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Amenable Groups and Stabilizers of Measures on the Boundary of a Hadamard Manifold

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1. Introduction

Let X be a symmetric space of noncompact type or more generally a Hadamard manifold, i.e. a complete simply connected Riemannian manifold of nonpositive sectional curvature. We consider a finite positive Borel-measure μ on the ideal boundary $X(\infty)$ (see Sect. 2 for the definitions). Let G be the isometry group of X and G_μ the subgroup which stabilizes the measure μ. In the case of a symmetric space we obtain the following result.

Theorem 1. *Let X be a symmetric space of nonpositive curvature and μ a finite positive measure on $X(\infty)$ such that the support of μ contains at least one regular point. Then the group G_μ is amenable and the identity component G_μ^0 has a normal cocompact solvable subgroup.*

Remark. One should compare our result with a theorem of Moore [M], who proved that G_μ is the group of real points of an algebraic $\mathbb{R}$-group and has a normal cocompact solvable subgroup in the case that μ is a positive measure on G/P for a minimal parabolic subgroup P of G. Our theorem is related to Moore's result in the following way: G/P can be viewed as a submanifold of the regular points of $X(\infty)$, and, hence, a measure on G/P induces a measure on $X(\infty)$ with regular support. Thus, by our theorem G_μ^0 is a compact extension of a solvable group. In Moore's case, the additional information that G/P is an algebraic variety enables one to prove that G_μ/G_μ^0 is finite. In fact, we can not prove this for our more general measures μ.

The proof of Theorem 1 is very geometric and we give a brief outline: the measure μ on $X(\infty)$ induces a measure μ_x on the unit tangent sphere at every point $x \in X$. The mean value of μ_x at each point gives a vectorfield $V(x)$ which is the gradient field of a G_μ-quasi invariant convex function F. If F assumes the minimum, then G_μ leaves the minimal set E of F invariant. It turns out, that E is a euclidean submanifold of X and G is amenable as a compact extension of the

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isometry group of E. If F does not assume the minimum, we prove that G_μ fixes a point $z \in X(\infty)$. In this case, G_μ is contained in the parabolic subgroup G_z of G and we can use an induction on the boundary component asymptotic to z to obtain the result.

Since the proof is geometric, we can use similar arguments to study groups stabilizing a measure on the boundary of an arbitrary Hadamard manifold. We obtain the following structure result for the action of amenable groups.

Theorem 2. *Let Γ be an amenable group of isometries operating on a Hadamard manifold X. Then one (or both) of the following holds:*

(1) Γ fixes a point $z \in X(\infty)$.

(2) Γ leaves a totally geodesic subspace $E \subset X$ invariant and E is isometric to an euclidean space.

From this theorem we derive the following results of Avez, Zimmer, and Anderson.

Corollary 1 (Avez [Av], Zimmer [Z]). *Let M be a complete Riemannian manifold of non positive curvature and finite volume with amenable fundamental group. Then, M is flat.*

Corollary 2 (Anderson [A]). *Let M be a compact manifold of non positive curvature, then every amenable subgroup of $\pi_1(M)$ is a Bieberbach group.*

2. Preliminaries

A. Amenable Groups (General Reference [P]*)*

A topological group H is amenable if for every continuous action of H on a compact topological space Y there exists a positive H-invariant measure on Y. We collect some well known properties of amenable groups.

a) Compact extensions of solvable topological groups are amenable (Kakutani-Markov).

b) A connected amenable Lie group is a compact extension of a solvable group [F].

c) If $f: G \to H$ is a continuous surjective homomorphism with kernel K then G is amenable if and only if both K and H are amenable.

d) Closed subgroups of locally compact amenable groups are amenable.

There exists a geometric interpretation of amenability in the case that Γ is the fundamental group of a compact manifold. Then let S be a finite set of generators and $\mathfrak{G}$ the corresponding graph. Then Γ is amenable if and only if $\inf_{A \subset \mathfrak{G}} \frac{\|\partial A\|}{\|A\|} = 0.$ This follows from Fõlner's condition [P]. Equivalently [B] the infinimum of the L^2-spectrum of the universal covering $\tilde{M}$ of M is 0.

