in the right-hand-side we can replace Y with Y', since they coincide on $[\![0, \tau[\![$ and the integral stops before τ. So we can assume without loss of generality that Y is $\mathbb{F}$-optional. Then, setting $A := e^{\Gamma} \cdot (Y^*)^2$, we see that A is $\mathbb{F}$-optional, non-decreasing and starting from 0. Thus, by (B.5),

$$\begin{aligned} \mathbb{E}\left[\int_0^T e^{\Gamma_s} \mathbb{1}_{\{s<\tau\}} d\left(Y_s^*\right)^2 \right] &= \mathbb{E}\left[A_T^{\tau-} \right] = \mathbb{E}'\left[\int_0^T e^{-\Gamma_s} dA_s \right] \\ &= \mathbb{E}'\left[\int_0^T d\left(Y_s^*\right)^2 \right] = \mathbb{E}'\left[\left(Y_T^*\right)^2 - Y_0^2 \right]. \end{aligned}$$

[3] Or we may simply say that Y is a process on $[0, \bar{\tau}]$ or on $[0, T]$.

Last, note that $\mathbb{E}\left[Y_0^2\right] = \mathbb{E}'\left[Y_0^2\right]$, because by Theorem B.13

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \mathcal{E}\left(\frac{1}{S_-}\boldsymbol{\cdot} Q\right)_0 = 1, \quad \text{on } \mathscr{F}_0,$$

and, as $\tau > 0$, $\mathscr{G}_0 \subseteq \overline{\mathscr{F}}_0 = \mathscr{F}_0$.

We also define the following spaces

$$\mathfrak{S}_2^\circ := \{Y \in \mathfrak{S}_2 \colon\; Y^\circ = Y \text{ and } Y_T = 0 \text{ on } \{T < \tau\}\}\,, \tag{2.4}$$

$$\mathbb{S}^2(\mathbb{F}) := \left\{ Y' \; \mathbb{F}\text{-optional process on } [0,T] \colon\; \mathbb{E}'\left[\sup_{t\in[0,T]} |Y'_t|^2\right] < \infty \right\}, \tag{2.5}$$

and

$$\mathfrak{S}'_2 := \left\{Y' \in \mathbb{S}^2(\mathbb{F}) \colon\; Y'_T = 0\right\}. \tag{2.6}$$

By (2.3), if $Y \in \mathfrak{S}_2$, then its $\mathbb{F}$-optional reduction Y' is in $\mathbb{S}^2(\mathbb{F})$, and if $X \in \mathbb{S}^2(\mathbb{F})$, then $X^\circ \in \mathfrak{S}_2$. This gives us a one-to-one correspondence between $\mathbb{S}^2(\mathbb{F})$ and the space of all $Y \in \mathfrak{S}_2$ stopped before τ. Moreover, if $X \in \mathfrak{S}'_2$, then $X^\circ_T = 0$ on $\{T < \tau\}$ and $X^\circ \in \mathfrak{S}_2^\circ$. Conversely, if $Y \in \mathfrak{S}_2^\circ$, then its $\mathbb{F}$-optional reduction Y' is in $\mathfrak{S}'_2$, because, taking the $(\mathbb{Q}, \mathscr{F}_T)$-conditional expectation, we have

$$0 = \mathbb{E}\big[Y_T \mathbb{1}_{\{T<\tau\}} \big| \mathscr{F}_T\big] = \mathbb{E}\big[Y'_T \mathbb{1}_{\{T<\tau\}} \big| \mathscr{F}_T\big] = Y'_T \underbrace{S_T}_{>0}.$$

So, there is also a one-to-one correspondence between $\mathfrak{S}_2^\circ$ and $\mathfrak{S}'_2$. Combining this one-to-one relations with Theorem B.18, we obtain the following important result.

Corollary 2.2. *The $\mathbb{F}$-optional reduction Y' of a $(\mathbb{G}, \mathbb{Q})$-martingale $Y \in \mathfrak{S}_2$ with no jump at τ is an $(\mathbb{F}, \mathbb{P})$-martingale in $\mathbb{S}^2(\mathbb{F})$, and, for any $(\mathbb{F}, \mathbb{P})$-martingale $X \in \mathbb{S}^2(\mathbb{F})$, the stopped process X° is a $(\mathbb{G}, \mathbb{Q})$-martingale in $\mathfrak{S}_2$ with no jump at τ. In other words, the map*

$$\mathcal{M}^T_{loc}(\mathbb{F}, \mathbb{P}) \cap \mathbb{S}^2(\mathbb{F}) \ni X \longmapsto X^\circ \in \mathcal{M}^{\tau-\wedge T}_{loc}(\mathbb{G}, \mathbb{Q}) \cap \mathfrak{S}_2$$

is a bijection with inverse the $\mathbb{F}$-optional reduction. Additionally, the exact same relation holds between the spaces $\mathcal{M}^T_{loc}(\mathbb{F}, \mathbb{P}) \cap \mathfrak{S}'_2$ and $\mathcal{M}^{\tau-\wedge T}_{loc}(\mathbb{G}, \mathbb{Q}) \cap \mathfrak{S}_2^\circ$.

Note that, by Proposition A.21, a local martingale is a uniformly integrable true martingale if and only if it is of class(D), which is guaranteed by the intersection with the spaces $\mathbb{S}^2(\mathbb{F})$ and $\mathfrak{S}_2$, respectively.

Remark 2.3. This corollary is a very important result that justifies the invariance valuation principle assumed in [AC18, Assumption 4.2].

Consider now the set

$$\mathfrak{L}_2 := \left\{ X \; \mathbb{G}\text{-progressive process on } [0,\bar{\tau}] \colon\; \mathbb{E}\left[\int_0^T e^{\Gamma_s} \mathbb{1}_{\{s<\tau\}} X_s^2 ds\right] < \infty \right\}. \tag{2.7}$$

If $X \in \mathfrak{L}_2$ is $\mathbb{F}$-adapted, we can see as above that

$$\mathbb{E}'\left[\int_0^T X_t^2 dt\right] = \mathbb{E}\left[\int_0^T e^{\Gamma_s} \mathbb{1}_{\{s<\tau\}} X_s^2 ds\right] < \infty,$$

this time by setting $A = (e^{\Gamma} X^2) \centerdot \lambda$. In case $X \in \mathfrak{L}_2$ is $\mathbb{G}$-predictable (resp. optional),

$$\mathbb{E}\left[\int_0^T e^{\Gamma_s} \mathbb{1}_{\{s<\tau\}} X_s^2 ds\right] = \mathbb{E}'\left[\int_0^T (X_t')^2 dt\right],$$

where X' is the $\mathbb{F}$-predictable (resp. optional) reduction of X. Now, we would like to find a similar result for $X \in \mathfrak{L}_2$ general. First, consider the $\mathbb{G}$-predictable projection ${}^pX$ of X, which exists because $\mathbb{G}$ is quasi-left-continuous (see Remark A.13). Then, we have

$$\begin{aligned}\int_0^T e^{\Gamma_t} \mathbb{1}_{\{t<\tau\}} X_t^2 dt &= \int_0^T e^{\Gamma_t} \mathbb{1}_{\{t\le\tau\}} X_t^2 dt = \left(\left(e^{\Gamma} X^2\right) \centerdot \lambda\right)_{\bar{\tau}} \\ &= \left(\left(e^{\Gamma}\, {}^p(X^2)\right) \centerdot \lambda\right)_{\bar{\tau}} = \int_0^T e^{\Gamma_t} \mathbb{1}_{\{t<\tau\}} {}^p(X_t^2) dt\end{aligned} \tag{2.8}$$

where the first and the last equalities hold because we integrate with respect to $dt = dA_t$ for the (deterministic) continuous function $A_t = t$, $t \ge 0$; the third equality follows by Remark A.17 and the smoothing property of the predictable projection(see Proposition A.14). By the Jensen's inequality, for all $t \ge 0$, $({}^pX_t)^2 = (\mathbb{E}[X_t|\mathscr{F}_t])^2 \le \mathbb{E}\left[X_t^2|\mathscr{F}_t\right] = {}^p(X_t^2)$. Therefore, we can conclude, that if X' denotes the $\mathbb{F}$-predictable reduction of ${}^pX$, then

$$\begin{aligned}\mathbb{E}'\left[\int_0^T (X_t')^2 dt\right] &= \mathbb{E}\left[\int_0^T e^{\Gamma_t} \mathbb{1}_{\{t<\tau\}} \left({}^pX_t\right)^2 dt\right] \\ &\le \mathbb{E}\left[\int_0^T e^{\Gamma_t} \mathbb{1}_{\{t<\tau\}} {}^p\left(X_t^2\right) dt\right] \\ &\overset{(2.8)}{=} \mathbb{E}\left[\int_0^T e^{\Gamma_t} \mathbb{1}_{\{t<\tau\}} X_t^2 dt\right] < \infty.\end{aligned} \tag{2.9}$$

From now on, we call the $\mathbb{F}$-predictable reduction of ${}^pX$ the $\mathbb{F}$-*progressive reduction* of X (if it exists).

Last, we define

$$\mathfrak{L}_2' := \left\{X' \ \mathbb{F}\text{-progressive process on } [0,T]\colon\ \mathbb{E}'\left[\int_0^T \left(X_t'\right)^2 dt\right] < \infty\right\}.$$

In a similar way as before, if $X' \in \mathfrak{L}_2'$, then $(X')^\circ \in \mathfrak{L}_2$, and, vice versa, if $X \in \mathfrak{L}_2$, $X' \in \mathfrak{L}_2'$, where X' can denotes the $\mathbb{F}$-optional, predictable, or progressive reduction of X, as applicable.

2.3 Discounting and valuation

From now on, all the introduced processes are assumed to be $\mathbb{G}$-adapted càdlàg semimartingales. We also assume that all the cash-flow processes are already discounted at the risk-free rate; that is, we use the risk-free asset, which is assumed to exist, as a numéraire.

Remark 2.4. As observed in [AC18], the problem here is “the existence of a publicly observable, reference rate for the remuneration of collateral.” In practice, the best approximation for such a rate is provided by the overnight index swap (OIS) rate; see [HW13] and [CBB14, Chapter 2].

In the sequel, when we refer to an amount paid (for example by the client to the bank), we mean that it is actually paid, if this amount is positive, and that it is received (by the client from the bank) if this is negative. The same convention applies to all cash-flows.

Since we have two filtrations and two measures, we now introduce our two pricing models, one with respect to $(\mathbb{G}, \mathbb{Q})$ and the other with respect to $(\mathbb{F}, \mathbb{P})$. More details about the interpretation of these two different valuations is given in the following chapters. Note that, in [AC18], "valuation is not price," in the sense that the valuation of some cash-flows given by a contract does not correspond to the price of the contract, as the latter also includes an additional risk premium, given by the XVAs.

Definition 2.5. Let $\mathcal{X}$ be a $\mathbb{Q}$-integrable $\mathbb{G}$-adapted process representing a cumulative cash-flow. The $(\mathbb{G}, \mathbb{Q})$ *value process* X of $\mathcal{X}$ is a $\mathbb{G}$-adapted process on $[0, \bar{\tau}]$ defined by

$$X_t := \mathbb{E}_t \left[\mathcal{X}_{\bar{\tau}} - \mathcal{X}_t \right], \quad t \in [0, \bar{\tau}].$$

Similarly, if $\mathcal{X}$ is $\mathbb{P}$-integrable and $\mathbb{F}$-adapted, the $(\mathbb{F}, \mathbb{P})$ *value process* X of $\mathcal{X}$ is an $\mathbb{F}$-adapted process on $[0, T]$ defined by

$$X_t := \mathbb{E}'_t \left[\mathcal{X}_T - \mathcal{X}_t \right], \quad t \in [0, T].$$

In other words, the $(\mathbb{G}, \mathbb{Q})$ value process of $\mathcal{X}$ is a $\mathbb{G}$-adapted process X on $[0, \bar{\tau}]$ such that $X_{\bar{\tau}} = 0$ and $X + \mathcal{X}$ is a $(\mathbb{G}, \mathbb{Q})$-martingale on $[0, \bar{\tau}]$; a similar characterisation holds for the $(\mathbb{F}, \mathbb{P})$ valuation on $[0, T]$. Therefore, by Proposition A.1, we can assume that the valuation process is always (a.s.) càdlàg.

Remark 2.6. Note that $\mathbb{Q}$ and $\mathbb{P}$ are risk-neutral pricing measures, calibrated to derivative market prices of fully collateralised transactions (so that no counterparty risk premium is included).[4] Since the market model in our setup is incomplete, there may be more than one risk-neutral measure. In [AC20] it is suggested to choose the so-called "truly risk-neutral measure", for which the priced risk factors have risk-neutral return rate and the non-prices ones maintain their physical return rate. See also [Bjö09, Chapter 15].

Furthermore, we estimate the historical probability measures $\widehat{\mathbb{Q}}$ and $\widehat{\mathbb{P}}$ by the pricing measures $\mathbb{Q}$ and $\mathbb{P}$, as in [AC18], since it is not possible to estimate them precisely over a long time horizon.[5] This approach is deemed conservative in [AC18], since "implied CDS spreads are typically larger than statistical estimates of default probabilities." This approximation generates model risk, which constitutes the so-called *additional value adjustment* AVA, which was introduced in the regulators [EBA15].[6] An approach to the model risk issue is given in [AC20, Section 7.6].

[4] See [AC20, Section 2.1].
[5] In the XVA context, the time horizon can be even fifty years.
[6] See also [Gre15, Section 8.8.5] for a short qualitative introduction to AVA.

3 Financial Setup and Cash Flows

In this chapter we introduce the financial setup and define the cash-flows of our model bank. We follow [AC18] for the choices of the model, and we base on [Gre15] for the definitions of the “banking terminology”. In Section 3.1, we describe what is counterparty credit risk, and what are the main ways to protect against it: hedging, collateralisation, and netting; in Section 3.2, we describe the model structure of the bank subdivided in different trading desks and various stakeholders, and we give a representation of the balance sheet of the bank; in Section 3.3, we illustrate what happens in case of default of the bank itself or of one of its clients; finally, in Section 3.4, we define the trading loss processes of the bank as a preparation for the next chapter in which we state the XVA equations.

We consider a *dealer bank*, which is a *market maker*, trading *bilateral* OTC derivatives with clients. The function of a market maker is to keep the financial market liquid by accepting trades proposed by the client. On the other hand, the bank clients are assumed to be price takers, the price being decided by the market maker. By clients we mean counterparties interested in trading derivatives with “our” bank; these can be, for instance, other banks, sovereigns, hedge funds, asset managers, pension funds, insurance companies, or corporates.

In this thesis we only focus on bilateral derivatives (i.e., traded between two parties: the bank and a client), as opposed to centrally clear. For more details on OTC derivatives, we refer the reader to [Gre15, Chapter 3]. A study of XVAs for centrally cleared derivatives can be found in [AAC20].

While exposing the model of the bank we progressively introduce the stochastic processes representing the relative cash-flows. Note that, for now, we work under the following simplifying assumption:

Assumption 3.1. The derivative portfolio of the bank is held on a run-off basis.

That is, the portfolio is fixed at time 0 and no new “unplanned trades” will enter the portfolio in the future. This assumption does not affect the financial setup we work on. Clearly, the portfolio of a bank is generally incremental. In Section 4.4 it is explained how to pass from a run-off setup to an incremental model.

3.1 Counterparty credit risk and its mitigation

Counterparty credit risk (or shortly, counterparty risk) is the risk that one of the two parties in the trade, the client or the bank itself, will fail to fulfil their contractual obligation towards the other party—for example, in case of default. For simplicity, to refer to such a credit event, we may simply say that the party (a client or the bank) defaulted.

As explained in more details in [Gre15, Chapter 4], it is important to note that counterparty risk is different than lending risk, which is the risk that one party (the borrower) does not pay back a borrowed amount to the other party (the lender). This can apply, for example, to loans and bonds. In such a case, the amount at risk is known approximatively well, and the risk is only taken by the lender. On the other hand, by bilateral derivative trading the value of the contract in the future is highly uncertain and can be both positive or negative—in this sense, counterparty risk is bilateral, because it is carried by both parties. As the uncertainty in the future value of a contract can be classified as market risk, we can see counterparty risk as a combination of market and credit risk.

Remark 3.2. In [Gre15, Section 4.1.2] counterparty risk is split into pre-settlement risk and settlement risk. The first one is what it is usually meant by counterparty risk, which is the risk that one of the two parties in the transaction defaults before the settlement of the trade. The second one results from the "timing differences" between the payments of the two parties at settlement of the transaction. The exposure to this risk could be substantial, for example, in a forward FX contract (as illustrated by the example-box in [Gre15, Section 4.1.2]). However, the likelihood of default before settlement is clearly higher. As we see in Section 3.1.2, netting is a good way to reduce the exposure to both types of risk. For simplicity, in this thesis we neglect settlement risk by assuming that the default of a counterparty (a client or the bank itself) is immediately known by the other party; in case of default at settlement of the contract,[7] we assume that no payments occur until the end of the liquidation period (as illustrated in Assumption 3.26).

The main measures a dealer bank can undertake to mitigate counterparty risk are hedging, netting, and collateralisation. Below we illustrate them in details. Note that these measures do not actually remove counterparty risk, but convert it in other forms of risk, such as operational or liquidity risk. Moreover, the counterparty risk mitigation is never perfect; the residual risk has to be covered by reserve capital, as expressed by the regulators [BCB15] and [EU12]. In this thesis we do not consider other contractual clauses, such as resets or additional termination events as in [Gre15, Chapter 5].

3.1.1 Hedging

A simple way for a bank to protect itself against the potential loss given by the default of a counterparty is to buy a credit default swap (CDS). A CDS works like an insurance: in case of default of the counterparty, any potential amount owed to the bank is covered by the CDS seller, so that the bank does not face any loss. Although CDS were widely used before the great financial crisis of 2007–2008, people now realise that they are "highly toxic."[8] Therefore, CDS are illiquid nowadays; this makes them a bad instrument to mitigate counterparty risk.

As mentioned above, counterparty risk is bilateral. So, it is important to also consider the default of the bank itself; however, this is even more problematic to hedge. In fact, to achieve this, the bank would need to trade its own debt, that is, going long its own bonds. The impossibility for the bank to do this is illustrated in details in [CF13, Section 10.7], by use of a simplified example. Furthermore, there is also an argument of scale: in [AC18] it is stated that "if all European banks were to be required to have capital equal to a third of their liabilities, then the total capitalisation of banks would be greater than the total capitalisation of the entire equity market as we know it today." Last, by the rules protecting the bondholders of the bank, it is also illegal for the bank to hedge its own jump to default (see also Section 4.2).

The rationale above justifies the following assumption.

Assumption 3.3. The bank cannot hedge its own jump-to-default exposure.

So, for now, to keep things more general, we admit the possibility for the bank to partially hedge against the counterparty's default; let $\mathcal{H}$ be the cumulative cash-flows process of the loss

[7] Actually, given our assumption that the default of the bank τ is totally inaccessible, the probability of the bank to default exactly at the settlement of the contract is 0 (note that a constant time is a predictable time); conversely, the clients of the bank may default at the settlement of the contract in our model (see Assumption 4.1).

[8] As expressed in [Gre15, Section 3.1.5].

(positive when payed by the bank)of this hedge. Recall that we assume that all cash-flows processes start from 0 at time 0, and are a $\mathbb{G}$-adapted stochastic processes.

Since hedging is not an effective measure against counterparty risk, netting and collateralisation (which are discussed below) should be used.

3.1.2 Netting

A bilateral derivative portfolio between two parties may contain a large number of transactions, which may partially offset one another. Netting consists in simplifying all (or some of) these transactions into one single (net) payment. So, the bank considers the default exposure relative to a so-called *netted set* (relative to one counterparty) instead of looking at each transaction separately.

Example 3.4. The benefit of netting can be easily illustrated by a simple example. Consider a bank B trading with a counterparty C. Assume, for simplicity, a one period model with deterministic cash flows and zero recovery at default. Say that B owes 1m CHF to C, and C owes 1.2m CHF to B. If at maturity B does not default and C does, then B pays 1m CHF and faces a loss (due to the counterparty's default) of 1.2m CHF. If the transactions is netted, that is, C owes 0.2m CHF to B and B nothing to C, then the credit exposure of B in case of default of C is (dramatically) reduced.

Clearly, the reality can be much more complicated than the given example, with large numbers of transactions in different currencies and at different maturities.[9] This makes netting an important way to reduce (counterparty) risk, as long as it is clearly defined what happens in case of default of one counterparty. If this is not the case, one may incur increasing legal risk. For more on this topic and on netting legislation, see [Gre15, Section 5.2]. This is beyond the scope of this thesis.

Remark 3.5. In [Gre15, Section 5.2] netting is split in "payment netting" and "close-out netting", which are related to settlement and pre-settlement risk, respectively, which are mentioned in Remark 3.2.

So, from now on, we split the portfolio of the bank in netted sets indexed by c (c like "contract", as, from the point of view of XVA, one can see a netted set like a single contract; see [AC18]). For any c, we let $\mathcal{P}^c$ denote a stochastic process describing the net contractually promised cumulative cash-flow (counted positively when received by the bank) related to the netting set c, and we denote by τ_c the time of default of the client of the netting set c, which is assumed to be a $\mathbb{G}$-stopping time.

Remark 3.6. It may seem confusing at first sight to treat the hedge cash-flow $\mathcal{H}$ as a loss and the promised cash-flows $\mathcal{P}^c$ as a gain, but we decided to maintain the same sign convention for the various cash-flows as in [AC18]. The reason for this, is that it is convenient to see the hedges as a loss (positive when payed by the bank) just as the loss processes $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{G}$ we introduce below, so that their valuation can be seen as (positive) liabilities on the balance sheet of the bank; on the other hand, it is also convenient to see the promised cash-flows $\mathcal{P}^c$ as gains (positive when received by the bank), so that their valuation can be seen as (positive) assets on the balance sheet of the bank.

[9]Note that, in general, trades in different currencies lie in different netting sets; see [Gre15, Section 5.2.3].

3.1.3 Collateralisation

After having reduced its counterparty risk exposure by netting, a bank can protect itself further by requiring from the client a *collateral*, which is an amount of cash or liquid assets posted as a guarantee against default. The collateral received by the bank stays the property of the client (the poster) up to default of the latter; after that it can be used by the bank (the receiver) to cover (part of) the losses due to liquidation of the netted set—clearly, the same holds from the point of view of the client with the bank posting the collateral. For simplicity, as in the setup of [AC18], we assume cash only collateral.

There are many possible collateral agreements, from no collateral posting to full or partial collateralisation; see the description of the Credit Support Annex (CSA) in [Gre15, Section 6.2.1]. As mentioned, the collateral agreements are in general bilateral: during the duration of the contract (which ends at maturity or at default of any of the two parties), in case the market valuation of the promised cash-flows $\mathcal{P}^c$ (the so-called mark-to-market, or clean valuation, see Definition 3.15) is positive, the bank may require the posting of a collateral by the client, and vice versa. We provisionally denote this collateral by a process C^c (positive if posted by the client), and we assume that it is continuously (instantaneously) updated to reflect the current mark-to-market and the collateral agreements between the two parties.[10] So, it may happen that the collateral receiver asks the other party for more collateral due to a fluctuation in the mark-to-market, or that some of the collateral is returned back to the poster.

One can distinguish between two different types of collateral: *variation margin* (VM) and *initial margin* (IM). The former is the one directly reflecting the fluctuations in mark-to-market of a netted set, as mentioned above. It can be rehypothecated by the receiving party; this means that the bank can use the VM received by clients for funding purposes (for example, to post collateral to a third party or to hedges). Even though the VM is (periodically) updated to reflect the mark-to-market of the netted set, during the liquidation period of a defaulted party there may be some gap between the two values (this is the so-called gap risk). Thus, an additional margin, the IM, may be required. As by the VM, the IM is periodically updated (in our model it is continuously updated), depending on the agreement between the two parties; but, in contrast, the IM is typically segregated, in the sense that it cannot be used by the receiving party. The advantage of segregating the IM is that, in case the receiving party defaults, the (non-defaulting) posting party can have back the IM. This significantly reduces counterparty risk. In practice, this is sometimes achieved by a "tri-party arrangement", where a third party holds the segregated collateral, to make sure the IM is in fact not used by the receiver (see [Gre15, Section 6.4.4]). The advantage of rehypothecated collateral (i.e., the VM) is that it generates less funding costs, since the received cash collateral can be used as funding for collateral posting in other transactions.

Remark 3.7. In the opposite direction of IM, the two parties in the transaction may agree to a *threshold* on the collateral agreement, which is a fixed amount below which no collateral is required (see [Gre15, Section 6.2.3]). Clearly, a threshold offers less counterparty risk mitigation than a full collateralisation (possibly with additional initial margin), but it helps reduce operational and liquidity costs.

Remark 3.8. In [AC18], there is no distinction between the valuation adjustments due to the cost of funding VM and IM—typically dubbed FVA and MVA, respectively. Here we generalise this approach, by separating the two value adjustments. Thus, for each netting set c, the process VM^c denotes the variation margin exchanged between the client and the bank (counted positive if posted by the client), the process RIM^c denotes the initial margin received by the bank from

[10] In practice this update is not done continuously due to operational costs; see [Gre15, Section 6.1].

the client (assumed non-negative), and the process PIM^c denotes the initial margin posted by bank to the client (assumed non-negative).[11] The total collateral posted by the client to the bank is, therefore, given by

$$C^{c,+} := \mathrm{VM}^{c,+} + \mathrm{RIM}^c, \tag{3.1}$$

and the collateral posted by the bank is

$$C^{c,-} := \mathrm{VM}^{c,-} + \mathrm{PIM}^c, \tag{3.2}$$

where $\mathrm{VM}^{c,+}$ and $\mathrm{VM}^{c,-}$ denote the positive and negative parts of VM^c, respectively.

Remark 3.9. Usually, depending on the assets used as collateral, any coupon payments or dividends are passed to the collateral poster (being the actual owner of the collateral, before its default). Because we assumed cash only collateral, the poster is remunerated at the risk-free rate. This generates a loss for the collateral poster, as the interest rate payed for funding is, in general, significantly higher than the risk-free rate (depending on the credit spread). This is what generates FVA and MVA.

Note that, as by netting, collateralisation reduces counterparty risk, but may give rise to other risks, such as market, operational or liquidity risk. The study of these risks is beyond the scope of this thesis; see [Gre15, Section 6] for more on this topic.

3.2 The structure of the bank

This section illustrates the structure of our model bank and defines the cash-flows before default.

3.2.1 Different stakeholders

In our model, the bank is split in two classes of stakeholders: *shareholders* and *creditors*. The shareholders are the one in control of the bank management before the bank defaults. Creditors are further split into senior and junior creditors. The words *senior* and *junior* mean that the two classes of creditors have different seniority: at the default of the bank the senior creditors are payed back first, i.e., they have priority over the junior creditors. In our model, the senior creditors are represented by an external funder that lend unsecured to the bank at some (exogenously given) risky funding spread over the risk-free rate. The junior creditors are represented by the *bondholders*; since they have no decision power before the bank's default, they need to be protected by *pari-passu laws* "forbidding certain trades that would trigger wealth away from them to shareholders" during the liquidation process of the bank.[12]For simplicity, we can assume that the senior and junior creditors are default-free. Before default, the trading cash-flows received by the bank go to the shareholders, while after that the shareholders are "wiped out" and any received cash-flows go to the bondholders. The latter are also responsible for bankruptcy costs, which are beyond the scope of this thesis.

To simplify our model, we assume a self-financed setup, that makes it a "closed-system":

Assumption 3.10. We assume that shareholders and bondholders hold nothing outside the bank.

[11]This is the same notation used in [ACC17].
[12]See [AC18, Section 2.2].

3.2.2 CA and clean desks

In our model, the management of the trading cash-flows of the bank is split in essentially three parts: the *CA desk*, the *clean desks*, and the *KVA desk*. The last one is the subject of the next section.

The CA desk could be further subdivided in CVA, FVA and MVA desks. The CVA desk is responsible for the cash-flow related to counterparty default, whereas the FVA and MVA desks are in charge of the funding expenditures for the variation margin and initial margin, respectively.

More precisely, let $\mathcal{C}$ be a stochastic process representing the total cash-flow triggered by credit events (of the clients or of the bank itself), called the *counterparty exposure cash-flow*. It depends on the agreements regarding the settlement of the contracts at default; see Section 3.3.

Remark 3.11. Note that, $\mathcal{C}^{\circ}$ is non-negative non-decreasing, and it represents the total cumulative losses of the bank before τ due to the defaults of the clients; $\mathcal{C}^{\bullet} = \mathcal{C}^{\circ} - \mathcal{C}$ is a bullet cash-flow occurring at time τ made of two parts: a positive windfall to the bank that corresponds to the total losses of the clients due to default of the bank, and a negative part (a shortfall for the bank) for the counterparty default losses due to clients' default occurring simultaneously with the bank's default—since we generally assume that the default of the bank may happen during the liquidation period of a netting set (see Section 3.3 for more details). We can see that, by consequence, $\mathcal{C}$ is a process of finite variation on $[0, \bar{\tau}]$.

Similarly, let $\mathcal{F}$ and $\mathcal{G}$ denote the cumulative risky funding cash-flows of the bank, to fund the variation and initial margin, respectively. These include the cumulative payments $\mathcal{F}^{\circ}$ and $\mathcal{G}^{\circ}$ to the external funder for the unsecured borrowing of funds before τ (so $\mathcal{F}^{\circ}$ and $\mathcal{G}^{\circ}$ are non-decreasing) and the (non-negative) windfalls to the bank $\mathcal{F}^{\bullet}$ and $\mathcal{G}^{\bullet}$, due to the non-remunerated unsecured funding at bank's default. Thus, $\mathcal{F}$ and $\mathcal{G}$ are of finite variation on $[0, \bar{\tau}]$ too. The actual form of $\mathcal{F}$ and $\mathcal{G}$ depends on the funding policy and collateralisation of the bank; see Section 6.1 for a concrete example.

Thus, the CVA , FVA, and MVA desks have to cover the cash-flows $\mathcal{C}^{\circ}$, $\mathcal{F}^{\circ}$, and $\mathcal{G}^{\circ}$. To do that, they (together as a unique desk: the CA desk) source from the clients an amount $\mathrm{CA} := \mathrm{CVA} + \mathrm{FVA} + \mathrm{MVA}$ (which is specified in more details in Chapter 4) at inception of the contract, and deposit this in an account called the *reserve capital* (RC) account, which they use to pay $\mathcal{C}^{\circ}$, $\mathcal{F}^{\circ}$, and $\mathcal{G}^{\circ}$ as they occur.

Remark 3.12. By our choice of the risk-free asset as numéraire, it is already clear, but it is important to stress that by risky-funding cash-flow, we exclude the risk-free accrual of the reserve capital account RC of the CA desk.

Thanks to the activity of the CA desk, the other trading desks of the bank, the clean desks,[13] can focus on the management of the market risk of the contracts, ignoring counterparty risk. In some sense, the CA desk acts as a filter, filtering out the counterparty exposure and risky funding cash-flows and leaving to the clean desks the "clean" contractually promised cash-flows $\mathcal{P}^c$. Then, the management of the market risk consists of a fully collateralised hedge of each $\mathcal{P}^c$. Specifically, for each netting set c, the following cash-flows occur before default of one of the two parties (that is, $0 < t < \bar{\tau} \wedge \tau_c$):

[13]In the setup of [AC18] there can be several clean desks; one for each business line of the bank's portfolio.

- The client and the CA desk exchange[14] the promised cash-flow $\mathcal{P}^c$ plus any required additional variation margin VM^c and initial margin RIM^c and PIM^c, depending on the fluctuations of the market value of the promised cash-flows and the collateral agreements. The initial margin PIM^c posted by the bank is provided by the MVA desk using unsecured fundings—this is what generates $\mathcal{G}$. The received initial margin RIM^c is not used by the CA desk (it is segregated in a separate account). Recall that all our cash-flows processes are cumulative, whereas the variation and initial margin processes denote the actual evolution of posted collateral.

- The CA desk passes to the clean desks the received cash-flows $\mathcal{P}^c$; additionally, the CA desk[15] posts a rehypothecable collateral to the clean desks that amounts to the current market value of the whole portfolio of the bank (the so-called mark-to-market MtM; see Definition 3.15 below).[16] Any positive difference between the needed collateral and the VM^c posted by the clients is covered by the CA desk using unsecured fundings—this is what generates $\mathcal{F}$. The above collateral is stored in a so-called *clean margin* (CM) account and remunerated at the risk-free rate by the receiver. We dub CM^c the collateral posted by the CA desk relative to the netting set c; so $\mathrm{CM} = \sum_c \mathrm{CM}^c$, where the sum (here and for the rest of the thesis) runs over all netting sets.

- Using the CM, the clean desks can finance a fully collateralised market hedge. For now we assume that this hedge may be imperfect, resulting in a hedging loss process $\mathcal{H}^c$ only partially offsetting $\mathcal{P}^c$.

Remark 3.13. Note that, in the last bullet point above, if the valuation of the contract is negative, the clean desks actually receive a full rehypothecable collateral from the hedges that they pass to the CA desk, which, in turn, can post it to the clients as variation margin. For this reason, the bank has no funding costs for the posting of variation margin to clients. Observe also that the variation margin received form the hedges counts negatively for the CM. On the other hand, when the valuation of a contract is positive, the CA desks has to fund the rehypothecable collateral that goes in the CM account and that the clean desks than use to post collateral to the hedges. In this sense, we can see FVA as the cost of funding the collateral for the hedges.[17] Additionally, note that with the "fully collateralised hedges" assumption we neglect the counterparty risk of the hedges.[18]

Thus, there are two portfolios in our setup: the *client portfolio* between the client and the CA desk, and the *clean portfolio* between the CA desk and the clean desks. Note that the promised cash-flows is the same for the two portfolios, but only the client portfolio has counterparty risk.

Remark 3.14. As in [AC18], we assume for simplicity that the hedges $\mathcal{H}$ of the CA desk and $\mathcal{H}^c$ of the clean desks are either swapped or trades through the repo market, without upfront payment. See [CBB14, Section 4.2.1] for more details.

Now one can better understand the difference between the two valuations in Definition 2.5: the CA desk uses the $(\mathbb{G}, \mathbb{Q})$-valuation, and the clean desks use the $(\mathbb{F}, \mathbb{P})$-valuation. From the

[14]With the usual convention: the client pays the amount to the bank if it is positive, and the bank pays if it is negative.

[15]In particular the FVA desk, sometimes also called the Tresury of the bank

[16]If the value is negative, the clean desks post the collateral to the CA desk.

[17]As explained in [AA14, Section 2].

[18]In practice, even if a position is fully collateralised, it is always subject to gap risk. So, in our model we may assume that the hedges have instantaneous liquidation period (or they post initial margin) so that we can ignore their counterparty risk.

next chapter (see Assumption 4.1), we will assume that the processes related to activities of the CA desk ($\mathcal{C}$, $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$) are $\mathbb{G}$-adapted and defined on $[0, \bar{\tau}]$ (taking into account the default of the bank τ, which is a $\mathbb{G}$-stopping time) and the ones related to activities of the clean desks ($\mathcal{P}^c$ and $\mathcal{H}^c$) are $\mathbb{F}$-adapted and defined on $[0, T]$. For simplicity, we also assume that they are $\mathbb{Q}$-integrable—by consequence, also $\mathbb{P}$-integrable if $\mathbb{F}$-adapted. So, we can define the $(\mathbb{F}, \mathbb{P})$-value process P^c of $\mathcal{P}^c$; we call this the *clean valuation* of the contractually promised cash-flows related to the netting set c. Clean valuation is additive over netting sets c, and, since it only considers promised cash-flows, it is independent of the creditworthiness of the involved parties, their funding, collateralisation and hedging policies.

Definition 3.15. The total clean valuation of the derivative portfolio of the bank is called *mark-to-market* MtM and it is given by

$$\mathrm{MtM} = \sum_c \mathrm{P}^c \mathbb{1}_{[\![0, \tau_c^\delta]\!]}, \tag{3.3}$$

where $\tau_c^\delta \geq \tau_c$ denotes the end of the liquidation (or close-out) period of the netting set c.

Thus, MtM vanishes on $[T, \infty[$, as each P^c does. Note that, we assume that the upper bound T includes the liquidation period of any netting set, that is, if $\tau_c \leq T$, then $\tau_c^\delta < T$ (and otherwise we can consider $\tau_c = \infty$, i.e., no default before maturity).

By (3.3), we see that a liquidated netting set does not contribute anymore to the MtM; in fact, at each time t, MtM_t corresponds to the clean valuation of the future (after t) promised cash-flows.

Remark 3.16. As in [AC18], we only consider European derivatives for simplicity. The valuation of other types of derivatives, such as American derivatives, would be more complex, but including them in the model would not change the resulting XVA equations.

3.2.3 KVA desk and shareholders capital

As anticipated at the start of the last section, in addition to the CA and clean desks, the bank also has a KVA desk in charge of the management of KVA. At inception of each contract, the clients pay an additional add-on called KVA that the KVA desk puts in the so-called *risk margin* (RM) account. The RM is then gradually released by the KVA desk into the shareholders dividend stream $\mathcal{D}$ (see Assumption 3.30), to remunerate shareholders for their capital at risk, which is contained in the *shareholders capital at risk* (SCR) account, at a certain so-called *hurdle rate h*. In other words, as we will see more rigorously in Definition 4.20, KVA is the expected (minimal) amount needed by the bank to be able to pay a rate of return h to the shareholders on their capital at risk SCR until maturity T, without the necessity of making new trades (in fact, for now we work under the run-off assumption). As we explain in Remark 4.24, in the incremental XVA approach, KVA is always computed in such a way that the bank could go into run-off at any time in the future, if wished, without giving up the interest payments to the shareholders at rate h on their capital at risk.[19]

The function of the SCR account is to cover any "exceptional" trading loss, beyond the expected (or estimated) losses, which are already covered by the RC. Last, there may be an additional account called *uninvested capital* (UC) account, whose amount is typically unknown. The RM, SCR, and UC accounts yield risk-free interest payments to shareholders.

