# RIEMANN SURFACES OF LARGE GENUS AND LARGE $\lambda_1$.

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## 0. Introduction.

Let $\Lambda_g = \sup\{\lambda_1(S) \mid S$ - a compact Riemann surface of genus $g\}$, where $\lambda_1(S)$ is the smallest positive element of the spectrum of the Riemann surface S. P. Buser [B1] posed a problem of determining whether the limit $L = \limsup_{g\to\infty} \Lambda_g$ is positive. He observed later [B2] that deep results of Selberg [S] and Jacquet-Langlands [JL] imply that this limit is positive and in fact is greater than or equal to 3/16. It follows easily from [C] that $L \leqslant 1/4$ and it is natural to conjecture that $L = 1/4$. In this paper we give a more concrete, geometric construction of Riemann surfaces of arbitrarily large genus and $\lambda_1 \geqslant c$, with c arbitrarily close to 3/16. As in [B2] we begin by considering the principal congruence subgroups $\Gamma_N$ of SL(2,Z) consisting of 2×2 matrices with integer entries congruent to the identity modulo N. If $N > 2$, $\Gamma_N$ has no torsion and acts on the upper half-plane U freely so that the quotient $S_0 = U/\Gamma_N$ is a Riemann surface of finite area with cusps. The number of cusps $\nu_\infty$ and the genus g of $S_0$ are given by [Sh]

$$\nu_\infty = \frac{N^2}{2}\prod_{p\mid N}\frac{p^2-1}{p^2}, \qquad g = 1 + \frac{N^2(N-6)}{24}\prod_{p\mid N}\frac{p^2-1}{p^2}$$

where the product is taken over all primes p dividing N. For our purpose it is important to note that the number of cusps is always even and that the genus tends to infinity when N grows. It is known that for every Riemann surface of finite area, the intersection of the spectrum of the Laplace operator $\Delta$ with [0,1/4) consists of finitely many eigenvalues of finite multiplicity. 1/4 is always in the continuous spectrum. Selberg [S] showed that for the surfaces $S_0$ the smallest positive element of the spectrum $\lambda_1(S_0)$ satisfies $\lambda_1(S_0) \geqslant 3/16$. He conjectured that $\lambda_1(S_0) = 1/4$. Given a fixed surface $S_0$ as above, we will show how to construct compact surfaces $S_t$ "approximating" $S_0$ as t tends to zero so that $\limsup_{t\to 0} \lambda_1(S_t) \geqslant \lambda_1(S_0) \geqslant 3/16$. The idea for constructing the surfaces $S_t$ is as follows. Consider a maximal set of simple, closed disjoint geodesics on $S_0$. Cutting along these geodesics we obtain a decomposition of $S_0$ into three-holed spheres. The boundary components of the three-holed spheres are either geodesics or punctures. Replace every three-holed sphere which contains one or two punctures with a three-holed sphere with geodesic boundaries of lengths defined as follows. If the boundary component was a geodesic we keep its length unchanged. If it was a puncture give it length $t > 0$. We use the same length t for all punctures and treat the number t as a parameter. Now reassemble the pieces using old identifications of the boundary components for the components which came from the geodesics of $S_0$. We obtain a surface with boundary consisting of an even number of geodesics of length t. Group these geodesics in pairs and identify each pair to form a compact surface $S_t$. Note that the new surface $S_t$ has genus larger than the genus of $S_0$. We remark that our strategy has a chance of succeeding, which can be seen as follows. The surfaces $S_t$ come equipped with a maximal set of disjoint, simple, closed geodesics. Some of these geodesics have lengths independent of t, while others have lengths equal to t. However, the number $L_1(t)$ equal to the minimum of the sum of lengths of geodesics belonging to our family and forming a chain separating $S_t$ is bounded away from zero by a constant which depends only on the choice of the dissection of $S_0$. By the theorem of [SWY] (cf. also [DR], [DPRS]) $\lambda_1(S_t)$ is bounded from below by a positive constant independent of t. We remark that in the actual construction we shall choose the parameter t in a different but equivalent way.

