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HOROCYCLE FLOW ON GEOMETRICALLY FINITE SURFACES

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Let $S = \Gamma\backslash D^2$ be a quotient of the Poincaré disc by a finitely generated discrete group Γ of orientation preserving isometries acting without fixed points on D^2. Topologically S can be obtained from a compact surface by removing a finite number of closed discs.

The group of orientation preserving isometries of D^2 is $PSL(2, \mathbb{R})$ and the unit tangent bundle $T_1 S$ of S is a homogeneous space of $PSL(2, \mathbb{R})$:

$$T_1 S = \Gamma\backslash PSL(2, \mathbb{R}).$$

In particular, the unipotent subgroup of $PSL(2, \mathbb{R})$

$$N = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$

acts on $T_1 S$.

It is our main goal to determine all N-invariant Radon measures on $T_1 S$. Our first remark is that if C is the cone of positive N-invariant Radon measures in the space $\mathcal{M}(T_1 S)$ of all Radon measures with the vague topology, then C is the closed convex hull of the union of its extremal generators [B, II No. 2]; moreover it is easily seen that a measure is on an extremal generator of C if and only if it is ergodic. This reduces the problem to the classification of all ergodic measures.

To proceed further we consider the following decomposition of $T_1 S$: Let S^1 be the ideal boundary of D^2 and $\Lambda \subset S^1$ be the limit set of Γ. Using the visual map:

$$\text{Vis}: T_1 D^2 \to S^1,$$

we obtain first a decomposition of $T_1 D^2$ as a union of two subsets

$$\mathscr{F}_c = \{p \in T_1 D^2 : \text{Vis}(p) \in \Lambda\}$$

$$\mathscr{F}_d = \{p \in T_1 D^2 : \text{Vis}(p) \in S^1 \backslash \Lambda\}.$$

This gives via the projection $T_1 D^2 \to T_1 S$ a decomposition of $T_1 S$ into two disjoint subsets

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$$T_1 S = \mathscr{F}_c \cup \mathscr{F}_d,$$

where $\mathscr{F}_c$ is closed, $\mathscr{F}_d$ open and both are invariant under the action of N and the action of the geodesic flow of $T_1 S$.

Recall at this point that the action of the geodesic flow in $T_1 S = \Gamma \backslash PSL(2, \mathbb{R})$ is given by the action of

$$A = \left\{ \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Now we can describe three families of N-invariant ergodic measures on $T_1 S$.

A. For each $g \in \mathscr{F}_d$ the orbit map:

$$\mathbb{R} \to T_1 S \qquad x \mapsto gn(x)$$

is a homeomorphism onto its image. The direct image of the Lebesgue measure dx on $\mathbb{R}$ under this orbit map gives an N-invariant ergodic measure supported on gN. Since all orbits of N on $\mathscr{F}_d$ are closed, this shows that each N-orbit on $\mathscr{F}_d$ is the support of an ergodic N-invariant measure which is unique up to scaling.

B. To each cusp of S there corresponds an immersed cylinder in $\mathscr{F}_c \subset T_1 S$ consisting of N-periodic points. Each of these periodic orbits carries a unique N-invariant probability measure.

C. Let μ_P be the Patterson measure on the limit set $\Lambda \subset S^1$ and let δ be the Hausdorff dimension of Λ. Using the origin $o \in D^2$ as a reference point we can identify canonically each fiber of the visual map

$$\text{Vis}: T_1 D^2 \to S^1$$

with the group AN. Via this identification we put on each fiber $\text{Vis}^{-1}(\zeta)$, $\zeta \in S^1$, the measure:

$$e^{\delta t}\, dt\, dx,$$

defined on AN.

Integrating along fibers of the visual map and integrating with respect to μ_P produces a measure on $\tilde{\mathscr{F}}_c$ which projects down to an N-invariant measure μ supported on $\mathscr{F}_c$. Note that if T_t denotes the action of the geodesic flow then $T_{t*}\mu = e^{t(1-\delta)}\mu$. In particular, if $\delta < 1$ this measure is infinite.

Hopefully any ergodic N-invariant measure is up to scaling a measure in the families listed above. In the case $\text{Vol}(S) < +\infty$, the measure constructed in C coincides with the $PSL(2, \mathbb{R})$-invariant probability measure on $T_1 S$ and $\mathscr{F}_d = \varnothing$. In this case the above description of N-invariant ergodic measures is complete as follows from work of Dani [D1], [D2].