[19] This possibility is the so-called "soft-landing-option".

Sometimes we will refer to the *shareholders capital* or *shareholders wealth* $\mathrm{SHC} := \mathrm{SCR} + \mathrm{UC}$. We also define the *core equity tier I capital* of the bank:

$$\mathrm{CET1} = \mathrm{RM} + \mathrm{SCR} + \mathrm{UC}. \tag{3.4}$$

Remark 3.17. From a regulatory point of view, the amount CET1 is usually seen as an indicator of the financial strength of a bank. Note that, as asserted in Section 3.3, we do not model the default of the bank τ as the first time $\mathrm{CET1} \leq 0$, which corresponds to a "structural default" (as it is called in [AC18]). In our setup, $\mathrm{CET1} < 0$ is allowed, and it is interpreted as recapitalisation (or equity dilution). In fact, the default of a bank is not simply insolvency, but it is a liquidity event, as stressed, for example, in [Duf10, Chapter 5].

3.2.4 Mark-to-model and balance sheet

As in [AC18], we make the following assumption in our model:

Assumption 3.18 (Mark-to-model)**.** The amounts RC, RM, CM^c, and SCR are continuously, instantaneously reset to some theoretical target level. In particular,

$$\mathrm{RC} = \mathrm{CA}, \quad \mathrm{RM} = \mathrm{KVA}, \quad \mathrm{CM}^c = \mathrm{P}^c \mathbb{1}_{[\![0,\tau_c{}^\delta]\!]}, \tag{3.5}$$

and SCR will be specified later (see (4.20)).

Hence, by (3.3),

$$\mathrm{CM} = \sum_c \mathrm{CM}^c = \sum_c \mathrm{P}^c \mathbb{1}_{[\![0,\tau_c{}^\delta]\!]} = \mathrm{MtM}. \tag{3.6}$$

To keep our model self-financed, the only account which is not mark-to-model, which is the UC account, plays the role of a residual "adjustment-variable"; so, this amount does not need to be computed explicitly. Note that all these amounts, including UC, only matter before the default of the bank τ; thus, we assume that they are all stopped before τ.

Remark 3.19. The setup described in the previous two subsections highlights the balance sheet approach to pricing and risk management of financial derivatives, as opposed to the hedging paradigm, which was used before the development of the XVA metrics. We can see in Figure 1 an analogous of the balance sheet in [AC18, Figure 1]. Note that we separate MVA from FVA in the liability side, and we separate MDA from FDA in the asset side; the rest is the same. We see that the balance sheet is structured in different "floors". In the lowest floor, the clean desks have the collateral CM^+ received from the clients and external funder (through the CA desk), which counts as a liability, and they expect the future promised cash-flows from the clients with valuation MtM^+, which counts as an asset (receivables). In case that $\mathrm{MtM} < 0$, the clean desks post the collateral CM^- that they expect to receive back from the clients (so it counts as an asset), and they owe the future promised cash-flows to the clients valued MtM^-, which is a liability (payables). On the second floor, the CA desk receives the reserve capital $\mathrm{RC} = \mathrm{CA}$ from the clients (an asset for them), and expect the losses due to counterparty credit risk and risky funding given by $\mathrm{CVA} + \mathrm{FVA} + \mathrm{MVA} = \mathrm{CA}$ (a liability). On the third floor, we have the KVA desk dealing with the KVA payments received from the clients that they use to build the risk margin. The risk margin is then gradually released as dividend payments to the shareholders. Since a failed dividend payment is no materiality of default, the risk margin is not considered a liability for the bank. Together, RM and SCR form the capital at risk $\mathrm{CR} := \mathrm{SCR} + \mathrm{RM}$, which is used by the shareholders to cover the losses beyond expectation (this is computed

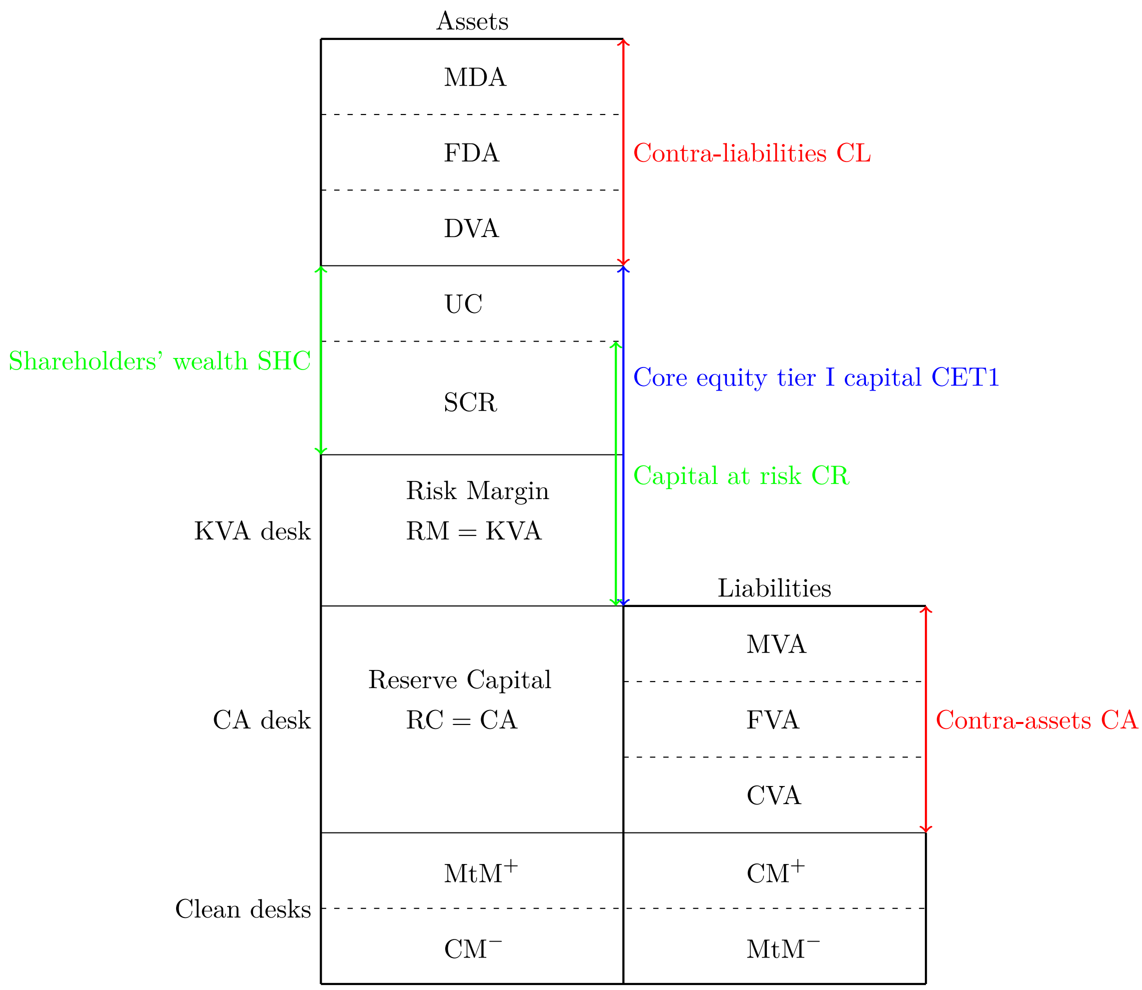

Figure 1: Balance sheet of the bank.

using an expected shortfall; see Section 4.3 for more details). Clearly, the shareholders' wealth SHC = SCR + UC is an asset for the bank, which together with the risk margin forms the core equity tier I capital CET1. The contra-liabilities CL = DVA + FDA + MDA (where DVA stands for *debs value adjustment*, FDA for *funding debt adjustment*, and MDA for *margin debt adjustment*), represent the valuation of the windfall to the bank occurring at τ, as described in Section 3.3. This is clearly an asset for the bank as a whole, but does not count as shareholders' capital, because shareholders do not actually benefit from it, as the management of the bank goes to bondholders at the time of default τ. Additionally, as illustrated in [AA14, Section 3.1], by regulatory requirements DVA should not be part of CET1. See also [AC18, Appendix A] for other considerations on the balance sheet of the bank.

Note that, even though we assumed the run-off assumption, the balance sheet of Figure 1 also represents the situation in the incremental XVA approach of Section 4.4.

3.3 The event of default

We have already introduced the notations τ and τ_c, representing the time of default of the bank and of the clients related to the netting set c, respectively. We stress that, default does not simply mean bankruptcy; by "event of default" we actually mean that, by one party's fault (the defaulted party), the trade must be terminated prematurely. This may include various type of events: see, for example the list of events of default in the ISDA Master Agreement [ISD19, Figure 2].

We model the default time of the bank as an exogenously given totally inaccessible $\mathbb{G}$ stopping time τ that has a $(\mathbb{G}, \mathbb{Q})$-intensity γ. The default of clients, given by τ_c, are assumed to be exogenously given $\mathbb{G}$-stopping times.

Remark 3.20. To calibrate τ, one may use the bank observed CDS, as in [Ces+09, Section 3.3].

Recall that at the default time of the bank the management of the bank passes from the shareholders to the bondholders. Moreover, we assume the following.

Assumption 3.21. At time τ the residual amounts RC and RM flow from the shareholders to the bondholders of the bank.

Remark 3.22. Note that any residual SHC = SCR + UC amount stays the property of the shareholders at default time τ. As pointed out in [AC18], the above assumption is justified by a no-arbitrage argument. Namely, in the limit event in which the bank defaults right after receiving the payments CA=RC and KVA=RM from the clients at inception (technically $\mathbb{Q}[\tau = 0] = 0$, as τ is totally inaccessible, but let us assume $\tau = 0$ for the sake of the argument), if some (strictly positive) part of the amount RC + RM goes to shareholders at default, then this generates a positive arbitrage[20] for them, since we assumed that no other cash-flows take place at time $t = 0$. Conversely, if in the same situation a (strictly positive) part of SCR flows to bondholders at default, this constitutes a (negative) arbitrage for the shareholders. The amount received by the bondholders is not a positive arbitrage for them either, as they have to face the bankruptcy costs, which are beyond the scope of this thesis.

Definition 3.23. We call *contra-assets* the process on $[0, \bar{\tau}]$ given by

$$\mathcal{S} := \mathcal{C}^\circ + \mathcal{F}^\circ + \mathcal{G}^\circ + \mathcal{H}^\circ + \mathbb{1}_{[\![\tau]\!]}\mathrm{RC}_\tau, \tag{3.7}$$

[20] We do not define arbitrage formally in our model. In this case, we simply mean a possible non-negative gain without any downside risk.

which denotes the cumulative cash-flows sourced from the shareholders by the activity of the CA desk. We call *contra-liabilities* the process on $[0, \bar{\tau}]$ given by

$$\mathcal{B} := \mathcal{C}^{\bullet} + \mathcal{F}^{\bullet} + \mathcal{G}^{\bullet} + \mathbb{1}_{[\![\tau]\!]}\mathrm{RC}_{\tau}, \tag{3.8}$$

which denotes the cumulative cash-flows received by the bondholders from the activity of the CA desk.

Note that both $\mathcal{S}$ and $\mathcal{B}$ include the bullet cash-flows RC_τ from the shareholders to the bondholders at default of the bank, accordingly to Assumption 3.21. The RM is not included, as we consider cash-flows triggered by the activity of the CA desk. The Contra-Assets give the name to the CA desk and to the process CA, which is the valuation of $\mathcal{S}$, as shown in Remark 4.12. Contra-assets and contra-liabilities can be thought of as "synthetic derivatives payoff,"[21] whose value process appears in the balance sheet of the bank respectively as special liabilities and assets (see Figure 1). As emphasised in [AC18], the purpose of the this capital structure model of the bank is to put in a balance sheet perspective the contra-assets and contra-liabilities.

Remark 3.24. As we illustrate in Section 4.2 (see, in particular, Definition 4.9 and Remark 4.12), the $(\mathbb{G}, \mathbb{Q})$-valuation of the contra-liabilities, which we denote by CL = DVA + FDA + MDA, represents a gain for the bank (an asset) due to its own default (see also Remark 4.14 for a more detailed discussion on this). We can see the contra-liabilities DVA, FDA, and MDA as counterparts of CVA, FVA, and MVA in the following sense. DVA accounts for the exposure of the clients to the risk of default of the bank itself, while FDA and MDA account for the "non-reimbursement" by the bank of its funding debt (due to its own default). In a complete market one should subtract these gains for the bank from the add-ons CA = CVA + FVA + MVA to obtain the (symmetric) "fair valuation of counterparty risk" FV = CA − CL.[22] However, as explained in more details in Remark 4.15, in our incomplete market setup, the bank needs to source the full amount CA from the clients to protect against counterparty risk. The extra gain given by CL will then flow to the bondholders as a recovery at the default of the bank. Thus, we do not interpret DVA, FDA, and MDA as part of the entry prices, but as a gains (or recoveries, since the bondholders also face a loss at bank's default) for the bondholders due to the incompleteness of the market.

Remark 3.25. Up to now we have implicitly assumed that all the payments after the default of the bank are instantaneously performed, with no delay between τ and the actual end of the liquidation period of the bank, that is, $\tau^\delta = \tau$. In practice, this period may last even many years, as in the case of Lehman Brothers showed in [Gre15, Figure 3.3] (see also [FS14]). The motivation for this assumption is that, taking the point of view of shareholders, we are only interested in what happens before τ. Indeed, as we can see in Theorem 6.10, the XVAs are computed unilaterally, only considering the cash-flows before τ (see also Section 6.2).

The following assumption illustrates the exchange of cash-flows occurring at the liquidation of each netting set. Note that, the liquidation (or close-out) period of a netting set c goes from the first default time $\tau_c \wedge \tau$ to the liquidation time $\tau_c^\delta \wedge \tau$.

Assumption 3.26. On the client portfolio side, at the time a party defaults (the bank or a client), the property of the variation margin posted by any of the two parties is transferred to the other one. The property of the initial margin posted by a defaulted party is transferred to the receiving party, if the latter is non-defaulted at the liquidation time. Moreover, at the liquidation time of a netting set c:

[21] This is how they are called in [AC18]

[22] This is what was done in [HW12].

- any positive value due by a non-defaulted party is paid in full to the other one;
- any positive value do by a defaulted party is only paid up to some exogenously given recovery rate.

On the clean portfolio side, during the liquidation period, the CA desk pays to the clean desks all the promised cash-flows, even if these are not received form a client (in case it defaults), and at each liquidation time the property of any amount $\mathrm{CM}^c = \mathrm{P}^c$ of rehypothecable collateral on the clean margin account goes from the CA to the clean desks.[23]

By "positive value", it is meant clean valuation P^c of the not yet payed promised cash-flows of the netting set c minus the corresponding collateral (which has already been transferred). Note that, the initial margin posted by the non-defaulted party stays its property, even if the counterparty defaults.

In particular, assume there is a netting set c that is liquidated before the default of the bank, that is, $\tau_c^\delta < \bar{\tau}$. Denoting by R_c the recovery rate of the client, at τ_c^δ the CA desk receives from the client the amount

$$R_c\left(\mathcal{P}^c_{\tau_c^\delta} - \mathcal{P}^c_{\tau_c-} + \mathrm{P}^c_{\tau_c^\delta} - \mathrm{VM}^c_{\tau_c-} - \mathrm{RIM}^c_{\tau_c-}\right)^+,$$

while the CA desk guarantees to the clean desks the full promised cash-flows $\mathcal{P}^c_{\tau_c^\delta} - \mathcal{P}^c_{\tau_c-}$ during the close-out period, as well as the rehypothecable collateral on the CM account

$$\mathrm{P}^c_{\tau_c^\delta} = \left(\mathrm{P}^c_{\tau_c^\delta} - \mathrm{VM}^c_{\tau_c-} - \mathrm{RIM}^c_{\tau_c-}\right) + \left(\mathrm{VM}^c_{\tau_c-} + \mathrm{RIM}^c_{\tau_c-}\right)$$

(the first part provided through unsecured funding and the second posted by the client) at the liquidation time τ_c^δ of the netting set. Hence, the credit exposure relative to the netting set c that the CA desk faces is given by

$$(1 - R_c)\left(\mathcal{P}^c_{\tau_c^\delta} - \mathcal{P}^c_{\tau_c-} + \mathrm{P}^c_{\tau_c^\delta} - \mathrm{VM}^c_{\tau_c-} - \mathrm{RIM}^c_{\tau_c-}\right)^+. \tag{3.9}$$

According to the assumption, any positive value due to the defaulted client is paid in full by the bank, which means that the client receives at time τ_c^δ

$$\left(\mathcal{P}^c_{\tau_c^\delta} - \mathcal{P}^c_{\tau_c-} + \mathrm{P}^c_{\tau_c^\delta} - \mathrm{VM}^c_{\tau_c-}\right)^-.$$

This payment does not constitute a loss for the CA desk, as this amount is actually owed to the client, even in case of no default. Moreover, in this case, any initial margin PIM^c posted by the (non-defaulted) bank is fully returned to the bank (the same would hold symmetrically in case the bank defaults before a client).

Now, following [AC20], we can make things more general, by allowing the close-out period of the bank and the clients to overlap, giving rise to the following credit default loss for the bank at the liquidation time $\tau_c^\delta \wedge \tau$ of the netting set c, in case $\tau_c \le \bar{\tau}$,[24]

$$(1 - R_c)\left(\mathcal{P}^c_{\tau_c^\delta \wedge \tau} - \mathcal{P}^c_{\tau_c-} + \mathrm{P}^c_{\tau_c^\delta \wedge \tau} - \mathrm{VM}^c_{\tau_c-} - \mathrm{RIM}^c_{\tau_c-}\right)^+. \tag{3.10}$$

[23] As usual, in case some of the promised cash-flows or posted collateral on the clean margin account is negative, the payments go in the opposite direction.

[24] Recall that, if $\tau_c \le T$, we assume $\tau_c^\delta < T$.

Similarly, we can compute the loss of a client associated to a netting set c in case the bank defaults before the liquidation time of the contract, i.e., $\tau \leq \tau_c^\delta \wedge T$, which is given by a "bullet" cash-flow at time $\tau_c^\delta \wedge \tau = \tau$ equals to

$$(1 - R_b)\left(\mathcal{P}_\tau^c - \mathcal{P}_{(\tau\wedge\tau_c)-}^c + \mathrm{P}_\tau^c - \mathrm{VM}_{(\tau\wedge\tau_c)-}^c + \mathrm{PIM}_{(\tau\wedge\tau_c)-}^c\right)^-, \tag{3.11}$$

where R_b represents the recovery rate of the bank.

Summing over all the netting sets, we obtain the credit default exposure cash-flows on $[0, \bar{\tau}]$:

$$\begin{aligned} \mathcal{C} = &\sum_{\substack{c \\ \tau_c \leq \bar{\tau}}} (1 - R_c)\left(\mathcal{P}_{\tau_c^\delta\wedge\tau}^c - \mathcal{P}_{\tau_c-}^c + \mathrm{P}_{\tau_c^\delta\wedge\tau}^c - \mathrm{VM}_{\tau_c-}^c - \mathrm{RIM}_{\tau_c-}^c\right)^+ \mathbb{1}_{[\![\tau_c^\delta\wedge\tau,\bar{\tau}]\!]} \\ &- \sum_{\substack{c \\ \tau\leq\tau_c^\delta\wedge T}} (1 - R_b)\left(\mathcal{P}_\tau^c - \mathcal{P}_{(\tau\wedge\tau_c)-}^c + \mathrm{P}_\tau^c - \mathrm{VM}_{(\tau\wedge\tau_c)-}^c + \mathrm{PIM}_{(\tau\wedge\tau_c)-}^c\right)^- \mathbb{1}_{[\![\tau]\!]}. \end{aligned} \tag{3.12}$$

Now, if we stop before τ, we obtain

$$\mathcal{C}^\circ = \sum_{\substack{c \\ \tau_c^\delta < \bar{\tau}}} (1 - R_c)\left(\mathcal{P}_{\tau_c^\delta\wedge\tau}^c - \mathcal{P}_{\tau_c-}^c + \mathrm{P}_{\tau_c^\delta\wedge\tau}^c - \mathrm{VM}_{\tau_c-}^c - \mathrm{RIM}_{\tau_c-}^c\right)^+ \mathbb{1}_{[\![\tau_c^\delta,\bar{\tau}]\!]}, \tag{3.13}$$

and

$$\begin{aligned} \mathcal{C}^\bullet &= \mathcal{C}^\circ - \mathcal{C} \\ &= -\sum_{\substack{c \\ \tau_c\leq\tau\leq\tau_c^\delta\wedge T}} (1 - R_c)\left(\mathcal{P}_\tau^c - \mathcal{P}_{\tau_c-}^c + \mathrm{P}_\tau^c - \mathrm{VM}_{\tau_c-}^c - \mathrm{RIM}_{\tau_c-}^c\right)^+ \mathbb{1}_{[\![\tau]\!]} \\ &\quad + \sum_{\substack{c \\ \tau\leq\tau_c^\delta\wedge T}} (1 - R_b)\left(\mathcal{P}_\tau^c - \mathcal{P}_{(\tau\wedge\tau_c)-}^c + \mathrm{P}_\tau^c - \mathrm{VM}_{(\tau\wedge\tau_c)-}^c + \mathrm{PIM}_{(\tau\wedge\tau_c)-}^c\right)^- \mathbb{1}_{[\![\tau]\!]}, \end{aligned} \tag{3.14}$$

where the first sum represents the counterparty default losses for the bank relative to netting sets c for which $\tau_c \leq \tau \leq \tau_c^\delta$ (that is, the bank defaults during the liquidation period of the netting set c), and the second sum represents the counterparty default losses for the clients due to bank's default (which is a windfall for the bank).

Remark 3.27. It is important to recall that the variation margin $\mathrm{VM}_{(\tau_c\wedge\tau)-}^c$ may be both positive or negative, depending if before $\tau_c \wedge \tau$ the clean valuation P^c of the future promised cash-flows is positive or negative. It may also happen that P^c changes sign during the liquidation period. In such a case, say at default of the client $\mathrm{P}_{\tau_c-}^c < 0$ and $\tau_c < \tau_c^\delta < \tau \wedge T$, the bank may have posted a variation margin $\mathrm{VM}_{\tau_c-}^{c,-} = -\mathrm{VM}_{\tau_c-}^c > 0$, and received no collateral (VM or IM) from the client at that point; if at the liquidation time we have $\mathrm{P}_{\tau_c^\delta}^c > 0$ (and, for simplicity, we assume that the trade consists in a single promised cash-flow at maturity T, so that $\mathcal{P}_{\tau_c^\delta}^c - \mathcal{P}_{\tau_c-}^c = 0$), then the bank looses

$$(1 - R_c)\left(\mathrm{P}_{\tau_c^\delta}^c + \mathrm{VM}_{\tau_c-}^{c,-}\right) > (1 - R_c)\mathrm{P}_{\tau_c^\delta}^c.$$

This approach is, therefore, a bit more general than the one in [AC20, Equation (27)] in this sense,[25] since we take into account this issue. However, it is in line with [ACC17, Lemma 3.1].

[25]But we assumed immediate liquidation for the bank, i.e., $\tau^\delta = \tau$.

3.4 The trading loss processes

Since the shareholders are only affected by the cash-flows before τ, it is important to distinguish the shareholders' cash-flows from that of the bank as a whole (shareholders and bondholders altogether), which also includes the default time of the bank. Therefore, we first introduce the cash-flows of the bank as a whole triggered respectively by the activities of the CA and clean desks:

Definition 3.28. The *accounting result of the CA desk* and the *accounting result of the clean desks* are respectively given by the following two processes:

$$\mathcal{A}^{ca} := -(\mathrm{RC} - \mathrm{RC}_0 + \mathcal{C} + \mathcal{F} + \mathcal{G} + \mathcal{H}), \tag{3.15}$$

$$\mathcal{A}^{cl} := \sum_c (\mathrm{CM}^c - \mathrm{CM}^c_0 + \mathcal{P}^c - \mathcal{H}^c)^{\tau_c{}^\delta}. \tag{3.16}$$

Definition 3.29. The *trading loss of the CA desk* and the *trading loss of the clean desks* are respectively defined by

$$\mathrm{L}^{cl} := -\mathcal{A}^{cl}, \quad \text{and} \quad \mathrm{L}^{ca} := -(\mathcal{A}^{ca})^\circ. \tag{3.17}$$

The *trading loss of the bank* is defined by

$$\mathrm{L} := -(\mathcal{A}^{cl} + \mathcal{A}^{ca})^\circ = (\mathrm{L}^{cl})^\circ + \mathrm{L}^{ca}. \tag{3.18}$$

The processes L^{ca} and L are stopped before τ, since we see them from the point of view of shareholders. On the other hand, L^{cl} is not stopped, because the clean desks do not consider the dafault of the bank in their modelling. Moreover, we can set $\mathrm{CM}^c = \mathrm{P}^c \mathbb{1}_{[\![0,\tau_c{}^\delta]\!]}$ and $\mathrm{RC} = \mathrm{CA}$ as in Assumption 3.18, and we obtain

$$\mathrm{L}^{cl} = -\sum_c (\mathrm{P}^c - \mathrm{P}^c_0 + \mathcal{P}^c - \mathcal{H}^c)^{\tau_c{}^\delta}, \tag{3.19}$$

$$\mathrm{L}^{ca} = \mathrm{CA} - \mathrm{CA}_0 + \mathcal{C}^\circ + \mathcal{F}^\circ + \mathcal{G}^\circ + \mathcal{H}^\circ, \tag{3.20}$$

$$\mathrm{L} = \mathrm{CA} - \mathrm{CA}_0 + \mathcal{C}^\circ + \mathcal{F}^\circ + \mathcal{G}^\circ + \mathcal{H}^\circ + (\mathrm{L}^{cl})^\circ. \tag{3.21}$$

Note that the initial amount on the reserve capital $\mathrm{RC}_0 = \mathrm{CA}_0$ is provided by the client at inception of the contract at $t = 0$,[26] via the CVA, FVA, and MVA add-ons, whereas the difference $\mathrm{RC}_t - \mathrm{RC}_0 = \mathrm{CA}_t - \mathrm{CA}_0$, for each $t \in]0, \tau[\cap]0, T]$ is covered by the shareholders in the following sense. As the losses $\mathcal{C}^\circ + \mathcal{F}^\circ + \mathcal{G}^\circ + \mathcal{H}^\circ$ occur, the amount in the RC account is used to cover these expenses, resulting in lower reserves. Simultaneously, by Assumption 3.18, at each time $t \in]0, \tau[\cap]0, T]$, RC_t is reset at a level sufficient to cover the expected future losses due to counterparty default ($\mathcal{C}$) and risky funding ($\mathcal{F}$ and $\mathcal{G}$), which is CA_t. In case that more reserve capital is needed, then the shareholders' capital is used. This can happen, for example, if the actual losses up to time t ($\mathcal{C}^\circ_t + \mathcal{F}^\circ_t + \mathcal{G}^\circ_t + \mathcal{H}^\circ_t$) are higher than expected, or if the credit quality of some clients worsens (and the additional risk is not covered by additional collateral posted by those clients). Conversely, it can also happen that at some point the reserve capital is higher than necessary, and some of it flows into SHC, as a gain for shareholders.

Regarding the loss of the clean desks, we have already seen that, before default, the difference $\mathrm{CM}^c - \mathrm{CM}^c_0 = \mathrm{P}^c - \mathrm{P}^c_0$, for each netting set c, is the additional rehypothecable collateral posted by the CA desk to the clean desks reflecting the fluctuations in the clean valuation of the promised

[26]Recall that we assumed run-off view; see Assumption 3.1.

cash-flows; one part of it may be posted by the client as VM and the rest is provided by the CA desk using risky funding.[27] The remainder terms composing $(\mathrm{L}^{cl})^{\circ}$ are the losses due to derivatives trading with clients, (possibly only partially) offset by the respective hedges,

$$-\sum_{c}(\mathcal{P}^{c}-\mathcal{H}^{c})^{\tau_{c}^{\delta}\wedge(\tau-)},$$

which flows out of (or "into", in case of a negative loss) the shareholders' capital.

Therefore, we have the following assumption.

Assumption 3.30. Before τ, the following equality between stochastic processes holds:

$$\mathrm{SHC}=\mathrm{SHC}_{0}+\mathcal{D}, \tag{3.22}$$

where the process

$$\mathcal{D}:=-(\mathrm{L}+\mathrm{RM}-\mathrm{RM}_{0}) \tag{3.23}$$

denotes the cumulative *dividends* to the shareholders.

As the processes SHC, RM, and L are all assumed to be without jump at τ, the equality holds on $[\![0,\bar{\tau}]\!]$. The difference $\mathrm{RM}_0-\mathrm{RM}$ above corresponds to the payments received from the KVA desk (see Subsection 3.2.3) that remunerate the shareholders for their capital at risk SCR. Note that negative dividends, which correspond to a decreasing process $\mathcal{D}$ (on a certain interval), are possible in this setup.

Lemma 3.31. *The following equality between stochastic processes holds on* $[0,\bar{\tau}]$:

$$\mathrm{CET1}=\mathrm{CET1}_{0}-\mathrm{L}. \tag{3.24}$$

Proof. By (3.22) and (3.4), we obtain

$$\mathrm{CET1}-\mathrm{CET1}_{0}=\mathrm{SHC}-\mathrm{SHC}_{0}+\mathrm{RM}-\mathrm{RM}_{0}=\mathcal{D}+\mathrm{RM}-\mathrm{RM}_{0}=-\mathrm{L}.$$

□

How exactly the gains and losses are distributed in the different accounts RM, SCR, and UC constituting CET1, will be more clear once we define the capital at risk and the economic capital in Section 4.3.

[27] In practice, the capital of shareholders may also be used for funding the collateral; see [CSS20]. We assume this is not the case in this thesis.

4 The XVA Equations

In this chapter we state the XVAs equations resulting from the financial setup exposed in Chapter 3. For now, we simply assume the well-posedness of these equations under the assumptions we describe in Section 4.1 and limit ourselves to study their relations and interpretations (Sections 4.2 and 4.3). The discussion on the well-posedness is postponed to the next chapter. Finally, in Section 4.4, we describe the incremental XVA approach and justify the run-off view of Assumption 3.1.

4.1 Invariance Valuation Setup

As anticipated in Section 3.2.2, the CA desk and clean desks use different filtration in their valuations; the former takes into account the default of the bank τ, which is a $\mathbb{G}$-stopping time, the latter does not and can simply use the "clean" filtration $\mathbb{F}$.

Assumption 4.1. All the processes describing cumulative cash-flows introduced in the previous chapter are càdlàg $\mathbb{Q}$-integrable $(\mathbb{G},\mathbb{Q})$-semimartingales on $[0,T]$. Additionally, the cash-flows related to the activity of the clean desks ($\mathcal{P}^c$ and $\mathcal{H}^c$) are also $\mathbb{F}$-adapted, and the cash-flows related to the activity of the CA desk ($\mathcal{C}$, $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$) are stopped at τ. Lastly, the client default times τ_c and liquidation times τ_c^δ are assumed to be $\mathbb{F}$-stopping times, for all netting sets c.

Remark 4.2. The rationale for the last assumption that the default times of the clients are $\mathbb{F}$-stopping times, is given by item *i*) of Lemma B.9, which states that for any τ_c (assuming it is a $\mathbb{G}$-stopping time, which is reasonable) there always exists an $\mathbb{F}$-stopping time τ_c' such that

$$\tau_c' \wedge \tau = \tau_c \wedge \tau.$$

Since we are only interested in the cash-flows up to time τ, passing to the reduction τ_c' is not a loss of generality. Regarding τ_c^δ, we can generally assume a measurable dependence from τ_c, e.g., $\tau_c^\delta = \tau_c + \delta$, for a fixed short period of time δ, such as two weeks.

Lemma 4.3. *An $\mathbb{F}$-adapted $(\mathbb{G},\mathbb{Q})$-semimartingale on $[0,T]$ is an $(\mathbb{F},\mathbb{P})$-semimartingale on $[0,T]$.*

Proof. Let $X = M + A$ be an $\mathbb{F}$-adapted $(\mathbb{G},\mathbb{Q})$-semimartingale on $[0,T]$, where M is the local martingale part and A the finite variation part. M is a $(\mathbb{G},\mathbb{Q})$-local martingale with no jump at τ, since M is $\mathbb{F}$-optional and it can only have jumps at $\mathbb{F}$-stopping times (see Proposition A.9). So, by item *iii*) of Lemma B.9, we know that $S_{-} \boldsymbol{\cdot} M' + [S, M']$, where $M' = M$ is the $\mathbb{F}$-optional reduction of M and S is the Azéma supermartingale, is an $(\mathbb{F},\mathbb{Q})$-local martingale on $[0,T]$. By Theorem B.16, this is equivalent to say that M is an $(\mathbb{F},\mathbb{P})$-local martingale on $[0,T]$. Since $\mathbb{P}$ is equivalent to $\mathbb{Q}$ on $\mathscr{F}_T$, A is also a finite variation process with respect to $\mathbb{P}$. So, $X = M + A$ is an $(\mathbb{F},\mathbb{P})$-semimartingale on $[0,T]$. □

So, the processes $\mathcal{P}^c$ and $\mathcal{H}^c$, for all netting sets c, are $(\mathbb{F},\mathbb{P})$-semimartingales on $[0,T]$, whereas the processes $\mathcal{C}$, $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ are $(\mathbb{G},\mathbb{Q})$-semimartingales on $[0,\bar{\tau}]$. In order to fully benefit from the invariance times theory of Appendix B, and, in particular, form Corollary 2.2, which gives us a one-to-one correspondence between $(\mathbb{G},\mathbb{Q})$-martingales in $\mathfrak{S}_2^\circ$ (so, with no jump at τ) and $(\mathbb{F},\mathbb{P})$-martingales on $[0,T]$ in $\mathfrak{S}_2'$, we make the following additional assumption.

Assumption 4.4. The following holds:

$$\mathcal{P}^c, \mathcal{H}^c \in \mathbb{S}^2(\mathbb{F}), \quad \text{for any netting set } c, \tag{4.1}$$
$$\mathcal{C}, \mathcal{F}, \mathcal{G}, \mathcal{H} \in \mathfrak{S}_2. \tag{4.2}$$

Remark 4.5. The rationale for the previous assumption is twofold: as mentioned, we want to use Corollary 2.2 to pass from $(\mathbb{F}, \mathbb{P})$-martingales on $[0, T]$ to $(\mathbb{G}, \mathbb{Q})$-martingales with no jump at τ, by stopping them before τ, and from $(\mathbb{G}, \mathbb{Q})$-martingales without jump at τ to $(\mathbb{F}, \mathbb{P})$-martingales on $[0, T]$, by $\mathbb{F}$-optional reduction; the second (and most important) reason to work in the spaces $\mathfrak{S}_2$ and $\mathbb{S}^2(\mathbb{F})$ is to be able to solve the XVA BSDEs using the theory of Appendix A.2.

As expressed in [AC18], in practice, the processes $\mathcal{H}^c$, $\mathcal{H}$, $\mathcal{F}$, and $\mathcal{G}$ are given by stochastic integrals of predictable ratios against "wealth processes of buy-and-hold strategies" in hedging and funding assets, respectively. For $\mathcal{F}$ and $\mathcal{G}$, this is made explicit in Section 6.1. These wealth processes are assumed to be local martingales with respect to $(\mathbb{G}, \mathbb{Q})$ or $(\mathbb{F}, \mathbb{P})$, depending if they are related to the default of the bank or not. Assuming that the predictable ratios mentioned above are integrable, the resulting stochastic integrals are local martingales as well. Thus, we have the following:

Assumption 4.6. The processes $\mathcal{H}^c$ are $(\mathbb{F}, \mathbb{P})$-local martingales on $[0, T]$, for all netting sets c. The process $\mathcal{H}$ is a $(\mathbb{G}, \mathbb{Q})$-local martingale on $[0, \bar{\tau}]$ with no jump at τ. The processes $\mathcal{F}$ and $\mathcal{G}$ are $(\mathbb{G}, \mathbb{Q})$-local martingales with finite variation on $[0, \bar{\tau}]$ with non-decreasing components given by $\mathcal{F}^\circ$ and $\mathcal{G}^\circ$, respectively.

Combining this with Assumption 4.4, each of the local martingales above is actually a true martingale bounded in L^2 (thus, uniformly integrable) with respect to $(\mathbb{F}, \mathbb{P})$ or $(\mathbb{G}, \mathbb{Q})$, as appropriate.[28]

The fact that $\mathcal{H}$ has no jump at τ reflects our Assumption 3.3 that it is not possible to hedge against the default of the bank itself. For the same reason, we assume that $\mathcal{F}^\circ$ and $\mathcal{G}^\circ$ are non-decreasing, that is, the bank can only borrow (not invest) at its own credit spread over risk-free rate. In fact, if the bank invests at its credit spread, at default time τ this spread would be infinity and cannot be payed back; so, the bank faces a loss (shortfall) at τ. This would be hedging the bank's jump-to-default exposure, which contradicts Assumption 3.3. For this reason, we can only have windfalls at τ, that is, $\mathcal{F}^\bullet \geq 0$ and $\mathcal{G}^\bullet \geq 0$.