Equivalent is also the condition $\inf_{B \subset \tilde{M}} \frac{\mathrm{Vol}(\partial B)}{\mathrm{Vol}(B)} = 0.$

B. *Manifolds of non Positive Curvature (General Reference* [E–O'N, BGS]*)*

Let X be a Hadamard manifold, this means a complete simply connected Riemannian manifold of nonpositive curvature. We denote by $d(\ ,\ )$ the distance function and by $\bar{X}=X\cup X(\infty)$ the Eberlein–O'Neill compactification of X. Thus $X(\infty)$ is the set of classes of asymptotic rays and homeomorphic to S^{n-1} where $n=\dim X$. For $p\in X$ and $q\in\bar{X}$, there is a unique unit speed geodesic from p to q. The initial vector at p of this geodesic is denoted by $V(p,q)$. For $p\in X$, $q_1,q_2\in\bar{X}$, let $\measuredangle_p(q_1,q_2)$ be the angle between $V(p,q_1)$ and $V(p,q_2)$.

For a unit speed geodesic c the Busemannfunction b_c is defined by $b_c(p)=\lim_{t\to\infty}(d(p,c(t))-t)$. Then b_c is a C^2-function with gradient $\nabla b_c(p)=-V(p,c(\infty))$ where $c(\infty)\in X(\infty)$ is the asymptotic class of c. Thus up to a constant, the Busemannfunction b_c depends only on $c(\infty)$. The function b_c is convex, this means that $b_c\circ h:\mathbb{R}\to\mathbb{R}$ is a convex function for every geodesic h.

Two geodesics c and h are called parallel, if $c(\infty)=h(\infty)$ and $c(-\infty)=h(-\infty)$. Parallel geodesics bound a totally geodesic flat euclidean strip in X. More generally, let P_c be the set of all points contained on parallels to c. Then P_c is a convex subset of X which splits isometrically as $P_c=P'_c\times\mathbb{R}$. If X is an analytic manifold, then P_c is a complete submanifold.

If h is a geodesic segment, $z\in X(\infty)$, then the function $\phi(t):=\measuredangle_{h(t)}(V(h(t),z),\dot{h}(t))$ is monotone increasing. If $\phi(t_1)=\phi(t_2)$, then $h|_{[t_1,t_2]}$ and the geodesics from $h(t_i)$ to z bound a flat strip.

C. *Symmetric Spaces (General Reference* [W, K, IH]*)*

Let X be a symmetric space of noncompact type with isometry group G. We fix a point $x_0\in X$ and identify X with the homogeneous space G/K where K is the isotropy group of x_0. We consider the Cartan-decomposition $\mathfrak{G}=\mathfrak{K}\oplus\mathfrak{P}$ of the Lie algebra of G and we identify $\mathfrak{P}$ with the tangent space $T_{x_0}X$.

A flat in X is a complete totally geodesic euclidean subspace in X of maximal dimension. This maximal dimension is the rank of the symmetric space. Let F be a flat with $x_0\in F$, then $T_{x_0}F$ is a maximal abelian subalgebra of $\mathfrak{P}$. Every tangent vector v in $\mathfrak{P}$ is contained in some maximal abelian subalgebra, hence every geodesic c is contained in some flat. A geodesic c (vector in $\mathfrak{P}$) is called regular, if it is contained in a unique flat (unique maximal abelian subalgebra). A point $z\in X(\infty)$ is called regular, if it is the endpoint $c(\infty)$ of a regular geodesic c. The singular vectors in a maximal abelian subalgebra $\mathfrak{B}$ of $\mathfrak{P}$ are contained in finitely many hyperplanes which divide $\mathfrak{B}$ into the Weylchambers. Correspondingly, the singular geodesics through x_0 in a pointed flat (F,x_0) divide the euclidean space into Weylchambers.

For a geodesic c through x_0 we consider the parallel set P_c. Let $v=\dot{c}(o)$ in $\mathfrak{P}$, then $T_{x_0}P_c=\{w\in\mathfrak{P}\,|\,[w,v]=0\}$. Thus v is regular, if the centralizer of v in $\mathfrak{P}$ is abelian, and c is regular if and only if P_c is an euclidean space. For the geodesic, we consider the transvection $\phi_t:=\exp_G tv\in G$. Then ϕ_t translates the geodesic c, i.e. $\phi_t c(s)=c(s+t)$ and the differential $d\phi_t$ realizes the parallel translation along c.

We define the horocyclic group

$$N_c := \left\{ \gamma \in G \,\middle|\, \lim_{t\to\infty} \phi_t^{-1} \gamma \phi_t = \mathrm{id} \right\}$$

(cf. [K, Sect. 4.2], [IH, Sect. 2]).