The paper is organized as follows. Section 1 contains a detailed construction of the surfaces $S_t$. In Section 2 we show that $\limsup_{t\to 0} \lambda_1(S_t) \geqslant \lambda_1(S_0)$ whenever surfaces $S_t$ are obtained as above from a given surface $S_0$ with an even number of cusps.

**1. Description of surfaces.**

In this section we give a more precise description of the geometry of a family of compact surfaces $S_t$, t>0, which approximate a given surface with cusps $S_0$ when t ⟶ 0. The surface $S_0$ has an even number of cusps and large genus, but to simplify the notation, we will consider the case of two cusps. Let $\gamma'$ be a simple closed curve enclosing the two cusps, i. e. such that one of the components of $S_0 - \gamma'$ is a twice punctured disk. The free homotopy class of $\gamma'$ contains a unique simple closed geodesic $\gamma$, and by [E], one of the components of $S_0 - \gamma$ is a twice punctured disk. Denote the closure this component $G_0$. We observe that $G_0$ can be dissected into four congruent quadrilaterals one of which is drawn in Figure 1.

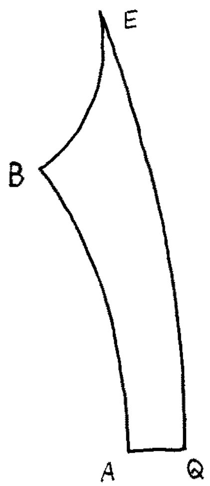
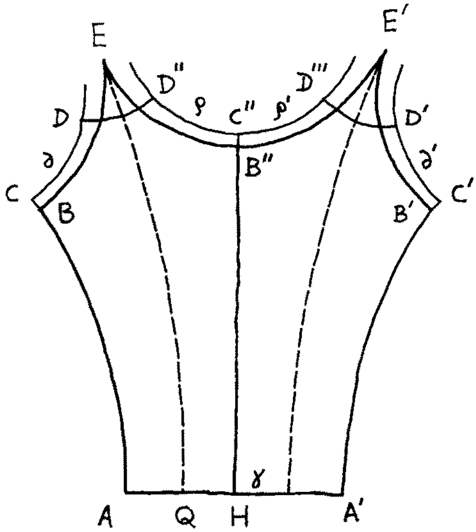



Figure 1

These quadrilaterals have angles π/2, π/2, π/2, and 0, and are uniquely determined by the length of the side AQ (cf. [Be], p. 156). To see that such a dissection exist, observe that $G_0$ is conformally equivalent to the unit disk in the complex plane punctured at -1/2 and 1/2. The four components of the complement of the real and imaginary axes are clearly conformally (or anti-conformally) equivalent, hence isometric with respect to the metric of $G_0$. The dissection of $G_0$ is accomplished by cutting the unit disk along the interval [-1/2,1/2] of the real axis and along the interval [-i,0] of the imaginary axis. Unrolling the resulting simply connected figure in the hyperbolic plane we obtain the following heptagon AA′B′E′B″EB. The figure also shows the four congruent quadrilaterals. $G_0$ is obtained by identifying AB with A′B′, BE with B″E, and B′E′ with B″E′.

Our next task is to "thicken" $G_0$ (identifying E and E′ we may interpret $G_0$ as an infinite handle) to form a finite handle. The parameter in this construction, a small real number $t > 0$, is different from the parameter in the introduction.

Figure 2




Let H be the midpoint of the segment AA′. Continue the geodesic segments AB, A′B′, and HB″ for distance t past B, B′, and B″ respectively. From the endpoints C, C′, C″ draw perpendiculars $\partial$, $\partial'$, $\rho$, $\rho'$ as shown in Figure 2. It follows from elementary hyperbolic geometry and continuity that for small $t > 0$ the common perpendicular DD″ of geodesics determined by $\partial$ and $\rho$ has length approaching 0. Moreover, by symmetry the lengths of DD″ and D′D‴ are equal. The handle $G_t$ is obtained now by identifying AC with A′C′, CD with C″D″, C′D′ with C″D‴, and finally DD′ with D″D‴ (see Figure 2). The image of DD″ in $G_t$ will be the disappearing geodesic $\kappa_t$, i.e. the length of $\kappa_t$ tends to zero as t approaches zero. Note that the handle $G_t$ is isometric to $G_0$ near the geodesic $\gamma$, so that $G_t$ can be attached to $S_0 - G_0$ along $\gamma$ by the same identification as $G_0$. The resulting surface will be denoted by $S_t$. As a consequence of the construction, we see that the handles $G_t$ are thicker than the cusps. More precisely, we have the following lemma.