Remark 4.7. By the martingale assumption of $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ we immediately see that their $(\mathbb{G}, \mathbb{Q})$ value process is zero.

Proposition 4.8. *The loss process of the clean desks* L^{cl} *is an* $(\mathbb{F}, \mathbb{P})$*-martingale on* $[0, T]$ *and it is bounded in* L^2 *with respect to* $\mathbb{P}$*, that is* $\mathrm{L}^{cl} \in \mathbb{S}^2(\mathbb{F}) \cap \mathcal{M}^T(\mathbb{F}, \mathbb{P})$.

Proof. By (3.19), L^{cl} is a finite sum of processes of the form $-(\mathrm{P}^c - \mathrm{P}_0^c + \mathcal{P}^c - \mathcal{H}^c)^{\tau_c^\delta}$, where $\mathcal{H}^c$ is an $(\mathbb{F}, \mathbb{P})$-martingale on $[0, T]$ by Assumption 4.6, $\mathrm{P}^c + \mathcal{P}^c$ is an $(\mathbb{F}, \mathbb{P})$-martingale on $[0, T]$ by the observation after Definition 2.5, P_0^c is a finite constant, and τ_c^δ is an $\mathbb{F}$-stopping time. The processes $\mathcal{H}^c$ and $\mathcal{P}^c$ are bounded in L^2 with respect to $\mathbb{P}$ by Assumption 4.6. The fact that the martingale $\mathrm{P}^c + \mathcal{P}^c = (\mathbb{E}'_t[\mathcal{P}_T^c])_{t \geq 0}$ is also bounded in L^2 follows by Doob's inequality:

$$\mathbb{E}'\left[\sup_{t \in [0,T]} \mathbb{E}'_t[\mathcal{P}_T^c]^2\right] \leq 4\mathbb{E}'\left[(\mathcal{P}_T^c)^2\right] \leq 4\mathbb{E}'\left[\sup_{t \in [0,T]} (\mathcal{P}_t^c)^2\right] < \infty. \tag{4.3}$$

This completes the proof. $\square$

[28]See Proposition A.21.

4.2 Contra-assets and contra-liabilities valuation

We now consider the equations of the contra-assets add-ons CVA, FVA, and MVA, as well as the relative contra-liabilities DVA, FDA, and MDA. In the previous section we have assumed a martingale condition on the cash-flows processes $\mathcal{H}$, $\mathcal{F}$, and $\mathcal{G}$ resulting from the activities of the CA desk. The last process associated to the activities of the CA desk still to consider is the counterparty exposure cash-flows process $\mathcal{C}$, which does not need to be a martingale. We have already assumed that $\mathcal{C}$ is a $(\mathbb{G},\mathbb{Q})$-semimartingale on $[0,\bar{\tau}]$ (see Assumption 4.1). Additionally, as we anticipated in Remark 3.11 and by looking at (3.12) we know that $\mathcal{C}$ is a process of finite variation, and $\mathcal{C}^\circ$ is non-decreasing (but not necessarily *the* non-decreasing component of $\mathcal{C}$, as we can see by (3.14)).

For now we assume that the given fixed-point problem of the next definition are well-posed in $\mathfrak{S}_2^\circ$;[29] see Section 5.3 for concrete well-posedness result.

Definition 4.9. We define the processes CA, CVA, FVA, MVA $\in \mathfrak{S}_2^\circ$ as the solutions to the following equations: for $t \in [0,\bar{\tau}]$,

$$\mathrm{CA}_t = \mathbb{E}_t\big[\mathcal{C}^\circ_{\bar{\tau}} + \mathcal{F}^\circ_{\bar{\tau}} + \mathcal{G}^\circ_{\bar{\tau}} - \mathcal{C}^\circ_t - \mathcal{F}^\circ_t - \mathcal{G}^\circ_t + \mathrm{CA}_\tau \mathbb{1}_{\{\tau<T\}}\big], \tag{4.4}$$

$$\mathrm{CVA}_t = \mathbb{E}_t\big[\mathcal{C}^\circ_{\bar{\tau}} - \mathcal{C}^\circ_t + \mathrm{CVA}_\tau \mathbb{1}_{\{\tau<T\}}\big], \tag{4.5}$$

$$\mathrm{FVA}_t = \mathbb{E}_t\big[\mathcal{F}^\circ_{\bar{\tau}} - \mathcal{F}^\circ_t + \mathrm{FVA}_\tau \mathbb{1}_{\{\tau<T\}}\big], \tag{4.6}$$

$$\mathrm{MVA}_t = \mathbb{E}_t\big[\mathcal{G}^\circ_{\bar{\tau}} - \mathcal{G}^\circ_t + \mathrm{MVA}_\tau \mathbb{1}_{\{\tau<T\}}\big]. \tag{4.7}$$

Let $\mathrm{CVA}^{\mathrm{CL}}$, $\mathrm{FVA}^{\mathrm{CL}}$, and $\mathrm{MVA}^{\mathrm{CL}}$ be defined by

$$\mathrm{CVA}^{\mathrm{CL}}_t = \mathbb{E}_t\big[\mathrm{CVA}_\tau \mathbb{1}_{\{\tau<T\}}\big], \tag{4.8}$$

$$\mathrm{FVA}^{\mathrm{CL}}_t = \mathbb{E}_t\big[\mathrm{FVA}_\tau \mathbb{1}_{\{\tau<T\}}\big], \tag{4.9}$$

$$\mathrm{MVA}^{\mathrm{CL}}_t = \mathbb{E}_t\big[\mathrm{MVA}_\tau \mathbb{1}_{\{\tau<T\}}\big], \quad t \in [0,\bar{\tau}], \tag{4.10}$$

which are the respective $(\mathbb{G},\mathbb{Q})$-valuation of the "bullet cash-flows" $\mathrm{CVA}_\tau \mathbb{1}_{\{\tau<T\}}$, $\mathrm{FVA}_\tau \mathbb{1}_{\{\tau<T\}}$, and $\mathrm{MVA}_\tau \mathbb{1}_{\{\tau<T\}}$ at time $\bar{\tau}$. Then, we define the processes DVA, FDA, and MDA as follows: for $t \in [0,\bar{\tau}]$,

$$\mathrm{DVA}_t = \mathbb{E}_t[\mathcal{C}^\bullet_{\bar{\tau}} - \mathcal{C}^\bullet_t] + \mathrm{CVA}^{\mathrm{CL}}_t, \tag{4.11}$$

$$\mathrm{FDA}_t = \mathbb{E}_t[\mathcal{F}^\bullet_{\bar{\tau}} - \mathcal{F}^\bullet_t] + \mathrm{FVA}^{\mathrm{CL}}_t, \tag{4.12}$$

$$\mathrm{MDA}_t := \mathbb{E}_t[\mathcal{G}^\bullet_{\bar{\tau}} - \mathcal{G}^\bullet_t] + \mathrm{MVA}^{\mathrm{CL}}_t. \tag{4.13}$$

Moreover, let CL := DVA + FDA + MDA denote the *contra-liabilities* process. Last, we call *fair valuation of counterparty risk*, denoted FV, the $(\mathbb{G},\mathbb{Q})$-valuation of the sum $\mathcal{C}+\mathcal{F}+\mathcal{G}+\mathcal{H}$.

Remark 4.10. Note that all the equations and fixed-point-problems of the previous definition, as well as for the rest of the thesis, are intended to hold $\mathbb{Q}$-.a.s. (or $\mathbb{P}$-a.s., if applicable), in the sense that there exists a measurable set $A \subseteq \Omega$ of probability one such that, for all $\omega \in A$, the equality holds at all times t (in $[0,\bar{\tau}(\omega)]$, or $[0,T]$, or as indicated).

Lemma 4.11. *The following equalities between stochastic processes hold:*

$$\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA} + \mathrm{MVA}, \tag{4.14}$$

$$\mathrm{FDA} = \mathrm{FVA}, \tag{4.15}$$

$$\mathrm{MDA} = \mathrm{MVA}, \quad \textit{and} \tag{4.16}$$

$$\mathrm{FV} = \mathrm{CA} - \mathrm{CL} = \mathrm{CVA} - \mathrm{DVA}. \tag{4.17}$$

[29] Meaning that they exist and are unique.

Furthermore, FV *is equal to the* $(\mathbb{G},\mathbb{Q})$*-valuation of* $\mathcal{C}$*.*

Proof. By the assumption of uniqueness of the fixed-point problems above, if we insert the process $X := \mathrm{CVA} + \mathrm{FVA} + \mathrm{MVA}$ in (4.4), we see that $X = \mathrm{CA}$. This yields (4.14). To see (4.15), note that for $t \in [0, \bar{\tau}]$ we have

$$\begin{aligned}\mathrm{FDA}_t &= \mathbb{E}_t\big[\mathcal{F}^\bullet_{\bar{\tau}} - \mathcal{F}^\bullet_t + \mathrm{FVA}_\tau \mathbb{1}_{\{\tau<T\}}\big]\\ &= \mathbb{E}_t[\mathcal{F}^\bullet_{\bar{\tau}} - \mathcal{F}^\bullet_t - (\mathcal{F}^\circ_{\bar{\tau}} - \mathcal{F}^\circ_t)] + \underbrace{\mathbb{E}_t\big[\mathcal{F}^\circ_{\bar{\tau}} - \mathcal{F}^\circ_t + \mathrm{FVA}_\tau \mathbb{1}_{\{\tau<T\}}\big]}_{=\mathrm{FVA}_t}\\ &= \mathrm{FVA}_t - \underbrace{\mathbb{E}_t[\mathcal{F}_{\bar{\tau}} - \mathcal{F}_t]}_{=0} = \mathrm{FVA}_t,\end{aligned}$$

since $\mathcal{F} = \mathcal{F}^\circ - \mathcal{F}^\bullet$ and $\mathcal{F}$ is a $(\mathbb{G},\mathbb{Q})$-martingale by Assumption 4.6. The exact same argument with $\mathcal{G}$ in place of $\mathcal{F}$ yields (4.16). By linearity of the conditional expectation and Remark 4.7, it is clear that FV is the $(\mathbb{G},\mathbb{Q})$-valuation of $\mathcal{C}$. Using the same idea as above, we write $\mathcal{C} = \mathcal{C}^\circ - \mathcal{C}^\bullet$ and obtain for all $t \in [0, \bar{\tau}]$:

$$\begin{aligned}\mathrm{FV}_t &= \mathbb{E}_t[\mathcal{C}^\circ_{\bar{\tau}} - \mathcal{C}^\circ_t - (\mathcal{C}^\bullet_{\bar{\tau}} - \mathcal{C}^\bullet_t)] + \mathrm{CVA}^{\mathrm{CL}}_t - \mathrm{CVA}^{\mathrm{CL}}_t\\ &= \mathbb{E}_t\big[\mathcal{C}^\circ_{\bar{\tau}} - \mathcal{C}^\circ_t + \mathrm{CVA}_\tau \mathbb{1}_{\{\tau<T\}}\big] - \Big(\mathbb{E}_t[\mathcal{C}^\bullet_{\bar{\tau}} - \mathcal{C}^\bullet_t] + \mathrm{CVA}^{\mathrm{CL}}_t\Big)\\ &= \mathrm{CVA} - \mathrm{DVA}\\ &= \mathrm{CVA} + \mathrm{FVA} + \mathrm{MVA} - (\mathrm{DVA} + \mathrm{FDA} + \mathrm{MDA}) = \mathrm{CA} - \mathrm{CL}.\end{aligned}$$

This gives (4.17) and concludes the proof. □

Remark 4.12. Note that the processes CA and CL are the respective $(\mathbb{G},\mathbb{Q})$-valuation of the contra-assets $\mathcal{S}$ and contra-liabilities $\mathcal{B}$ of Definition 3.23. In fact, for any $t \in [0, \bar{\tau}]$

$$\begin{aligned}\mathbb{E}_t[\mathcal{S}_{\bar{\tau}} - \mathcal{S}_t] &= \mathbb{E}_t\big[\mathcal{C}^\circ_{\bar{\tau}} + \mathcal{F}^\circ_{\bar{\tau}} + \mathcal{G}^\circ_{\bar{\tau}} + \mathrm{CA}_\tau \mathbb{1}_{\{\tau<T\}} - \mathcal{C}^\circ_t - \mathcal{F}^\circ_t - \mathcal{G}^\circ_t - \mathrm{CA}_t \mathbb{1}_{\{t=\tau<T\}}\big] + \mathbb{E}_t[\mathcal{H}^\circ_{\bar{\tau}} - \mathcal{H}^\circ_t]\\ &= \mathbb{E}_t\big[\mathcal{C}^\circ_{\bar{\tau}} + \mathcal{F}^\circ_{\bar{\tau}} + \mathcal{G}^\circ_{\bar{\tau}} - \mathcal{C}^\circ_t - \mathcal{F}^\circ_t - \mathcal{G}^\circ_t + \mathrm{CA}_\tau \mathbb{1}_{\{\tau<T\}}\big] \overset{(4.4)}{=} \mathrm{CA}_t,\end{aligned}$$

using that $\mathrm{RC} = \mathrm{CA}$, $\mathcal{H} = \mathcal{H}^\circ$ is a martingale, and $\mathbb{Q}[\tau = t] = 0$ since τ is totally inaccessible. For the contra-liabilities we have, similarly,

$$\begin{aligned}\mathbb{E}_t[\mathcal{B}_{\bar{\tau}} - \mathcal{B}_t] &= \mathbb{E}_t\big[\mathcal{C}^\bullet_{\bar{\tau}} + \mathcal{F}^\bullet_{\bar{\tau}} + \mathcal{G}^\bullet_{\bar{\tau}} + \mathrm{CA}_\tau \mathbb{1}_{\{\tau<T\}} - \mathcal{C}^\bullet_t - \mathcal{F}^\bullet_t - \mathcal{G}^\bullet_t - \mathrm{CA}_t \mathbb{1}_{\{t=\tau<T\}}\big]\\ &= \mathbb{E}_t[\mathcal{C}^\bullet_{\bar{\tau}} - \mathcal{C}^\bullet_t] + \mathbb{E}_t\big[\mathrm{CVA}_\tau \mathbb{1}_{\{\tau<T\}}\big] + \mathbb{E}_t[\mathcal{F}^\bullet_{\bar{\tau}} - \mathcal{F}^\bullet_t] + \mathbb{E}_t\big[\mathrm{FVA}_\tau \mathbb{1}_{\{\tau<T\}}\big]\\ &\quad + \mathbb{E}_t[\mathcal{G}^\bullet_{\bar{\tau}} - \mathcal{G}^\bullet_t] + \mathbb{E}_t\big[\mathrm{MVA}_\tau \mathbb{1}_{\{\tau<T\}}\big]\\ &= \mathrm{DVA}_t + \mathrm{FDA}_t + \mathrm{MDA}_t = \mathrm{CL}_t,\end{aligned}$$

using that $\mathrm{CA} = \mathrm{CVA} + \mathrm{FVA} + \mathrm{MVA}$ and Definition 4.9.

Proposition 4.13. *The trading loss process of the CA desk* L^{ca} *is a* $(\mathbb{G},\mathbb{Q})$*-martingale on* $[0, \bar{\tau}]$ *without jump at* τ*, and it is bounded in* L^2 *with respect to* $\mathbb{Q}$*, that is* $\mathrm{L}^{ca} \in \mathfrak{S}_2 \cap \mathcal{M}^{\tau-\wedge T}(\mathbb{G},\mathbb{Q})$*. Therefore, the trading loss process* L *is also a* $(\mathbb{G},\mathbb{Q})$*-martingale on* $[0, \bar{\tau}]$ *without jump at* τ *bounded in* L^2 *with respect to* $\mathbb{Q}$*.*

Proof. The process $\mathcal{H} = \mathcal{H}^\circ$ is a $(\mathbb{G},\mathbb{Q})$-martingale with no jump at τ by Assumption 4.6. The process given by

$$\mathrm{CA}_t + \mathcal{C}^\circ_t + \mathcal{F}^\circ_t + \mathcal{G}^\circ_t \overset{(4.4)}{=} \mathbb{E}_t\big[\mathcal{C}^\circ_{\bar{\tau}} + \mathcal{F}^\circ_{\bar{\tau}} + \mathcal{G}^\circ_{\bar{\tau}} + \mathrm{CA}_\tau \mathbb{1}_{\{\tau<T\}}\big], \quad t \in [0, \bar{\tau}],$$

is a $(\mathbb{G},\mathbb{Q})$-martingale on $[0,\bar{\tau}]$, since the random variable $\mathcal{C}^\circ_{\bar{\tau}} + \mathcal{F}^\circ_{\bar{\tau}} + \mathcal{G}^\circ_{\bar{\tau}} + \mathrm{CA}_\tau \mathbb{1}_{\{\tau<T\}}$ is a $\mathbb{Q}$-integrable random variable (since everything is bounded in L^2 by our assumptions). As CA_0 is constant, we see from (3.20) that L^{ca} is actually a $(\mathbb{G},\mathbb{Q})$-martingale on $[0,\bar{\tau}]$. Since $\mathrm{CA} \in \mathfrak{S}^\circ_2$, CA is without jump at τ; all the other summands in (3.20) are also stopped before τ, so L^{ca} is without jump at τ. As all the summands in L^{ca} are in $\mathfrak{S}_2$, so is L^{ca}. By Definition 3.29, $\mathrm{L} = \mathrm{L}^{ca} + (\mathrm{L}^{cl})^\circ$, where $(\mathrm{L}^{cl})^\circ$ a $(\mathbb{G},\mathbb{Q})$-martingale in $\mathfrak{S}_2$ by Proposition 4.8 and Corollary 2.2. □

Remark 4.14. We observe that the contra-assets CVA, FVA, and MVA also take into account for the impact of the default of the bank itself, besides that of the clients. Indeed, the residual amount $\mathrm{CA}_\tau \mathbb{1}_{\{\tau<T\}} = \mathrm{RC}_\tau \mathbb{1}_{\{\tau<T\}}$ on the reserve capital account flows from shareholders to bondholders at τ, as per Assumption 3.21. However, as we see in Section 6.2, the resulting XVAs are unilateral in a certain sense. The contra-liabilities DVA, FDA, and MDA accounts for the windfalls $\mathcal{C}^\bullet$, $\mathcal{F}^\bullet$, and $\mathcal{G}^\bullet$ at bank's default, which depends on the agreements of the bank with the clients and the external funder (see Section 6.1 for a more concrete example); they also include the contra-liability components $\mathrm{CVA}^{\mathrm{CL}}$, $\mathrm{FVA}^{\mathrm{CL}}$, and $\mathrm{MVA}^{\mathrm{CL}}$. The latter quantities can be seen as a gain for the bank (particularly for the bondholders) coming from the residual amount on the reserve capital

$$\mathrm{CVA}_\tau \mathbb{1}_{\{\tau<T\}} + \mathrm{FVA}_\tau \mathbb{1}_{\{\tau<T\}} + \mathrm{MVA}_\tau \mathbb{1}_{\{\tau<T\}} = \mathrm{RC}_\tau \mathbb{1}_{\{\tau<T\}},$$

which is not payed to the clients and external funder after the default of the bank.

Remark 4.15. The reason for the name "fair valuation of counterparty risk" of the process FV given in [AC18, Section 5.3] is that, under the assumption that the bank would be able to hedge against its own default by selling the contra-liabilities cash-flow $\mathcal{B} = \mathcal{C}^\bullet + \mathcal{F}^\bullet + \mathcal{G}^\bullet + \mathrm{CA}_\tau \mathbb{1}_{[\![\tau]\!]}$ (in the sense that the bank would pay the amount $\mathcal{C}^\bullet_\tau + \mathcal{F}^\bullet_\tau + \mathcal{G}^\bullet_\tau + \mathrm{CA}_\tau$ at τ to a third party), the bank would receive the $(\mathbb{G},\mathbb{Q})$ valuation CL of this cash-flow and use it to (partially) cover the needed amount CA. In such a case, the only amount required by the CA desk to reach RC = CA would be FV = CA − CL. Observe that this amount would be initially provided by the clients, with the payment FV_0 at time $t = 0$, and then by the shareholders (using their capital), covering the difference $\mathrm{FV}_t - \mathrm{FV}_0$ at each $t > 0$ (in a similar way as described after Definition 3.29). Moreover, note that in this situation the bondholders do not receive the contra-liabilities cash-flow $\mathcal{B}$ at default. This would significantly reduce their recovery at default and, by consequence, it is against the "pari passu" rules protecting bondholders.

Since by Assumption 3.3 the bank cannot hedge its own default exposure, in reality we have that the clients and shareholders pay together the whole amount $\mathrm{CA}_0 + (\mathrm{CA} - \mathrm{CA}_0) = \mathrm{CA}$ to the CA desk. This is the (minimal) amount needed to cover the expected future counterparty default and risky funding costs, even though this also includes the additional amount CL, which can be seen as the portion of CA (always in conditionally expected terms) that will not be used to actually cover counterparty default and funding costs, due to the default of the bank. So this is a "gift" to the bondholders (from clients and shareholders); however, as explained in [AC18, Section 5.3], it is "not necessarily a positive arbitrage," since bondholders have to cover the bankruptcy costs after the default of the bank.

Requiring this additional component as value adjustment form the clients is what makes the resulting XVAs unilateral, as opposed to bilateral. See the relative discussion in Section 6.2.

4.3 Capital at risk and cost of capital

As discussed in Section 3.1.1, for the bank it is not only problematic to hedge against its own default exposure, but also against that of the clients. So, by regulatory requirement, the bank has to to protect itself—and, by consequence, protecting other clients making the default of the bank less likely—by setting at risk an amount called *capital at risk* (CR). As the expected losses due to counterparty default and risky funding are already covered by the reserve capital RC = CA, the CR is used for the losses beyond expectation. The CR is formed by the amounts SCR and RM:

$$\mathrm{CR} = \mathrm{RM} + \mathrm{SCR}. \tag{4.18}$$

Recall that the RM=KVA is provided by the clients at inception via KVA_0 with the shareholders covering the difference $\mathrm{KVA} - \mathrm{KVA}_0$ afterwords. Hence, we assume (as in [AC18]) that the RM is "loss-absorbing", and hence it is part of CR. Indeed, in Subsection 3.2.3, we declared that the function of RM is to compensate the shareholders for their capital at risk SCR at a certain given hurdle rate h. But this is a decision of the bank's management;[30] a failed RM payment to shareholders is not a materiality for default (while failed counterparty default and risky funding payments are), so RM is not an actual liability for the bank.

The actual minimum amount of capital at risk needed by regulation is the *economic capital* (EC), which we assume[31] to be the 97.5% expected shortfall of the negative of the variation over a one-year period of CET1.[32] By (3.24), we know that the variation of CET1 is $-\mathrm{L}$. Typically, this capital calculation are made under the assumption that the bank cannot default, that is "on a going concern". This means, in our setup, that the $(\mathbb{F}, \mathbb{P})$ valuation is used:

Definition 4.16. The stochastic process EC, denoting the *economic capital*, is defined at each time $t \in [0,T]$ as the $(\mathscr{F}_t, \mathbb{P})$ conditional 97.5% expected shortfall of $\left(\mathrm{L}'_{(t+\Delta)\wedge T} - \mathrm{L}'_t\right)$, where L' is the $\mathbb{F}$-optional reduction of L and Δ denotes a period of one year.

Lemma 4.17. EC *is non-negative.*

Proof. By Corollary 2.2, the $\mathbb{F}$-optional reduction L' of the loss process L is an $(\mathbb{F}, \mathbb{P})$-martingale on $[0,T]$, and thus $\mathbb{E}'_t[\mathrm{L}'_{(t+\Delta)\wedge T} - \mathrm{L}'_t] = 0$, for all $t \in [0,T]$. Then, the result follows from Lemma C.4. □

Thus, once we have determined CA by (4.4), we can obtain the processes L and L', we calculate EC as above, and then we require that $\mathrm{CR} \geq \mathrm{EC}$. Additionally, we said that CR includes the risk margin RM, which by Assumption 3.18 is mark-to-model to KVA. In the following we turn our focus to this last valuation adjustment. First, let

$$\mathrm{CR} = \max(\mathrm{EC}, \mathrm{KVA}). \tag{4.19}$$

Note that, once KVA is computed, we can determine the mark-to-model of SCR as per Assumption 3.18 as

$$\mathrm{SCR} = \mathrm{CR} - \mathrm{RM} = (\mathrm{EC} - \mathrm{KVA})^+. \tag{4.20}$$

Next, to establish the KVA equation, we need to know the target hurdle rate h.

[30] In practice, where there are many banks competing with each other, the level of compensation to shareholders for their capital at risk is strongly influenced by the market.
[31] As in [AC18, Section 5.4].
[32] See Appendix C for a summary on the expected shortfall.

Assumption 4.18. The hurdle rate $h \geq 0$ is constant and exogenously give.

Goal: We want that the payments from the KVA desk to the shareholders remunerating them for their capital at risk are as follows: at any $t \in [0, \bar{\tau}]$,

$$\text{“shareholders' instantaneous average return”} = h\mathrm{SCR}_t. \tag{4.21}$$

For the cost-of-capital computation, we assume again that the bank is default prone and we use the $(\mathbb{G}, \mathbb{Q})$ valuation. The shareholders return is given by the dividends cash-flow $\mathcal{D} = -(\mathrm{L} + \mathrm{KVA} - \mathrm{KVA}_0)$ defined in Assumption 3.30. Given our general assumptions, it is reasonable to assume that KVA is a $(\mathbb{G}, \mathbb{Q})$-semimartingale on $[0, \bar{\tau}]$, that is, there is a $(\mathbb{G}, \mathbb{Q})$-local martingale M and a process of finite variation A, both defined on $[0, \bar{\tau}]$ and started from zero, such that $\mathrm{KVA} = \mathrm{KVA}_0 + M - A$. If we assume $\mathrm{KVA} \in \mathfrak{S}_2^\circ$, then KVA is a special semimartingale bounded in L^2 and we can assume that the decomposition above is the unique canonical decomposition (see Proposition-Definition A.26). Let us assume that the finite variation part A of KVA is given by

$$dA_t = h\mathrm{SCR}_t dt.$$

In such a case we simply say that KVA has the *drift coefficient* hSCR. Then, A is continuous non-decreasing and, noting that

$$\mathrm{SCR} = \mathrm{CR} - \mathrm{KVA} = \max(\mathrm{EC}, \mathrm{KVA}) - \mathrm{KVA} = (\mathrm{EC} - \mathrm{KVA})^+ \leq \mathrm{EC}, \tag{4.22}$$

we have

$$\begin{aligned} \mathbb{E}\left[\sup_{\substack{0 \leq t \leq T \\ t < \tau}} |A_t|^2\right] &= \mathbb{E}\left[|A_{\bar{\tau}}|^2\right] = \mathbb{E}\left[\left(\int_0^{\bar{\tau}} h\mathrm{SCR}_t dt\right)^2\right] \\ &\leq h^2 T^2 \mathbb{E}\left[\int_0^T \mathbb{1}_{\{t<\tau\}} \mathrm{EC}_t^2 dt\right] \leq h^2 T^2 \mathbb{E}\left[\int_0^T e^{\Gamma_t} \mathbb{1}_{\{t<\tau\}} \mathrm{EC}_t^2 dt\right]. \end{aligned}$$

Therefore, if we assume $\mathrm{EC} \in \mathfrak{L}_2$ (recall its definition in (2.7)), then A is bounded in L^2 and so is $M = \mathrm{KVA} - \mathrm{KVA}_0 - A$. Thus, M is a true martingale, and KVA is a supermartingale with drift coefficient A, which means that the “shareholders' instantaneous average return” is equal to $h\mathrm{SCR}_t$, at any $t \in [0, \bar{\tau}]$, as we wanted.

So, to finally write the KVA equation, we need the following assumption:

Assumption 4.19. $\mathrm{EC} \in \mathfrak{L}_2$.

As in Definition 4.9, we assume that the next problem is well-posed in $\mathfrak{S}_2^\circ$ (see Section 5.2 for the well-posedness).

Definition 4.20. We define the process $\mathrm{KVA} \in \mathfrak{S}_2^\circ$ as the $(\mathbb{G}, \mathbb{Q})$-special semimartingale on $[0, \bar{\tau}]$ with canonical decomposition $\mathrm{KVA} = \mathrm{KVA}_0 + M - A$ such that

$$dA_t = h(\mathrm{EC}_t - \mathrm{KVA}_t)^+ dt, \qquad t \in [0, \bar{\tau}]. \tag{4.23}$$

Theorem 4.21. *Under the assumption that the fixed-point problem of Definition 4.20 is well-posed in $\mathfrak{S}_2^\circ$, the shareholders' wealth* $\mathrm{SHC} = \mathrm{SCR} + \mathrm{UC}$ *is a $(\mathbb{G}, \mathbb{Q})$-submartingale on $[0, \bar{\tau}]$ without jump at τ given by*

$$\mathrm{SHC} = \mathrm{SHC}_0 - (\mathrm{L} + \mathrm{KVA} - \mathrm{KVA}_0) = \mathrm{SHC}_0 - \mathrm{L} - M + A, \tag{4.24}$$

where M is a $(\mathbb{G}, \mathbb{Q})$-martingale and $A_t = \int_0^t h\mathrm{SCR}_s\, ds$, $t \in [0, \bar{\tau}]$; that is, SHC *has the drift coefficient* hSCR. *Moreover,* $\mathrm{SHC} \in \mathfrak{S}_2$.

Proof. By (3.22), $\mathrm{SHC} = \mathrm{SHC}_0 - (\mathrm{L} + \mathrm{KVA} - \mathrm{KVA}_0)$, where SHC_0 is a constant and $\mathrm{KVA} - \mathrm{KVA}_0 = M - A$ is the canonical decomposition of the $(\mathbb{G}, \mathbb{Q})$-special semimartingale $\mathrm{KVA} - \mathrm{KVA}_0$ as in Definition 4.20. In the discussion above we have seen that, under Assumption 4.19 and with (4.23), $M \in \mathfrak{S}_2$ is a true martingale on $[0, \bar{\tau}]$ with no jump at τ, and the non-decreasing component A is as wanted. Since $\mathrm{KVA} \in \mathfrak{S}_2^\circ$, KVA has no jump at τ, and, by Proposition 4.13, we conclude that SHC is a $(\mathbb{G}, \mathbb{Q})$-submartingale on $[0, \bar{\tau}]$ with no jump at τ that is in $\mathfrak{S}_2$. □

Remark 4.22. As stressed in [AC18], the main purpose of this concept of KVA is to have a "sustainable dividend release strategy," where sustainable means that, even without new trades (since we assumed a run-off view; see also the incremental approach in the next section), the shareholders can be repaid at the hurdle rate h for their capital at risk.

Remark 4.23. Note that, if $h = 0$, then $\mathrm{KVA} = 0$ and SHC is a martingale. Such a situation could only be accepted by risk-neutral shareholders. So we can see h as a risk-aversion parameter of the shareholders; see [AC18, Appendix B.4] for a discussion in a one-period static setup.

4.4 Incremental XVA approach

Up to now we worked under the run-off assumption (see Assumption 3.1). In this subsection we expose how to introduce new trades in our model, maintaining the equilibrium given by the "shareholders' balance conditions" of the previous subsections, which means that the following equalities hold on $[0, \bar{\tau}]$:

$$\mathrm{CM} = \mathrm{MtM}, \quad \mathrm{RC} = \mathrm{CA}, \quad \mathrm{RM} = \mathrm{KVA}, \tag{4.25}$$

where MtM, CA, and KVA are given respectively by (3.3), (4.4), and (4.23). Of course, a new trade at a certain time $\theta > 0$ may alterate the values of MtM, CA, and KVA, and thus the bank would have to charge the new client the right add-ons, to preserve the above equilibrium. This has to be done in such a way that the shareholders, which are in control of the bank before default, are indifferent to the new deal, in the sense that the shareholders' wealth SHC is unchanged (in particular, no losses for the shareholders) and the hurdle rate h is preserved. More precisely:

Goal: The entry price of the new deal should be such that the shareholders' balance condition (4.25) is preserved—with the same, constant hurdle rate $h \geq 0$ of Assumption 4.18—without modification of SHC.

Remark 4.24. The advantage of the run-off view in the computation of the MtM, CA, and KVA is that they can be interpreted as the target amounts to be maintained on the clean margin, reserve capital, and risk margin accounts (see Assumption 3.18), so that the bank, if desired, could go into run-off at any time (not only at time 0) without giving up the interest payments to the shareholders at the hurdle rate h. This possibility is referred to in [AC18] as the "soft landing option". In other words, this means that the bank does not necessarily need new trades to sustain the already existing business (and, in particular, the shareholders' interest payments), like in a "Ponzi scheme".

Thus, we now introduce a scheme that describes how to manage the new deal, in order to achieve the above goal, but then, between one new deal and the other, we always work under the run-off assumption, in the sense that the values of the processes (MtM, XVA, and so on) after the new deal are computed using the same rules as in the previous chapters. This guarantees the soft landing option after each new deal.

Suppose that at a positive time $\theta > 0$ a new deal is made, and denote by $\mathcal{P}$ the cumulative contractually promised cash-flows on $[\theta, T]$. The clean valuation of the new deal at time θ is

$$\mathrm{P}_\theta := \mathbb{E}'_\theta \left[\mathcal{P}_T - \mathcal{P}_\theta\right].$$

Clearly, the new deal is not part of a netting set already involved in a liquidation procedure, in the sense that either a new netting set is created, or the new deal is part of a netting set c with $\theta < \tau_c$. For a process Y (e.g. $Y = \mathrm{MtM}$ or XVA), we denote by $\Delta_\theta Y$ the difference of the values of Y at time θ computed with and without the new deal. Hence, $\Delta_\theta \mathrm{MtM} = \mathrm{P}_\theta$.

Assumption 4.25. We assume that the following pricing and accounting scheme takes place when a new deal is made at time $\theta > 0$.

- If $\mathrm{P}_\theta > 0$, the clean desks pay P_θ to the client and the CA desk provides a rehypothecable cash collateral of value P_θ (partially posted by the client of the new deal as a VM, the rest being covered through unsecured funding) that flows into the CM account. If $\mathrm{P}_\theta < 0$, the client pays $-\mathrm{P}_\theta$ to the clean desks and an amount of value $-\mathrm{P}_\theta$ is withdrawn from the CM account by the CA desk and posted as collateral to the client.[33]
- If $\Delta_\theta \mathrm{CA} > 0$, the client pays $\Delta_\theta \mathrm{CA}$ to the CA desk that puts this amount in the RC account; if $\Delta_\theta \mathrm{CA} < 0$, an amount of value $-\Delta_\theta \mathrm{CA}$ is withdrawn from the RC account and is passed from the CA desk to the client;
- if $\Delta_\theta \mathrm{KVA} > 0$, the client pays $\Delta_\theta \mathrm{KVA}$ to the KVA desk that puts this amount in the RM account; if $\Delta_\theta \mathrm{KVA} < 0$, an amount of value $-\Delta_\theta \mathrm{KVA}$ is withdrawn from the RM account and is passed from the KVA desk to the client.

At the same time, we assume that the clean desks may set up a hedge, (partially) offsetting $\mathcal{P}$, and any required initial margin is posted by the client or the bank (depending on the sign of P_θ).

Using this scheme, the goal above is achieved:

Proposition 4.26. *Under Assumption 4.25, the shareholders' balance condition* (4.25) *and the hurdle rate h are preserved, and* $\Delta_\theta \mathrm{SHC} = 0$.

Proof. It is easy to see that, following the scheme of Assumption 4.25, we have

$$\Delta_\theta \mathrm{MtM} = \mathrm{P}_\theta = \Delta_\theta \mathrm{CM}, \quad \Delta_\theta \mathrm{CA} = \Delta_\theta \mathrm{RC}, \quad \Delta_\theta \mathrm{KVA} = \Delta_\theta \mathrm{RM}.$$

Hence, the shareholders' balance condition (4.25) is preserved, and, since the KVA with and without the new trade is computed using the same h (see (4.23)), the hurdle rate h is also preserved. As the amounts $\Delta_\theta \mathrm{MtM}$, $\Delta_\theta \mathrm{CA}$ and $\Delta_\theta \mathrm{KVA}$ on the clean margin, reserve capital, and risk margin accounts are provided by[34] the client (and partially by the CA desk through risky funding, in case that the collateral posted by the client is less than $\Delta_\theta \mathrm{MtM} = \mathrm{P}_\theta$), there is no loss for the shareholders, in the sense that $\Delta_\theta \mathrm{SHC} = 0$. □

[33] Possibly, the amount posted as rehypothecable collateral to the client in this case may not be the whole $-P_\theta$ but only a part of it, depending on the collateral agreements. In any case, the amount withdrawn from the CM account stays exactly $-\mathrm{P}_\theta$.

[34] Or "received by", if negative.

Definition 4.27. The all-inclusive XVA add-on that the client of the new deal pays to the bank at time θ is called *funds transfer price* (FTP) and is given by

$$\begin{aligned} \mathrm{FTP} :&= \Delta_\theta \mathrm{CA} + \Delta_\theta \mathrm{KVA} \\ &= \Delta_\theta \mathrm{CVA} + \Delta_\theta \mathrm{FVA} + \Delta_\theta \mathrm{MVA} + \Delta_\theta \mathrm{KVA} = \Delta_\theta \mathrm{FV} + \Delta_\theta \mathrm{CL} + \Delta_\theta \mathrm{KVA}. \end{aligned} \tag{4.26}$$

Remark 4.28. We see from (4.26) that the only XVAs that affect the entry price of the new deal are CVA, FVA, MVA, and KVA. Note also that the current portfolio of the bank before the new deal influences the FTP; it may happen that the new trade reduces the risk of the total portfolio of the bank, resulting in a negative price: $\mathrm{FTP} < 0$.