If $\mathrm{Vol}(S) = +\infty$ it follows from recent results of M. Ratner [R] that the only N-invariant ergodic probability measures are supported on periodic orbits of N. In particular, if S has no cusps there are no invariant probability measures. Here we want to show that if S is geometrically finite without cusps and the Hausdorff dimension δ of the limit set verifies $\delta > 1/2$, then the above description of N-invariant ergodic measures is complete. This follows immediately from the following:

THEOREM 1. *Assume S is geometrically finite without cusps and $\delta > 1/2$. Then there is, up to a scalar multiple, a unique N-invariant Radon measure supported on $\mathscr{F}_c$.*

To put the hypothesis on the Hausdorff dimension in the context of our method we recall the following facts about the Laplacian of S. The Laplace-Beltrami operator Δ of S acts in the space of C^∞ functions with compact support $C_K^\infty(S)$, and has a unique self-adjoint extension to an unbounded operator on $L^2(S)$. The spectrum of Δ in $(-1/4, 0]$ consists only of eigenvalues with finite multiplicity and the essential spectrum of Δ is contained in $(-\infty, -1/4]$ [DPRS]. It follows from work of Patterson [P], [S, Th. 2.17], that $\delta > 1/2$ if and only if $\mathrm{Spec}\, \Delta_S \cap (-1/4, 0] \neq \varnothing$ in which case $\lambda_0 = \delta(\delta - 1)$ is the highest eigenvalue of Δ_S. This eigenvalue has multiplicity one and any associated eigenfunction is of constant sign on S. Patterson showed that such an eigenfunction can be obtained in the following way: Let L be the Lebesgue measure of S^1 and $j(g)$ the Radon-Nikodym derivative of $g_* L$ with respect to L, where $g \in PSL(2, \mathbb{R})$. Then

$$\varphi_0(h) = \int_{S^1} d\mu_P(\zeta) j(h, \zeta)^\delta$$

is a Γ-invariant eigenfunction on D^2 of eigenvalue $\delta(\delta - 1)$. If $\delta > 1/2$ it is in $L^2(S)$. On the other hand, a straightforward computation shows that the direct image of the measure μ via the map $T_1 S \to S$ is the measure

$$\varphi_0(h)\, dh,$$

where dh is the area element of S. In particular, the function φ_0 viewed on $T_1 S$ is in $L^1(T_1 S, \mu)$. Theorem 1 shows now that $\varphi_0(h)\, dh$ has a topological characterization in terms of the action of N on $T_1 S$. Concerning the proof of Theorem 1 we study how the probability measure on $PSL(2, \mathbb{R})$:

$$m_T(\varphi) = \frac{1}{2T} \int_{-T}^{T} \varphi(n(t))\, dt$$

acts in the space of a unitary representation of $PSL(2, \mathbb{R})$. We obtain that m_T acts as a contraction in the space of C^∞-vectors when measured in a suitable norm (see Proposition 1). Moreover, the contraction constant tends to zero as $T \to \infty$. This and a certain conservativity property of the action of N on $\mathscr{F}_c$ enables us to show Theorem 1. As a corollary of Theorem 1 and the method of proof we obtain the following equidistribution result.

COROLLARY. *Under the assumptions of Theorem* 1 *we have for all* $\varphi \in C_K(T_1 S)$

$$\lim_{\tau \to \infty} \frac{\int_1^\tau dx\, x^{-\delta-1} \int_{-x}^{x} \varphi(gn(t))\, dt}{\int_1^\tau dx\, x^{-\delta-1} \int_{-x}^{x} \varphi_0(gn(t))\, dt} = \frac{\int_{T_1 S} \varphi(p)\, d\mu(p)}{\|\varphi_0\|_2^2}$$

uniformly on compact sets in $\mathscr{F}_c$.

In the case where S is compact we have a more precise version of this corollary where we control the rate of uniform distribution of horocycles with respect to the Lebesgue measure on $T_1 S$ (see Theorem 2 of §1).

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1. Unitary action of a unipotent subgroup of $PSL(2, \mathbb{R})$.

In 1.1 and 1.2 we recall some classical facts concerning the representation theory of $PSL(2, \mathbb{R})$. Standard references are [L], [D]. In 1.3 we state the main proposition (Proposition 1) and derive some corollaries for hyperbolic surfaces (Theorem 2). §1.4 is devoted to the proof of Proposition 1.