<u>Lemma 1.1</u>. For every $R > 0$ there exist $T > 0$ and $\iota > 0$ so that $inj(x) \geq \iota$ whenever $x \in S_t$, $0 \leq t \leq T$, and $dist(x, S_0 - G_0) \leq R$.

## 2. Eigenvalues and eigenfunctions.

In this section we investigate the behavior of the first positive eigenvalue of the surface $S_t$ as t tends to zero. We introduce the following notation. For $t > 0$, $\lambda_t = \lambda_1(S_t)$ and $\varphi_t$ is a normalized eigenfunction belonging to $\lambda_t$. Thus

$$\Delta\varphi_t + \lambda_t \varphi_t = 0,$$

$$\int_{S_t} \varphi_t \, dA = 0, \qquad \int_{S_t} \varphi_t^2 \, dA = 1.$$

For $t = 0$, $\lambda_0$ will denote the smallest positive element of the spectrum of $S_0$. If $\lambda_0 < 1/4$, then it is an eigenvalue and we write $\varphi_0$ for the corresponding normalized eigenfunction (cf. [DPRS], p. 106).
If $\lambda_0 = 1/4$, then it may not be an eigenvalue.

<u>Theorem 2.1</u>. $\limsup_{t\to 0} \lambda_t \geq \lambda_0$.

<u>Proof</u>: By the theorem of [SWY] (see also [DR] and [DPRS]) $\liminf_{t\to 0} \lambda_t > 0$. Therefore $\lambda = \limsup_{t\to 0} \lambda_t$ is positive. By construction the diameter of $S_t$ tends to infinity as t approaches 0. Hence, by a theorem of Cheng [C], $\lambda \leq 1/4$. Let $F = S - G_0 = S_t - G_t$. Fix a point $x_0 \in F$. For $t \geq 0$ let $p_t : U \to S_t$ be the universal covering map of the upper half-plane U onto $S_t$ normalized so that

a) $p_t(z_0) = x_0$ for a fixed point $z_0 \in U$ and all $t \geq 0$.

b) $p_t(z) = p_{t'}(z)$ for all t, t' and all z in a neighborhood of $z_0$.

We define $\psi_t = p_t \circ \varphi_t$. By the inequalities of Sobolev and Gårding [BJS] for every integer $k \geq 0$

$$|\nabla^k \psi_t(x)| \leq c(r,k) \sum_{\ell=0}^{N(k)} |\Delta^\ell \psi_t|_{L^2(B_r(x))},$$

where the constant depends only on the radius r and the number k of derivatives. It follows from Lemma 1.1, that for x in a compact set there exists $r > 0$, so that the covering maps $p_t$ are injective on the balls $B_r(x)$. Therefore, the inequality above implies that

$$|\nabla^K \psi_t(x)| \leq c(r,K) \sum_{\ell=0}^{N(K)} \lambda_t^{\ell} .$$

It follows that the derivatives of all orders of the functions $\psi_t$ are bounded uniformly in t on compact subsets of the upper half-plane. Thus we can choose a subsequence $\psi_{t(i)}$, $t(i) \longrightarrow 0$ as $i \longrightarrow \infty$ which converges to a limiting function $\psi$ on U with derivatives of all orders uniformly on compacta. Furthermore, we can arrange that $\psi$ satisfies $\Delta\psi + \lambda\psi = 0$. From now on we shall write only t as a subscript when we mean t(i). Observe that there is nothing to prove if $\lambda = 1/4$. Thus we only consider the case $\lambda < 1/4$. To prove the theorem, it will suffice to show that $\psi = \varphi \circ p_0$ for a smooth function $\varphi$ in $L^2(S_0)$ which does not vanish identically. Indeed such a function $\varphi$ would be an eigenfunction belonging to the eigenvalue $\lambda > 0$ so that $\lambda \geq \lambda_0$.