As anticipated above, between one deal and the other we always work under the run-off assumption. More precisely:

Assumption 4.29. We assume the following *cost-of-capital XVA rolling strategy:*

- between time 0 and the next deal, we assume the run-off view in the computation of the amounts MtM, CA, and KVA, obtaining the shareholders' balance condition (4.25);
- at the time of the next deal (whenever finite), which we assume to be a $\mathbb{G}$-stopping time $\theta > 0$, we proceed as in Assumption 4.25;
- after that, we assume again the run-off view, and iterate the same strategy at every new trade.

So, after each new trade all the processes and amounts in the different banking accounts (in particular, the trading process L and the XVAs) are recalculated relatively to the new portfolio of the bank, which includes the new trade. Observe that we may have more the one new trade at the same time θ. This does not change the rolling strategy; in fact, at each time new deals are made, the portfolio is updated accordingly, taking into account all the new deals. Hence, we can assume without loss of generality that the successive times at which new trades occur form an increasing sequence of $\mathbb{G}$-stopping times $(\theta_i)_{\{i\geq 1\}}$ such that $\theta_i < \theta_{i+1}$ on $\{\theta_i < \infty\}$, for $i \geq 1$.

For the statement of the next theorem, let us extend the processes introduced up to now (in particular, L, KVA, and SHC) from $[0, \bar{\tau}]$ to $[0, \tau] \cap [0, \infty[$ by constancy from time T onward.

Theorem 4.30. *Under the Assumption 4.29 and assuming that the processes* L *and* KVA *relative to each successive portfolio do not jump at the time of the following deal,* SHC *is a* $(\mathbb{G}, \mathbb{Q})$-*submartingale on* $[0, \tau] \cap [0, \infty[$ *with drift coefficient* hSCR.

Remark 4.31. If we assume that the times of the new deals are $\mathbb{G}$-predictable stopping times (and recall that the filtration $\mathbb{G}$ is quasi-left-continuous), then the martingales L and

$$M := \mathrm{KVA} - \int_0^{\cdot} h\mathrm{SCR}_s \, ds$$

(compare with Definition 4.20) do not jump at these times, by Proposition A.10. Since the finite variation part of KVA is continuous, KVA does not jump at predictable times too. The assumption that the new deals happen at predictable times appears intuitively reasonable.

Proof of Theorem 4.30. Let us first focus on the period between time 0 and the first new deal (assumed to be just one, without lost of generality), happening at the $\mathbb{G}$-predictable time $\theta_1 > 0$.

We also assume that $\theta_1 < \tau$,[35] since otherwise nothing would have to be done. For a generic process Y (that can be L, KVA, or SHC), let us denote by $\widetilde{Y}$ the process on $[0, \tau \wedge \theta_1]$ without the new deal, while the notation Y is the "true" process considering all the successive new deals, as per Assumption 4.29. Thus, $Y = \widetilde{Y}$ on $[0, \tau] \cap [0, \theta_1[$, and at time θ_1 it holds $Y_{\theta_1} = \widetilde{Y}_{\theta_1} + \Delta_{\theta_1} Y$ (using the notation $\Delta_\theta \cdot$ introduced before the Assumption 4.25 with $\theta = \theta_1$). By Theorem 4.21, we know that on $[0, \tau] \cap [0, \theta_1[$

$$\mathrm{SHC} = \widetilde{\mathrm{SHC}} = \widetilde{\mathrm{SHC}}_0 - \left(\widetilde{\mathrm{L}} + \widetilde{\mathrm{KVA}} - \widetilde{\mathrm{KVA}}_0\right) = \mathrm{SHC}_0 - (\mathrm{L} + \mathrm{KVA} - \mathrm{KVA}_0)$$

is a $(\mathbb{G}, \mathbb{Q})$-submartingale with drift $h\mathrm{SCR} = h\widetilde{\mathrm{SCR}}$. By our assumption, $\widetilde{\mathrm{L}}$, and $\widetilde{\mathrm{KVA}}$ do not jump at θ_1, and, by Proposition 4.26, $\Delta_{\theta_1}\mathrm{SHC} = 0$. Thus, on $[0, \tau \wedge \theta_1]$, SHC is a $(\mathbb{G}, \mathbb{Q})$-submartingale with drift $h\widetilde{\mathrm{SCR}}$, and

$$\mathrm{SHC} = \mathrm{SHC}_0 - \left(\widetilde{\mathrm{L}} + \widetilde{\mathrm{KVA}} - \mathrm{KVA}_0\right).$$

Since at time θ_1 the processes SHC, L, and KVA (as well as the others) are re-calibrated to the new portfolio including the new trade (in particular, KVA is set to make sure that (4.24) still holds after θ_1), from time θ_1 onward we have

$$\mathrm{SHC} = \mathrm{SHC}_{\theta_1} - (\mathrm{L} - \mathrm{L}_{\theta_1} + \mathrm{KVA} - \mathrm{KVA}_{\theta_1}),$$

where $\mathrm{L} - \mathrm{L}_{\theta_1}$ and $\mathrm{KVA} - \mathrm{KVA}_{\theta_1}$ are respectively a $(\mathbb{G}, \mathbb{Q})$-martingale and supermartingale on $[\theta_1, \tau] \cap [0, \theta_2[$ and $\mathrm{KVA} - \mathrm{KVA}_{\theta_1}$ has the drift coefficient $h\mathrm{SCR}$. By "glueing together" the submartingale SHC before and after θ_1, we obtain that SHC is a $(\mathbb{G}, \mathbb{Q})$-submartingale on $[0, \tau] \cap [0, \theta_2[$, with drift coefficient $h\mathrm{SCR}$ and without jump at θ_1. Iterating the same argument for each successive new trade, we obtain the result. ☐

Remark 4.32. From the above proof we can observe that the (as we called them) "true" processes L and KVA considering all the successive new trades may actually jump at θ_1, since, for example,[36]

$$\Delta\mathrm{L}_{\theta_1} = \mathrm{L}_{\theta_1} - \mathrm{L}_{\theta_1-} = \mathrm{L}_{\theta_1} - \widetilde{\mathrm{L}}_{\theta_1-} = \mathrm{L}_{\theta_1} - \widetilde{\mathrm{L}}_{\theta_1} = \Delta_{\theta_1} L,$$

as $\widetilde{\mathrm{L}}$ does not jump at θ_1. Including the new trade in the portfolio may alter the loss process L of the bank, so in general $\Delta\mathrm{L}_{\theta_1} \neq 0$. The same applies to KVA. Now, as $\mathrm{SHC}_{\theta_1} = \widetilde{\mathrm{SHC}}_{\theta_1}$, on $[\theta_1, \tau] \cap [0, \theta_2[$ it holds

$$\begin{aligned}
\mathrm{SHC} &= \mathrm{SHC}_{\theta_1} - (\mathrm{L} - \mathrm{L}_{\theta_1} + \mathrm{KVA} - \mathrm{KVA}_{\theta_1}) \\
&= \mathrm{SHC}_0 - \left(\widetilde{\mathrm{L}}_{\theta_1} - \mathrm{L}_0 + \widetilde{\mathrm{KVA}}_{\theta_1} - \mathrm{KVA}_0\right) - (\mathrm{L} - \mathrm{L}_{\theta_1} + \mathrm{KVA} - \mathrm{KVA}_{\theta_1}) \\
&= \mathrm{SHC}_0 - (\mathrm{L} - \mathrm{L}_0 - \Delta\mathrm{L}_{\theta_1} + \mathrm{KVA} - \mathrm{KVA}_0 - \Delta\mathrm{KVA}_{\theta_1}),
\end{aligned}$$

where the contributions ΔL_{θ_1} and $\Delta\mathrm{KVA}_{\theta_1}$ are provided by the client via the FTP. Last, note that while SHC does not jump at θ_1, $\mathrm{SCR} = (\mathrm{EC} - \mathrm{KVA})^+$ may.

Remark 4.33. The same as in Theorem 4.30 holds with $(\mathbb{F}, \mathbb{P})$ instead of $(\mathbb{G}, \mathbb{Q})$ if we replace all the processes with their $\mathbb{F}$-optional reduction and set $\tau = \infty$. To see this, note that, by the uniqueness of the $\mathbb{F}$-optional reduction (see Lemma B.11), the $\mathbb{F}$-optional reduction of a sum of

[35]This implies assuming $\theta_1 < \infty$.

[36]Note that the notation ΔX, for a generic process X, is the usual notation we use for the jump process $\Delta X_t = X_t - X_{t-}$, $t \geq 0$; see (A.4).

processes is the sum of the $\mathbb{F}$-optional reductions. Furthermore, the $\mathbb{F}$-optional reduction of the process $A = h\mathrm{SCR}_{\bullet}\lambda$ is $A' := h\mathrm{SCR}'_{\bullet}\lambda$. Indeed, A' is an $\mathbb{F}$-optional process on $[0, T]$ with

$$A'\mathbb{1}_{[\![0,\tau[\![} = \int_0^{\cdot} \mathbb{1}_{\{s<\tau\}}\mathrm{SCR}'_s ds\, \mathbb{1}_{[\![0,\tau[\![} = \int_0^{\cdot} \mathbb{1}_{\{s<\tau\}}\mathrm{SCR}_s ds \mathbb{1}_{[\![0,\tau[\![} = A\mathbb{1}_{[\![0,\tau[\![}.$$

Hence, by repeating the same argument as in the proof of Theorem 4.30, if the $\mathbb{F}$-optional reductions[37] L' and KVA' relative to each successive portfolio have no jump at the time of the following deal, then SHC' (the $\mathbb{F}$-optional reduction of SHC) is an $(\mathbb{F}, \mathbb{P})$-submartingale on $[0, \infty[$ with drift coefficient $h\mathrm{SCR}'$.

[37] By Corollary 2.2, L' is an $(\mathbb{F}, \mathbb{P})$-martingale on $[0, T]$ and $\mathrm{KVA}' = M' - h\mathrm{SCR}'_{\bullet}\lambda$ is an $(\mathbb{F}, \mathbb{P})$-submartingale on $[0, T]$ with drift $h\mathrm{SCR}'$.

5 Well-Posedness of the XVA Equations

In this chapter we prove the well-posedness of the XVA equations derived in the previous chapter. In Section 5.1 we simplify slightly the general assumptions we used so far, and afterwards we show the well-posedness of the KVA equation (Section 5.2) and CVA, FVA, and MVA equations (Section 5.3), by first writing them in an "$\mathbb{F}$-reduced form", and then using the theory of invariance times developed in Appendix B to come back to the well-posedness of the original equation.

5.1 Fine-Tuning of the Assumptions

In the incremental XVA approach we have seen that, at each new trade, the XVA computations of the new portfolio are performed under the run-off assumption. Hence, in this chapter we continue to work under Assumption 3.1 and study the well-posedness of the XVA equations of Section 4.2 and 4.3, which were also constructed using this assumption.

Moreover, we add some further simplifying assumptions. Firstly, we assume that the clean desks perfectly hedge all the market risk of the portfolio, resulting in a vanishing clean desks trading loss process $\mathrm{L}^{cl} = 0$. As explained in [AC20], this is a natural assumption; in fact, by the Volcker rule,[38] a dealer bank is not allowed to do "proprietary trading." Secondly, we also conservatively assume that the CA desk does not hedge against counterparty risk, that is, $\mathcal{H} = 0$.

Assumption 5.1. We assume that $\mathrm{L}^{cl} = 0$ and that $\mathcal{H} = 0$.

We immediately get by Definition 3.28:

$$\mathrm{L} = \mathrm{L}^{ca} = \mathrm{CA} - \mathrm{CA}_0 + \mathcal{C}^\circ + \mathcal{F}^\circ + \mathcal{G}^\circ. \tag{5.1}$$

So the computation of the loss process L, which is used to determine the economic capital EC in the KVA equation (see Section 4.3), needs the process CA. On the other hand, the cash-flows $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{G}$, which are used in the CA computation, take as input the clean valuation of the portfolio of the bank, that is, the processes P^c and MtM. This is intuitively clear for the credit exposure cash-flows $\mathcal{C}$, since the credit exposure of a trade depends on the market value of the trade itself (see (3.12)-(3.13)-(3.14)). The risky funding processes $\mathcal{F}$ and $\mathcal{G}$ arise to finance collateral posting, which directly depends on the MtM of the portfolio of the bank (see also Section 6.1 for a concrete example). Therefore, we see a connection between MtM, CA = CVA+FVA+MVA, and KVA, which are the processes representing the target values on the clean margin, reserve capital, and risk margin accounts, respectively (as per shareholders' balance condition (4.25)), and as such they describe the bank's derivative portfolio pricing problem as a whole. This connection makes the MtM, CA, and KVA equations a "self-contained problem under the cost-of-capital XVA approach."[39]

In this thesis we are more interested in the valuation of XVAs, so we leave aside the valuation of MtM and we assume it to be given, in the sense that we assume that $\mathbb{Q}$ is the risk-neutral measure used for pricing the promised cash-flows $\mathcal{P}^c$ to obtain the (market consistent) clean valuations of the derivative trades of the bank.

Finally, we assume, as in [AC18], that the capital CET1= CR + UC is not used for funding purposes:

[38] See, for example, [FR].
[39] As defined in [AC18].

Assumption 5.2. We assume that the bank does not use its equity capital CET1 for funding purposes.

In (6.32) we describe how we can change this assumption in the concrete setup of the Section 6.1. See also [CSS20].

5.2 The KVA equation

5.2.1 The case of a default-free bank

In this subsection we assume that the bank cannot default, that is, $\tau = \infty$ and $(\mathbb{F}, \mathbb{P}) = (\mathbb{G}, \mathbb{Q})$.[40] In the next subsection we extend the result to the general case of a defaultable bank. To differentiate between the general results and the ones obtained under the no-default assumption, where $\mathbb{F} = \mathbb{G}$, we use the notation "$\cdot'$". The results of this subsection will be then interpreted as an $\mathbb{F}$ reduction (optional or predictable, as appropriate) of the ones of the next subsection, and will help us solve the general case.

Since $\tau = \infty$, we have that the Azéma supermartingale S is constant equal 1, so $Q = 1$ and $D = 0$ (using the notation of Appendix B.1). Thus $\Gamma = \gamma' \bullet \lambda = \frac{1}{S_-} \bullet D = 0$ and we easily see by (2.1) that $\mathfrak{S}_2^\circ = \mathfrak{S}_2'$. By Definition 4.20 and Theorem 4.21, $\mathrm{KVA}' \in \mathfrak{S}_2'$ is an $(\mathbb{F}, \mathbb{P})$-supermartingale on $[0, T]$ with finite variation component given by

$$-A_t := -\int_0^t h\left(\mathrm{EC}_s - \mathrm{KVA}'_s\right)^+ ds, \quad t \in [0, T],$$

if it exists. So $\mathrm{KVA}' = \mathrm{KVA}'_0 + M - A$, where M is an $(\mathbb{F}, \mathbb{P})$-martingale of class(D) on $[0, T]$ started from 0. Note that we do not write EC', since the economic capital is already assumed to be $\mathbb{F}$-adapted in Definition 4.16. Moreover, by Assumption 4.19 and since $\tau = \infty$, $\mathrm{EC} \in \mathfrak{L}_2 = \mathfrak{L}_2'$. Assuming existence of such a supermartingale KVA', we have

$$\mathrm{KVA}'_t = \mathbb{E}'_t\big[\mathrm{KVA}'_t\big] = \mathbb{E}'_t\big[\mathrm{KVA}'_t - \underbrace{\mathrm{KVA}'_T}_{=0}\big] = -\underbrace{\mathbb{E}'_t[M_T - M_t]}_{=0} + \mathbb{E}'_t[A_T - A_t],$$

for any $t \in [0, T]$. Hence, if KVA' exists, it solves the following fixed-point problem:

$$\mathrm{KVA}'_t = \mathbb{E}'_t\left[\int_t^T h\left(\mathrm{EC}_s - \mathrm{KVA}'_s\right)^+ ds\right], \quad t \in [0, T]. \tag{5.2}$$

Conversely, if we assume that $\mathrm{KVA}' \in \mathfrak{S}_2'$ satisfies (5.2) and

$$M := \mathrm{KVA}' + \int_0^\cdot h\left(\mathrm{EC}_s - \mathrm{KVA}'_s\right)^+ ds \in \mathbb{S}^2(\mathbb{F}), \tag{5.3}$$

then

$$\mathbb{E}'_t[M_T - M_t] = \mathbb{E}'_t\left[\mathrm{KVA}'_T - \mathrm{KVA}'_t + \int_t^T h\left(\mathrm{EC}_s - \mathrm{KVA}'_s\right)^+ ds\right] = 0, \quad \forall\, t \in [0, T],$$

[40] Indeed, by (B.4), if $\tau = \infty$ we have $\frac{d\mathbb{P}}{d\mathbb{Q}} = 1$, as the Azéma supermartingale S would be constant equal 1.

that is, M is a true $(\mathbb{F},\mathbb{P})$-martingale of class(D) and $\mathrm{KVA}' = M - \int_0^{\cdot} h(\mathrm{EC}_s - \mathrm{KVA}'_s)^+ ds$ satisfies Definition 4.20. We conclude that, in the case $\tau = \infty$, solving the fixed-point problem in the definition of KVA′ is equivalent to find a process $\mathrm{KVA}' \in \mathfrak{S}'_2$ with (5.2)-(5.3).[41] To put this problem in the form of a BSDE as in Section A.2, we write the martingale M as in Proposition A.32 (with W one-dimensional standard Brownian motion and μ a homogeneous Poisson random measure on $\Omega \times \mathscr{B}([0,\infty[) \times \mathscr{B}(\mathcal{U})$, $\mathcal{U} \subseteq \mathbb{R}^m \setminus \{0\}$, as in Section A.2):

$$M = \int_0^{\cdot} Z_s dW_s + \int_0^{\cdot} \int_{\mathcal{U}} \psi_s(u)\tilde{\mu}(du, ds) + N,$$

where $Z \in \mathrm{L}^2_{loc}(W)$, $\psi \in \mathcal{G}(\mu)$, and N is an $(\mathbb{F},\mathbb{P})$-local martingale on $[0,T]$ orthogonal to W and μ, in the sense of (A.15). Note that, if $\mathrm{KVA}' \in \mathfrak{S}'_2$ satisfies (5.2)-(5.3), then, $\mathbb{P}$-a.s. for all $t \in [0,T]$,

$$\begin{aligned} \mathrm{KVA}'_t &= -\left(\mathrm{KVA}'_T - \mathrm{KVA}'_t\right) = \int_t^T h\left(\mathrm{EC}_s - \mathrm{KVA}'_s\right)^+ ds - (M_T - M_t) \\ &= \int_t^T h\left(\mathrm{EC}_s - \mathrm{KVA}'_s\right)^+ ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathcal{U}} \psi_s(u)\tilde{\mu}(du, ds) - \int_t^T dN_s. \end{aligned} \tag{5.4}$$

Last, before we focus on the well-posedness of the latter BSDE, we introduce a similar fixed-point problem: for given $C' \in \mathfrak{L}'_2$ such that $C' \geq \mathrm{EC} \geq 0$, $K' \in \mathfrak{S}'_2$ satisfies

$$K'_t = \mathbb{E}'_t\left[\int_t^T h\left(C'_s - K'_s\right) ds\right], \quad t \in [0,T]. \tag{5.5}$$

As above, this problem can be written as a linear BSDE of the form

$$K'_t = \int_t^T h\left(C'_s - K'_s\right) ds - \int_t^T Z_s dW_s + \int_t^T \int_{\mathcal{U}} \psi_s(u)\tilde{\mu}(du, ds) - \int_t^T dN_s. \tag{5.6}$$

Lemma 5.3. *i) The BSDE* (5.6) *is well posed in* $\mathfrak{S}'_2$ *with solution*

$$K'_t = h\mathbb{E}'_t\left[\int_t^T e^{-h(s-t)} C'_s ds\right], \quad t \in [0,T]; \tag{5.7}$$

ii) the BSDE (5.4) *is well posed in* $\mathfrak{S}'_2$.

Recall that by well-posedness we mean both existence of a unique solution in $\mathfrak{S}'_2$ and comparison (in the sense of Proposition A.35).

Proof. We only prove the well-posedness for BSDE (5.4), as the other one can be proved similarly. By Proposition A.34 and Proposition A.35,we need to check that the function

$$f \colon \Omega \times [0,T] \times \mathbb{R} \longrightarrow \mathbb{R},\ (\omega, t, y) \mapsto h\left(\mathrm{EC}_t - y\right)^+$$

satisfies (H1), (H2), and (H4) (the measurability of f is clear, as EC is $\mathbb{F}$-progressive). As usual, we omit the dependence on $\omega \in \Omega$. It is easy to see that f is continuous in y and, for any $y, y' \in \mathbb{R}$,

$$\big(f(t,y) - f(t,y')\big)(y - y') \leq h(y - y')^2,$$

[41] This is comparable to [AC20, Definition 4.1].

which gives (H1) with $\alpha = h$. Let $r > 0$. Then,

$$\begin{aligned}\mathbb{E}'\left[\int_0^T \sup_{|y|\le r} |f(t,y) - f(t,0)| dt\right] &= h\mathbb{E}'\left[\int_0^T \sup_{|y|\le r} \left|(\mathrm{EC}_t - y)^+ - \mathrm{EC}_s\right| dt\right] \\ &\le h\mathbb{E}'\left[\int_0^T (2\mathrm{EC}_t + r) dt\right] \\ &\le h\left(Tr + 2\sqrt{T}\mathbb{E}'\left[\left(\int_0^T \mathrm{EC}_t^2 dt\right)^{\frac{1}{2}}\right]\right) < \infty,\end{aligned}$$

using Hölder's inequality and that $\mathrm{EC} \in \mathfrak{L}_2'$. This gives (H2). Since $f(\cdot, 0) = (\mathrm{EC}_\cdot - 0)^+ = \mathrm{EC} \in \mathfrak{L}_2'$, (H4) is also satisfied. This shows that a unique solution $\mathrm{KVA}' \in \mathbb{S}^2(\mathbb{F})$ to the BSDE exists (that is, KVA' is a càdlàg adapted process bounded in L^2), and that the comparison result holds. Finally, since $\mathrm{KVA}'_T = 0$, we can conclude that the unique solution of the BSDE (5.4) is in $\mathfrak{S}_2'$. This shows the well-posedness of the KVA' BSDE in $\mathfrak{S}_2'$. Let us now show that the solution of (5.6) is actually of the form of (5.7). Assume that (K', Z, ψ, N) is the unique L^2-solution of (5.6), and let $\beta_t := e^{-ht}$, for $t \in [0, T]$. Then, by Itô's formula,[42]

$$d(\beta_t K_t') = -he^{-ht} K_t' dt - e^{-ht} h(C_t' - K_t') dt + e^{-ht}\left[\int_{\mathcal{U}} \psi_t(u)\tilde{\mu}(du, dt) + Z_t dW_t + dN_t\right],$$

where the last summand is a true martingale by Remark A.33, because $(Z, \psi, N) \in \mathbb{L}^2 \times \mathbb{L}^2_\mu \times \mathbb{M}^{2,\perp}$ by Proposition A.34 (recall the notation in (A.19)-(A.20)-(A.21)). Therefore,

$$e^{-ht} K_t' = M_t - \int_0^t he^{-hs} C_s' ds, \quad t \in [0, T],$$

for some $(\mathbb{F}, \mathbb{P})$-martingale M; using $K_T' = 0$, we obtain

$$e^{-ht} K_t' = -\mathbb{E}_t'\left[e^{-hT} K_T' - e^{-ht} K_t'\right] = \mathbb{E}_t'\left[\int_t^T he^{-hs} C_s' ds\right],$$

From which it immediately follows (5.7). This concludes the proof. □

Remark 5.4. It is easy to see that, for the unique L^2-solutions of the BSDEs (5.4) and (5.6), the processes KVA' and K' respectively solve the fixed-point problems (5.2) and (5.5). In particular, KVA' uniquely solves the problem given by (5.2)-(5.3), which is equivalent to say that the KVA equation of Definition 4.20 (in the case $\tau = \infty$) is well-posed.

Note that, since $C' \ge 0$, K' is also non-negative. Moreover, by uniqueness of the solution we can write $K' = K'(C')$. Let us define the set

$$\mathfrak{C}' := \left\{C' \in \mathfrak{L}_2' \,\middle|\, C' \ge \max\big(\mathrm{EC}, K'(C')\big)\right\}. \tag{5.8}$$

This can be interpreted as a set of admissible capital at risk processes, since $C' \ge \mathrm{EC}$ satisfies the capital requirements of Section 4.3 and $K'(C')$ plays the role of the risk margin, which is part of the capital at risk. Recall that, by (4.19) (and since we are assuming $\tau = \infty$), we have $\mathrm{CR}' = \max(\mathrm{EC}, \mathrm{KVA}')$, where KVA' is the unique solution of (5.4).

[42]See Theorem A.29.

Lemma 5.5. *We have* $\mathrm{KVA}' = K'(\mathrm{CR}')$*, that is*

$$\mathrm{KVA}'_t = h\mathbb{E}'_t\left[\int_t^T e^{-h(s-t)} \max\left(\mathrm{EC}_s, \mathrm{KVA}'_s\right) ds\right], \quad t \in [0,T]. \tag{5.9}$$

Proof. By definition of CR', we have that $\left(\mathrm{EC} - \mathrm{KVA}'\right)^+ = \mathrm{CR}' - \mathrm{KVA}'$, and thus, for $t \in [0,T]$,

$$\mathrm{KVA}'_t = \mathbb{E}'_t\left[\int_t^T h\left(\mathrm{CR}'_s - \mathrm{KVA}'_s\right) ds\right],$$

which means that $\mathrm{KVA}' \in \mathfrak{S}'_2$ solves the linear BSDE (5.6) for $C' = \mathrm{CR}' \in \mathfrak{L}'_2$. By uniqueness of the solution, it follows $\mathrm{KVA}' = K'(\mathrm{CR}')$, and (5.9) is obtained by inserting $C' = \mathrm{CR}'$ in (5.7). ☐

Proposition 5.6. *Using the above notation, on the interval* $[0,T]$ *the following holds:*

i) $\mathrm{CR}' = \min \mathfrak{C}'$*;*

ii) $\mathrm{KVA}' = \min_{C' \in \mathfrak{C}'} K'(C')$*;*

iii) The process KVA' *is non-decreasing in the hurdle rate* h*.*

Remark 5.7. The minimum here is intended with respect to the usual order relation "$\leq$", where, for two stochastic processes X and Y,

$$X \leq Y \quad \Leftrightarrow \quad \left\{(\omega,t) \in \Omega \times [0,\infty[\,|\, X_t(\omega) > Y_(\omega)\right\} \text{ is an evanescent set},$$

in line with [HWY92, Definition 4.9] (see also the definition of evanescent set in Definition A.4).

Proof of Proposition 5.6. Since $\mathrm{EC} \in \mathfrak{L}'_2$ and $\mathrm{KVA}' \in \mathfrak{S}'_2 \subseteq \mathfrak{L}'_2$, $\mathrm{CR}' \in \mathfrak{L}'_2$, and hence $\mathrm{CR}' \in \mathfrak{C}'$, because

$$\mathrm{CR}' = \max(\mathrm{EC}, \mathrm{KVA}') = \max(\mathrm{EC}, K'(\mathrm{CR}')).$$

Let $C' \in \mathfrak{C}'$. Then, $\mathbb{P}$-a.s. for any $t \in [0,T]$,

$$f(t, K'_t(C')) = h\big(\mathrm{EC}_t - K'_t(C')\big)^+ \leq h\big(C'_t - K'_t(C')\big)^+ = h\big(C'_t - K'_t(C')\big).$$

Hence, the generator of the KVA' BSDE (5.4) is always smaller or equal to the one of the K' linear BSDE (5.6) if both generators are evaluated at the solution $K'(C')$ of the second BSDE. Therefore, by the comparison principle of Proposition A.35, we have that $\mathrm{KVA}' \leq K'(C')$. As $\mathrm{CR}' \in \mathfrak{C}'$ and $\mathrm{KVA}' = K'(\mathrm{CR}')$, item ii) is proved. This also yields item i), as

$$C' \geq \max(\mathrm{EC}, K'(C')) \geq \max(\mathrm{EC}, \mathrm{KVA}') = \mathrm{CR}', \quad \forall\, C' \in \mathfrak{C}'.$$

The last item follows again by the comparison principle applied to the KVA' BSDE (5.4) with two different hurdle rates $h_1 \leq h_2$. Indeed, as $\mathrm{EC} \geq 0$, the coefficient of the BSDE is non-decreasing in h. This completes the proof. ☐

5.2.2 The case of a defaultable bank

Now we remove the assumption $\tau = \infty$ and study the KVA equation in the general invariance times setup. Recall that, by Definition 4.20 and Theorem 4.21, the process $\mathrm{KVA} \in \mathfrak{S}_2^\circ$, if it exists, is a $(\mathbb{G}, \mathbb{Q})$-supermartingale on $[0, \bar{\tau}]$ without jump at τ and with finite variation part given by $-A$, where

$$A_t := \int_0^t h \left(\mathrm{EC}_s - \mathrm{KVA}'_s\right)^+ ds, \quad t \in [0, \bar{\tau}].$$

In the case KVA exists, as $\mathrm{KVA}_T \mathbb{1}_{\{T<\tau\}} = 0$, with a similar argument as before (5.2) we obtain[43]

$$\mathrm{KVA}_t = \mathbb{E}_t\left[\int_t^{\bar{\tau}} h \left(\mathrm{EC}_s - \mathrm{KVA}_s\right)^+ ds + \mathrm{KVA}_\tau \mathbb{1}_{\{\tau<T\}}\right], \quad t \in [0, \bar{\tau}]. \tag{5.10}$$

As in the case of default-free bank of Subsection 5.2.1, we can see that solving the KVA fixed-point problem of Definition 4.20 is equivalent to find a process $\mathrm{KVA} \in \mathfrak{S}_2^\circ$ satisfying (5.10) and such that

$$M := \mathrm{KVA} + \int_0^{\cdot} h \left(\mathrm{EC}_s - \mathrm{KVA}_s\right)^+ ds \in \mathfrak{S}_2. \tag{5.11}$$

Again, we could rewrite this problem as a BSDE, and then try to directly solve it.[44]Instead, we follow [AC18] (which uses an approach that is generalised in [AC20, Theorem 4.1]). This consists in writing the fixed-point problem in differential form. First, the general (that is, with τ general invariance time) KVA problem can be written as follows: the $(\mathbb{G}, \mathbb{Q})$-special semimartingale $\mathrm{KVA} \in \mathfrak{S}_2^\circ$ satisfies

$$d\mathrm{KVA}_t = -h \left(\mathrm{EC}_t - \mathrm{KVA}_t\right)^+ dt + dn_t, \quad t \in]0, \bar{\tau}], \tag{5.12}$$

where $n \in \mathfrak{S}_2$ is some $(\mathbb{G}, \mathbb{Q})$- martingale on $[0, \bar{\tau}]$ without jump at τ. Similarly, the same problem in the case $\tau = \infty$ can be written as follows: the $(\mathbb{F}, \mathbb{P})$-special semimartingale $\mathrm{KVA}' \in \mathfrak{S}'_2$ satisfies

$$d\mathrm{KVA}'_t = -h \left(\mathrm{EC}_t - \mathrm{KVA}'_t\right)^+ dt + dm_t, \quad t \in]0, T], \tag{5.13}$$

where $m \in \mathbb{S}^2(\mathbb{F})$ is some $(\mathbb{F}, \mathbb{P})$-martingale on $[0, T]$.

Lemma 5.8. *The KVA problem* (5.12) *in* $\mathfrak{S}_2^\circ$ *is equivalent to the* KVA' *problem* (5.13) *in* $\mathfrak{S}'_2$ *via* $\mathbb{F}$*-optional reduction. Specifically, if the process* KVA *solves* (5.12)*, then its* $\mathbb{F}$*-optional reduction solves* (5.13)*, and if* KVA' *solves* (5.13)*, then* $(\mathrm{KVA}')^\circ$ *solves* (5.12).

Proof. Assume first that $\mathrm{KVA} \in \mathfrak{S}_2^\circ$ solves (5.12). By (2.3) (see also the discussion before Corollary 2.2), its $\mathbb{F}$-optional reduction KVA' is in $\mathfrak{S}'_2$. We can show as in Remark 4.33 that the $\mathbb{F}$-optional reduction of $h(\mathrm{EC} - \mathrm{KVA})^+ \bullet \lambda$ is given by $h(\mathrm{EC} - \mathrm{KVA}')^+ \bullet \lambda$. Take the $\mathbb{F}$-optional reduction $m := n' \in \mathfrak{S}'_2$, which is an $(\mathbb{F}, \mathbb{P})$-martingale on $[0, T]$, by Corollary 2.2. Then, (5.13) holds on $[\![0, \tau[\![\cap [0, T]$. In other words, before τ the following indistinguishable equality between $\mathbb{F}$-optional processes holds

$$\mathrm{KVA}' = -\int_0^{\cdot} h \left(\mathrm{EC}_s - \mathrm{KVA}'_s\right)^+ ds + m.$$

Hence, by Lemma B.11, the same equality holds on $[0, T] \subseteq \{S_- > 0\}$, which means that (5.13) holds. Conversely, if $\mathrm{KVA}' \in \mathfrak{S}'_2$ solves (5.13), then we define $\mathrm{KVA} := (\mathrm{KVA}')^\circ$, which is in $\mathfrak{S}_2^\circ$,

[43]This corresponds to the "shareholder valuation" of [AC20, Section 4.1].

[44]Possibly, one could try using the BSDE transfer properties of [CS18, Section 9].

by (2.3). In a similar way as above, we take $n := m^\circ \in \mathfrak{S}_2$, which is a $(\mathbb{G}, \mathbb{Q})$-martingale on $[0, \bar\tau]$ with no jump at τ, by Corollary 2.2. Then, (5.12) holds before τ. Since all the processes involved have no jump at τ, the equation also holds on $\rrbracket 0, \bar\tau \rrbracket$. ☐

Theorem 5.9. *The KVA problem of Definition 4.20 is well-posed in* $\mathfrak{S}_2^\circ$.

Proof. The KVA problem is equivalent to the problem given in (5.12), which is equivalent to (5.13) by Lemma 5.8. The latter problem is well-posed by Lemma 5.3 (in particular, see Remark 5.4), and hence the theorem is proved. ☐

Therefore, to compute the process KVA, one should first solve the KVA′ equation in the case of a default-free bank, and then take $\mathrm{KVA} = (\mathrm{KVA}')^\circ$.

Before turning our attention to the other XVAs, we prove the same minimality result of Proposition 5.6 in the general case of a defaultable bank. To do that, we introduce again an alternative problem: given $C \in \mathfrak{L}_2$ such that $C \geq \mathrm{EC} \geq 0$, we want to find $K \in \mathfrak{S}_2^\circ$ such that

$$K_t = \mathbb{E}_t\left[\int_t^{\bar\tau} h\,(C_s - K_s)\,ds + K_\tau \mathbb{1}_{\{\tau<T\}}\right], \quad t \in [0, \bar\tau]. \tag{5.14}$$

This equation is a generalised version of (5.5). Noting that, for $C \in \mathfrak{L}_2$, its $\mathbb{F}$-progressive reduction C' is in $\mathfrak{L}_2'$, by (2.9), it can be proved in the same way as in Lemma 5.8 (passing through a differential form first) that this problem is equivalent to (5.5).[45]In particular, for any $C \in \mathfrak{L}_2$, if a solution K to (5.14) exists, then its $\mathbb{F}$-optional reduction K' solves (5.5) for C'. This means that $K' = K'(C')$ (recall that $K'(C')$ denotes the unique solution to (5.5) for C'). Conversely, the process $K := (K'(C'))^\circ$ solves (5.14) for C. Therefore, we have the following result:

Lemma 5.10. *The problem* (5.14) *is well-posed in* $\mathfrak{S}_2^\circ$*; that is, for each* $C \in \mathfrak{L}_2$ *there is a unique* $K = K(C) \in \mathfrak{S}_2^\circ$ *that solves the equation.*

Recall that the capital at risk process is given by $\mathrm{CR} = \max(\mathrm{EC}, \mathrm{KVA})$ (see (4.19)). We define the set

$$\mathfrak{C} := \Big\{ C \in \mathfrak{L}_2 \,\Big|\, C \geq \max\big(\mathrm{EC}, K(C)\big) \Big\}, \tag{5.15}$$

which can be seen as a set of admissible capital at risk processes (compare it with $\mathfrak{C}'$ in (5.8)).

Theorem 5.11. *Using the above notation, on the interval* $[0, \bar\tau]$ *the following holds:*

i) $\mathrm{CR} = \min \mathfrak{C}$*;*

ii) $\mathrm{KVA} = \min_{C \in \mathfrak{C}} K(C)$*;*

iii) The process KVA *is non-decreasing in the hurdle rate* h.

[45]The proof of this equivalence is skipped, since it would be exactly the same as the proof of Lemma 5.8. The only difference is that, unlike for EC, in general $C \neq C'$ (not even on $\llbracket 0, \tau \llbracket$, since C' is defined as the $\mathbb{F}$-predictable reduction of the $\mathbb{G}$-predictable projection ${}^pC$ of C). This apparent issue can be solved by substituting C with its $\mathbb{G}$-predictable projection ${}^pC$. Since C only appears inside an integral with respect to dt, this substitution does not change the problem.