Then N_c is a subgroup of the parabolic group G_z, the stabilizer of $z = c(\infty)$. Indeed N_c is a maximal nilpotent normal subgroup of G_z. The orbit $N_c x_0$ is the horocycle determined by c. We have the decomposition of X as a disjoint union $X = \bigcup_{n\in N_c} nP_c$. This decomposition has the following geometric description: we consider all geodesics asymptotic to z and call two of them equivalent, if they are parallel. Then, an equivalence class of geodesics can be written as $n \cdot P_c$ for a unique $n \in N_c$. This is the viewpoint in [K]. In correspondence to this decomposition of X we have the Harish-Chandra decomposition $G_z = K'A'N$ of the parabolic group G_z. An element $g \in G_z$ can be written as $g = k'a'n$, where $k' \in K$, $n \in N_c$ and $a' \in \mathrm{Isom}(P_c)$.

We have the projection maps

$$p: X \to P_c$$
$$x \mapsto n^{-1}x, \quad \text{if} \quad x \in nP_c$$

and

$$q_1: G_z \to G_z/N_c = K'A'$$
$$g = k'a'n \mapsto k'a' = \lim_{t\to\infty} \phi_t^{-1} g \phi_t .$$

Let $q_2: K'A' \to \mathrm{Isom}(P_c)$ be the restriction map. The kernel of q_2 is a compact normal subgroup of $K'A'$. Let $q: G_z \to \mathrm{Isom}(P_c)$ be the composition $q = q_2 \circ q_1$.

Lemma. a) *The projection $p: X \to P_c$ extends to a Borel measurable projection $p: X(\infty) \to P_c(\infty)$.*

b) *p(regular points in $X(\infty)$) $\subset$ regular points in $P_c(\infty)$.*

To prove a) we remark that for a point $w \in X(\infty)$ there exists a geodesic c' asymptotic to z with $w \in P_{c'}(\infty)$ [IH].

Then, $P_{c'} = nP_c$ for a suitable $n \in N_c$ and the projection map extends by $p(w) = n^{-1}w$.

To prove that p is well defined, it suffices to show that if $n(y) \in P_c(\infty)$ for some $y \in P_c(\infty)$, then $n(y) = y$. To prove this choose a one parameter group of transvections ϕ_t such that $c(t) = \phi_t(x)$, $x = c(0)$. Then $y = \lim_{t\to\infty} \phi_t^{-1} n \phi_t(y) = n(y)$ since ϕ_t fixes every point of $P_c(\infty)$. We note that the projection $p: X(\infty) \to P_c(\infty)$ is not continuous even in the case that X is the hyperbolic plane. However p is clearly Borel measurable.

b) Let $w \in nP_c(\infty)$ be a regular point of $X(\infty)$ and let h be a geodesic in nP_c asymptotic to w. Then P_h is an euclidean space and $n^{-1}h$ is a geodesic in P_c asymptotic to $n^{-1}w = p(w)$. Since $P_{n^{-1}h}$ is euclidean, also $P_{n^{-1}h} \cap P_c$ is euclidean and thus $p(w)$ is a regular point of $P_c(\infty)$.

3. The Function F

Let μ be a finite positive measure on the boundary $X(\infty)$ of a Hadamard manifold X. We associate to μ a convex function on X (comp. [Z]). For $z \in X(\infty)$ and $p \in X$, let $b_p(\cdot, z)$ be the Busemannfunction of z normalized such that $b_p(p, z) = 0$.

We define

$$F_p^\mu(q) := \int_{X(\infty)} b_p(q, z)\, d\mu(z).$$

As a positive mean of convex functions F_p^μ is convex. Since Busemann functions are C^2 it is not difficult to check that F_p^μ is C^2 with

$$\nabla F_p^\mu(q) = \int_{X(\infty)} \nabla b_p(q, z)\, d\mu(z) = -\int_{X(\infty)} V(p, z)\, d\mu(z).$$

Lemma 1. *The function F_p^μ is quasi-invariant under the action of the stabilizer G_μ of μ, i.e. for $\gamma \in G_\mu$ there exists a constant $c(\gamma)$ such that*

$$F_p^\mu(\gamma q) = F_p^\mu(q) + c(\gamma).$$

Proof.

$$\begin{aligned} F_p^\mu(\gamma q) &= \int_{X(\infty)} b_p(\gamma q, z)\, d\mu(z) \\ &= \int_{X(\infty)} b_{\gamma^{-1}p}(q, \gamma^{-1} z)\, d\mu(z) \\ &= \int_{X(\infty)} b_{\gamma^{-1}p}(q, z)\, d\mu(z), \end{aligned}$$

where the last equality holds since $\gamma \in G_\mu$.