We first show that $\psi$ is the pull-back of a function defined on $S_0$. Choose a fundamental domain K for the action of $\pi_1(S_0)$ with the property that the inverse image $\tilde{\gamma}$ of the geodesic $\gamma$ disconnects K in such a way that one of the components is congruent to the heptagon in Figure 2. Call this component $\tilde{G}_0$. Let $\tilde{F}$ be the other component of $K - \tilde{\gamma}$. It is clear that the functions $\varphi_t$ converge uniformly with all derivatives to a limiting function on F. Consider two boundary points of $\tilde{G}_0$, say x and x′, which lie over the same point in $G_0$. If x lies on AB and x′ lies on A′B′ (see Figure 3) then it is clear that

$$(2.1) \qquad \psi_t(x) = \psi_t(x'), \qquad \frac{\partial\psi_t}{\partial n}(x) = -\frac{\partial\psi_t}{\partial n}(x').$$

Therefore the same equalities hold for the function $\psi$. We can show that the same is true if x, x′ lie on BE, B″E respectively as follows. Let s be the distance between x and B (which is the same as the distance from x′ to B″). If $\tilde{G}_t$ is the lift of $G_t$ containing $\tilde{G}_0$ and congruent to the octagon ACDD″D‴D′C′A′ we define $x_t$ and $x_t'$ as follows. $x_t$ is chosen as the point on CD at distance s from C, $x_t'$ is on C″D″ at distance s from C″. This is shown in Figure 3.
Clearly,

$$\psi_t(x_t) = \psi_t(x_t'), \qquad \frac{\partial\psi_t}{\partial n}(x_t) = -\frac{\partial\psi_t}{\partial n}(x_t').$$

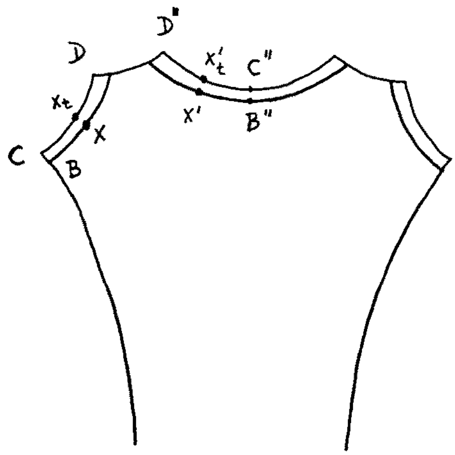



Figure 3

As t approaches zero $x_t$ and $x_t'$ converge to x and x' respectively. It follows that (2.1) holds in this situation as well. Thus $\varphi|K$ is the pullback of a function $\varphi$ on $S_0$. A straightforward application of Green's formula and (2.1) shows that

$$\int (\Delta u + \lambda u)\varphi dA = 0$$

for every compactly supported smooth function u. By elliptic regularity [BJS] $\varphi$ is real analytic on $S_0$. Similarly $\psi$ is analytic on U. Since $\psi$ and $\varphi \circ p_0$ agree on an open set in U they are equal everywhere.

We show next that $\varphi$ is square-integrable. Let $S_0(r)$ be the subset of $S_0$ consisting of points whose distance from $F = S_0 - G_0$ is less than or equal to r. Define $S_t(r) \subset S_t$ in a similar way. If we exclude sets of measure 0, $S_0(r)$ is contained in $S_t(r)$ for small $t > 0$ in the sense that the corresponding $G_0(r)$ is contained in $G_t(r)$ and $S_t - G_t = S_0 - G_0 = F$. Therefore

$$\int_{S_0(r)} \varphi^2 dA \leqslant \lim_{t \to 0} \int_{S_t(r)} \varphi_t^2 \, dA \leqslant 1.$$

Since this is true for arbitrarily large r, $\varphi \in L^2(S_0)$.