Proof. First, we show that $\mathrm{KVA} = K(\mathrm{CR})$. To this end, note that $\mathrm{KVA} = (\mathrm{KVA}')^\circ$, which implies that $\mathrm{CR} = \max\big(\mathrm{EC}, (\mathrm{KVA}')^\circ\big) = (\mathrm{CR}')^\circ$ on $[0, \bar\tau]$. By Lemma 5.5, $\mathrm{KVA}' = K'(\mathrm{CR}')$. By the argument before Lemma 5.10, $\mathrm{KVA} = \big(K'(\mathrm{CR}')\big)^\circ$ solves (5.14) for $C = \mathrm{CR}$. Thus, by uniqueness, $\mathrm{KVA} = K(\mathrm{CR})$. Now, since $\mathrm{CR} \in \mathfrak{L}_2$ and $\mathrm{CR} = \max(\mathrm{EC}, K(\mathrm{CR}))$, $\mathrm{CR} \in \mathfrak{C}$. On the interval $[0, \tau[\cap [0, T]$, we have the following:

$$\mathrm{KVA} = \mathrm{KVA}' \le \min_{C \in \mathfrak{C}} K'(C') = \min_{C \in \mathfrak{C}} K(C)$$

where the inequality is given by item ii) of Proposition 5.6, using that $\{C' \,|\, C \in \mathfrak{L}_2\} \subseteq \mathfrak{C}'$. Since $\mathrm{KVA} = K(\mathrm{CR})$ with $\mathrm{CR} \in \mathfrak{C}$, the inequality is actually an equality. As both sides of the equality have no jump at τ, item ii) is proved. As in the proof of Proposition 5.6, item i) follows easily. The last item can be proved with a similar argument as above: by Proposition 5.6, KVA is non-decreasing in h before τ, and, having no jump at τ, the same holds on $[0, \bar\tau]$. ☐

5.3 The other XVA equations

We now study the well-posedness of the equations (4.5), (4.6), and (4.7) in a similar way as we did for KVA. First, we write $\mathcal{C}'$, $\mathcal{F}'$, and $\mathcal{G}'$ for the $\mathbb{F}$-optional reductions of $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{G}$, respectively. So, $\mathcal{C}^\circ = \mathcal{C}'$, $\mathcal{F}^\circ = \mathcal{F}'$, and $\mathcal{G}^\circ = \mathcal{G}'$ before τ. Note that, by Assumption 4.4 and (2.3), we have

$$\mathcal{C}', \mathcal{F}', \mathcal{G}' \in \mathbb{S}^2(\mathbb{F}).$$

We also add a further assumption of the risky funding cash-flow $\mathcal{F}^\circ$, as proposed in [AC18].

Assumption 5.12. Whenever the CVA and MVA process are already well defined in $\mathfrak{S}_2^\circ$, we assume that

$$d\mathcal{F}^\circ_t = f_t(\mathrm{FVA}_t)dt, \tag{5.16}$$

for a $\mathscr{P}(\mathbb{F}) \otimes \mathscr{B}(\mathbb{R})$-measurable function $f \colon \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$.[46]

Remark 5.13. See Section 6.1 for a concrete example, where this holds. This assumption does not change the qualitative results we have obtained in the previous chapters.

Assume for a moment, as we did for KVA, that the bank is default free, that is $\tau = \infty$ and $(\mathbb{F}, \mathbb{P}) = (\mathbb{G}, \mathbb{Q})$. Then the CVA, FVA, and MVA equations become

$$\mathrm{CVA}'_t = \mathbb{E}'_t[\mathcal{C}'_T - \mathcal{C}'_t], \tag{5.17}$$

$$\mathrm{FVA}'_t = \mathbb{E}'_t[\mathcal{F}'_T - \mathcal{F}'_t] = \mathbb{E}'_t\left[\int_t^T f_s(\mathrm{FVA}'_s)ds\right], \text{ and} \tag{5.18}$$

$$\mathrm{MVA}'_t = \mathbb{E}'_t[\mathcal{G}'_T - \mathcal{G}'_t], \quad t \in [0, T]. \tag{5.19}$$

Lemma 5.14. *The* CVA *and* MVA *equations* (4.5) *and* (4.7) *in* $\mathfrak{S}_2^\circ$ *are equivalent to the* CVA' *and* MVA' *equations* (5.17) *and* (5.19) *in* $\mathfrak{S}'_2$*, respectively. Assuming* (5.17) *and* (5.19) *are well posed, the* FVA *equation* (4.6) *in* $\mathfrak{S}_2^\circ$ *is equivalent to the* FVA' *equation* (5.18) *in* $\mathfrak{S}'_2$*.*

[46] As usual, we omit the dependence on $\omega \in \Omega$.

Proof. The idea of the proof is exactly the same as we did for KVA, so we only give the main steps of the proof of the CVA statement. First, write the two problems in differential form: solving (4.5) in $\mathfrak{S}_2^\circ$ is equivalent to find a $(\mathbb{G},\mathbb{Q})$-special semimartingale CVA $\in \mathfrak{S}_2^\circ$ such that

$$d\mathrm{CVA}_t = -d\mathcal{C}_t^\circ + dn_t, \quad t \in]0,\bar{\tau}], \tag{5.20}$$

where $n \in \mathfrak{S}_2$ is some $(\mathbb{G},\mathbb{Q})$-martingale on $[0,\bar{\tau}]$ without jump at τ. Solving (5.17) in $\mathfrak{S}_2'$ is equivalent to find an $(\mathbb{F},\mathbb{P})$-special semimartingale $\mathrm{CVA}' \in \mathfrak{S}_2'$ such that

$$d\mathrm{CVA}_t' = -d\mathcal{C}_t' + dm_t, \quad t \in]0,T], \tag{5.21}$$

where $m \in \mathbb{S}^2(\mathbb{F})$ is some $(\mathbb{F},\mathbb{P})$-martingale on $[0,T]$. Since we have $\mathcal{C}^\circ = \mathcal{C}'$ before τ, applying the exact same argument as in the proof of Lemma 5.8 yields the result. For FVA and MVA we can repeat the same argument. Note that the particular form of $\mathcal{F}^\circ$ does not change anything. □

Therefore, as in the KVA case, if CVA', FVA', and MVA' are well defined, the processes CVA, FVA, and MVA are obtained by stopping before τ:

$$\mathrm{CVA} = (\mathrm{CVA}')^\circ, \quad \mathrm{FVA} = (\mathrm{FVA}')^\circ, \text{ and } \quad \mathrm{MVA} = (\mathrm{MVA}')^\circ. \tag{5.22}$$

Lemma 5.15. *i) By our assumptions, the* CVA' *and* MVA' *eqautions* (5.17) *and* (5.19) *are well-posed in* $\mathfrak{S}_2'$*;*

ii) A sufficient condition for the well-posedness in $\mathfrak{S}_2'$ *of the* FVA' *equation* (5.18) *is that the function f of Assumption 5.12 satisfies the conditions (H1), (H2), and (H4) of Section A.2.*

Proof. As the process $\mathcal{C}'$ is well-defined and integrable, the process CVA' is also well-defined and unique (up to modification). We can write, for every $t \in [0,T]$,

$$\mathrm{CVA}_t' = \mathbb{E}_t'[\mathcal{C}_T'] - \mathcal{C}_t',$$

where the first summand is an $(\mathbb{F},\mathbb{P})$-martingale. Thus, by Proposition A.1, we can choose a modification of this martingale that is càdlàg. Since $\mathcal{C}'$ is also càdlàg, there exists a unique (up to indistinguishability) càdlàg process CVA' satisfying (5.17). Clearly, $\mathrm{CVA}_T' = 0$, and, by the boundedness in L^2 of $\mathcal{C}'$ and the Doob's inequality, we have

$$\begin{aligned}
\mathbb{E}'\left[\sup_{t\in[0,T]} \left(\mathrm{CVA}_t'\right)^2\right] &= \mathbb{E}'\left[\sup_{t\in[0,T]} \left(\mathbb{E}_t'[\mathcal{C}_T'] - \mathcal{C}_t'\right)^2\right] \\
&\le 2\mathbb{E}'\left[\sup_{t\in[0,T]} \mathbb{E}_t'[\mathcal{C}_T']^2\right] + 2\mathbb{E}'\left[\sup_{t\in[0,T]} \left(\mathcal{C}_t'\right)^2\right] \\
&\le 8\mathbb{E}'\left[\left(\mathcal{C}_T'\right)^2\right] + 2\mathbb{E}'\left[\sup_{t\in[0,T]} \left(\mathcal{C}_t'\right)^2\right] < \infty.
\end{aligned}$$

This proves that $\mathrm{CVA}' \in \mathfrak{S}_2'$. The exact same argument applies to the MVA' equation and yields item *i*). To prove the well-posedness of the FVA' equation, we proceed as in the proof on Lemma 5.3, writing the problem as a BSDE:

$$\mathrm{FVA}_t' = -\int_t^T f_s(\mathrm{FVA}_s')ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathcal{U}} \psi_s(u)\tilde{\mu}(du,ds) - \int_t^T dN_s \tag{5.23}$$

for $t \in [0,T]$. By Proposition A.34 and Proposition A.35, the BSDE is well-posed in $\mathfrak{S}_2'$. This proves item *ii*) and finishes the proof. □

Putting all the pieces together, we obtain the following.

Theorem 5.16. *The* CVA *and* MVA *equations* (4.5) *and* (4.7) *are well-posed in* $\mathfrak{S}_2^\circ$*. If* f *in Assumption 5.12 satisfies (H1), (H2), and (H4) form Section A.2, then the* FVA *equation* (4.6) *is well-posed in* $\mathfrak{S}_2^\circ$*.*

Remark 5.17. Note that if $f\colon \Omega \times [0,T] \times \mathbb{R} \ni (\omega, t, y) \mapsto f_t(y) \in \mathbb{R}$ is Lipschitz in y uniformly in (ω, t) and $f_\cdot(0) \in \mathfrak{L}_2'$, then the conditions (H1), (H2), (H4) are fulfilled. Thus, item *ii*) of Lemma 5.15 can be seen a slight generalisation of the second part of [AC20, Proposition 4.2]. On the other hand, for the well-posedness of the CVA$'$ equation in [AC20, Proposition 4.2] only the $\mathbb{P}$-square integrability of $\mathcal{C}_T'$ is required. Since $\mathcal{C}^\circ$ is non-decreasing, the process $\mathcal{C}'$ can be assumed to be non-decreasing on $[0,T]$ (this is also the idea in [AC18]), and hence, the $\mathbb{P}$-square integrability of $\mathcal{C}_T'$ implies that $\mathcal{C}' \in \mathbb{S}^2(\mathbb{F})$, which is what we used (since this is equivalent to $\mathcal{C} \in \mathfrak{S}_2$ in Assumption 4.4). The MVA equation is treated in a similar way as the CVA one, even if in [AC18] MVA is part of FVA. See the concrete example in the next section, where this choice becomes more clear. We obtain an MVA equation in line with the one in [ACC17].

6 Concrete Setup

In the previous two chapters we set up the assumptions on the cash-flows processes, we defined the XVAs through fixed point problems and proved their well-posedness. In this chapter we now give a more concrete example of the cash-flows processes obtaining implementable XVA equations. In Section 6.1 we slightly enhance the example in [AC18, Section 7.4] (which corresponds to the one in [AC20, Section 5]), by generalising some of the simplifying assumptions. In Section 6.2 we discuss about the fact that our resulting XVAs are in fact computed unilaterally, even if we do consider the default of the bank in our model. Lastly, in Section 6.3 we give an even more concrete example of bank's derivative portfolio that could be used in an implementation.

6.1 An example

Let us denote by U and $\bar{U}$ the processes describing the (discounted) value of the risky funding assets that are assumed to be used by the bank to repay the external funder for the provided unsecured funding for the variation and initial margin, respectively. We assume that some exogenously given constant recovery rates $R, \bar{R} \in [0,1]$ exist, representing the portion of debt recovered by the external funder at time τ in case of default of the bank. So, we assume that U and $\bar{U}$ satisfy the following.

$$U_0 = 1, \text{ and } dU_t = \varphi_t U_t dt + (1-R)U_{t-}dJ_t = U_{t-}(\varphi_t dt + (1-R)dJ_t), \tag{6.1}$$

$$\bar{U}_0 = 1, \text{ and } d\bar{U}_t = \bar{\varphi}_t \bar{U}_t dt + (1-\bar{R})\bar{U}_{t-}dJ_t = \bar{U}_{t-}(\bar{\varphi}_t dt + (1-\bar{R})dJ_t), \quad t \in [0,\bar{\tau}], \tag{6.2}$$

where $\varphi, \bar{\varphi} \geq 0$ are processes on $[0,\bar{\tau}]$, representing the spreads over the risk-free rate (for us, the OIS rate, as expressed in Remark 2.4) that the external funder charges to the bank for the unsecured funding (relative to the variation and initial margin, respectively). At the default time of the bank τ, the external funder faces a loss amounting to

$$(1-R)U_{\tau_-} + (1-\bar{R})\bar{U}_{\tau-},$$

which is interpreted as a windfall for the bank. This is expressed by the differential

$$dJ_t = d(\mathbb{1}_{[\![0,\tau[\![}),$$

where the process J is always constant except at τ, where it has a jump $\Delta J_\tau = -1$.

Remark 6.1. Note that the bank can only be short in U and $\bar{U}$, as otherwise it would sell protection against its own default, which is not allowed, by our Assumption 3.3.

Assumption 6.2. We assume that the value processes U and $\bar{U}$ are a $(\mathbb{G},\mathbb{Q})$-martingales on $[0,\bar{\tau}]$.

The reason for this assumption is that, if the price process of an asset is uniformly bounded, it should be a martingale with respect to the pricing measure $\mathbb{Q}$.[47] If φ is uniformly bounded on $[0,\tau[\,\cap\,[0,T]$, then so is U. Even though theoretically the spread φ should rise as the credit quality of the bank worsens approaching default, in practice it is reasonable to assume that it will never exceed some fixed threshold.

[47] This would be in line with the fundamental theorem of asset pricing; see, for example, [DS06, Theorem 9.1.1].

Thus, assuming $R \neq 1$ and $\bar{R} \neq 1$, the processes χ and $\bar{\chi}$ given by

$$(1-R)d\chi_t = \varphi_t dt + (1-R)dJ_t \quad \text{and} \quad (1-\bar{R})d\bar{\chi}_t = \bar{\varphi}_t dt + (1-\bar{R})dJ_t$$

are two $(\mathbb{G}, \mathbb{Q})$-martingales. By Proposition A.18, this holds if and only if $\varphi_t = (1-R)\gamma_t$ and $\bar{\varphi}_t = (1-\bar{R})\gamma_t$, where $\gamma_t dt$ is the compensator of $\mathbb{1}_{[\![\tau,\infty[\![} = 1 - J$, that is, γ is the intensity of τ. By our assumption that, without lost of generality, γ is $\mathbb{F}$-predicable, it follows that φ and $\bar{\varphi}$ are also $\mathbb{F}$-predictable. Hence, $d\chi_t = d\bar{\chi}_t = \gamma_t dt + dJ_t$ is the $(\mathbb{G}, \mathbb{Q})$ *compensated jump-to-default martingale* of the bank, and γ is interpreted as *default intensity* of the bank.

Now we focus on the cash-flow $\mathcal{F}$, and then we see how to treat $\mathcal{G}$. Recall that VM^c denotes of the exchanged variation margin between the bank and the clients (relative to a netting set c), which can be positive (if posted by the client) or negative (if posted by the bank). Then, we have

$$\mathrm{VM} = \sum_c \mathrm{VM}^c. \tag{6.3}$$

Let

$$D := \mathrm{MtM} - \mathrm{VM} \tag{6.4}$$

denote the difference between the valuation of the derivative portfolio of the bank and the total exchanged variation margin. That is, when MtM is positive, D is positive (since $|\mathrm{VM}| \leq |\mathrm{MtM}|$) and it represents the difference between the collateral posted by the CA desk to the clean desks and the collateral posted by the clients; when MtM is negative, D is negative and it represents the difference between the collateral posted by the clean desks to the CA desk and the collateral posted by the CA desk to the clients. Since the variation margin VM^c of each netting set c generally follows the clean valuation of the trade P^c, it is reasonable to assume that it is given as a measurable function of the promised cash-flow $\mathcal{P}^c$. In other words, it make sense to assume that the processes VM^c, and hence VM, are $\mathbb{F}$-adapted.[48] Thus, D is also assumed to be $\mathbb{F}$-adapted; as such, it has no jump at τ.

Using this notation, we have the following characterisation of the risky funding cash-flows $\mathcal{F}$.

Lemma 6.3. *We have, for* $t \in [0, \bar{\tau}]$,

$$d\mathcal{F}^{\circ}_t = \varphi_t(D_t - \mathrm{CVA}'_t - \mathrm{FVA}'_t - \mathrm{MVA}'_t)^+ dt, \tag{6.5}$$

$$d\mathcal{F}^{\bullet}_t = (1-R)(D_t - \mathrm{CVA}_t - \mathrm{FVA}_t - \mathrm{MVA}_t)^+(-dJ_t), \tag{6.6}$$

and

$$d\mathcal{F}_t = (1-R)(D_t - \mathrm{CVA}_t - \mathrm{FVA}_t - \mathrm{MVA}_t)^+ d\chi_t. \tag{6.7}$$

Remark 6.4. By consequence, $\mathcal{F}^{\circ}$ is of the form of Assumption 5.12 with

$$f_t(y) = \varphi_t(D_t - \mathrm{CVA}'_t - \mathrm{MVA}'_t - y)^+, \quad t \in [0, T], y \in \mathbb{R}.$$

It is easy to verify that f is in fact a $\mathscr{P}(\mathbb{F}) \otimes \mathscr{B}(\mathbb{R})$-measurable function satisfying the assumptions of Lemma 5.15, since $\mathcal{P}^c \in \mathbb{S}^2(\mathbb{F})$ for each netting set c, by Assumption 4.4. Indeed, the continuity in (H1) is clear, and (H2) and (H4) follow because in this case, by (3.3), we have $\mathrm{MtM} \in \mathbb{S}^2(\mathbb{F})$, and thus $D \in \mathbb{S}^2(\mathbb{F})$, as well.

Proof of Lemma 6.3. Note that, by Theorem 5.16, the CVA and MVA equations are well posed in $\mathfrak{S}^{\circ}_2$. Recall that with Assumption 5.2 we rule out the possibility for the bank to use its

[48] See also Section 6.3.

own capital for funding purposes. In this case, the funding strategy of the CA desk consists in borrowing form the external funder any amount of cash needed to be able to post the collateral MtM to the clean desks. Thus, we can split the reserve capital RC = CA in the following way:

$$\mathrm{CA} = D + (\mathrm{CA} - D)^+ - (\mathrm{CA} - D)^-. \tag{6.8}$$

The first term is posted collateral (from the CA desk, net of the collateral already posted by the clients) remunerated at the risk-free rate,[49] the second one is cash surplus that the CA desk can invest at the risk-free rate, and the third one is the amount that needs to be unsecurely funded to be able to post all the necessary rehypothecable collateral in the clean margin account. By our definition, the risky funding cash-flow $\mathcal{F}^\circ$ before τ consists in the interest payments to the external funder for this rehypothecable collateral, that is,

$$d\mathcal{F}^\circ_t = (\mathrm{CA}_t - D_t)^- \varphi_t dt, \quad t \in [0, \tau[\cap [0, T], \tag{6.9}$$

where $(\mathrm{CA} - D)^-$ is the borrowed amount and φ is the interest rate—which also corresponds to a spread over risk-free rate, since we use the risk-free asset as numéraire. Note that, before τ, we have $(\mathrm{CA} - D)^- = (D - \mathrm{CA})^+ = (D - \mathrm{CVA}' - \mathrm{FVA}' - \mathrm{MVA}')^+$. Since both sides of (6.9) have no jump at τ, we obtain (6.5). On the other hand, the cash-flow $\mathcal{F}^\bullet$ represents the windfall to the bank at default due to the unpaid borrowed amount to the external funder (beyond the recovery rate). This means that the process $\mathcal{F}^\bullet$ vanishes on $[0, \tau[\cap [0, T]$ and has a (positive) jump at τ equals to $(1 - R)$ times the unsecurely funded rehypothecable collateral before time τ, that is,

$$\Delta \mathcal{F}^\bullet_\tau = (1 - R)(\mathrm{CA}_{\tau-} - D_{\tau-})^- = (1 - R)(D_\tau - \mathrm{CA}_\tau)^+,$$

as both D and CA have no jump at τ. Since $dJ_t = -\mathbb{1}_{\{t=\tau\}}$, this yields (6.6). It follows that, for $t \in [0, \bar{\tau}]$,

$$\begin{aligned} d\mathcal{F}_t &= d\mathcal{F}^\circ_t - d\mathcal{F}^\bullet_t \\ &= (D_t - \mathrm{CVA}_t - \mathrm{FVA}_t - \mathrm{MVA}_t)^+ \varphi_t dt + (1 - R)(D_t - \mathrm{CVA}_t - \mathrm{FVA}_t - \mathrm{MVA}_t)^+ dJ_t \\ &= (D_t - \mathrm{CVA}_t - \mathrm{FVA}_t - \mathrm{MVA}_t)^+ (\varphi_t dt + (1 - R) dJ_t) \\ &= (1 - R)(D_t - \mathrm{CVA}_t - \mathrm{FVA}_t - \mathrm{MVA}_t)^+ d\chi_t. \end{aligned}$$

This finishes the proof. ☐

Remark 6.5. Note that we used a slightly different proof than the one in [AC18, Lemma 7.6], following the idea of [ACC17, Lemma 3.3], which seems more intuitive.

Remark 6.6. Note that, in this case, $\mathcal{F} = \mathcal{F}^\circ - \mathcal{F}^\bullet$ is actually a process of finite variation, being the difference of two non-decreasing processes, and it is also a $(\mathbb{G}, \mathbb{Q})$-martingale, since it is the stochastic integral with respect to the bounded martingale χ and the integrand is bounded by

$$0 \le (1 - R)(D - \mathrm{CA})^+ \le D^+ \le \mathrm{MtM}^+,$$

which is bounded in L^2. Thus, the requirement of Assumption 4.6 for $\mathcal{F}$ is fulfilled.

With a similar (but simpler) argument, we obtain the cash-flow $\mathcal{G}$ for the unsecured funding of the initial margin. Recall that, for each netting set c, the initial margin posted by the bank to the client is given by $\mathrm{PIM}^c \ge 0$, and it is assumed to be $\mathbb{F}$-adapted, with a similar argument

[49] As usual, if $D < 0$, this means that D is received collateral.

as for VM^c (see before Lemma 6.3). Thus, if we want to define a process PIM describing the total initial margin posted by the bank at each time, we may assume with no loss of generality that $\tau = \infty$. If the client defaults before maturity, at the liquidation time $\tau_c^\delta < T$ the segregated initial margin PIM^c is returned to the bank (see Assumption 3.26 and the discussion following it). Thus, the process describing the total initial margin posted by the bank at each time is

$$\mathrm{PIM} := \sum_c \mathrm{PIM}^c \mathbb{1}_{[\![0,\tau_c{}^\delta]\!]}, \quad \text{on } [0,T]. \tag{6.10}$$

Since there is no capital account to fund the initial margin, the whole PIM is borrowed by the external funder at the spread over risk-free rate $\bar{\varphi} = (1-\bar{R})\gamma$. Therefore, we obtain the following.

Lemma 6.7. *For $t \in [0, \bar{\tau}]$,*

$$d\mathcal{G}^\circ_t = \bar{\varphi}_t \mathrm{PIM}_t dt, \tag{6.11}$$

$$d\mathcal{G}^\bullet_t = (1-\bar{R})\mathrm{PIM}_t d(-J_t), \tag{6.12}$$

and

$$d\mathcal{G}_t = (1-\bar{R})\mathrm{PIM}_t d\chi_t. \tag{6.13}$$

Proof. As in the case of $\mathcal{F}$ in Lemma 6.3, before τ the cash-flow $\mathcal{G}^\circ$ consists in interest payments to the external funder for the needed initial margin fundings, that is,

$$d\mathcal{G}^\circ_t = \mathrm{PIM}_t \bar{\varphi}_t dt, \quad t \in [0, \bar{\tau}[\cap [0,T].$$

Since both sides of the equality have no jump at τ, we obtain (6.11). On the other hand, the cash-flow $\mathcal{G}^\bullet$ represents the windfall to the bank at default due to the received funding for the initial margin that is not returned to the external funder (beyond the recovery rate). Thus,

$$\Delta\mathcal{G}^\bullet_\tau = (1-\bar{R})\mathrm{PIM}_{\tau-} = (1-\bar{R})\mathrm{PIM}_\tau,$$

and (6.12) follows. As in Lemma 6.3, (6.13) follows form the fact that

$$(1-\bar{R})d\chi_t = \bar{\varphi}_t dt + (1-\bar{R})dJ_t.$$

This concludes the proof. □

Remark 6.8. It is reasonable to assume that the initial margin posted by the bank never exceed the MtM of the derivative portfolio. Thus, we can assume that PIM is bounded in L^2. Then, one can prove that $\mathcal{G} \in \mathfrak{S}_2$ is a $(\mathbb{G}, \mathbb{Q})$-martingale of finite variation, as required in Assumption 4.4 and Assumption 4.6.

We now move our attention to the counterparty exposure cash-flows $\mathcal{C}$. For each netting set c, let

$$J^c := \mathbb{1}_{[\![0,\tau_c{}^\delta[\![}$$

denote the survival indicator process of the netting set up to liquidation time $\tau_c{}^\delta$, where $\tau_c{}^\delta = \tau_c + \delta$, for a fixed short period of time $\delta > 0$, such as two weeks.[50] Recall that we denote by $R_c \in [0,1]$ the recovery rate that the client relative to the netting set c guarantees to the bank in case of default, and by $R_b \in [0,1]$ the fixed recovery rate that all the (not yet defaulted) clients receive in case of bank's default.

By looking at (3.12), (3.13), and (3.14), we immediately get the following.

[50] Recall the we assumed after Definition 3.15 that if $\tau_c \leq T$, then $\tau_c{}^\delta < T$ as well.

Lemma 6.9. *For $t \in [0, \bar{\tau}]$,*

$$d\mathcal{C}^{\circ}_t = \sum_{\substack{c \\ \tau_c{}^{\delta} < \bar{\tau}}} (1 - R_c)\big(\mathcal{P}^c_{\tau_c{}^{\delta}} - \mathcal{P}^c_{\tau_c-} + \mathrm{P}^c_{\tau_c{}^{\delta}} - \mathrm{VM}^c_{\tau_c-} - \mathrm{RIM}^c_{\tau_c-}\big)^+(-dJ^c_t), \tag{6.14}$$

and

$$\begin{aligned} d\mathcal{C}^{\bullet}_t = &- \sum_{\substack{c \\ \tau_c \leq \tau < \tau_c{}^{\delta}}} (1 - R_c)\big(\mathcal{P}^c_{\tau} - \mathcal{P}^c_{\tau_c-} + \mathrm{P}^c_{\tau} - \mathrm{VM}^c_{\tau_c-} - \mathrm{RIM}^c_{\tau_c-}\big)^+(-dJ_t) \\ &+ (1 - R_b) \sum_{\substack{c \\ \tau \leq \tau_c{}^{\delta} \wedge T}} \Big(\mathcal{P}^c_{\tau} - \mathcal{P}^c_{(\tau\wedge\tau_c)-} + \mathrm{P}^c_{\tau} - \mathrm{VM}^c_{(\tau\wedge\tau_c)-} + \mathrm{PIM}^c_{(\tau\wedge\tau_c)-}\Big)^-(-dJ_t). \end{aligned} \tag{6.15}$$

To simplify the notation we let

$$\Delta^+_c := (\mathcal{P}^c_{\tau_c{}^{\delta}} - \mathcal{P}^c_{\tau_c-} + \mathrm{P}^c_{\tau_c{}^{\delta}} - \mathrm{VM}^c_{\tau_c-} - \mathrm{RIM}^c_{\tau_c-})^+, \tag{6.16}$$

$$\Delta^+_{c,\tau} := (\mathcal{P}^c_{\tau_c{}^{\delta}\wedge\tau} - \mathcal{P}^c_{\tau_c-} + \mathrm{P}^c_{\tau_c{}^{\delta}\wedge\tau} - \mathrm{VM}^c_{\tau_c-} - \mathrm{RIM}^c_{\tau_c-})^+, \quad \text{and} \tag{6.17}$$

$$\Delta^-_{c,\tau} := \Big(\mathcal{P}^c_{\tau} - \mathcal{P}^c_{(\tau\wedge\tau_c)-} + \mathrm{P}^c_{\tau} - \mathrm{VM}^c_{(\tau\wedge\tau_c)-} + \mathrm{PIM}^c_{(\tau\wedge\tau_c)-}\Big)^-. \tag{6.18}$$

Recall that the processes P^c, VM^c, RIM^c, and PIM^c are all $\mathbb{F}$-adapted. Moreover, by Assumption 4.1, each clients' default time τ_c is an $\mathbb{F}$-stopping time. Therefore, all the random variables Δ^+_c above are $\mathscr{F}_{\tau_c{}^{\delta}}$-measurable.[51] This means that we can easily find the $\mathbb{F}$-optional reduction $\mathcal{C}'$ of $\mathcal{C}$ by sending τ to infinity; we obtain the following process on $[0, T]$:

$$\mathcal{C}' = \sum_{\substack{c \\ \tau_c \leq T}} (1 - R_c)\Delta^+_c \mathbb{1}_{[\![\tau_c{}^{\delta}, T]\!]}. \tag{6.19}$$

Note that, then $\mathcal{C}^{\circ} = (\mathcal{C}')^{\circ}$. Additionally, since by Assumption 4.4 $\mathcal{P}^c$ is bounded in L^2, all the Δ^+_c and Δ^-_c above are $\mathbb{Q}$-square integrable, and hence $\mathcal{C}'$ is also bounded in L^2, being a finite sum of those; that is, $\mathcal{C}' \in \mathbb{S}^2(\mathbb{F})$. By (2.1), we see that $\mathcal{C} \in \mathfrak{S}_2$ and Assumption 4.4 is satisfied.

Putting all the pieces together, we can use the concrete form of the cash-flows $\mathcal{C}$, $\mathcal{F}$, and $\mathcal{G}$ developed in this example to compute the XVAs.

Theorem 6.10. *Under the setup of this section, the* CVA, FVA, *and* MVA *equations* (4.5), (4.6), *and* (4.7) *are well-posed in $\mathfrak{S}^{\circ}_2$ and*

$$\mathrm{CVA} = (\mathrm{CVA}')^{\circ}, \quad \mathrm{FVA} = (\mathrm{FVA}')^{\circ}, \quad \textit{and} \quad \mathrm{MVA} = (\mathrm{MVA}')^{\circ}, \tag{6.20}$$

where, for $t \in [0, T]$,

$$\mathrm{CVA}'_t = \sum_c (1 - R_c)\mathbb{E}'_t\big[\mathbb{1}_{\{t < \tau_c{}^{\delta} < T\}}\Delta^+_c\big], \tag{6.21}$$

$$\mathrm{MVA}'_t = (1 - \bar{R})\mathbb{E}'_t\left[\int_t^T \gamma_s \mathrm{PIM}_s ds\right], \tag{6.22}$$

$$\mathrm{FVA}'_t = \mathbb{E}'_t\left[\int_t^T (1 - R)\gamma_s(D_s - \mathrm{CVA}'_s - \mathrm{MVA}'_s - \mathrm{FVA}'_s)^+ ds\right]. \tag{6.23}$$

[51] In other words, we can see Δ^+_c as an "$\mathbb{F}$-reduction" version of $\Delta^+_{c,\tau}$.

Moreover, we have, for $t \in [0, \bar{\tau}]$,

$$\begin{aligned}\mathrm{DVA}_t &= -\sum_c (1-R_c)\mathbb{E}_t\big[\mathbb{1}_{\{\tau_c \le \tau \le \tau_c^\delta \wedge T\}}\Delta^+_{c,\tau}\big] + \sum_c (1-R_b)\mathbb{E}_t\big[\mathbb{1}_{\{\tau \le \tau_c^\delta \wedge T\}}\Delta^-_{c,\tau}\big] \\ &\quad + \mathbb{E}_t\big[\mathrm{CVA}_\tau \mathbb{1}_{\{\tau \le T\}}\big],\end{aligned} \tag{6.24}$$

$$\mathrm{MDA}_t = \mathbb{E}_t\big[\mathbb{1}_{\{\tau \le T\}}(1-\bar{R})\mathrm{PIM}_\tau\big] + \mathbb{E}_t\big[\mathrm{MVA}_t \mathbb{1}_{\{\tau \le T\}}\big], \tag{6.25}$$

$$\mathrm{FDA}_t = \mathbb{E}_t\big[\mathbb{1}_{\{\tau \le T\}}(1-R)(D_\tau - \mathrm{CVA}_\tau - \mathrm{MVA}_\tau - \mathrm{FVA}_\tau)^+\big] + \mathbb{E}_t\big[\mathrm{FVA}_\tau \mathbb{1}_{\{\tau \le T\}}\big], \tag{6.26}$$

$$\begin{aligned}\mathrm{FV}_t &= \sum_c (1-R_c)\mathbb{E}_t\big[\mathbb{1}_{\{t < \tau_c^\delta < \bar{\tau}\}}\Delta^+_{c,\tau}\big] + \sum_c (1-R_c)\mathbb{E}_t\big[\mathbb{1}_{\{\tau_c \le \tau \le \tau_c^\delta \wedge T\}}\Delta^+_{c,\tau}\big] \\ &\quad - \sum_c (1-R_b)\mathbb{E}_t\big[\mathbb{1}_{\{\tau \le \tau_c^\delta \wedge T\}}\Delta^-_{c,\tau}\big],\end{aligned} \tag{6.27}$$

$$\begin{aligned}d\mathrm{L}_t &= \sum_c (1-R_c)\Delta^+_c(-dJ^c_t) + d\mathrm{CVA}_t + (1-\bar{R})\gamma_t \mathrm{PIM}_t dt + d\mathrm{MVA}_t \\ &\quad + (1-R)\gamma_t(D_t - \mathrm{CVA}_t - \mathrm{FVA}_t - \mathrm{MVA}_t)^+ dt + d\mathrm{FVA}_t.\end{aligned} \tag{6.28}$$

Proof. The well-posedness and (6.20) follow respectively from Theorem 5.16 combined with Remark 6.4, and (5.22). The equations (6.21), (6.22), and (6.23) follow directly from (5.17), (5.19), and (5.18) inserting the correct values of $\mathcal{C}'$, $\mathcal{G}'$, and $\mathcal{F}'$. The first one is directly given by (6.19). By (6.11), $\mathcal{G}' = \int_0^\cdot \bar{\varphi}_t \mathrm{PIM}_t dt$ on $[0, T]$, as both γ and PIM are $\mathbb{F}$-adapted on $[0, T]$ and by the uniqueness of the $\mathbb{F}$-optional reduction, form Lemma B.11. Similarly, by (6.5), $\mathcal{F}' = \int_0^\cdot \varphi_t(D_t - \mathrm{CA}'_t)^+ dt$ on $[0, T]$. The equations (6.24), (6.25), and (6.26) follow by Definition 4.9, inserting the cash-flows $\mathcal{C}^\bullet$, $\mathcal{G}^\bullet$, and $\mathcal{F}^\bullet$ we derived in this section (see (6.15), (6.12), and (6.6)). Similarly, (6.27) follows from Lemma 4.11, inserting the cash-flow $\mathcal{C} = \mathcal{C}^\circ - \mathcal{C}^\bullet$. Lastly, (6.28) follows from (5.1), writing CA = CVA + MVA + FVA. □

We see that, once one has all the promised cash-flows $\mathcal{P}^c$, for all netting sets c, and the default times τ_c of the clients and τ (together with its intensity γ) of the bank, then one can compute all the XVAs (including KVA) in the following way. First, one has to compute the clean valuation of the promised cash-flows P^c and of the total portfolio MtM, using a risk neutral measure $\mathbb{Q}$. Then, depending on the collateral agreements of each trade, one can also calculate the corresponding VM^c, RIM^c, and PIM^c on $[0, T]$ (seethe example in Section 6.3). Having these, it is now easy to compute the $\mathbb{F}$-reductions of the credit exposure cash-flow $\mathcal{C}'$ and initial margin funding cash-flow $\mathcal{G}'$. By calculating a conditional expectation, we directly obtain CVA$'$ and MVA$'$. Then, FVA$'$ is obtained by solving its BSDE (6.23). Subsequently, one obtains the $\mathbb{F}$-optional reduction of the trading loss process:

$$L'_t = \mathrm{CVA}'_t - \mathrm{CVA}'_0 + \mathcal{C}'_t + \mathrm{MVA}'_t - \mathrm{MVA}'_0 + \mathcal{G}'_t + \mathrm{FVA}'_t - \mathrm{FVA}'_0 + \mathcal{F}'_t, \quad 0 \le t \le T, \tag{6.29}$$

which can be used to compute the economic capital EC; in fact, by Definition 4.16,

$$\mathrm{EC}_t = \mathrm{ES}_t\Big[\mathrm{L}'_{(t+1)\wedge T} - \mathrm{L}'_t\Big], \quad 0 \le t \le T, \tag{6.30}$$

where ES_t represents the $(\mathscr{F}_t, \mathbb{P})$ conditional 97.5% expected shortfall. Finally, we have all ingredients to write down the KVA BSDE (5.9)

$$\mathrm{KVA}'_t = h\mathbb{E}'_t\left[\int_t^T e^{-h(s-t)} \max\big(\mathrm{EC}_s, \mathrm{KVA}'_s\big)\, ds\right], \quad t \in [0, T]. \tag{6.31}$$

By stopping before τ we then obtain all the XVAs.