Note that $b_p(\cdot, z)$ and $b_{\gamma^{-1}p}(\cdot, z)$ differ by the constant $b_{\gamma^{-1}p}(p, z)$. Thus

$$\begin{aligned} F_p^\mu(\gamma q) &= \int_{X(\infty)} (b_p(q, z) + b_{\gamma^{-1}p}(p, z))\, d\mu(z) \\ &= F_p^\mu(q) + c(\gamma) \end{aligned}$$

with

$$c(\gamma) = c_p^\mu(\gamma) = \int_{X(\infty)} b_{\gamma^{-1}p}(p, z)\, d\mu(z).$$

Remark. The quasi invariance of F_p^μ is equivalent to the invariance of the vector field ∇F_p^μ.

We now study the case that F_p^μ does not assume the minimum.

Lemma 2. *Let f be a convex function on X quasi-invariant under the action of a group Γ of isometries on X. Let us assume that f does not assume a minimum. Then there is a point $z \in X(\infty)$ fixed by all $\gamma \in \Gamma$.*

Proof. Let $a := \inf f \in \mathbb{R} \cup \{-\infty\}$ and let $a_i > a$ be a monotone decreasing sequence converging to a. We consider the convex subsets $A_i = f^{-1}((-\infty, a_i])$ of X, then $\cap A_i = \emptyset$. We fix a point $p \in X$ and let $p_i := \pi_{A_i}(p)$ be the projected points, where π_{A_i} is the orthogonal projection onto A_i. Then the sequence p_i has no accumulation point in X and by choosing a subsequence we can assume that $p_i \to z \in X(\infty)$. We prove that $\gamma z = z$ for $\gamma \in \Gamma$. Note that $\gamma A_i = A_i'$, where A_i' is another sublevel of f. Without loss of generality let $A_i' \supset A_i$ (in the opposite case we consider γ^{-1} instead of γ). Let $b := d(p, \gamma p)$, $t_i := d(p, p_i) = d(\gamma p, \gamma p_i)$. $c_i := d(\gamma p, p_i)$, $s_i := d(p_i, \gamma p_i)$, $\alpha_i := \sphericalangle_{\gamma p}(p_i, \gamma p_i)$.

We will prove that $\alpha_i \to 0$. Since $p_i \to z$, this implies that also $\gamma p_i \to z$ and hence $\gamma z = z$. By the triangle inequality we have $c_i \leqq t_i + b$. Since $\gamma p_i = \pi_{A_i'}(\gamma p)$ and $p_i \in A_i'$ we have $\sphericalangle_{\gamma p_i}(p_i, \gamma p) \geqq \frac{\pi}{2}$. Thus, by the law of cosine [E–O'N], we have $c_i^2 \geqq t_i^2 + s_i^2$. Hence $(t_i + b)^2 \geqq t_i^2 + s_i^2$ and $2t_i b + b^2 \geqq s_i^2$.

Moreover, $s_i^2 \geqq c_i^2 + t_i^2 - 2c_i t_i \cos\alpha_i \geqq 2c_i t_i (1 - \cos\alpha_i)$ by the law of cosines applied to the triangle γp, p_i, γp_i and hence $\alpha_i \to 0$.

To investigate the case that F_p^μ does assume the minimum, we study subsets, on which convex functions are linear.

Definition. Let f be a convex function on X and let M be a convex subset of X. Then f is called linear on M, if the function $t \mapsto f \circ c(t)$ is affine for all geodesic segments c in M.

Lemma 3. *Let $M \subset X$ be a convex subset on which F_p^μ is linear. Then the functions $b_p(\cdot, z)$ are linear on M for all $z \in \operatorname{supp}(\mu)$.*

Proof. Let us assume to the contrary that $b_p(\cdot, z)$ is not linear on M for $z \in \operatorname{supp}(\mu)$. Then there exists a geodesic segment $c : [0, 1] \to M$ such that

$$b_p(c(1/2), z) < 1/2(b_p(c(0), z) + b_p(c(1), z)).$$

Since the function $(p, q, z) \mapsto b_p(q, z)$ is continuous, we have

$$b_p(c(1/2), w) < 1/2(b_p(c(0), w) + b_p(c(1), w))$$

for all w in an open neighborhood U of z.

Since $\mu(U) > 0$, we have

$$F_p^\mu(c(1/2)) < 1/2(F_p^\mu(c(0)) + F_p^\mu(c(1)))$$

a contradiction to the linearity of F_p^μ.