1.1. Let $G = PSL(2, \mathbb{R})$, $\mathfrak{g}$ its Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$ and $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$. To a continuous unitary representation π of G in a separable Hilbert space $\mathscr{H}$ one associates the derived representation $d\pi$ of $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ which acts in the space of C^∞-vectors:

$$\mathscr{H}^\infty = \{v \in \mathscr{H} : g \to \pi(g)v \text{ is a } C^\infty \text{ map from } G \text{ to } \mathscr{H}\}.$$

The center of $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ is generated by the Casimir element w:

$$w = \tfrac{1}{4}(2iW - W^2 + E_+ E_-)$$

where $W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $E_+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$, $E_- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$, is a basis of $\mathfrak{g}_{\mathbb{C}}$. If $(\mathscr{H}, \pi)$ is irreducible, $d\pi(w)$ acts as scalar multiplication on $\mathscr{H}^\infty$.

1.2. Let

$$K = \left\{ k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, 0 \le \theta \le \pi \right\}$$

be a maximal compact subgroup of G. We can state the classification of irreducible unitary representations of $PSL(2, \mathbb{R})$ in the following way [L, p. 123]:

(a) For each $\lambda \in (-\infty, 0]$ there is a unique irreducible unitary representation $(\mathscr{H}_\lambda, \pi_\lambda)$ which has a K-invariant vector and such that the action of the Casimir operator on $\mathscr{H}_\lambda^\infty$ is $d\pi(w) = \lambda \cdot \mathrm{Id}$. The trivial representation corresponds to $\lambda = 0$.

(b) For each even integer $m \geqslant 2$ there is a unique irreducible representation $\mathscr{H}(m)$ having a lowest weight vector of weight m with respect to K and a unique irreducible one $\mathscr{H}(-m)$ having a highest weight vector of weight $-m$. These are the discrete series of $PSL(2, \mathbb{R})$ and $d\pi(w) = (m/2 - 1)m/2 \cdot \mathrm{Id}$ on $\mathscr{H}^\infty(m) \oplus \mathscr{H}^\infty(-m)$.

This classification enables us to identify the dual space $\hat{G}$ of G with the topological space

$$(-\infty, 0] \cup Z, \qquad \text{where } Z = \{\pm m;\, m \geqslant 2, \text{even}\}.$$

If $(\mathscr{H}, \pi)$ is a continuous unitary representation of G in a separable Hilbert space $\mathscr{H}$, then $(\mathscr{H}, \pi)$ is a direct sum of multiplicity free representations

$$(\mathscr{H}, \pi) = \bigoplus_{n=1}^{\infty} (\mathscr{L}_n, \alpha_n),$$

see [D, 8.6.6]. Moreover, each multiplicity free representation $(\mathscr{L}_n, \alpha_n)$ is defined via a Borel measure μ_n on $\hat{G}$. We define the support of π, $\operatorname{supp} \pi \subset \hat{G}$ by

$$\operatorname{supp} \pi = \bigcup_{n=1}^{\infty} \operatorname{supp} \mu_n .$$

1.3. Let $(\mathscr{H}, \pi)$ be a continuous unitary representation of $PSL(2, \mathbb{R})$ in a separable Hilbert space $\mathscr{H}$. We assume that we are given a norm N on the space of C^∞ vectors $\mathscr{H}^\infty$ satisfying the following properties:

(a) $N(\pi(g)v) = N(v)$ for all $g \in G$, $v \in \mathscr{H}^\infty$.

(b) There is a finite subset $S \subset \mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ and a constant $c > 0$ such that

$$N(v) \leqslant c \max_{L \in S} \|d\pi(L)v\|$$

for all $v \in \mathscr{H}^\infty$.

Consider the following one parameter family of probability measures on $PSL(2, \mathbb{R})$

$$m_T(f) = \frac{1}{2T} \int_{-T}^{T} f(n(t))\, dt, \qquad T > 0.$$

The next proposition shows how the map

$$T \to N(\pi(m_T)v)$$

vanishes at infinity for $v \in \mathscr{H}^\infty$. In order to state the proposition we introduce some notation. Let

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

If $S \subset \mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ is a finite subset

$$\|v\|_S = \max_{L \in S} \|d\pi(L)v\|, \qquad v \in \mathscr{H}^\infty.$$

PROPOSITION 1. *Let $(\mathscr{H}, \pi)$ be a continuous unitary representation of $PSL(2, \mathbb{R})$ in a separable Hilbert space $\mathscr{H}$ and let N be a norm on $\mathscr{H}^\infty$ satisfying properties* (a), (b) *above.*

(1) *If $(\mathscr{H}, \pi)$ has no nonzero fixed vector then*

$$\lim_{T \to \infty} N(\pi(m_T)v) = 0$$

for every $v \in \mathscr{H}^\infty$.