Remark 6.11. From the equation (6.21) we easily see that CVA is additive over netting sets. We can draw the same conclusion from (6.22) about MVA, by noting that PIM is additive over netting sets, by (6.10). On the contrary, FVA is clearly not additive over single netting sets. In fact, according to [AA14] FVA is only additive over so-called "funding sets", which are sub-portfolios of the bank's portfolio for which the VM can be rehypothecated across all the trades of the sub-portfolio; in our case, we only have one "funding set", since we assumed that VM can be rehypothecated across all trades of the bank's portfolio.

Note that our results in Theorem 6.10 are a slight generalisation of [AC18, Proposition 7.2][52] for the following two reasons:

1. we considered a portfolio with more than one client, allowing the liquidation of the clients to be non-instantaneous, i.e. $\tau_c^\delta > \tau_c$ (but the one of the bank is still instantaneous, i.e. $\tau^\delta = \tau$), also with the possibility of contractual promised cash-flows occurring during the liquidation period;
2. we did not assume all rehypothecable collateral, but considered a situation in which the bank can both post and receive initial margin, resulting in an improved CVA formula and the addition of an MVA formula.

Therefore, our results are more akin to formulas in [ACC17, Proposition 4.1], although there the positive liquidation period is modelled through an approximation (see [ACC17, Equation (11)]). Moreover, in [ACC17, Equations (28)-(29)] we can see how the system of "decoupled" equations above transforms into a coupled forward-backward SDE (FBSDE) if we allow the capital of shareholders to be used as a funding source (which we previously excluded in Assumption 5.2). Using the setup of this thesis, if we allow the FVA desk to use the capital at risk of the shareholders

$$\mathrm{CR} = \max(\mathrm{EC}, \mathrm{KVA}) \quad \big(\text{or } \mathrm{CR}' = \max(\mathrm{EC}, \mathrm{KVA}')\big)$$

to fund the variation margin, the equations (6.23), (6.29), and (5.2) build the following FBSDE:[53]

$$\begin{cases} d\mathrm{L}'_t = d\mathrm{CVA}'_t + \sum_c (1-R_c)\Delta_c^+(-dJ_t^c) + d\mathrm{MVA}'_t + (1-\bar{R})\gamma_t \mathrm{PIM}_t dt \\ \qquad\quad + d\mathrm{FVA}'_t + \varphi_t\big(D_t - \mathrm{CVA}'_t - \mathrm{MVA}'_t - \mathrm{FVA}'_t - \max(\mathrm{EC}_t(\mathrm{L}'), \mathrm{KVA}'_t)\big)^+ dt, \\ \mathrm{FVA}'_t = \mathbb{E}'_t\Big[\int_t^T \varphi_s\big(D_s - \mathrm{CVA}'_s - \mathrm{MVA}'_s - \mathrm{FVA}'_s - \max(\mathrm{EC}_s(L'), \mathrm{KVA}'_s)\big)^+ ds\Big], \\ \mathrm{KVA}'_t = \mathbb{E}'_t\Big[\int_t^T h\left(\mathrm{EC}_s - \mathrm{KVA}'_s\right)^+ ds\Big], \end{cases} \tag{6.32}$$

for $0 < t \le T$ and with initial and terminal conditions $L'_0 = 0$ and $\mathrm{FVA}'_T = \mathrm{KVA}'_T = 0$, respectively. The obtained FBSDE is comparable with [CSS20, Equation (27)], where there is no MVA and the cash-flows are not already discounted. Note that, in this case we assumed that CR is only used to fund VM, not PIM; therefore, CR does not influence MVA. Under the assumption of L^2-boundedness of $\mathcal{P}^c$ of this thesis, it should be possible to prove that the above FBSDE is well-posed in a similar way as [CSS20, Theorem 4.1].

[52] Which also corresponds to the one in [AC20, Proposition 5.1].

[53] We can use the $\mathbb{F}$-optional reduction L' instead of L, since then $\mathrm{L} = (\mathrm{L}')^\circ$. Note that we also write $\mathrm{EC}(\mathrm{L}')$, to stress that the economic capital depends on L'.

6.2 First to default, bilateral, and unilateral XVAs

We can see from (6.27) that the fair valuation of counterparty risk FV = CVA − DVA can be written as the difference between the so-called *first-to-default* CVA and DVA:

$$
\begin{aligned}
\mathrm{FV}_t &= \mathbb{E}_t[\mathcal{C}^{\circ}_{\bar{\tau}} - \mathcal{C}^{\circ}_t] - \mathbb{E}_t[\mathcal{C}^{\bullet}_{\bar{\tau}} - \mathcal{C}^{\bullet}_t] \\
&= \underbrace{\mathrm{CVA}_t - \mathrm{CVA}^{\mathrm{CL}}_t}_{=:\mathrm{FTDCVA}_t} - \underbrace{\Big(\mathrm{DVA}_t - \mathrm{CVA}^{\mathrm{CL}}_t\Big)}_{=:\mathrm{FTDDVA}_t}
\end{aligned}
\tag{6.33}
$$

for $0 \le t \le \bar{\tau}$. So, in the concrete setup of Theorem 6.10,

$$
\mathrm{FTDCVA}_t = \sum_c (1 - R_c)\mathbb{E}_t\big[\mathbb{1}_{\{t<\tau_c{}^{\delta}<\bar{\tau}\}}\Delta^{+}_{c}\big]\,, \quad \text{and} \tag{6.34}
$$

$$
\mathrm{FTDDVA}_t = -\sum_c (1 - R_c)\mathbb{E}_t\big[\mathbb{1}_{\{\tau_c\le\tau\le\tau_c{}^{\delta}\wedge T\}}\Delta^{+}_{c,\tau}\big] - \sum_c (1 - R_b)\mathbb{E}_t\big[\mathbb{1}_{\{\tau\le\tau_c{}^{\delta}\wedge T\}}\Delta^{-}_{c,\tau}\big], \tag{6.35}
$$

$0 \le t \le \bar{\tau}$.

In particular, FTDCVA and FTDDVA only value the credit exposure cash-flows between the bank and its clients up to time τ, without considering the contra-liability cash-flow $\mathrm{CVA}_{\tau}\mathbb{1}_{[\![\tau]\!]}$ that is passed from the shareholders to the bondholders at the default of the bank if $\tau < T$,[54] and has $(\mathbb{G}, \mathbb{Q})$-valuation given by $\mathrm{CVA}^{\mathrm{CL}}$ (see (4.8)). As discussed in Remark 4.15, the resulting FV is a fair and symmetric valuation of counterparty risk between the bank and its clients, which is consistent with the so-called "law of one price". Given our market incompleteness assumption (for which the bank is not able to hedge its own jump to default; see Assumption 3.3), the actual add-on (relative to counterparty exposure) needed by the bank also comprises the contra-liability component $\mathrm{CVA}^{\mathrm{CL}}$. In this sense, we can say that CVA is computed unilaterally, while FTDCVA is bilateral, since it also compensates the clients for the risk of default of the bank itself. Actually, a unilateral computation of CVA is not only necessary due to market incompleteness, but also by a regulatory point of view. In fact, as summarised in [AA14, Section 3.1], the capital CET1 of shareholders is not supposed to increase as an effect of the sole deterioration of the bank's own creditworthiness. Suppose that we replace CVA by FTDCVA in the computation of the trading loss of the bank L (recall that L describes the change in CET1 by (3.24)); then, a deterioration of the credit quality of the bank (the rest staying unchanged) would result in a higher valuation of the contra-liability component $\mathrm{CVA}^{\mathrm{CL}}$, and hence a lower FTDCVA = CVA − $\mathrm{CVA}^{\mathrm{CL}}$ and a lower loss L. Therefore, a bilateral computation of CVA would not fulfil the above mentioned regulatory requirement.

Analogously, FVA, MVA, and KVA are also computed unilaterally in this setup. In fact, since we assumed a run-off view, the add-ons payed by the client are given by $\mathrm{XVA}_0 = \mathrm{XVA}'_0$, since $\mathbb{Q}[\tau > 0] = 1$ and X can be C, F, M, or K, where XVA' is the $\mathbb{F}$-optional reduction version of XVA and is computed under the assumption that $\tau = \infty$, that is, the bank is default-free. In [AC18, Section 8] is illustrated how one could obtain bilateral KVA and FVA (MVA could be done similarly, but in [AC18] MVA and FVA are merged) by changing some of the assumptions regarding the default of the bank. Specifically, if one assumes that the residual amounts KVA_{τ} on the risk margin account and FVA_{τ} on the reserve capital account go to the shareholders instead of the bondholders at the default time of the bank, one obtains bilateral KVA and FVA. Such an assumption would be less conservative and against the "no-arbitrage argument" of Remark 3.22. However, a bilateral FVA would not be problematic regarding the regulatory requirement we pointed out above for CVA, as expressed in [AA14, Section 3.3].

[54] Recall Assumption 3.21 and Definition 3.23.

6.3 Portfolio Case Study

In Theorem 6.10 and the discussion following it, we have seen that knowing the contractually promised cash-flows $\mathcal{P}^c$ of each netting set c of the bank's derivative portfolio, we can compute first MtM, and then CVA, FVA, MVA, and KVA. In this section we consider the following simple example of a derivative portfolio of the bank.[55]

We consider ten different clients buying in-arrears interest rate swaps (IRS) from the bank at time $t = 0$; no other trades occur after $t = 0$ (that is, we assume a run-off situation). We assume that the clients comes from different economies using a different interest rates and currencies; this can be model as in [Ces+09]: one simulates ten different interest rate models with separable volatility terms as in [Ces+09, Section 2.3], and nine foreign exchange models as in [Ces+09, Section 2.4]. This gives the prices of a zero coupon bonds at time $s \geq 0$ with maturity $s \leq t \leq T$ (in the ten different currencies) $D^i_{s,t}$, $i = 1, \ldots, 10$, and exchange rates (taking the currency of the first economy as reference currency) χ^i_t, $i = 2, \ldots, 10$, $0 \leq t \leq T$. We denote the (simply compounded) forward rate of the economy $i \in \{1, \ldots, 10\}$ at time $t \geq 0$ for the period $[S_1, S_2]$, for $t \leq S_1 \leq S_2$, by

$$F^i(t; S_1, S_2) := \alpha^{-1} \left(\frac{D^i_{t,S_1}}{D^i_{t,S_2}} - 1 \right), \tag{6.36}$$

where α denotes the distance in years between S_1 and S_2 (see [BM06, Definition 1.2.1]). Here, we can simply assume $\alpha = S_2 - S_1$. When valued at time $t = S_1$, this rate gives the simply compounded interest rate

$$L^i(S_1, S_2) := F^i(S_1; S_1, S_2) = \alpha^{-1} \left(\frac{1}{D^i_{S_1,S_2}} - 1 \right). \tag{6.37}$$

Then, one can model the payoff of the IRSs in the following way—see also [BM06]. Let $T_0, \ldots, T_{20}, T_{21}$ be fixed dates distanced six months one another with $T_0 = 0$ and $T_{20} = 10$ (years). The last date is only used to compute the last interest rate payment occurring at time T_{20}. So, we can assume $T = T_{20} + \delta$ be the final maturity of the portfolio, including a potential liquidation period δ. Let us denote by $B = (B_t)_{0 \leq t \leq T}$ the bank account of the economy $i = 1$ that we use as numéraire (in line with our setup; see Section 2.3). Recall that $\mathbb{Q}$ is the risk-neutral measure with respect to the numéraire B. Then, the cumulative discounted payoff of the first IRS (counted positive when received by the bank) is given by[56]

$$\mathrm{IRS}^1_t := N \sum_{\substack{j \geq 1 \\ T_j \leq t}} \frac{1}{B_{T_j}} \frac{1}{2} \left(L^1(T_j, T_{j+1}) - K^1 \right), \quad 0 \leq t \leq T, \tag{6.38}$$

where K^1 is the fixed-leg rate for the IRS relative to client 1 and N is a fixed notional amount (we can assume, for example, $N = 10'000$). Analogously, the cumulative discounted payoff of the other IRSs expressed in the reference currency is given by

$$\mathrm{IRS}^i_t := N \sum_{\substack{j \geq 1 \\ T_j \leq t}} \frac{1}{B_{T_j}} \frac{1}{2} \chi^i_{T_j} \left(L^i(T_j, T_{j+1}) - K^i \right), \quad 0 \leq t \leq T, \tag{6.39}$$

[55] Inspired from the swap portfolio case study of [Alb+19, Section 5].

[56] The multiplication times $\frac{1}{2}$ corresponds to the distance between the coupon payments $\alpha_j = T_{j+1} - T_j = \frac{1}{2}$.

for fixed-leg rates K^i, $i = 2, \dots, 10$. Let us denote by π^i the clean valuation at time 0 (i.e., the mark-to-market) of the i-th IRS, that is,

$$\pi^i := \mathbb{E}'\left[\mathrm{IRS}^i_T\right], \quad i = 1, \dots, 10, \tag{6.40}$$

where $\mathbb{E}'$ denotes the expectation with respect to the invariance measure $\mathbb{P}$, as usual. This measure can be computed by (B.4), once one has the default time of the bank τ (see Remark **??** below).

We can assume that the prices π^i, $i = 1, \dots, 10$, are payed by the bank to the respective clients (if negative, this means that a positive amount flows from the client to the bank, as usual) at time $t = T_1$. This makes the contract fair at inception.

Once one has all this, it is possible to compute all the cash-flows building the portfolio of the bank. For each netting set $c = c_1, \dots, c_{10}$, the cumulative promised cash-flow is given by

$$\mathcal{P}^{c_i}_t = -\pi^i \mathbb{1}_{[\![T_1, \infty[\![} + \mathrm{IRS}^i_t, \quad 0 \le t \le T, \tag{6.41}$$

and $\mathcal{P} = \sum_{i=1}^{10} \mathcal{P}^{c_i}$. Then, the clean valuation of the promised cash-flows are

$$\mathrm{P}^{c_i}_t = \mathbb{E}'_t[\mathcal{P}^{c_i}_T - \mathcal{P}^{c_i}_t], \quad 0 \le t \le T. \tag{6.42}$$

Before one can start computing the XVAs, one has to fix a way to compute explicitly the collateral quantities VM and IM. Up to now, these are assumed to be exogenously given by the collateral agreements between the bank and its clients. According to [Gre15, Section 6.2.2], the possibilities range from "no CSA" to "one-way CSA" or "two-way CSA"; moreover, if collateral is exchanged, we could assume either a threshold or an initial margin (which work in the opposite direction; see [Gre15, Section 6.4.2]). For simplicity, we assume as in [Alb+19] that we only have two possibilities: either no collateral is exchanged ("no CSA") or there is full collateralisation with posting of an initial margin ("CSA"). In the latter case, for each netting set c the variation margin is given by $\mathrm{VM}^c = \mathrm{P}^c$, and the received and posted initial margins may be computed respectively as an α_{RIM} and α_{PIM} value-at-risk of the *gap risk*, which is the risk that the valuation of the transaction may fluctuate considerably during a short period of time.[57] Let δ denote a fixed short period of time, which may correspond to the liquidation period of a contract; then, we set, for each netting set c and $0 \le t \le T$,

$$\mathrm{RIM}^c_t = \mathrm{VaR}^{\alpha_{\mathrm{RIM}}}_t \left[\mathrm{P}^c_{t+\delta} + \left(\mathcal{P}^c_{t+\delta} - \mathcal{P}^c_{t-}\right) - \mathrm{P}^c_t\right], \tag{6.43}$$

$$\mathrm{PIM}^c_t = \mathrm{VaR}^{\alpha_{\mathrm{PIM}}}_t \left[-\left(\mathrm{P}^c_{t+\delta} + \left(\mathcal{P}^c_{t+\delta} - \mathcal{P}^c_{t-}\right) - \mathrm{P}^c_t\right)\right], \tag{6.44}$$

where, for $\alpha \in [0,1]$, VaR^α_t denotes the level α value-at-risk with respect to $\mathbb{P}$ conditional on $\mathscr{F}_t$.[58] In the case with "no CSA" we have $\mathrm{VM}^c = \mathrm{RIM}^c = \mathrm{PIM}^c = 0$.

The last thing to model before we state the XVA equations are the default times $\tau_1, \dots, \tau_{10}$ of the clients and τ of the bank itself. Hence, we have a total of eleven stopping times to model. One way to do this is described in [CBB14, Chapter 8] as the "common-shock" model, which is a generalisation of the Marshall Olkin model (see [MO67]). For example, in [Alb+19] this common-shock model is used with Cox-Ingersoll-Ross default intensities, as per [CBB14, Example 8.2.12 (ii)].Alternatively, one can also use a Gaussian copula model, as presented in [Ces+09, Section 2.7.2].

[57] This is the same as in [Alb+19], and it is in line with [Gre15, Chapter 6].

[58] Where $(\mathscr{F}_t)_t$ is the filtration that contains all the informations except for the default time τ of the bank.

To implement the common shock model for default times in our case study, we let $\mathcal{Z}$ be the set of all subsets of $I := \{1, \dots, 10, b\}$ and

$$\mathcal{Y} := \{\{1\}, \dots, \{10\}, \{b\}, I_1, \dots, I_m\} \subseteq \mathcal{Z},$$

where $I_1, \dots, I_m$ are so-called multi-name groups that contain at least two elements and represent a shock causing the default of all their contained elements. For example one could choose, in a similar way as in [CBB14, Section 8.4.3], $I_l = \{1, \dots, l+1\}$, for $1 \le l \le 9$, and $I_{10} = I$. Then, for each $Y \in \mathcal{Y}$, one can model the intensity γ^Y by an extended CIR model

$$d\gamma_t^Y = a(b_Y(t) - \gamma_t^Y)dt + c\sqrt{\gamma_t^Y}dW_t^Y, \quad 0 \le t \le T, \tag{6.45}$$

for non-negative constants a, c independent of Y, a non-negative function b_Y, and a $(\mathbb{G}, \mathbb{Q})$ standard Brownian motion W^Y (all the W^Y, $Y \in \mathcal{Y}$ are independent of each other). Once one has the γ^Y, for all $Y \in \mathcal{Y}$, one takes $\Gamma^Y := \int_0^\cdot \gamma_s^Y ds$, and define

$$\eta_Y := \inf\{t > 0 \colon \Gamma_t^Y > \varepsilon_Y\}, \tag{6.46}$$

where ε_Y, $Y \in \mathcal{Y}$, are i.i.d. exponentially distributed with parameter 1. Finally, the default time of each $i \in I$, is given by

$$\tau_i := \min\{\eta_Y \colon i \in Y \in \mathcal{Y}\}. \tag{6.47}$$

Thus, the default of the bank is given by $\tau := \tau_b$. By [CBB14, Lemma 8.2.2], we know that the intensity γ of the $\mathbb{G}$-stopping time τ (in the sense of Definition A.20) is given by

$$\gamma := \sum_{\substack{Y \in \mathcal{Y} \\ i \in Y}} \gamma^Y. \tag{6.48}$$

Remark 6.12. Theoretically, once we know τ, we can compute the Azéma supermartingale S as the $(\mathbb{F}, \mathbb{Q})$-optional projection of $\mathbb{1}_{[\![0,\tau]\!]}$, and then compute its Doob-Meyer decomposition $S = S_0 + Q - D$, where Q is a uniformly integrable $(\mathbb{F}, \mathbb{Q})$-martingale started from 0 and D is an $\mathbb{F}$-predictable $\mathbb{Q}$-integrable increasing process (in the sense of (A.7)). With this, we can compute the invariance measure $\mathbb{P}$ on $\mathscr{F}_T$ by (B.4)

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathscr{F}_T} = \mathcal{E}\left(\frac{1}{S_-} \bullet Q\right)_T.$$

We do not discuss here how one could implement such an invariance measure. One way to bypass this issue is to consider a simpler immersion setup, as described in Remark B.12.

Remark 6.13. Note that, modelling the default times of the clients and the bank completely independently of the cash-flows $\mathcal{P}^c$ would result in an underestimation of the so-called "wrong-way-risk", which is the risk that the credit exposure increases when the counterparty is more likely to default.[59]

Now we have all ingredients to compute the XVAs. First, recall that we only need to compute the $\mathbb{F}$-optional reduction of each XVA, and then stop it before τ to obtain the desired add-on.[60]

[59] See [Gre15, Chapter 14].

[60] Compare with Theorem 6.10 and Theorem 5.9.

We start with CVA. Let $R_1, \dots, R_{10} \in [0,1]$ denote the recovery rates relative to the default of each client. Then, the process CVA$'$ is given by

$$\mathrm{CVA}'_t = \sum_{i=1}^{10}(1-R_i)\mathbb{E}'_t\left[\mathbb{1}_{\{t<\tau_i\le T_{20}\}}\left(\mathrm{P}^{c_i}_{\tau_i^\delta} + \left(\mathcal{P}^{c_i}_{\tau_i^\delta} - \mathcal{P}^{c_i}_{\tau_i-}\right) - \mathrm{VM}^{c_i}_{\tau_i-} - \mathrm{RIM}^{c_i}_{\tau_i-}\right)^+\right], \tag{6.49}$$

for $0 \le t \le T$. Note that, if $\tau_i > T_{20}$, the default of the client happens after the last payment, so the default does not impact on CVA. Thus, once we have implemented all the promised cash-flows, initial margins and default times, we directly obtain CVA by computing a conditional expectation. To simplify the notation in the sequel we write

$$\Delta_i^+ := \left(\mathrm{P}^{c_i}_{\tau_i^\delta} + \left(\mathcal{P}^{c_i}_{\tau_i^\delta} - \mathcal{P}^{c_i}_{\tau_i-}\right) - \mathrm{VM}^{c_i}_{\tau_i-} - \mathrm{RIM}^{c_i}_{\tau_i-}\right)^+, \quad i = 1, \dots, 10. \tag{6.50}$$

Recall that the stopping times τ_i^δ denote the liquidation time after default of the client i; we may assume $\tau_i^\delta := \tau_i + \delta$, for a fixed short period of time δ, such as two weeks.

Next, we can compute the $\mathbb{F}$-optional reduction of MVA, which is also given by a conditional expectation. Let γ be the default intensity of the bank (that is given by (6.48)), and $\bar{R}$ the recovery rate to the external funder in case of default of the bank relative to the funding of the initial margin (below we use a different recovery rate R that is related to the funding of variation margin). Then, we have, for $0 \le t \le T$,

$$\mathrm{MVA}'_t = (1-\bar{R})\mathbb{E}'_t\left[\int_t^T \gamma_s \mathrm{PIM}_s ds\right], \tag{6.51}$$

where PIM is the total initial margin posted by the bank and it is given by[61]

$$\mathrm{PIM}_t := \sum_{i=1}^{10} \mathrm{PIM}^{c_i}_t \mathbb{1}_{[\![0,\tau_i^\delta]\!]}, \quad 0 \le t \le T. \tag{6.52}$$

Next, we compute the $\mathbb{F}$-optional reduction of FVA by solving a BSDE. Let R denote the recovery rate related to the funding of variation margin, let MtM and VM be the total clean valuation of the portfolio of the bank and total exchanged variation margin, which are respectively given by[62]

$$\mathrm{MtM}_t = \sum_{i=1}^{10} \mathrm{P}^{c_i}_t \mathbb{1}_{[\![0,\tau_i^\delta]\!]}, \quad \text{and} \quad \mathrm{VM}_t = \sum_{i=1}^{10} \mathrm{VM}^{c_i}_t, \quad 0 \le t \le T. \tag{6.53}$$

Then, once CVA$'$ and MVA$'$ are determined, FVA$'$ satisfies the following BSDE on $[0,T]$:

$$\begin{aligned}\mathrm{FVA}'_t = &\int_t^T (1-R)\gamma_s \left(\mathrm{MtM}_s - \mathrm{VM}_s - \mathrm{CVA}'_s - \mathrm{MVA}'_s - \mathrm{FVA}'_s\right)^+ ds \\ &- \int_t^T Z_s dW_s - \int_t^T \int_{\mathcal{U}} \psi_s(u)\tilde{\mu}(du, ds),\end{aligned} \tag{6.54}$$

where W is a one dimensional standard Brownian motion and $\tilde{\mu}$ is a compensated homogeneous Poisson random measure; see Appendix A.2.

[61]Compare with (6.10).
[62]Compare with (3.3) and (6.3).

Remark 6.14. Note that this is a simplified version of the more general BSDE (5.23). Here we assume that any local martingale M can be represented as in Proposition A.32 with $N = M_0$, which is equivalent to say that the semimartingale X with continuous component $X^c = W$ and associated jump-measure $\mu^X = \mu$ has the weak property of predictable representation, according to [HWY92, Definition 13.13].

After the computation of CVA′, MVA′, and FVA′ it is possible to compute the clean loss L′, which corresponds to the $\mathbb{F}$-optional reduction of the trading loss process L. In fact, we have, for $0 \leq t \leq T$,[63]

$$\begin{aligned} \mathrm{L}'_t = \mathrm{CVA}'_t - \mathrm{CVA}'_0 + \sum_{i=1}^{10}(1-R_i)\Delta_i^+ \mathbb{1}_{\{\tau_i^\delta \leq t\}} + \mathrm{MVA}'_t - \mathrm{MVA}'_0 + (1-\bar{R})\int_0^t \gamma_s \mathrm{PIM}_s ds \\ + \mathrm{FVA}'_t - \mathrm{FVA}'_0 + (1-R)\int_0^t \gamma_s \big(\mathrm{MtM}_s - \mathrm{VM}_s - \mathrm{CVA}'_s - \mathrm{MVA}'_s - \mathrm{FVA}'_s\big)^+ ds. \end{aligned} \tag{6.55}$$

Then, we can use L′ to compute the economic capital EC_t, as in (6.30).

Lastly, we can write and solve the BSDE for the $\mathbb{F}$-optional reduction of KVA, which is given by (5.4) (with $dN = 0$):

$$\mathrm{KVA}'_t = \int_t^T h\left(\mathrm{EC}_s - \mathrm{KVA}'_s\right)^+ ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathcal{U}} \psi_s(u)\tilde{\mu}(du, ds), \tag{6.56}$$

where h is an exogenously given constant hurdle rate.

Once we have obtained all the $\mathbb{F}$-optional versions of each XVA, we can easily compute

$$\mathrm{CVA} = (\mathrm{CVA}')^\circ,\ \mathrm{MVA} = (\mathrm{MVA}')^\circ,\ \mathrm{FVA} = (\mathrm{FVA}')^\circ,\ \mathrm{KVA} = (\mathrm{KVA}')^\circ. \tag{6.57}$$

However, if one is just interested in computing the entry prices that the clients pay at inception of the contract, it is sufficient to compute the reduced processes XVA′, and value them at time 0, since $\mathrm{XVA}_0 = \mathrm{XVA}'_0$.

The next step would be to actually implement this case study portfolio. After this, it would be interesting to enhance the given example to the level of a true banking portfolio, with thousands of transactions and different types of derivatives.

[63] Compare with (6.29).

A Semimartingale Theory and BSDEs

A.1 Semimartingales theory and stochastic calculus

In this section we introduce the necessary basic results of the theory of semimartingales and stochastic calculus in continuous time. Our main reference is [HWY92].

A.1.1 Stochastic processes and stopping times

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a general probability space with a right-continuous filtration $\mathbb{F} = (\mathscr{F}_t)_{t\in[0,\infty[}$, such that $\mathscr{F}_0$ contains all the $\mathbb{P}$-nullsets, that is, $\mathbb{F}$ satisfies the usual condition.[64] Any stochastic object in this section is assumed to be defined on this probability space. We denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$.

We say that a stochastic process $X\colon \Omega \times [0,\infty[\to \mathbb{R}$ is right-continuous (resp. left-continuous, resp. càdlàg) if there is a measurable set $A \in \mathscr{F}$ with probability one such that, for any $\omega \in A$, the trajectory $X_.(\omega)$ is right-continuous (resp. left-continuous, resp. càdlàg).[65] Two processes X and Y are *modification of each other* if, for any $t \in [0,\infty[$, $X_t = Y_t$ a.s.. They are *indistinguishable* if there is a set $A \in \mathscr{F}$ of probability one such that, for any $\omega \in A$, $X_.(\omega) = Y_.(\omega)$. Any equality between two stochastic processes is intended in the sense of indistinguishability, and when we speak of uniqueness of a process, we mean up to indistinguishability.

We denote by $\mathscr{E}(\mathbb{F})$, $\mathscr{O}(\mathbb{F})$ and $\mathscr{P}(\mathbb{F})$ the *progressive*, *optional* and *predictable* σ-fields on $\Omega \times [0,\infty[$ with respect to the filtration $\mathbb{F}$, respectively,[66] as defined in [HWY92, Section 3.2]. We say that a stochastic process X is *progressively measurable* (respectively *optional*, respectively *predictable*) if it is measurable with respect to $\mathscr{E}(\mathbb{F})$ (respectively $\mathscr{O}(\mathbb{F})$, respectively $\mathscr{P}(\mathbb{F})$). In particular, a stochastic process X is progressively measurable if, for all $t \geq 0$, X restricted on $\Omega \times [0,t]$ is $\mathscr{F}_t \otimes \mathscr{B}([0,t])$-measurable, where $\mathscr{F}_t \otimes \mathscr{B}([0,t])$ denotes the product σ-field of $\mathscr{F}_t$ and the Borel σ-field on the interval $[0,t]$. Note that $\mathscr{P}(\mathbb{F}) \subseteq \mathscr{O}(\mathbb{F}) \subseteq \mathscr{E}(\mathbb{F})$. In general, when we introduce a stochastic process X, we only assume that it is *measurable*, in the sense that $X\colon \Omega \times [0,\infty[\to \mathbb{R}$ is $\mathscr{F} \otimes \mathscr{B}([0,\infty[)$-measurable.

Putting together Theorem 2.46 and 2.47 in [HWY92], we have the following result.

Proposition A.1. *If X is an $\mathbb{F}$-supermartingale such that the map $t \mapsto \mathbb{E}[X_t]$ is right-continuous, then there exists a modification of X which is a càdlàg $\mathbb{F}$-supermartingale.*

Definition A.2. For an $\mathbb{F}$-stopping time $\theta\colon \Omega \to [0,\infty]$, we define

$$\mathscr{F}_\theta := \{A \in \mathscr{F}_\infty \colon\ \forall t \geq 0\ A \cap \{\theta \leq t\} \in \mathscr{F}_t\}; \tag{A.1}$$

$$\mathscr{F}_{\theta-} := \sigma\left(\mathscr{F}_0, \{A \cap \{t < \theta\} \colon\ A \in \mathscr{F}_t,\ t \geq 0\}\right), \tag{A.2}$$

where $\mathscr{F}_\infty := \sigma\left(\mathscr{F}_t,\ t \geq 0\right)$.

Note that, for a constant stopping time $\theta = t > 0$, $\mathscr{F}_{\theta-} = \mathscr{F}_{t-} = \sigma(\mathscr{F}_s,\ s < t)$, and $\mathscr{F}_{0-} = \mathscr{F}_0$. See Definition 3.3 and Theorem 3.4 of [HWY92] for more details on the definition and some basic properties of stopping times.

[64] The filtration $\mathbb{F}$ in this section may represent both the filtrations $\mathbb{G}$ and $\mathbb{F}$ introduced in Chapter 2.

[65] Note that this is different than in [HWY92], where a right-continuous process has right-continuous trajectory for each $\omega \in \Omega$; our definition corresponds to the "a.s. right-continuity" in [HWY92].

[66] See [HWY92, Definition 3.10 and 3.15] for more details.

Definition A.3. A *stochastic set* is a set $B \subseteq \Omega \times [0,\infty[$ such that the indicator process $\mathbb{1}_B = (\mathbb{1}_B(\cdot,t))_{t\geq 0}$ is $\mathbb{F}$-progressively measurable. A particular type of stochastic sets we are interested in are *stochastic intervals*, which are defined as follows: for two given stopping times $\theta \leq \eta$, put

$$[\![\theta,\eta]\!] := \{(\omega,t) \in \Omega \times [0,\infty[\colon\ \theta(\omega) \leq t \leq \eta(\omega)\}\,, \tag{A.3}$$

and analogously $[\![\theta,\eta[\![$, $]\!]\theta,\eta]\!]$, and $]\!]\theta,\eta[\![$. When $\theta = \eta$, we write $[\![\theta]\!] := [\![\theta,\theta]\!]$, which is called the *graph of* θ.

Definition A.4. A stochastic set B is said to be *thin* if there exist countable many stopping times $(\theta_n)_{n\in\mathbb{N}}$ such that $B = \bigcup_n [\![\theta_n]\!]$. It is said to be *evanescent* if its projection

$$\pi(B) := \{\omega \in \Omega \,|\, \exists\, t \geq 0 \colon (\omega,t) \in B\}$$

is a $\mathbb{P}$-null set. A process X is said to be thin (respectively evanescent) if the set

$$\{(\omega,t) \in \Omega \times [0,\infty[\,|\, X_t(\omega) \neq 0\}$$

is thin (respectively evanescent).

Definition A.5. A stopping time θ is called *predictable time* if $[\![\theta,\infty[\![\in \mathscr{P}(\mathbb{F})$. A sequence of stopping times $(\theta_n)_{n\in\mathbb{N}}$ is said to *foretell* a stopping time θ if, on $\{\theta > 0\}$, $\theta_n < \theta$ for each n and $\theta_n \uparrow \theta$. Then, θ is called *fortellable.* A stopping time θ is *totally inaccessible* if $\mathbb{P}[\theta = \eta < \infty] = 0$, for any predictable time η. A stopping time θ is called *accessible* if there exists a sequence of predictable times $(\theta_n)_n$ such that $[\![\theta]\!] \subseteq \bigcup_n [\![\theta_n]\!]$.

Definition A.6. The filtration $\mathbb{F}$ is said to be *quasi-left-continuous* if $\mathscr{F}_\theta = \mathscr{F}_{\theta-}$, for any predictable time θ.

By [HWY92, Theorem 3.27], a fortellable stopping time is a predictable time. Conversely, since we assume the completeness of $\mathbb{F}$, by [HWY92, Theorem 4.34], any predictable time is foretellable. Moreover, by [HWY92, Theorem 3.40.1)], the filtration $\mathbb{F}$ is quasi-left-continuous if and only if all accessible times are predictable times. Hence, in our setup we have the following:

Proposition A.7. *A stopping time is foretellable if and only if it is a predictable time. Assuming that the filtration $\mathbb{F}$ is quasi-left-continuous, a stopping time is accessible if and only if it is a predictable time.*

Proposition A.8. *Let θ be a stopping time. Then, there exists a (a.s. unique) set $B \in \mathscr{F}_{\theta-}$ with $A \subseteq \{\theta < \infty\}$ such that $\theta^a := \theta \mathbb{1}_B + (+\infty)\mathbb{1}_{\Omega\setminus B}$ is an accessible stopping time and $\theta^i := (+\infty)\mathbb{1}_B + \theta\mathbb{1}_{\Omega\setminus B}$ is a totally inaccessible stopping time. θ^a and θ^i are respectively called* accessible part *and* totally inaccessible part *of θ.*

Proof. See [HWY92, Theorem 4.20]. □

So, we can see that the graph of the stopping time θ can be (uniquely) decomposed as the disjoint union of the graphs of θ^a and θ^i: $[\![\theta]\!] = [\![\theta^a]\!] \cup [\![\theta^i]\!]$. Also, if $\mathbb{F}$ is quasi-left-continuous, we can say that the accessible part θ^a is a predictable time. From this, we also have the following result.

Proposition A.9. *For any càdlàg adapted process X, one can find a sequence of strictly positive stopping times $(\theta_n)_n$ that are either predictable or totally inaccessible, have disjoint graphs (in the sense that $[\![\theta_n]\!] \cap [\![\theta_m]\!] = \emptyset$ for $n \neq m$), and* exhaust the jumps of X, *in the sense that*

$$\{(\omega, t) \in \Omega \times [0, \infty[\mid \Delta X_t(\omega) \neq 0\} \subseteq \bigcup_n [\![\theta_n]\!].$$

Proof. See [HWY92, Theorem 4.21]. □

Note that, by ΔX_t we denote the jump of X and time $t > 0$, that is

$$\Delta X_t := X_t - X_{t-} = X_t - \lim_{s \uparrow t} X_s. \tag{A.4}$$

If $t = 0$, we use the convention $X_{0-} = X_0$ and $\Delta X_0 = 0$.

Proposition A.10. *Let X be a càdlàg martingale stopped at some positive time $T > 0$, and assume that the filtration $\mathbb{F}$ is quasi-left-continuous. Then, X has only totally inaccessible jumps.*[67]

Proof. By [HWY92, Theorem 4.41], we have

$$\mathbb{E}[X_\theta \mid \mathscr{F}_{\theta-}] = X_{\theta-} \quad \text{a.s.},$$

for any $\mathbb{F}$-predictable time θ. Since $\mathbb{F}$ is quasi-left-continuous, $\mathscr{F}_\theta = \mathscr{F}_{\theta-}$, and thus $X_\theta = \mathbb{E}[X_\theta \mid \mathscr{F}_\theta] = X_{\theta-}$ a.s.. Hence, X cannot have predictable jumps, but it only has totally inaccessible jumps. □

Another important result form the classical semimartingale theory that we need, is the Doob-Meyer decomposition of supermartingales. We say that a process X is *of class(D)*, if the set $\{X_\theta \mathbb{1}_{\{\theta < \infty\}} \mid \theta \text{ stopping time}\}$ is uniformly integrable.