Lemma 4. *Let b be a Busemann function on X for a point $z \in X(\infty)$. Let $A \subset X$ be a convex subset on which b is linear. Then the vectorfield $V(\cdot, z)$ is parallel on A. If X is analytic, then A is contained in a parallel set P_c for a geodesic c asymptotic to z.*

Proof. Let $g : [0, 1] \to A$ be a geodesic segment. Since b is affine on g, we have

$$\langle \nabla b(g(t)), \dot{g}(t) \rangle = -\langle V(g(t), z), \dot{g}(t) \rangle$$

is constant. Thus, by the last remark in Sect. 2 *B*, the points $g(0)$, $g(1)$ and z span a totally geodesic euclidean strip. It follows that $V(g(t), z)$ is tangent to this strip and parallel.

If X is analytic, then let c be the geodesic with $c(0) = p \in A$ and $c(\infty) = \mathfrak{z}$. The argument above shows that, for a geodesic segment g from p to $q \in A$, the ray $c|_{[0,\infty)}$ is the boundary of a flat strip containing q. By analyticity, this lies in a flat plane and hence $q \in P_c$.

Lemma 5. *Let X be analytic and let us assume that F_p^μ assumes the minimum with $Y := \{q \in X \mid F_p^\mu(q) \text{ minimum}\}$.*

Let $q \in Y$, then $Y = \bigcap_{z \in \operatorname{supp}(\mu)} P_{\overline{qz}}$, where $\overline{qz}$ is the geodesic with $c(0) = q$, $c(\infty) = z$.

Proof. By Lemma 3, $b_p(\cdot, z)$ is linear on Y for all $z \in \operatorname{supp}(\mu)$ and thus $Y \subset \bigcap_{z \in \operatorname{supp}(\mu)} P_{\overline{qz}}$ by Lemma 4. To prove the opposite inclusion, note that $V(\cdot, z)$ is a parallel vectorfield on $P_{\overline{qz}}$. Thus $\nabla F_p^\mu = \int_{X(\infty)} V(\cdot, z)\, d\mu(z)$ is a parallel vectorfield on $M = \bigcap_{z \in \operatorname{supp}(\mu)} P_{\overline{qz}}$. Since $\nabla F_p^\mu(q) = 0$, we have $\nabla F_p^\mu \equiv 0$ on M and thus $M \subset Y$.

Proof of Theorem 1. We prove the theorem by induction on the dimension of X where the case $\dim X = 1$ is trivial.

We first consider the case that X has a non-trivial euclidean de Rham factor. We write $X = \mathbb{R}^s \times X^*$, where $s \geqq 1$ and X^* is symmetric of noncompact type. We assume that X^* has positive dimension for otherwise the result is immediately true. If μ is a finite positive Borel measure on $X(\infty)$ whose support contains a point of $R(\infty)$, the set of regular points of $X(\infty)$, then the restriction of μ to $R(\infty)$ is nonzero.

We define a continuous projection $p: R(\infty) \to R^*(\infty) \subset X^*(\infty)$ in the following way. For a point $x \in R(\infty)$ and a geodesic c of X with $c(\infty) = x$ we write $c(t) = (c_1(\alpha t), c_2(\beta t))$ where α and β are positive constants with $\alpha^2 + \beta^2 = 1$ and c_1, c_2 unit speed geodesics of $\mathbb{R}^s$, X^* respectively. Since c is a regular geodesic in X it follows that $\beta \neq 0$ and c_2 is a regular geodesic in X^*. We define $p(x) := c_2(\infty)$. One checks that this definition is independent of c.

We define a measure μ^* on $X^*(\infty)$ by $\mu^*(A^*) = \mu(p^{-1}A^*)$ for $A^* \subset X^*(\infty)$. By the hypothesis on μ the measure μ^* is nonzero and its support lies in $R^*(\infty)$. Let $G^* = \operatorname{Isom}(X^*)$ and $G = \operatorname{Isom}(X) = \operatorname{Isom}(\mathbb{R}^s) \times G^*$. If we define $q: G \to G^*$ to be the obvious projection then $H = q^{-1}(G^*_{\mu^*})$ is a closed subgroup of G. The homomorphism $q: H \to G^*_{\mu^*}$ is continuous and surjective with amenable kernel $H \cap \operatorname{Isom}(\mathbb{R}^s)$. By induction $G^*_{\mu^*}$ is amenable thus H is amenable by property c) of 2A. It follows from property d) of 2A that G is amenable as a closed subgroup of H.