To conclude the proof we show that $\varphi$ is not identically zero. Roughly speaking, the only way for the limiting function $\varphi$ to vanish is to have the functions $\varphi_t$ concentrating in the degenerating handle. This is shown to be impossible unless $\lambda = 1/4$. Thus assume that $\varphi \equiv 0$. Then the functions $\varphi_t$ and their gradients converge to zero uniformly on compact subsets of U. Again consider the configuration in Figure 2. Choose a point P on BE. This choice is independent of t. Let $L_1$ be the locus through P of points equidistant from the geodesic segment DD″, and let $L_2$ be the equidistant locus unit distance closer to DD″. Consider also the symmetric pair $L_1'$, $L_2'$ of curves equidistant from D‴D′ (see Figure 4). When t approaches 0 these equidistant loci converge to horocycles centered at E (cf. [Be], p. 163). Let $M_t$ be the intersection of the bands between the two pairs of equidistant loci with $\tilde{G}_t$.

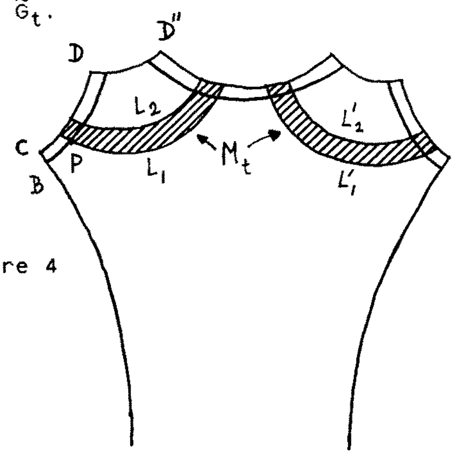
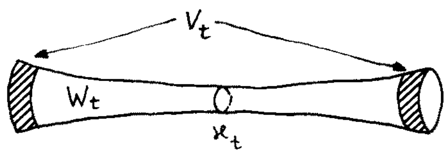



Figure 4




The union of sets $M_t$ for $t \leq t_0$ is compact. It follows that $\varphi_t$ and $\nabla\varphi_t$ are uniformly small on $V_t = p_t(M_t)$ and in fact on the complement of the tubular neighborhood $W_t$ of the geodesic $\kappa_t$ between the images of $L_1$ and $L_1'$. Since the area of $S_t$ is constant, it follows that

$$\int_{W_t} \varphi_t^2 dA \approx 1$$

$$\int_{V_t} (\varphi_t^2 + |\nabla\varphi_t|^2) dA \approx 0,$$

for small t. By the argument of Lemma 3.3 of [DPRS] the energy of $\varphi_t$ is bounded from below by a constant arbitrarily close to 1/4. It follows that $\lambda = \limsup \lambda_t = \limsup \int |\nabla\varphi_t|^2 dA = 1/4$. This contradicts our assumption that $\lambda < 1/4$ and concludes the proof.

Remarks. (a) If the surface $S_0$ has an even number of cusps we can group them in pairs and thicken every pair into a handle as described in Section 1 using the same parameter t for every pair of cusps. This yields a family of surfaces $S_t$. Theorem 2.1 remains true with an almost identical proof.

(b) The following example suggests that the possibility that the eigenfunctions $\varphi_t$ converge to zero cannot be ruled out. Let $S_0$ be the sphere with three punctures. Consider the family of surfaces $S_t$, where $S_t$ is a three-holed sphere and each boundary component is a geodesic of length t. Let $\lambda_t$ be the first positive eigenvalue for the Neumann problem on $S_t$, and let $\varphi_t$ be a normalized eigenfunction belonging to $\lambda_t$. It is easy to see that $\lim_{t\to 0} \lambda_t = 1/4$. On the other hand, the spectrum of $S_0$ is contained in $[1/4,\infty)$ and 1/4 is not an eigenvalue. The argument in the proof of Theorem 2.1 implies that a subsequence of $\{\varphi_t\}$ converges to a limiting function. This limit has to be zero, for otherwise the limiting function would be an eigenfunction for $S_0$ with eigenvalue 1/4.

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