Theorem A.11 (Doob-Meyer Decomposition)**.** *Let X be a right-continuous supermartingale of class(D). Then, X can be uniquely decomposed as $X = M - A$, where M is a uniformly integrable martingale and A is a predictable integrable increasing process (in the sense of* (A.7)*) starting from zero.*

Proof. See [HWY92, Theorem 5.48]. □

A.1.2 Optional and predictable projections

We now define the important concept of optional and predictable projection of a process. By [HWY92, Definition 1.15], a random variable ξ is said *σ-integrable* with respect to a sub-σ-field $\mathscr{G} \subseteq \mathscr{F}$ if there exists a sequence $(\Omega_n)_{n \in \mathbb{N}} \subseteq \mathscr{G}$ with $\Omega_n \uparrow \Omega$ such that, for each n, $\xi \mathbb{1}_{\Omega_n} \in \mathrm{L}^1(\Omega, \mathscr{F}, \mathbb{P})$.

[67] In [HWY92, Definition 4.22] a process with only totally inaccessible jumps is called *quasi-left-continuous*.

Proposition-Definition A.12. By [HWY92, Theorem 5.1], if a measurable process X is such that, for all stopping time θ, $X_\theta \mathbb{1}_{\{\theta<\infty\}}$ is σ-integrable with respect to $\mathscr{F}_\theta$, then there exists a unique optional process ${}^oX$ such that

$$\mathbb{E}[X_\theta \mathbb{1}_{\{\theta<\infty\}} \,|\, \mathscr{F}_\theta] = {}^oX_\theta \mathbb{1}_{\{\theta<\infty\}} \text{ a.s., } \forall\, \theta \text{ stopping time.} \tag{A.5}$$

The process ${}^oX$ is called *optional projection* of X. Similarly, by [HWY92, Theorem 5.2], if $X_\theta \mathbb{1}_{\{\theta<\infty\}}$ is σ-integrable with respect to $\mathscr{F}_{\theta-}$ for all predictable times θ, then there exists a unique predictable process ${}^pX$ such that

$$\mathbb{E}[X_\theta \mathbb{1}_{\{\theta<\infty\}} \,|\, \mathscr{F}_{\theta-}] = {}^pX_\theta \mathbb{1}_{\{\theta<\infty\}} \text{ a.s., } \forall\, \theta \text{ predictable time.} \tag{A.6}$$

The process ${}^pX$ is called *predictable projection* of X.

Remark A.13. Note that, if X is a progressive measurable process, then X_θ is $\mathscr{F}_\theta$-measurable, for any stopping time θ. Thus, we can see that $X_\theta \mathbb{1}_{\{\theta<\infty\}}$ is actually σ-integrable with respect to $\mathscr{F}_\theta$, by choosing $\Omega_n := \{|X_\theta \mathbb{1}_{\{\theta<\infty\}}| \le n\}$ in the definition of σ-integrability. So, if we have a progressive process X, the optional projection always exists. Assuming that the filtration $\mathbb{F}$ is quasi-left-continuous, we can conclude the same regarding the existence of the predictable projection.

We also recall the following "smoothing property" of [HWY92, Theorem 5.4].

Proposition A.14. *If the predictable projection of a process X exists and Y is a predictable process, then ${}^p(XY) = {}^pXY$.*

We now define the dual predictable projection of a non-decreasing non-negative right-continuous measurable process A which is integrable,[68] in the sense that

$$\mathbb{E}[A_\infty] = \lim_{t\to\infty} \mathbb{E}[A_t] < \infty. \tag{A.7}$$

In this thesis, we simply call any process like A a ($\mathbb{P}$-)*integrable increasing* process—note that we do not require adaptedness. We define the integral of a process H with respect to A as in [HWY92, Definition 3.45], and we denote it as

$$H\centerdot A = \int_0^\cdot H_s dA_s = \int_{]0,\cdot]} H_s dA_s.$$

Note that the integral starts after zero, that is $\int_{[0,t]} H_s dA_s = H_0 A_0 + (H\centerdot A)_t$.

Summing up Definition 5.10, 5.17, and 5.21 of [HWY92], we get the following.

Definition A.15. The *dual predictable projection*, or *compensator* A^p of A is the unique non-decreasing predictable process such that, for any bounded measurable process X, it satisfies

$$\mathbb{E}\left[\int_{[0,\infty[} {}^pX_s dA_s\right] = \mathbb{E}\left[\int_{[0,\infty[} X_s dA^p_s\right]. \tag{A.8}$$

Note that, if A is already predictable, then $A^p = A$ (this is an easy consequence of [HWY92, Theorem 5.13]).

In Appendix B we make use of the following two important propositions.

[68] Actually, local integrability in the sense of [HWY92, Definition 5.18] is enough, but in this thesis we only see dual predictable projections of integrable processes.

Proposition A.16. *Let A be a predictable integrable increasing process, and H a non-negative process such that ${}^pH$ exists and $H\centerdot A$ is an integrable non-decreasing process. Then,*

$$(H\centerdot A)^p = {}^pH\centerdot A.$$

Proof. See [HWY92, Theorem 5.25.2)]. □

Remark A.17. Integrating with respect to the deterministic process given by $A_t = t$, $t \geq 0$, is equivalent to integrating with respect to the Lebesgue measure on $[0,\infty[$, which we denote by λ. For a process H that is integrable with respect to A, we write

$$H\centerdot\lambda := H\centerdot A = \int_0^\cdot H_s ds.$$

This integral is continuous (for each $\omega \in \Omega$), and thus it is predictable. As the dual predictable projection of a predictable process is the process itself, by the previous proposition, we get that

$$H\centerdot\lambda = (H\centerdot\lambda)^p = {}^pH\centerdot\lambda.$$

Proposition A.18. *Let A be an adapted integrable increasing process and B a predictable integrable increasing processes. Then, B is the compensator of A if and only $A - B$ is a uniformly integrable martingale starting from zero.*

Proof. See [HWY92, Corollary 5.31.1)] (recalling that when we say process we already mean progressively measurable process). □

Remark A.19. As in [CS17], we call *compensator of a stopping time θ*, the compensator of the increasing process $A = \mathbb{1}_{[\![\theta,\infty[\![}$. In [CS17] a stopping time θ is defined to be totally inaccessible if $\theta > 0$ and its compensator is continuous on $[0,\theta]$. To show that this is equivalent to our definition, we use [HWY92, Theorem 5.27.2)], which states the following:

$$\Delta A^p_\eta \mathbb{1}_{\{\eta<\infty\}} = \mathbb{E}[\Delta A_\eta \mathbb{1}_{\{\eta<\infty\}} \,|\, \mathscr{F}_{\eta-}] \text{ a.s., for all predictable times } \eta.$$

For simplicity, we also assume that $\mathbb{F}$ is quasi-left-continuous. The only jump of A happens at the stopping time θ. If θ is totally inaccessible, then for any predictable time η, $\Delta A_\eta = 0$, so A^p is continuous, and one direction is proved. Let θ^a and θ^i denote the accessible and totally inaccessible parts of θ as in Proposition A.8. Then, we can decompose

$$A = \mathbb{1}_{[\![\theta^a,\infty[\![} + \mathbb{1}_{[\![\theta^i,\infty[\![} =: A^a + A^i.$$

Assume that θ is not totally inaccessible, that is $\mathbb{P}[\theta^a < \infty] > 0$. As A^i has only a jump at the totally inaccessible time θ^i, for any predictable time η, $\Delta A^i_\eta = 0$. Thus, for the predictable time $\eta = \theta^a$ (which is predictable by Proposition A.7), we have

$$\Delta A^p_{\theta^a} = \mathbb{E}[\mathbb{1}_{\{\theta^a<\infty\}} \,|\, \mathscr{F}_{\theta^a-}] = \mathbb{1}_{\{\theta^a<\infty\}} \text{ a.s.,}$$

that is, A^p is not continuous on $[0,\theta]$. This shows the equivalence of the two definitions.

Definition A.20. Let θ by a totally inaccessible stopping time and $A = \mathbb{1}_{[\![\theta,\infty[\![}$. If A^p is also absolutely continuous with respect to the Lebesgue measure, in the sense that there exists a non-negative process γ on $[0,\theta]$ such that

$$dA^p_t = \gamma_t dt, \quad 0 \leq t \leq \theta,$$

then θ is said to have an *intensity* (and γ is the intensity of θ).

A.1.3 Semimartingales and stochastic integrals

We define the class of *local martingales* $\mathcal{M}_{loc}$ as in [HWY92, Definition 7.11], that is, as the localised class[69] of uniformly integrable martingales, denoted by $\mathcal{M}$. Thus, any local martingale is already assumed to be adapted and càdlàg.

Proposition A.21. *A local martingale is a uniformly integrable true martingale if and only if it is of class(D).*

Proof. See [HWY92, Theorem 7.12]. □

Proposition A.22. *A local martingale M can be uniquely decomposed as $M = M_0 + M^c + M^d$, where M^c is a continuous local martingale started from zero and M^d is a purely discontinuous local martingale.*

Proof. See [HWY92, Theorem 7.25]. □

Definition A.23. A *process of finite variation* A is defined as the difference of two non-negative non-decreasing right-continuous processes. The class of *semimartingales* $\mathcal{S}$ contains all the processes $X = M + A$, where $M \in \mathcal{M}_{loc}$ (called the local martingale part) and A is a process of finite variation (called the finite variation part). For any $X \in \mathcal{S}$ with decomposition $X = M + A$, we set $X^c = M^c$, the continuous part of the semimartingale X. Note that the continuous part X^c does not depend on the decomposition, but it is uniquely determined by X, because a continuous local martingale that is also purely discontinuous has to be equal to zero (see [HWY92, Definition 7.21 and Lemma 7.22]). We also define the *quadratic covariation* of two semimartingales X and Y as a finite variation process started from zero given by

$$[X, Y]_t := \langle X^c, Y^c \rangle_t + \sum_{s \leq t} \Delta X_s \Delta Y_s, \quad t \geq 0, \tag{A.9}$$

where the *predictable quadratic covariation* term $\langle X^c, Y^c \rangle$ is defined as the unique locally integrable predictable process of finite variation such that $X^c Y^c - \langle X^c, Y^c \rangle$ is a locally square integrable martingale.[70]

Remark A.24. Note that the above definition of quadratic covariation of two semimartingales is well-defined, since the continuous part of a semimartingale does not depend on the decomposition and the sum converges a.s. for all $t \geq 0$, by [HWY92, Lemma 7.27]. Moreover, note that we used a different convention than [HWY92] here, as we assumed the quadratic covariation starts from zero, removing the term $X_0 Y_0$ from the definition. This is in line with [CS17].

Definition A.25. If the finite variation part of a semimartingale is locally integrable, the semimartingale is called *special semimartingale*.

Proposition-Definition A.26. For any special semimartingale X, there is exactly one so-called *canonical decomposition* $X = M + A$ such that $M \in \mathcal{M}_{loc}$ and A is a predictable process of finite variation starting from zero. A sufficient (and also necessary) condition for a semimartingale X to be a special semimartingale is that the "running maximum" $X_t^* := \sup_{0 \leq s \leq t} |X_s|$, $t \geq 0$, is locally integrable.

[69] See also [HWY92, Definition 7.1].

[70] Which exists by [HWY92, Theorem 7.28], since continuous local martingales are locally square integrable.

Proof. See [HWY92, Theorem 8.5 and 8.6]. □

An important tool in the theory of invariance times in [CS17] are the so-called *sets of interval type*. These are sets $B \in \Omega \times [0, \infty[$ for which there exists a random variable $\theta\colon \Omega \to [0, \infty]$ such that for each $\omega \in \Omega$ the section $B_\omega = \{t \in [0, \infty[\,|\, (\omega, t) \in B\}$ is either given by $[0, \theta(\omega)[$ or $[0, \theta(\omega)]$ and $B_\omega \neq \emptyset$. In particular, [CS17] focuses on predictable sets of interval type, which can be characterised as follows.

Proposition A.27. *Let $B \in \Omega \times [0, \infty]$. Then B is a predictable set of interval type if and only if there exists an increasing sequence of stopping times $(\theta_n)_{n\in\mathbb{N}}$ (called a fundamental sequence of B) such that $B = \bigcup_n [0, \theta_n]$.*

Proof. See [HWY92, Theorem 8.16]. □

For a given predictable set of interval type B with fundamental sequence $(\theta_n)_{n\in\mathbb{N}}$, we say that a process X defined on B is a semimartingale on B if, for each n, the stopped process X^{θ_n} is a semimartingale (on $[0, \infty[$). Hence, as pointed out in [CS17], the stochastic calculus on B simply reduces to standard stochastic calculus on $[0, \infty[$ for each X^{θ_n}.

For the definitions and basic properties of stochastic integrals with respect to local martingales and semimartingales we refer to [HWY92, Chapter 9]. In particular, for a predictable process H integrable with respect to a local martingale M, we denote the stochastic integral by

$$H_{\bullet}M = \int_{]0,\cdot]} H_s dM_s = \int_0^{\cdot} H_s dM_s.$$

By [HWY92, Theorem 7.32], a sufficient condition on H such that the local martingale $H_{\bullet}M$ is a true martingale, is that the quadratic variation of the stochastic integral $[H_{\bullet}M] = H^2{}_{\bullet}[M]$ is integrable, that is, $\mathbb{E}\big[\int_0^\infty H_s^2 d[M]_s\big] < \infty$ (actually, in this case $H_{\bullet}M$ is also bounded in L^2).

Definition A.28. For a semimartingale X, we denote by $\mathcal{E}(X)$ the *stochastic exponential* of X, which is the unique semimartingale Z such that a.s. for all $t \geq 0$[71]

$$Z_t = 1 + \int_0^t Z_{s-} dX_s.$$

Lastly, we recall the important Itô's formula (see [HWY92, Theorem 9.35]).

Theorem A.29. *Let $d \in \mathbb{N}$, $X^1, \dots, X^d$ be semimartingales, and $F\colon \mathbb{R}^d \to \mathbb{R}$ be C^2. Writing $X = (X^1, \dots, X^d)$, we have, $\mathbb{P}$-a.s. for all $t \geq 0$,*

$$\begin{aligned} F(X_t) - F(X_0) = &\sum_{j=1}^d \int_0^t D_j F(X_{s-}) dX_s^j + \sum_{0<s\leq t} \left(\Delta F(X_s) - \sum_{j=1}^d D_j F(X_{s-}) \Delta X_s^j \right) \\ &+ \sum_{i,j=1}^d \int_0^t D_{i,j} F(X_{s-}) d\langle (X^i)^c, (X^j)^c \rangle_s, \end{aligned}$$

where D_j and $D_{i,j}$ denote partial derivatives.

[71]See [HWY92, Theorem 9.39].

A.1.4 The representation property

To understand the results in [KP15] on the theory of BSDEs, we need to introduce the concept of martingale representation property. To this end, we first give a brief summary on random measures and characteristics of semimartingales from [HWY92, Chapter 11].

Let $\mathcal{U} \subseteq \mathbb{R}^m \setminus \{0\}$, for some $m \in \mathbb{N}$, and denote by $\mathscr{B}(\mathcal{U})$ the Borel σ-field on $\mathcal{U}$.

Definition A.30. We call a map $\mu \colon \Omega \times (\mathscr{B}([0,\infty[) \otimes \mathscr{B}(\mathcal{U})) \to [0,\infty]$,[72] a *random measure* if

i) $\forall\, \omega \in \Omega \colon\; \mu(\omega, \cdot)$ is a σ-finite measure on $\mathscr{B}([0.\infty[) \otimes \mathscr{B}(\mathcal{U})$;

ii) $\forall\, \hat{B} \in \mathscr{B}([0,\infty[) \otimes \mathscr{B}(\mathcal{U}) \colon \mu(\cdot, \hat{B}) \colon \Omega \to [0,\infty]$ is an $\mathscr{F}$-measurable random variable.

A random measure μ generates the following measuregiven by

$$M_\mu(\tilde{B}) := \mathbb{E}\left[\int_{[0,\infty[\times\mathcal{U}} \mathbb{1}_{\tilde{B}}(\cdot, t, u)\mu(\cdot, dt, du)\right], \quad \tilde{B} \in \widetilde{\mathscr{F}} \tag{A.10}$$

on the measurable space $\Big(\Omega \times [0,\infty[\times\mathcal{U}, \quad \widetilde{\mathscr{F}} := \mathscr{F} \otimes \mathscr{B}([0,\infty[) \otimes \mathscr{B}(\mathcal{U})\Big)$.

We say that μ is *predictably* (respectively *optionally*) *σ-integrable* if M_μ is σ-finite on the product σ-field $\widetilde{\mathscr{P}} := \mathscr{P}(\mathbb{F}) \otimes \mathscr{B}(\mathcal{U})$ (respectively $\widetilde{\mathscr{O}} := \mathscr{O}(\mathbb{F}) \otimes \mathscr{B}(\mathcal{U})$). Let ψ be a measurable real-valued function on $\Omega \times [0,\infty[\times\mathcal{U}$ such that, for all $t \geq 0$, $\int_{[0,t[\times\mathcal{U}} |\psi| d\mu < \infty$; then, the process defined by

$$\psi * \mu_t := \int_{[0,t[\times\mathcal{U}} \psi(\cdot, s, u)\mu(\cdot, ds, du), \quad t \geq 0,$$

is a process of finite variation. We may simply denote the integral with respect to M_μ by $M_\mu[\psi] := \mathbb{E}[\psi * \mu_\infty]$. If μ is predictably σ-finite, we can define the notion of "conditional expectation relative to M_μ" with respect to the σ-field $\widetilde{\mathscr{P}}$, as in [JS03, No. 3.16 of Chapter III]: for a non-negative measurable function ψ on $\Omega \times [0,\infty[\times\mathcal{U}$, the "conditional expectation" $\psi' := M_\mu[\psi \,|\, \widetilde{\mathscr{P}}]$ is the M_μ-a.e. unique $\widetilde{\mathscr{P}}$-measurable function such that

$$M_\mu[\psi\varphi] = M_\mu[\psi'\varphi], \text{ for all non-negative } \widetilde{\mathscr{P}}\text{-measurable functions } \varphi.$$

Then, we can easily generalise this to all M_μ-integrable functions ψ.

A random measure μ is said *predictable* (respectively *optional*) if, for all $\widetilde{\mathscr{P}}$-measurable (respectively $\widetilde{\mathscr{O}}$-measurable) functions ψ such that $\psi * \mu$ exists, $\psi * \mu$ is a predictable (respectively optional) process.

Definition A.31. Let μ be a random measure. The *compensator* (or *dual predictable projection*) $\nu = \mu^p$ of μ (if exists) is a predictable random measure which is predictably σ-integrable and such that $M_\nu = M_\mu$ on $\widetilde{\mathscr{P}}$.

By [HWY92, Theorem 11.5], we know that the compensator of a random measure is unique (up to indistinguishability), and by [HWY92, Theorem 11.8] we also know that it exists if and only if μ is predictably σ-integrable.

[72]The notation $(\mathscr{B}([0,\infty[) \otimes \mathscr{B}(\mathcal{U}))$ denotes the product σ-field between the Borel σ-fields on $[0,\infty[$ and $\mathcal{U}$.

An optional and optionally σ-integrable random measure μ is called *integer-valued*, if it takes values in $\mathbb{N}_0 \cup \{+\infty\}$ and $\mu(\{t\} \times \mathcal{U}) \leq 1$, for all $t \geq 0$. From now on, let us assume that μ is an integer-valued random measure that has a compensator $\nu := \mu^p$ and such that $\mu(\{0\} \times \mathcal{U}) = 0$ a.s.. We now define the stochastic integral of a predictable function ψ with respect to $\mu - \nu$ as in [HWY92, Definition 11.16]. To simplify the notation, one denotes $\hat{\nu}_t(du) := \nu(\cdot, \{t\}, du)$, for $t \geq 0$. For a $\widetilde{\mathscr{P}}$-measurable function ψ such that $\mathbb{P}$-a.s., for all $t \geq 0$, $\int_{\mathcal{U}} |\psi(\cdot, t, u)| \hat{\nu}_t(du) < \infty$, we also define

$$\hat{\psi}_t := \int_{\mathcal{U}} \psi(\cdot, t, u) \hat{\nu}_t(du), \quad t \geq 0, \tag{A.11}$$

and

$$\tilde{\psi}_t := \int_{\mathcal{U}} \psi(\cdot, t, u) \mu(\cdot, \{t\}, du) - \hat{\psi}_t, \quad t \geq 0. \tag{A.12}$$

Let

$$\mathcal{G}(\mu) := \Bigg\{ \psi \ \widetilde{\mathscr{P}}\text{-measurable} \mid \mathbb{P}\text{-a.s.} \forall\, t \geq 0 \ \int_{\mathcal{U}} |\psi(\cdot, t, u)| \hat{\nu}_t(du) < \infty \\ \text{and } \sqrt{\Sigma(\tilde{\psi})^2} \text{ is locally integrable} \Bigg\}, \tag{A.13}$$

where the *summation process* ΣX of a thin process X is defined as $\Sigma X_t = \sum_{s \leq t} X_s$, $t \geq 0$, if $\sum_{s \leq t} |X_s| < \infty$ for all $t \geq 0$.[73] Then, if $\psi \in \mathcal{G}(\mu)$, there exists a unique totally discontinuous local martingale M such that $\Delta M = \tilde{\psi}$.[74] We write $M = \psi * (\mu - \nu)$ and call it the *stochastic integral of* W *with respect to* $\mu - \nu$.

In [KP15], the only random measure needed is a *homogeneous Poisson random measure*, which is an integer-valued random measure μ such that[75]

i) the measure $\pi \colon \mathscr{B}([0, \infty[) \otimes \mathscr{B}(\mathcal{U}) \to [0, \infty]$, $B \mapsto \pi(B) := \mathbb{E}[\mu(B)]$ is σ-finite and given by $\pi(dt, du) = g(du)dt$, for a non-negative σ-finite measure g on $\mathcal{U}$;

ii) for every $s \geq 0$ and any Borel-measurable set $B \subseteq]s, \infty[\times \mathcal{U}$ with $\pi(B) < \infty$, the random variable $\mu(\cdot, B)$ is independent of $\mathscr{F}_s$.

The measure π is called *intensity measure* of μ. By [JS03, Proposition II.1.21], the compensator of a homogeneous Poisson random measure μ always exists, and it is given by the intensity measure π, that is, $\mu^p = \pi$.

The next result is a martingale representation property, which is crucial to show the well-posedness of BSDEs in [KP15]. We denote by W a one-dimensional standard Brownian motion with respect to $(\mathbb{F}, \mathbb{P})$, and by $\tilde{\mu}(dt, du) = \mu(dt, du) - g(du)dt$ a compensated homogeneous Poisson random measure. We also denote by $\mathrm{L}^2_{loc}(W)$ the set of (real-valued) predictable processes Z such that $Z^2 \bullet \langle W \rangle = Z^2 \bullet \lambda$ is locally integrable.

Proposition A.32. *Every local martingale M has the decomposition*

$$M = Z \bullet W + \psi * \tilde{\mu} + N, \tag{A.14}$$

[73] See [HWY92, Definition 7.39]. The process $\tilde{\psi}^2$ is thin, because the integer-valued random measure μ has thin support by [HWY92, Theorem 11.13] and $(\hat{\nu}_t(\mathcal{U}))_{t \geq 0}$ is a thin process by [HWY92, Theorem 11.14].

[74] This follows by [HWY92, Theorem 7.42], since $\tilde{\psi}$ is a thin process with ${}^p\psi = 0$ by [HWY92, Theorem 11.11].

[75] See, for example, [JS03, Definition II.1.20].

where $Z \in \mathrm{L}^2_{loc}(W)$, $\psi \in \mathcal{G}(\mu)$, and N is a local martingale that is orthogonal to W and $\tilde{\mu}$, in the following sense:

$$[N^c, W] = 0 \text{ and } \quad M_\mu[\Delta N \,|\, \widetilde{\mathscr{P}}] = 0. \tag{A.15}$$

Proof. This is a particular case of [JS03, Lemma III.4.24] for a semimartingale X with continuous martingale component $X^c = W$ and associated jump-measure $\mu^X = \mu$ (as defined in [JS03, Proposition II.1.16], or equivalently in [HWY92, Theorem 11.15]). □

A.2 Theory of BSDEs

We summarise here the results on BSDEs from [KP15], to which we refer in Chapter 5. We continue to assume that $(\Omega, \mathscr{F}, \mathbb{P})$ is a probability probability space with a filtration $\mathbb{F}$ satisfying the usual condition. Additionally, we assume that $\mathbb{F}$ is quasi-left-continuous.

Let W denote a one-dimensional Brownian motion with respect to $(\mathbb{F}, \mathbb{P})$, and μ be a homogeneous Poisson random measure on $\Omega \times (\mathscr{B}([0,\infty[) \otimes \mathscr{B}(\mathcal{U}))$ (recall that $\mathcal{U} \subseteq \mathbb{R}^m \setminus \{0\}$) with intensity $\pi(dt, du) = g(du)dt$, for some non-negative σ-finite measure g on $\mathcal{U}$ such that

$$\int_{\mathcal{U}} (1 \wedge |u|^2) g(du) < \infty. \tag{A.16}$$

Let us denote by $\mathbb{L}^2_g$ the set of Borel-measurable functions $\varphi\colon \mathcal{U} \to \mathbb{R}$ such that

$$\int_{\mathcal{U}} \varphi(u)^2 g(du) < \infty.$$

We fix a time finite time $T > 0$, and denote the progressive σ-field on $\Omega \times [0,T]$ by $\mathscr{P}(0,T)$. We consider the following BSDE:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s, \psi_s) ds - \int_t^T Z_s dW_s - \int_t^T \int_{\mathcal{U}} \psi(s,u) \tilde{\mu}(ds, du) - \int_t^T dN_s, \tag{A.17}$$

where ξ is an $\mathscr{F}_T$-measurable random variable, called the *terminal condition*, and

$$f\colon \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{L}^2_g \to \mathbb{R}$$

is a $\mathscr{P}(0,T) \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^k) \otimes \mathscr{B}(\mathbb{L}^2_g)$-measurable random function (we omit to write the dependence on ω), which we call *driver* or *generator* of the BSDE. The unknowns of the BSDE are (Y, Z, ψ, N), where Y is a real-valued càdlàg adapted process, $Z \in \mathrm{L}^2_{loc}(W)$, $\psi \in \mathcal{G}(\mu)$, and N is a local martingale orthogonal to W and $\tilde{\mu}$.

For the purpose of this thesis, we can assume that f is independent of (Z, ψ), that is, our generator is of the form $f\colon \Omega \times [0,T] \times \mathbb{R} \to \mathbb{R}$, $(\omega, t, y) \mapsto f(t, y)$, and we also have $\xi = 0$. Hence, the assumptions on f and ξ in [KP15] reduce to the following:

(H1) for every $t \in [0,T]$, the map $y \mapsto f(t,y)$ is continuous, and there exists a constant α such that $\mathbb{P}$-a.s., for all $t \in [0,T]$ and for all $y, y' \in \mathbb{R}$,

$$\big(f(t,y) - f(t,y')\big)(y - y') \le \alpha (y - y')^2;$$

(H2) $\forall\, r > 0$, $\mathbb{E}\Big[\int_0^T \sup_{|y| \le r} |f(t,y) - f(t,0)| dt\Big] < \infty$;

(H4) $\mathbb{E}\Big[\int_0^T f(t,0)^2 dt\Big] < \infty$.

We define the following spaces of processes:

$$\mathbb{S}^2 := \left\{ Y \text{ càdlàg adapted process} \,\middle|\, \mathbb{E}\left[\sup_{t\in[0,T]} Y_t^2\right] < \infty \right\}, \tag{A.18}$$

$$\mathbb{L}^2 := \left\{ Z \text{ predictable process} \,\middle|\, \mathbb{E}\left[\int_0^T Z_t^2 dt\right] < \infty \right\}, \tag{A.19}$$

$$\mathbb{L}^2_\mu := \left\{ \psi \in \mathcal{G}(\mu) \,\middle|\, \mathbb{E}\left[\int_0^T \int_{\mathcal{U}} \psi(t,u)^2 g(du) dt\right] < \infty \right\}, \tag{A.20}$$

$$\mathbb{M}^{2,\perp} := \left\{ N \text{ martingale orthogonal to } W \text{ and } \tilde{\mu} \mid \mathbb{E}[[N]_T] < \infty \right\}. \tag{A.21}$$

Remark A.33. Note that, if $(Z,\psi,N) \in \mathbb{L}^2 \times \mathbb{L}^2_\mu \times \mathbb{M}^{2,\perp}$, then the process

$$\int_0^\cdot Z_t dW_t + \int_0^\cdot \int_{\mathcal{U}} \psi(t,u)\tilde{\mu}(dt,du) + \int_0^\cdot dN_t$$

is a true martingale bounded in L^2 on $[0,T]$. Indeed, the first integral $Z\centerdot W$ is a local martingale with $\mathbb{E}\left[[Z\centerdot W]_T\right] = \mathbb{E}\left[Z^2\centerdot [W]_T\right] < \infty$, which by [HWY92, Theorem 7.32] is equivalent to the boundedness in L^2 of $Z\centerdot W$.[76] Similarly, N is a local martingale bounded in L^2, and so a true martingale as well. Consider now the second summand $M := \int_0^\cdot \int_{\mathcal{U}} \psi(t,u)\tilde{\mu}(dt,du)$. Note that, as the compensator of μ is $\pi(dt,du) = g(du)dt$, we have $\pi(\{t\}\times\mathcal{U}) = 0$ and $\hat{\psi}_t = 0$, using the notation in (A.11). So, the process

$$C(\psi) := (\psi - \hat{\psi})^2 * \pi_t + \sum_{s\le t}\big(1 - \pi(\{s\}\times\mathcal{U})\big)\hat{\psi}_s^2 = \psi^2 * \pi_t, \qquad 0 \le t \le T,$$

defines an integrable increasing process on $[0,T]$ (in the sense of (A.7)), since

$$\mathbb{E}\left[\psi^2 * \pi_T\right] = \mathbb{E}\left[\int_0^T \int_{\mathcal{U}} \psi(t,u)^2 g(du) dt\right] < \infty.$$

By [JS03, Theorem II.1.33a)], this proves that M is a true martingale bounded in L^2 on $[0,T]$

Proposition A.34. *Suppose that $\xi = 0$ and f is independent of (Z,ψ). If the generator f satisfies the assumptions (H1), (H2), and (H4), then the BSDE* (A.17) *has a unique solution* $(Y,Z,\psi,N) \in \mathbb{S}^2 \times \mathbb{L}^2 \times \mathbb{L}^2_\mu \times \mathbb{M}^{2,\perp}$.

Proof. See [KP15, Theorem 1]. □

Note that, by solution we mean that (A.17) is satisfied $\mathbb{P}$-a.s. for all $t \in [0,T]$ (this is equivalent to [KP15, Definition 1] that the equation holds $\mathbb{P}\times\lambda$-a.e. in $\Omega\times[0,T]$, since both sides of the equation are càdlàg), and by uniqueness we mean up to indistinguishability, as usual.

We also have the following comparison principle.

[76] Recall that, by Proposition A.21, a local martingale bounded in L^2 is a true martingale.

Proposition A.35. *Assume that f_1 and f_2 are two generators both satisfying (H1), (H2), and (H4). Denote by (Y^i, Z^i, ψ^i, N^i) the corresponding unique solutions in $\mathbb{S}^2 \times \mathbb{L}^2 \times \mathbb{L}^2_\mu \times \mathbb{M}^{2,\perp}$, for $i = 1, 2$. If $\mathbb{P}$-a.s. for all $t \in [0, T]$ we have $f_1(t, Y^1_t) \leq f_2(t, Y^1_t)$, then $Y^1 \leq Y^2$.*

As in [AC18], when we say that a BSDE is well-posed, we mean that a unique solution exists in $\mathbb{S}^2 \times \mathbb{L}^2 \times \mathbb{L}^2_\mu \times \mathbb{M}^{2,\perp}$ and that the comparison principle holds.

B Invariance Times

In this section we introduce the theory of invariance time of [CS17], which is crucial to study the well-posedness of the XVA equations in Chapter 5. We only give the proofs that are omitted or only sketched in [CS17]; for all other results, we refer to the original paper.

We assume that a probability space $(\Omega, \mathcal{A}, \mathbb{Q})$ is given, together with two filtrations $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ with $\mathcal{F}_t \subseteq \mathcal{G}_t$, for all $t \geq 0$ (that is, $\mathbb{F}$ is a subfiltration of $\mathbb{G}$). Both filtrations are assumed to satisfy the usual condition. Let τ be a $\mathbb{G}$-stopping time that is not an $\mathbb{F}$-stopping time. For the purposes of this thesis it is enough to assume that τ is totally inaccessible and has an intensity γ with respect to $\mathbb{G}$ (see Definition A.20). This simplifies some of the arguments in [CS17], where a general stopping time is considered.

B.1 The condition (B)

Condition(B) For any $\mathbb{G}$-predictable process L, there exists an $\mathbb{F}$-predictable process L' such that $L\mathbb{1}_{]\!]0,\tau]\!]} = L'\mathbb{1}_{]\!]0,\tau]\!]}$.

The process L' is called $\mathbb{F}$-*predictable reduction* of L.

Assumption B.1. From now on in this section, we assume that the condition(B) holds.

Remark B.2. As shown in [CS17, Lemma 2.1], the condition(B) is satisfied if and only if $\mathbb{G}$ is a subfiltration of $\overline{\mathbb{F}} := \left(\overline{\mathcal{F}}_t\right)_{t\geq 0}$, where

$$\overline{\mathcal{F}}_t := \{B \in \mathcal{A} \,|\, \exists\, A \in \mathcal{F}_t \colon\; B \cap \{t < \tau\} = A \cap \{t < \tau\}\}\,, \quad \forall\, t \geq 0.$$

We can easily verify that $\overline{\mathbb{F}}$ is right-continuous, and, being larger than $\mathbb{F}$, it is also complete; hence, it satisfies the usual condition. Note that, τ is always an $\overline{\mathbb{F}}$-stopping time; indeed, for any $t \geq 0$, $B := \{t < \tau\} = A \cap \{t < \tau\}$, if $A := \Omega \in \mathcal{F}_t$. Therefore, if we assume that $\mathbb{G}$ is the smallest filtration larger than $\mathbb{F}$ and such that τ is a $\mathbb{G}$-stopping time, then $\mathbb{G}$ is a subfiltration of $\overline{\mathbb{F}}$ and condition(B) holds.

Definition B.3. Let $J := \mathbb{1}_{[\![0,\tau[\![}$. We define the *Azéma supermartingale of* τ as the $(\mathbb{F}, \mathbb{Q})$-optional projection $S := {}^o\!J$ (which exists, because J is bounded).

Thus, S is an $\mathbb{F}$-optional process such that $S_t = \mathbb{Q}[\tau > t|\mathcal{F}_t]$ a.s. for all $t \geq 0$. To see that S is actually a supermartingale, note that, since $\mathbb{1}_{\{\tau>t\}} \leq \mathbb{1}_{\{\tau>s\}}$ for $s \leq t$, by the tower property we have

$$\mathbb{E}[S_t|\mathcal{F}_s] = \mathbb{E}[\mathbb{1}_{\{\tau>t\}}|\mathcal{F}_s] \leq \mathbb{E}[\mathbb{1}_{\{\tau>s\}}|\mathcal{F}_s] = S_s,\ \mathbb{Q}\text{-a.s. for } s \leq t.$$

Since S is bounded and $\mathbb{F}$-adapted, it is an $(\mathbb{F}, \mathbb{Q})$-supermartingale. Observing that the function $t \mapsto \mathbb{Q}[\tau > t]$ is right-continuous (by the continuity of measures), by Proposition A.1 we can choose S to be a càdlàg $\mathbb{F}$-supermartingale. By the Doob-Meyer decomposition (see Theorem A.11), we can write $S = S_0 + Q - D$, where Q is a uniformly integrable $(\mathbb{F}, \mathbb{Q})$-martingale with $Q_0 = 0$ and D is an $\mathbb{F}$-predictable $\mathbb{Q}$-integrable increasing process (in the sense of (A.7)) starting from zero. Since τ is totally inaccessible, we have that $\tau > 0$; thus, we know that $S_0 = 1$.

Proposition B.4. *The process D is the $(\mathbb{F}, \mathbb{Q})$-compensator of* $\mathbb{1}_{\{0<\tau\}}\mathbb{1}_{[\![\tau,\infty[\![} = \mathbb{1}_{[\![\tau,\infty[\![}$.