Thus we can assume that X is a symmetric space of noncompact type. Let μ be a positive measure on $X(\infty)$ and F_p^μ the corresponding convex function as in Sect. 3. We have to consider two cases:

(1) If F_p^μ assumes the minimum, then G_μ leaves the minimal set Y of F_p^μ invariant. The set Y can be written as $Y = \bigcap_{z \in \operatorname{supp}(\mu)} P_{\overline{qz}}$ by Lemma 5. Since by assumption at least one $z_0 \in \operatorname{supp}(\mu)$ is regular, Y is an euclidean submanifold of X. This shows that G_μ modulo a compact subgroup is a closed subgroup of the isometries of the euclidean space and hence amenable.

(2) If F_p^μ does not assume the minimum, then G_μ is contained in a parabolic subgroup G_z for some $z \in X(\infty)$ by Lemma 2. We choose a geodesic c asymptotic to z and consider the associated projections $p: X(\infty) \to P_c(\infty)$ and $q: G_z \to G^*$ described in Sect. 2C with $G^* = \operatorname{Isom}(P_c)$.

Then μ^* defined by $\mu^*(A^*) = \mu(p^{-1}A^*)$ is a positive measure on $P_c(\infty)$. Since the map p is q-equivariant, we have $q(G_\mu) \subset G^*_{\mu^*}$. If $A^* = \operatorname{supp}(\mu^*)$ then $B := \operatorname{closure}(p^{-1}A^*)$ contains the support of μ. If A^* contains only singular points of $P_c(\infty)$ then B would contain only singular points of $X(\infty)$ by the lemma in 2C which contradicts the hypothesis on μ. Thus the support of μ^* contains at least one regular point. Note that $\dim P_c < \dim X$ since X does not contain an euclidean factor. Then by induction $G^*_{\mu^*}$ is amenable. But $q^{-1}(G^*_{\mu^*})$ is a closed subgroup of G_z and $q^{-1}(G^*_{\mu^*})/\operatorname{Ker} q \cap q^{-1}(G^*_{\mu^*})$ is topologically isomorphic to $G^*_{\mu^*}$. But $\operatorname{Ker} q$ is a compact extension of a nilpotent group, hence amenable, and $q^{-1}(G^*_{\mu^*})$ is amenable as an amenable extension of an amenable group. Thus G_μ is amenable as a closed subgroup of $q^{-1}(G^*_{\mu^*})$.

4. Proof of Theorem 2 and the Corollaries

We will prove Theorem 2 in larger generality only assuming that X is a closed convex subset of a Hadamard manifold (we need this generalization for an induction argument, cf. [BGS, Sect. 6]).

Let W be a closed convex Γ-invariant subset of X which is minimal under these conditions. If there does not exist such a minimal set W, then there is a sequence $W_1 \supset W_2 \supset W_3 \ldots$ of Γ-invariant closed convex subsets with $\cap W_i = \emptyset$ and the group Γ has a fixed point $z \in X(\infty)$ by the arguments of Lemma 2 and we are in case (1) of Theorem 2. Thus we can assume that such a minimal set W exists.

If W is compact, then by a "center of mass" construction (see e.g. [BGS, Sect. 1]), Γ has a fixed point $p \in W$ and we are in case (2) of Theorem 2 with a trivial subspace E.

If W is not compact, we consider $W(\infty) \subset X(\infty)$, $W(\infty)$ is compact. The amenable group Γ operates on $W(\infty)$ and thus leaves a probability measure μ on $W(\infty)$ invariant. Thus we can consider the Γ-quasi-invariant convex function F^μ_p on W as defined in Sect. 3 for a point $p \in W$.

If F^μ_p does not assume the minimum we are in case (1) by Lemma 2. If F^μ_p assumes the minimum, then F^μ_p and the set where F^μ_p is minimal are Γ-invariant. By the minimality of W, F^μ_p is constant on W. Let q be a point in the interior of W and let $z_0 \in W(\infty)$ be a point in the support of μ. Let $c_q : [0, \infty) \mapsto W$ be the geodesic from q to z_0. We claim, that we can extend c_q to a complete geodesic $\bar{c}_q : \mathbb{R} \to W$. Let us assume to the contrary that $r = \bar{c}_q(s_0) \in \partial W$ for a point $s_0 < 0$. Since W is convex, there is a vector $v \in T_r W$ with $\sphericalangle(v, w) \leq \frac{\pi}{2}$ for all vectors $w \in T_r W$ pointing inside W. In particular, $\sphericalangle(v, V(r, z)) \leq \frac{\pi}{2}$ for all $z \in W(\infty)$. Note that $\sphericalangle(v, V(r, z_0)) = \sphericalangle(v, \dot{\bar{c}}_q(s_0)) < \frac{\pi}{2}$ since $c_q(0)$ is in the interior of W. Thus

$$\langle \nabla F^\mu_p(r), v \rangle = - \int_{W(\infty)} \langle V(r, z)\, d\mu(z), v \rangle < 0$$

which is a contradiction to the fact that F^μ_p is constant.