Proof sketch. This can be seen by looking at the construction of D in the proof of its existence in [HWY92, Theorem 5.47]. The bounded non-negative right-continuous supermartingale $Z = (S_t - \mathbb{E}[S_\infty|\mathscr{F}_t])_{t\geq 0}$ is a potential of class(D), since $\mathbb{E}[Z_t] = \mathbb{E}[S_t] - \mathbb{E}[S_\infty]$ and $S_t \to S_\infty$ in L^1 by [HWY92, Theorem 2.50].[77] By inserting Z in the definition of the measure μ in [HWY92, Equation (46.2)], we see that μ is the measure generated by the non-decreasing process $A := 1 - J = \mathbb{1}_{[\![\tau,\infty[\![}$ (that is, $\mu = \mu_A$ using the notation of [HWY92, Definition 5.10]), using that $\mathbb{E}[S_\theta] = \mathbb{E}[J_\theta]$, for any $\mathbb{F}$-stopping time θ. Then, it follows from the above mentioned proof, that D is actually the $(\mathbb{F}, \mathbb{Q})$-compensator of A.[78] □

Lemma B.5. *The left limit process S_- is equal to the $\mathbb{F}$-predictable projection ${}^p(J_-)$ of $J_- = \mathbb{1}_{[\![0,\tau]\!]}$.*

Proof. S_- is a predictable process, and, since J_- is bounded, a predictable projection exists; thus, by [HWY92, Remark 5.3.2)] it is enough to check that

$$\mathbb{E}[J_-(\theta)\mathbb{1}_{\{0<\theta<\infty\}}] = \mathbb{E}[S_{\theta-}\mathbb{1}_{\{0<\theta<\infty\}}]$$

for any $\mathbb{F}$-predictable time θ. Let θ be an $\mathbb{F}$-predictable time. Since S is càdlàg, we have $S_{\theta-} = \lim_{n\to\infty} S_{\theta_n}$ $\mathbb{Q}$-a.s., where θ is an $\mathbb{F}$-predictable time and $(\theta_n)_n$ a sequence of $\mathbb{F}$-stopping times foretelling θ (which exists by Proposition A.7). Thus, using that $S = {}^oJ$ and that $\theta_n < \infty$ on $\{\theta > 0\}$ for all n, yields

$$S_{\theta-} = \lim_{n\to\infty} S_{\theta_n} = \lim_{n\to\infty} \mathbb{E}[J_{\theta_n}|\mathscr{F}_{\theta_n}] = \lim_{n\to\infty} \mathbb{Q}[\theta_n < \tau|\mathscr{F}_{\theta_n}] \quad \text{a.s. on } \{\theta > 0\}.$$

By dominated convergence theorem,

$$\mathbb{E}[S_{\theta-}\mathbb{1}_{\{\theta>0\}}|\mathscr{F}_{\theta_m}] = \lim_{n\to\infty} \mathbb{Q}[\theta_n < \tau|\mathscr{F}_{\theta_m}]\mathbb{1}_{\{\theta>0\}} = \mathbb{Q}[0 < \theta \leq \tau|\mathscr{F}_{\theta_m}], \quad \text{for any } m \in \mathbb{N}.$$

By [HWY92, Theorem 3.4.11)], $\mathscr{F}_{\theta-} = \sigma\left(\bigcup_{m\in\mathbb{N}} \mathscr{F}_{\theta_m}\right)$, which implies that

$$S_{\theta-}\mathbb{1}_{\{\theta>0\}} = \mathbb{Q}[0 < \theta \leq \tau|\mathscr{F}_{\theta-}].$$

Therefore, since $\{\theta < \infty\} \in \mathscr{F}_{\theta-}$,

$$\begin{aligned}\mathbb{E}[S_{\theta-}\mathbb{1}_{\{0<\theta<\infty\}}] &= \mathbb{E}\big[\mathbb{Q}[0 < \theta < \infty,\ \theta \leq \tau|\mathscr{F}_{\theta-}]\big] \\ &= \mathbb{Q}[0 < \theta < \infty,\ \theta \leq \tau] = \mathbb{E}[J_-(\theta)\mathbb{1}_{\{0<\theta<\infty\}}].\end{aligned}$$

□

Furthermore, we have by [HWY92, Remark 3.5.1)], that the $\mathbb{F}$-predictable projection of the uniformly integrable martingale Q is Q_-, and D is already predictable; thus,

$${}^pS = Q_- - D = S - \Delta Q = S_- - \Delta D \leq S_-.$$

In particular, ${}^pS = S_-$, if D is continuous.

[77] Note: S is a potential $\Leftrightarrow$ $S_\infty = 0$ $\Leftrightarrow$ $\mathbb{Q}[\tau = \infty] = 0$.

[78] Note that A is not $\mathbb{F}$-adapted, but this is not necessary in the definition of the dual predictable projection in [HWY92]; indeed, only measurability and local integrability are required. In particular, since τ is not an $\mathbb{F}$-stopping times, we cannot say that D is the compensator of τ.

Lemma B.6. *The* $(\mathbb{G},\mathbb{Q})$*-compensator* v *of* τ *satisfies*

$$v = \int_0^{\cdot} \frac{1}{S_{s-}} dD_s = \int_0^{\cdot\wedge\tau} \frac{1}{S_{s-}} dD_s. \tag{B.1}$$

Moreover, $D = \gamma' S_- \centerdot \lambda$ *on* $\{S_- > 0\}$*, where* γ' *is the* $\mathbb{F}$*-predictable reduction of the* $(\mathbb{G},\mathbb{Q})$*-intensity* γ *of* τ*.*

Proof. The process $v := \frac{1}{S_-} \centerdot D$ is $\mathbb{F}$-predictable (see [HWY92, Theorem 3.46]), so we need to show that, for any bounded measurable process H, it holds[79]

$$\mathbb{E}\left[\int_0^\infty H_s dv_s\right] = \mathbb{E}\left[\int_0^\infty {}^pH_s dA_s\right], \tag{B.2}$$

where $A = \mathbb{1}_{[\![\tau,\infty[\![}$ and ${}^pH$ denotes the $(\mathbb{G},\mathbb{Q})$-predictable projection of H. Let H be a bounded measurable process. By the predictability of v, we have

$$\mathbb{E}\left[\int_0^\infty H_s dv_s\right] = \mathbb{E}\left[\int_0^\infty {}^pH_s dv_s\right] = \mathbb{E}\left[\int_0^\infty {}^pH_s \frac{1}{S_{s-}} dD_s\right].$$

Let us denote by H' the $\mathbb{F}$-predictable reduction of ${}^pH$. Since $H' = {}^pH$ on $]\!]0,\tau]\!]$ and D is stopped at τ, we have

$$\mathbb{E}\left[\int_0^\infty {}^pH_s \frac{1}{S_{s-}} dD_s\right] = \mathbb{E}\left[\int_0^\infty H'_s \frac{1}{S_{s-}} dD_s\right] = \mathbb{E}\left[\int_0^\infty H'_s dD_s\right],$$

where the last equality follows by the fact that $\{S_- = 1\} = \{{}^p(J_-) = 1\}$ is the largest $\mathbb{F}$-predictable set in $[\![0,\tau]\!]$ (this follows by [Jeu80, Lemme (4,3)]) and D is $\mathbb{F}$-predictable. Now, since D is the $(\mathbb{F},\mathbb{Q})$-compensator of A (see Proposition B.4) and H' is $\mathbb{F}$-predictable,

$$\mathbb{E}\left[\int_0^\infty H'_s dD_s\right] = \mathbb{E}\left[\int_0^\infty H'_s dA_s\right] = \mathbb{E}[H'_\tau] = \mathbb{E}[{}^pH_\tau],$$

which yields (B.2). This shows the first equality in (B.1), and that v is indeed the compensator of τ. By Proposition A.18, the difference $A - v$ is a martingale. Since A is constant after τ and v is non-decreasing, v is also stopped at τ. This yields the second equality in (B.1).

Since τ has a $(\mathbb{G},\mathbb{Q})$-intensity γ, it holds

$$v = \frac{1}{S_-} \centerdot D = \gamma \centerdot \lambda, \quad \text{on } [0,\tau].$$

Since v is stopped at τ, we can assume that γ vanishes on $]\!]\tau,\infty[\![$, and obtain the above equality on $[0,\infty[$. Thus, $D = \gamma S_- \centerdot \lambda$ on $[0,\infty[$. Let γ' denote the $\mathbb{F}$-predictable reduction of γ, which is identical to γ on $]\!]0,\tau]\!]$. Then, $D = \gamma' S_- \centerdot \lambda$ on $[\![0,\tau]\!]$. By Lemma B.11 below, the equality holds on $\{S_- > 0\}$. □

Remark B.7. Therefore, D is continuous and $S_- = {}^pS$. Also, we have

$$\mathcal{E}\left(\pm\frac{1}{S_-} \centerdot D\right) = e^{\pm\frac{1}{S_-}\centerdot D} = e^{\pm\gamma'\centerdot\lambda}, \quad \text{on } \{S_- > 0\},$$

[79] Note that both v and A start from zero, so it is enough to integrate on $]0,\infty[$, instead of $[0,\infty[$.

where $\mathcal{E}(\cdot)$ denotes the stochastic exponential (see Definition A.28). This is an important simplification of the general setup in [CS17] due to our assumption that τ is totally inaccessible and has an intensity.

Since we are not interested in what happens after τ, we assume with no loss of generality that γ is $\mathbb{F}$-predictable.

Next, we define the following stopping times:

$$\varsigma := \inf\{s > 0\colon\ S_s = 0\}, \text{ and } \varsigma_n := \inf\Big\{s > 0\colon\ S_s \le \frac{1}{n}\Big\},\ n \ge 1.$$

Then, we immediately have $\varsigma = \sup_n \varsigma_n$, and, since S is a non-negative supermartingale, we also know that $\varsigma = \inf\{s > 0\colon\ S_{s-} = 0\}$ (see, for example, [HWY92, Theorem 2.62] or [DM82, No. 17 Chapter VI]). Note that, by right-continuity $\{S > 0\} = [\![0, \varsigma[\![$, whearas it holds

$$\{S_- > 0\} = \bigcup_n [\![0, \varsigma_n]\!].$$

To see this, note that for $\omega \in \{S_{\varsigma-} > 0\}$, that is, where the trajectory $S(\omega)$ has a jump at $\varsigma(\omega)$, one has $\{s \ge 0\colon\ S_{s-}(\omega) > 0\} = [0, \varsigma(\omega)]$ and there exists a (random) $n = n(\omega)$ such that $S_{\varsigma-}(\omega) > \frac{1}{n}$, which means that $\varsigma_n(\omega) = \varsigma(\omega)$ and $\bigcup_n [0, \varsigma_n(\omega)] = [0, \varsigma(\omega)]$. Conversely, for $\omega \in \{S_{\varsigma-} = 0\}$ the trajectory $S(\omega)$ has no jump at $\varsigma(\omega)$ and $\varsigma(\omega)$ is not an attained maximum. So, $\{s \ge 0\colon\ S_{s-}(\omega) > 0\} = [0, \varsigma(\omega)[= \bigcup_n [0, \varsigma_n(\omega)]$. To sum up, we have[80]

$$[\![0, \varsigma[\![= \{S > 0\} \subseteq \{S_- > 0\} = \bigcup_n [\![0, \varsigma_n]\!] \subseteq [\![0, \varsigma]\!].$$

Thus, on $]\!]\varsigma, \infty[\![$, it holds $S = S_- = 0$; hence, D and Q are constant on $[\varsigma, \infty[$. Additionally, note that, by [Yor78, Lemme 0], $S_{\tau-} > 0$ on $\{\tau < \infty\}$.

Remark B.8. Note that, although $J_\tau = 0$, S is not necessarily zero at τ, being S the $\mathbb{F}$-optional (not $\mathbb{G}$) projection of J. If τ were also an $\mathbb{F}$-stopping time, by the definition of optional projection we would have $S_\tau = \mathbb{E}[J_\tau \,|\, \mathcal{F}_\tau] = 0$. This would imply that $\tau = \varsigma$. Since we assumed that this is not the case, in general we only have $\varsigma \le \tau$.

Lemma B.9. *i) For any $\mathbb{G}$-stopping time θ, there exists an $\mathbb{F}$-stopping time θ', such that $\{\theta < \tau\} = \{\theta' < \tau\} \subseteq \{\theta = \theta'\}$; θ' is called $\mathbb{F}$-reduction of θ.*

ii) Let $(E, \mathcal{E})$ be a measurable space. Any $\mathcal{P}(\mathbb{G}) \otimes \mathcal{E}$ (resp. $\mathcal{O}(\mathbb{G}) \otimes \mathcal{E}$) measurable function $g\colon \Omega \times \mathbb{R}_+ \times E \to \mathbb{R}$ admits a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$-(resp. $\mathcal{O}(\mathbb{F}) \otimes \mathcal{E}$-) reduction, that is, a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{E}$-(resp. $\mathcal{O}(\mathbb{F}) \otimes \mathcal{E}$-) measurable function $g'\colon \Omega \times \mathbb{R}_+ \times E \to \mathbb{R}$ such that $\mathbb{1}_{]\!]0,\tau]\!]} g = \mathbb{1}_{]\!]0,\tau]\!]} g'$ (resp. $\mathbb{1}_{[\![0,\tau[\![} g = \mathbb{1}_{[\![0,\tau[\![} g'$).

iii) Let $T > 0$ and assume $S_T > 0$. Let M be a $(\mathbb{G}, \mathbb{Q})$-local martingale on $[0, T]$ with no jump at τ. For any $\mathbb{F}$-optional reduction M' of M (which exists by item ii); see also Remark B.10), M' is an $\mathbb{F}$-semimartingale on $[0, T]$ and

$$S_- \centerdot M' + [S, M'] \text{ is an } (\mathbb{F}, \mathbb{Q})\text{-local martingale on } [0, T]. \tag{B.3}$$

Conversely, for any $\mathbb{F}$-semimartingale X on $[0, T]$ such that $S_- \centerdot X + [S, X]$ is an $(\mathbb{F}, \mathbb{Q})$-local martingale on $[0, T]$, the stopped process $X^{\tau-}$ is a $(\mathbb{G}, \mathbb{Q})$-local martingale.

[80] Note that all the inclusion and equalities between stochastic sets are intended up to evanescent set.

iv) Let $T > 0$ and assume that $S_T > 0$. The Azéma supermartingale S admits a predictable multiplicative decomposition $S = S_0\mathcal{Q}\mathcal{D}$ on $[0,T]$, where $\mathcal{D} = e^{-\gamma\centerdot\lambda}$ is an $\mathbb{F}$-predictable process of finite variation and $\mathcal{Q} = \mathcal{E}\left(\frac{1}{S_-}\centerdot Q\right) > 0$ is an $(\mathbb{F},\mathbb{Q})$-local martingale on $[0,T]$.

Proof. For the first item,[81] let θ be a $\mathbb{G}$-stopping time, and $L := \mathbb{1}_{]\!]0,\theta]\!]}$. Then, L is $\mathbb{G}$-predictable and $\theta' := \inf\{t \geq 0 \,|\, L'_t = 0\}$, where L' denotes the $\mathbb{F}$-predictable reduction of L, is an $\mathbb{F}$-stopping time. It is easy to see that, on $\{\theta < \tau\}$, $L' = \mathbb{1}_{]\!]0,\theta]\!]}$, so $\theta = \theta'$ and $\theta' < \tau$. Similarly, on $\{\theta' < \tau\}$, we have $\theta = \theta'$ and $\theta < \tau$.

For item ii), see [CS17, Lemma 2.2 (2)]. For item iii) see [Son16, Lemma 6.5 and 6.8]. Item iv) is a special case of [CS17, Lemma 2.2 (5)] with $S_T > 0$, combined with the fact that $\frac{1}{S_-}\centerdot D = \gamma\centerdot\lambda$ on $[0,T]$ (see Remark B.7). □

Remark B.10. Therefore, by item ii) in Lemma B.9, any $\mathbb{G}$-optional process L has an $\mathbb{F}$-optional reduction L', such that $L\mathbb{1}_{[\![0,\tau[\![} = L'\mathbb{1}_{[\![0,\tau[\![}$.

Lemma B.11. *Two $\mathbb{F}$-predictable (resp. optional) processes indistinguishable on $[\![0,\tau]\!]$ (resp. on $[\![0,\tau[\![$) are indistinguishable on $\{S_- > 0\}$ (resp. $\{S > 0\}$).*

Proof. See [CS17, Lemma 2.3]. □

Let $T > 0$. By the above lemma, if $S_T > 0$, the $\mathbb{F}$-predictable and optional reductions of processes are unique (up to indistinguishability) on $[0,T]$. In this case, if a process L is $\mathbb{G}$-predictable, its $\mathbb{F}$-optional and $\mathbb{F}$-predictable reductions are indistinguishable on $[0,T]$. Nevertheless, it is crucial to always carefully distinguish where the notation $\cdot'$ indicates a predictable or an optional reduction (and with respect to which filtration).

B.2 The condition (A)

We assume the same setup of section B.1, also assuming condition(B) and, as before, that τ has a $(\mathbb{G},\mathbb{Q})$-intensity γ (which can be assumed $\mathbb{F}$-predictable by reduction). We also fix $T > 0$.

Condition(A) There exists a probability measure $\mathbb{P}$ on $\mathscr{A}$ equivalent to $\mathbb{Q}$ on $\mathscr{F}_T$ such that

$$\forall\, X \in \mathcal{M}_{loc}(\mathbb{F},\mathbb{P}):\; X^{\tau-} \in \mathcal{M}_{loc,[0,T]}(\mathbb{G},\mathbb{Q}),$$

where $\mathcal{M}_{loc,[0,T]}(\mathbb{G},\mathbb{Q})$ denotes the set of $(\mathbb{G},\mathbb{Q})$-local martingales on the interval $[0,T]$. If the condition holds, we call τ an *invariance time* and $\mathbb{P}$ an *invariance measure.*

Remark B.12. As explained in [CS16], this setup goes beyond the basic immersion setup, where $\mathbb{P} = \mathbb{Q}$ and the $\mathbb{F}$-local martingales are $\mathbb{Q}$-local martingales without jump at τ.

We now give some of the results in [CS17] on the condition(A). To simplify things, we always assume that $S_T > 0$, that is, $\varsigma > T$. Recall that the supermartingale S has Doob-Meyer decomposition $S = S_0 + Q - D$, where Q is a uniformly integrable $(\mathbb{F},\mathbb{Q})$-martingale with $Q_0 = 0$ and D is an $\mathbb{F}$-predictable $\mathbb{Q}$-integrable increasing process (in the sense of (A.7)) starting from zero.

[81] For this argument, we follow [DMM92, Chapitre XX, no. 75 b)].

Theorem B.13. *Assume $S_T > 0$. The condition(A) holds if and only if the predictable process $\frac{1}{S_-}$ is $(\mathbb{F},\mathbb{Q})$ integrable with respect to Q on $[0,T]$ and $\mathcal{Q} = \mathcal{E}(\frac{1}{S_-} \centerdot Q)$ is a positive $(\mathbb{F},\mathbb{Q})$ true martingale on $[0,T]$. In this case, a probability measure $\mathbb{P}$ on $\mathscr{A}$ is an invariance measure if and only if*

$$\left.\frac{d\mathbb{P}}{d\mathbb{Q}}\right|_{\mathscr{F}_T} = \mathcal{E}\left(\frac{1}{S_-} \centerdot Q\right)_T. \tag{B.4}$$

Proof. See [CS17, Theorem 3.2]. □

So we have uniqueness of the invariance probability measure $\mathbb{P}$ on $\mathscr{F}_T$; note that we only need to know $\mathbb{P}$ on $\mathscr{F}_T$, as in the sequel we will only use it for the valuation of $\mathbb{F}$-adapted processes on $[0,T]$.

Theorem B.14. *If $S_T > 0$ and that $\mathbb{E}[\mathcal{E}(\frac{1}{S_-} \centerdot D)_{\tau\wedge T}] < \infty$, then the condition(A) holds.*

Proof. See [CS17, Theorem 3.5]. □

Remark B.15. By Remark B.7, the condition of Theorem B.14 reduces to a condition on γ. In fact,

$$\mathcal{E}\left(\frac{1}{S_-} \centerdot D\right)_{\tau\wedge T} = e^{(\gamma\centerdot\lambda)_{\tau\wedge T}}.$$

So, under the assumptions $S_T > 0$ (and that τ has a $(\mathbb{G},\mathbb{Q})$-intensity γ), it is enough that $e^{\int_0^{\tau\wedge T}\gamma_s ds}$ is $\mathbb{Q}$-integrable, to make sure that condition (A) holds.

In addition, we give the following characterisation of $(\mathbb{F},\mathbb{P})$-local martingales on $[0,T]$ in case the condition(A) holds, which is needed in the proof of Theorem B.18.

Theorem B.16. *If $S_T > 0$ and the condition(A) holds with an invariance measure $\mathbb{P}$, then*

$$X \in \mathcal{M}_{loc,[0,T]}(\mathbb{F},\mathbb{P}) \iff S_- \centerdot X + [S,X] \in \mathcal{M}_{loc,[0,T]}(\mathbb{F},\mathbb{Q}).$$

Moreover, in this case $\{S_- > 0\} = \{S > 0\} = [\![0,\varsigma[\![$.

B.3 The condition (C)

As in [CS18], in order to benefit from all the results of the previous subsections, we can assume the following condition.

Condition(C) The condition (B) holds, $S_T > 0$, and $\mathbb{E}[e^{(\gamma\centerdot\lambda)_{\tau\wedge T}}] < \infty$.

During the thesis we make use of the following "expectation transfer formulas" to pass from an expectation with respect to $\mathbb{Q}$ to one with respect to $\mathbb{P}$, and vice versa.

Theorem B.17. *Let A be an $\mathbb{F}$-optional non-decreasing process starting from 0. Then, the following holds:*

$$\mathbb{E}\big[A_T^{\tau-}\big] = \mathbb{E}'\left[\int_0^T e^{-\Gamma_s} dA_s\right]. \tag{B.5}$$

Proof. See [CSS20, Theorem 5.1] or [CS18, Theorem 3.1]. □

The following theorem corresponds to the first part of [CS18, Theorem 4.1].

Theorem B.18. *Denote by* $\mathcal{M}^T_{loc}(\mathbb{F},\mathbb{P})$ *the set of all* $(\mathbb{F},\mathbb{P})$*-local martingales (on* $[0,\infty[$*) stopped at* T*, and by* $\mathcal{M}^{\tau-\wedge T}_{loc}(\mathbb{G},\mathbb{Q})$ *the set of all* $(\mathbb{G},\mathbb{Q})$*-local martingales (on* $[0,\infty[$*) stopped at* $\tau-\wedge T$*, that is, stopped at* T *and before* τ*. Then, the map*

$$\mathcal{M}^T_{loc}(\mathbb{F},\mathbb{P}) \ni L \longmapsto L^{\tau-} \in \mathcal{M}^{\tau-\wedge T}_{loc}(\mathbb{G},\mathbb{Q}) \tag{B.6}$$

is a bijection with inverse the $\mathbb{F}$*-optional reduction*

$$\mathcal{M}^{\tau-\wedge T}_{loc}(\mathbb{G},\mathbb{Q}) \ni L \longmapsto L' \in \mathcal{M}^T_{loc}(\mathbb{F},\mathbb{P}) \tag{B.7}$$

Proof. Let $X \in \mathcal{M}^T_{loc}(\mathbb{F},\mathbb{P})$. By the condition(A), $X^{\tau-} \in \mathcal{M}^{\tau-\wedge T}_{loc}(\mathbb{G},\mathbb{Q})$. Conversely, if $M \in \mathcal{M}^{\tau-\wedge T}_{loc}(\mathbb{G},\mathbb{Q})$ and M' denotes its $\mathbb{F}$-optional reduction, $S_{-}\centerdot M' + [S,M']$ is an $(\mathbb{F},\mathbb{Q})$-local martingale on $[0,T]$, by Lemma B.9 iii), which is equivalent to $M' \in \mathcal{M}^T_{loc}(\mathbb{F},\mathbb{P})$, by Theorem B.16. This shows that the two maps are well defined. To prove that they are bijective, we need to show that for any $X \in \mathcal{M}^T_{loc}(\mathbb{F},\mathbb{P})$, $(X^{\tau-})' = X$, and for any $M \in \mathcal{M}^{\tau-\wedge T}_{loc}(\mathbb{G},\mathbb{Q})$, $(M')^{\tau-} = M$. For the first one, we have that $(X^{\tau-})' = X^{\tau-} = X$ on $[\![0,\tau[\![$. By the optional version of Lemma B.11, the (indistinguishable) equality holds also on $[0,T]$, and, since everything is stopped at T, we have equality on $[0,\infty[$. For the second one, we have that $M' = (M')^{\tau-} = M$ on $[\![0,\tau[\![$, and, since M is stopped before τ, the second equality holds on $[0,\infty[$. This completes the proof. □

Therefore, in our setup, an $(\mathbb{F},\mathbb{P})$-local martingale on $[0,T]$ stopped before τ defines a unique (up to indistinguishability) $(\mathbb{G},\mathbb{Q})$-local martingale on $[0,\tau\wedge T]$ with no jump at τ. Conversely, the $\mathbb{F}$-optional reduction of a $(\mathbb{G},\mathbb{Q})$-local martingale on $[0,\tau\wedge T]$ with no jump at τ defines a unique $(\mathbb{F},\mathbb{P})$-local martingale on $[0,T]$.

C Value at Risk and Expected Shortfall

In this section we give a short summary on value at risk and expected shortfall. Our references are [FS11] and [MFE05].

Suppose a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is given, and let $\mathbb{E}$ denote the expectation with respect to $\mathbb{P}$. Let $\alpha \geq 50\%$ be a given confidence level, and L be a real-valued random variable on $(\Omega, \mathscr{F}, \mathbb{P})$ representing a loss.

Definition C.1. The *value at risk* at the confidence level α of L is defined by

$$\mathrm{VaR}_\alpha(L) := \inf \{x \in \mathbb{R} \,|\, \mathbb{P}[L \leq x] \geq \alpha\}. \tag{C.1}$$

Note that we take a "loss view" as in [MFE05, Definition 2.10]; in [FS11, Definition 4.45] another convention is used, where the random variable in the value at risk denotes a gain.

Definition C.2. Assuming $\mathbb{E}[|L|] < \infty$, the *expected shortfall* at the confidence level α of L is defined by[82]

$$\mathrm{ES}_\alpha(L) := \frac{1}{1-\alpha} \int_\alpha^1 \mathrm{VaR}_u(L) du. \tag{C.2}$$

This corresponds to the definition of *average value at risk* (AV@R) in [FS11, Definition 4.48], in the sense that,

$$\mathrm{AV@R}_{1-\alpha}(-L) = \mathrm{ES}_\alpha(L).$$

Lemma C.3. *Assume* $\mathbb{E}[L] \geq 0$. *Then,* $\mathrm{ES}_\alpha(L) \geq 0$.

Proof. By [FS11, Theorem 4.52], we know that ES_α is a coherent risk measure; hence, it can be written as

$$\mathrm{ES}_\alpha(L) = \max_{\mathbb{Q} \in \mathcal{Q}_\alpha} \mathbb{E}_\mathbb{Q}[L],$$

where $\mathcal{Q}_\alpha$ is the set containing all probability measures $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$ with density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ bounded by $\frac{1}{1-\alpha}$ ($\mathbb{E}_\mathbb{Q}$ denote the expectation with respect to $\mathbb{Q}$). Since $\mathbb{P} \in \mathcal{Q}_\alpha$ and $\mathbb{E}[L] \geq 0$, the maximum is also non-negative. □

By [FS11, Equations (11.21)-(11.22)], we see that we can define conditional versions of value at risk and expected shortfall. Let $\mathbb{F} = (\mathscr{F}_t)_{t \geq 0}$ be a filtration on $(\Omega, \mathscr{F}, \mathbb{P})$. By adapting to our notation, we get

$$\mathrm{VaR}_{\alpha,t}(L) := \operatorname{ess\,inf} \big\{ x_t \in \mathrm{L}^\infty(\mathscr{F}_t) \,\big|\, \mathbb{P}[L \leq x_t \,|\, \mathscr{F}_t] \geq \alpha \big\} \quad \text{and} \tag{C.3}$$

$$\mathrm{ES}_{\alpha,t}(L) := \operatorname*{ess\,sup}_{\mathbb{Q} \in \mathcal{Q}_t^\alpha} \mathbb{E}_\mathbb{Q}[L \,|\, \mathscr{F}_t], \quad \text{for } t \geq 0, \tag{C.4}$$

where $\mathrm{L}^\infty(\mathscr{F}_t)$ is the set of all bounded $\mathscr{F}_t$-measurable random variables and $\mathcal{Q}_t^\alpha$ denotes the set of all probability measures $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$ such that $\mathbb{Q} = \mathbb{P}$ on $\mathscr{F}_t$ and the density $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is bounded by $\frac{1}{1-\alpha}$.

With a similar argument as above we have the following lemma.

Lemma C.4. *Let* $t \geq 0$ *and assume* $\mathbb{E}[L \,|\, \mathscr{F}_t] \geq 0$. *Then,* $\mathrm{ES}_{\alpha,t}(L) \geq 0$.

[82] See also [MFE05, Definition 2.15].

References

[AA14] C. Albanese and L. Andersen. “Accounting for OTC derivatives: Funding Adjustments and Re-hypothecation Option”. In: (Sept. 14, 2014). URL: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2482955.

[AAC20] C. Albanese, Y. Armenti, and S. Crépey. “XVA metrics for CCP Optimisation”. In: (Feb. 17, 2020). URL: https://math.maths.univ-evry.fr/crepey/papers/CCP-NEW.pdf.

[ACC17] C. Albanese, S. Caenazzo, and S. Crépey. “Credit, funding, margin, and capital valuation adjustment for bilateral portfolios”. In: (June 26, 2017). URL: https://link.springer.com/content/pdf/10.1186/s41546-017-0019-2.pdf.

[AC20] C. Albanese and S. Crépey. “The Cost-of-Capital XVA Approach in Continuous Time”. In: (Mar. 22, 2020). URL: https://math.maths.univ-evry.fr/crepey/papers/cost-of-capital-XVA-CT.pdf.

[AC18] C. Albanese and S. Crépey. “XVA Analysis From The Balance Sheet”. In: (Sept. 25, 2018). URL: https://www.maths.univ-evry.fr/crepey/papers/KVA-FVA-I-REVISED.pdf.

[Alb+19] C. Albanese et al. “XVA Analysis From The Balance Sheet”. In: (Oct. 22, 2019). URL: https://math.maths.univ-evry.fr/crepey/papers/xva-analysis-balance.pdf.

[BCB15] Basel Commetee of Banking Supervision (BCBS). *Margin requirements for non-centrally cleared derivatives*. 2015. URL: https://www.bis.org/bcbs/publ/d317.pdf.

[Bjö09] T. Björk. *Arbitrage Theory in Continuous Time*. Third Edition. Oxford University Press, 2009. ISBN: 978-0-19-957474-2.

[BM06] D. Brigo and F. Mercurio. *Interest Rate Models – Theory and Practice*. Second Edition. Springer Verlag Berlin Heidelberg, 2006. ISBN: 978-3-540-22149-4.

[CF13] A. Castagna and F. Fede. *Measuring and Managing Liquidity Risk*. Wiley, 2013.

[Ces+09] G. Cesari et al. *Modelling, Pricing, and Hedging Counterparty Credit Exposure*. Springer Verlag Berlin Heidelberg, 2009. ISBN: 978-3-642-04453-3.

[CBB14] S. Crépey, T. Bielecki, and D. Brigo. *Counterparty Risk and Funding. A Tale of Two Puzzles*. Chapman and Hall/CRC Financial Mathematics Series, 2014. ISBN: 978-1-4665-1646-5.

[CSS20] S. Crépey, W. Sabbagh, and S. Song. “When Capital is a Funding Source: The Anticipated Backward Stochastic Differential Equations of X-Value Adjustment”. In: (Jan. 21, 2020). URL: https://math.maths.univ-evry.fr/crepey/papers/capital-funding-source.pdf.

[CS16] S. Crépey and S. Song. “Counterparty Risk and Funding: Immersion and Beyond”. In: Fincance Stoch.20 (2016), pp. 901–930. DOI: https://doi.org/10.1007/s00780-016-0305-3.

[CS17] S. Crépey and S. Song. “Invariance Times”. In: (Feb. 3, 2017). URL: https://arxiv.org/abs/1702.01045.

[CS18] S. Crépey and S. Song. “Invariance Times Transfer Properties”. In: (Nov. 16, 2018). URL: https://math.maths.univ-evry.fr/crepey/papers/Reduction.pdf.

[DS06] F. Delbaen and W. Schachermayer. *The Mathematics of Arbitrage*. Springer Verlag Berlin Heidelberg, 2006. ISBN: 978-3-540-21992-7.

[DMM92] C. Dellacherie, B. Maisonneuve, and P. Meyer. *Probabilitiés et potentiel. Chapitres XVII à XXIV. Processus de Markov Compléments de calcul stochastique*. Hermann Éditeurs des Sciences et des Arts, 1992. ISBN: 2 7056 1434 8.

[DM82] C. Dellacherie and P. Meyer. *Probabilities and Potential B*. Noth-Holland Publishing Company, 1982. ISBN: 0444865268.

[Duf10] D. Duffie. *How Big Banks Fail And What to Do about It*. Princeton University Press, 2010. ISBN: 978-0691148854.

[EBA15] European Banking Authority (EBA). *EBA FINAL draft Regulatory Technical Standards on prudent valuation under Article 105(14) of Regulation (EU) No 575/2013 (Capital Requirements Regulation) – CRR)*. Jan. 23, 2015. URL: `https://eba.europa.eu`.

[EU12] Official Journal of the European Union (EU). *Regulation (EU) No 648/2012 of the European Parliament and of the Council of 4 July 2012 on OTC derivatives, central counterparties and trade repositories Text with EEA relevance*. 2012. URL: `https://eur-lex.europa.eu/legal-content/EN/TXT/?uri=CELEX%5C%3A32012R0648`.

[FS14] M. Fleming and A. Sarkar. "The Failure Resolution of Lehman Brothers". In: (Apr. 10, 2014). URL: `https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2422433`.

[FS11] H. Föllmer and A. Schied. *Stochastic Finance, an introduction in discrete time*. Berlin: De Gruyter, 2011.

[FR] Federal Reserve Board (FR). *Volcker Rule*. URL: `https://www.federalreserve.gov/supervisionreg/volcker-rule.htm` (visited on 07/05/2020).

[Gre15] J. Gregory. *The xVA Challenge. Counterparty Credit Risk, Funding, Collateral and Capital*. Third Edition. Wiley Finance Series, 2015. ISBN: 978-1-119-10941-9.

[HWY92] S. He, J. Wang, and J. Yan. *Semimartingale And Stochastic Calculus*. Science Press and CRC Press Inc., 1992. ISBN: 978-0849377150.

[HW13] J. Hull and A. White. "LIBOR vs. OIS: The Derivatives Discounting Dilemma". In: (Apr. 2013). URL: `https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2211800`.

[HW12] J. Hull and A. White. "The FVA debate, followed by The FVA debate continued". In: Risk Magazine.July 83–85 and October 52 (2012).

[ISD19] International Swaps and Derivative Association (ISDA). *Legal guidelines for smart derivatives contracts: The ISDA Master Agreement*. Feb. 2019. URL: `https://www.isda.org/a/23iME/Legal-Guidelines-for-Smart-Derivatives-Contracts-ISDA-Master-Agreement.pdf`.

[JS03] J. Jacod and A. Shiryaev. *Limit Theorems for Stochastic Processes*. Second Edition. Springer-Verlag Berlin Heidelberg GmbH, 2003. ISBN: 978-3-642-07876-7.

[Jeu80] T. Jeulin. *Semi-martingales et grossissement d'une filtration. Lecture Notes in Mathematics*. 883. Springer-Verlag Berlin Heidelberg, 1980. ISBN: 978-3-540-10265-6.

[KP15] T. Kruse and A. Popier. "BSDEs with monotone generator driven by Brownian and Poisson noises in a general filtration". In: (June 10, 2015). URL: `https://arxiv.org/abs/1412.4622`.

[MO67] A. Marshall and I. Olkin. "A Multivariate Exponential Distribution". In: Journal of American Statistical Association.62:317 (1967), pp. 30–44. URL: http://dx.doi.org/10.1080/01621459.1967.10482885.

[MFE05] A. McNeil, R. Frey, and P. Embrechts. *Quantitative Risk Management. Concepts, Techniques and Tools*. Princeton University Press, 2005. ISBN: 0-691-12255-5.

[Son16] S. Song. "Local martingale deflators for asset processes stopped at a default time S^τ or right before $S^{\tau-}$". In: (July 20, 2016). URL: https://arxiv.org/abs/1405.4474.

[Yor78] M. Yor. "Grossisement d'une filtration et semi-martingales: theoremes generaux". In: Séminaire de Probabilités XII. Lecture Notes in Mathematics.649 (1978), pp. 61–69. DOI: https://doi.org/10.1007/BFb0064595.

ETH

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Declaration of originality

The signed declaration of originality is a component of every semester paper, Bachelor's thesis, Master's thesis and any other degree paper undertaken during the course of studies, including the respective electronic versions.

Lecturers may also require a declaration of originality for other written papers compiled for their courses.

I hereby confirm that I am the sole author of the written work here enclosed and that I have compiled it in my own words. Parts excepted are corrections of form and content by the supervisor.

Title of work (in block letters):

XVA Analysis for Bilateral Derivatives in Continuous Time

Authored by (in block letters):
For papers written by groups the names of all authors are required.

Name(s):	**First name(s):**
Gallo	Vito

With my signature I confirm that
- I have committed none of the forms of plagiarism described in the 'Citation etiquette' information sheet.
- I have documented all methods, data and processes truthfully.
- I have not manipulated any data.
- I have mentioned all persons who were significant facilitators of the work.

I am aware that the work may be screened electronically for plagiarism.

Place, date
Zürich, 12.09.2020

Signature(s)
Vito Gallo

For papers written by groups the names of all authors are required. Their signatures collectively guarantee the entire content of the written paper.