The proof of Lemma 4 shows that c_p and c_q are parallel for p, q in the interior of W. Thus W is contained in a parallel set P_c for a geodesic c. This argument applied for all $z \in \operatorname{supp}(\mu)$ shows that $W = W' \times \mathbb{R}^s$ and $\operatorname{supp}(\mu) \subset \mathbb{R}^s(\infty)$. Thus Γ operates on W and respects the splitting, i.e. every $\gamma \in \Gamma$ operates as (γ', γ'') on $W' \times \mathbb{R}^s$. We consider the projections $p : \gamma \to \gamma'$. Then the amenable group $p(\Gamma)$ operates on W' with $\dim W' < \dim W$. By induction on W', we obtain the result.

Proof of Corollary 1. (Compare the proof of the corollary in [S].) Since $M = X/\Gamma$ has finite volume, the fundamental group Γ satisfies the duality condition [C–E], thus for any geodesic $c : \mathbb{R} \to X$ there is a sequence $\gamma_n \in \Gamma$ such that $\gamma_n x$ converges to $c(\infty)$ and $\gamma_n^{-1} x$ converges to $c(-\infty)$ for any $x \in X$. This implies easily that there does not exist a proper Γ-invariant convex subset $W \subset X$. Thus, if we are in case (2) of Theorem 2, then X is flat and Γ a Bieberbach group. If Γ has a fixed point $z \in X(\infty)$, then by [C–E, Theorem 4.2], X splits Γ-invariantly as $X = X_0 \times X'$, where X_0 is an euclidean space of positive dimension and every $\gamma \in \Gamma$ splits as (γ_0, γ') with a translation γ_0. Let $p: \gamma \mapsto \gamma'$ be the projection. Then $p(\Gamma)$ is an amenable group on X' which satisfies the duality condition. By induction, X' is flat and we obtain the desired result.

Remark. The proof of the corollary shows more generally that X is flat, if it allows the operation of an amenable group Γ satisfying the duality condition.

Proof of Corollary 2. Let $M = X/\Gamma$ and let $A \subset \Gamma$ be an amenable subgroup. We apply Theorem 2 to our group A. If we are in case (2), then A leaves an euclidean space $E \subset X$ invariant and thus is a Bieberbach group. Let us therefore assume that we are in case (1) and A fixes a point $z \in X(\infty)$. Let us furthermore assume that A is finitely generated by elements $\alpha_1, \ldots, \alpha_s$. Let $c: \mathbb{R} \to X$ be a geodesic with $c(\infty) = z$. For an isometry γ on X, let $d_\gamma: p \mapsto d(\gamma p, p)$ be the convex displacement function. Since the elements α_j fix the point $z \in X(\infty)$, $d_{\alpha_j} \circ c$ is monotone decreasing. We choose elements $\gamma_i \in \Gamma$ such that $\gamma_i(c(i)) \in D$, where D is a fixed compact fundamental domain of Γ. By choosing a subsequence we can assume $\gamma_i(c(i)) \to p \in D$ and $\gamma_i(c) \to g$, where g is a geodesic with $g(0) = p$. Note that $d_{\gamma_i \alpha_j \gamma_i^{-1}}(\gamma_i(c(i))) = d_{\alpha_j}(c(i)) < C$ with $C := \max_{1 \leq j \leq s} (d_{\alpha_i}(c(0)))$. Thus the displacement of $\gamma_i \alpha_j \gamma_i^{-1}$ is universally bounded at p. Since the group Γ is discrete, there exists an element $\gamma \in \Gamma$ such that, for all j, we have $\gamma_i \alpha_j \gamma_i^{-1} = \gamma \alpha_j \gamma^{-1}$ for infinitely many i. By the same argument $d_{\gamma_i \alpha_j \gamma_i^{-1}}$ is bounded by C on $\gamma_i(c(0, \infty)) \to g(-i, \infty)$.

This implies that $d_{\gamma \alpha_j \gamma^{-1}}$ is a bounded function on g and thus α_j is bounded on $h = \gamma^{-1} g$. Thus the α_j are constant on h as bounded convex functions and hence the group A leaves the parallel set $P_h = P'_h \times \mathbb{R}$ invariant. Every $\alpha \in A$ splits as (α', translation).

Now let $r \geq 1$ be the largest integer such that there exists an r-flat F in X with d_α bounded on F for all $\alpha \in A$. Let P_F denote the union of all r-flats F' that are parallel to F, that is, $F(\infty) = F'(\infty)$. Then $P_F = W' \times \mathbb{R}^r$, where W' is a closed convex subset of X and $F = \{p'\} \times \mathbb{R}^r$ for some $p' \in W'$. If F' is an r-flat parallel to F, then d_α is constant on F' for all $\alpha \in A$ by convexity. Hence A leaves P_F and the splitting above invariant and for each $\alpha \in A$ one may write $\alpha = \alpha' \times$ translation where α' is an isometry of W'. Now consider the action of the amenable group $A' = \{\alpha' \mid \alpha \in A\}$ on W'. By Theorem 2 either A' leaves invariant some flat subspace E' of W' or A' fixes a point of $W'(\infty)$. It suffices to show that the second case does not occur. If A' and hence also A fixes some point z' in $W'(\infty)$, then by the same argument as in the first part of the proof one can show that there exists an element $\gamma \in \Gamma$, an r-flat F^* and a geodesic c^* orthogonal to F^* with $c^* \subset P_{F^*}$ such that $d_{\gamma \alpha_j \gamma^{-1}}$ is bounded on $c^*(\mathbb{R}) \cup F^*$ for all $1 \leq j \leq s$. Hence $d_{\gamma \alpha_j \gamma^{-1}}$ is bounded on the $(r+1)$-flat $c^*(\mathbb{R}) \times F^*$ for all j, which contradicts the definition of the integer r.

Thus we have proved, that every finitely generated subgroup A_0 of A leaves an euclidean space E invariant. It is not difficult to prove now that also A itself leaves an euclidean space invariant and is a Bieberbach group.

Remark. We used the compactness of M only to show that the points $c(i)$ can be translated into a compact fundamental domain. Note that $d_{\alpha_j} \circ c(t) \leq C$ for $t \geq 0$. Thus the injectivity radius of $\pi(c(t))$ for $t \geq 0$ and $\pi: X \to M$ is bounded by $C/2$. Thus the proof works also in the case that the set $\{p \in M \mid \text{injectivity radius } (p) \leq R\}$ is compact for all R. This condition is satisfied, for example, if M is noncompact with curvature $K \leqq -a^2 < 0$ and finitely many expanding ends. This case was also considered by Anderson [A].

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References

[A] Anderson, M.T.: On the fundamental group of nonpositively curved manifolds. Math. Ann. **276**, 269–278 (1987)

[Av] Avez, A.: Variétés riemanniennes sans points focaux. C. R. Acad. Sci. Paris **270**, 188–191 (1970)

[BGS] Ballmann, W., Gromov, M., Schroeder, V.: Manifolds of nonpositive curvature. Prog. Math. Boston **61** (1985)

[B] Brooks, R.: The fundamental group and the spectrum of the Laplacian. Comment. Math. Helv. **56**, 581–598 (1981)

[C–E] Chen, S., Eberlein, P.: Isometry groups of simply connected manifolds of nonpositive curvature. Ill. J. Math. **24**, 73–103 (1980)

[E–O'N] Eberlein, P., O'Neill, B.: Visibility manifolds. Pac. J. Math. **46**, 45–109 (1973)

[F] Fürstenberg, H.: A Poisson formula for semisimple Lie groups. Ann. Math. **77**, 335–383 (1963)

[G] Gromov, M.: Structures métriques pour les variétés riemanniennes, rédigé par J. Lafontaine et P. Pansu. Paris: Cedic/Nathan 1981

[IH] Im Hof, H.C.: Die Geometrie der Weylkammern in symmetrischen Räumen vom nichtkompakten Typ. Habilitationsschrift, Bonn (1979)

[K] Karpelevič, F.I.: The geometry of geodesics and the eigenfunctions of the Beltrami-Laplace operator on symmetric spaces. Tr. Mosc. Mat. O.-va. **14**, 48–185 (1965); Trans. Mosc. Math. Soc. **14**, 51–199 (1965)

[M] Moore, C.: Amenable subgroups of semisimple groups and proximal flows. Israel J. Math. **34**, 121–138 (1979)

[P] Pier, J.-P.: Amenable locally compact groups. New York: Wiley

[S] Schroeder, V.: A splitting theorem for spaces of nonpositive curvature. Invent. math. **79**, 323–327 (1985)

[W] Warner, G.: Harmonic analysis on semisimple Lie groups. I. Berlin, Heidelberg, New York: Springer 1972

[Z] Zimmer, R.: Curvature of leaves in amenable foliations. Am. J. Math. **105**, 1011–1022 (1983)

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