(2) *Let $0 < \alpha \leqslant 1/2$ and assume that*

$$\operatorname{supp} \pi \subset (-\infty, \alpha(\alpha - 1)] \cup Z$$

then we have for all $v \in \mathscr{H}^\infty$ and $T \geqslant 1$

$$N(\pi(m_T)v) \leqslant c \frac{T^{-\alpha} - T^{\alpha - 1}}{1 - 2\alpha} \{\|v\|_S + \|d\pi(H)v\|_S + \|d\pi(X_-)v\|_S\},$$

where $c > 0$ is some positive constant.

Let us show how this result applies in concrete situations: Let $S = \Gamma \backslash D^2$ be any hyperbolic surface. We consider the unitary representation π of $PSL(2, \mathbb{R})$ on $\mathscr{H} = L^2(\Gamma \backslash PSL(2, \mathbb{R}))$ given by right translations. On the space of C^∞ vectors $\mathscr{H}^\infty$ we would like to take the norm

$$N(f) = \sup_{x \in T_1 S} |f(x)|.$$

The case of surfaces with cusps shows that N is not always defined on $\mathscr{H}^\infty$. However, assume that there is a positive lower bound on the injectivity radius of S and choose a left invariant Riemannian metric on $PSL(2, \mathbb{R})$ whose projection on D^2 is the hyperbolic metric. It then follows from [A, 2.10 and 2.2.1] that the Sobolev imbedding theorem holds for the Riemannian manifold $T_1 S = \Gamma \backslash PSL(2, \mathbb{R})$. In particular there is a constant $c > 0$ and a finite subset $L \subset \mathscr{U}(\mathfrak{g})$ of polynomials of degree at most two such that for all $f \in C_K^\infty(T_1 S)$ we have

$$\sup_x |f(x)| \leqslant c \cdot \|f\|_L.$$

We can furthermore identify $\mathscr{H}^\infty$ with a subspace of the space of bounded C^∞ functions on $T_1 S$.

Now we can apply Proposition 1 to the norm $N(f) = \sup_x |f(x)|$ defined on $\mathscr{H}^\infty$ to obtain

THEOREM 2. *Let $S = \Gamma \backslash D^2$ be a hyperbolic surface whose injectivity radius has a positive lower bound and let $\|f\|_{H^2_3}$ be the Sobolev L^2 norm involving all the derivatives of f up to the third order.*

(A) *For every continuous function f on $T_1 S$ vanishing at infinity*

$$\lim_{T\to\infty} \sup_{g \in T_1 S} \frac{1}{2T} \int_{-T}^{T} f(gn(t))\, dt = 0.$$

(*B*) *Assume that the spectrum of the Laplacian of S is contained in $(-\infty, \alpha(\alpha - 1)]$, where α is some number satisfying $0 < \alpha \leqslant 1/2$. Then we have for all $f \in C_K^\infty(T_1 S)$ and $T \geqslant 1$*

$$\sup_{g \in T_1 S} \left| \frac{1}{2T} \int_{-T}^{T} f(gn(t))\, dt \right| \leqslant c \frac{T^{-\alpha} - T^{\alpha - 1}}{1 - 2\alpha} \|f\|_{H^2_3}.$$

(C) *Assume that S is compact. Let $\lambda_1 < 0$ be the first nonzero eigenvalue of the Laplacian of S and let $0 < \alpha \leqslant 1/2$ satisfy $\alpha(\alpha - 1) \geqslant \lambda_1$. Then we have for all $f \in C_K^\infty(T_1 S)$ and $T \geqslant 1$*

$$\sup_{g \in T_1 S} \left| \frac{1}{2T} \int_{-T}^{T} f(gn(t))\, dt - \int_{T_1 S} f(h)\, dh \right| \leqslant c \frac{T^{-\alpha} - T^{\alpha - 1}}{1 - 2\alpha} \|f\|_{H^2_3}.$$

Proof. (A) and (B) are direct consequences of Proposition 1, (1), (2). To obtain (C) we apply Proposition 1, (2) to the restriction of π to the subspace of functions $f \in L^2(\Gamma \backslash PSL(2, \mathbb{R}))$ orthogonal to the constants. □

Let us give two examples of surfaces satisfying the hypothesis of Theorem 2 (B):
(1) Let $S_0 = \Gamma \backslash D^2$ be a compact surface and $\Gamma' \lhd \Gamma$ be a normal subgroup of Γ such that Γ/Γ' is not amenable. Then $S = \Gamma' \backslash D^2$ satisfies the hypothesis of Theorem 2 (B) (see [Br] for instance).
(2) Any geometrically finite surface of infinite volume and without cusps [DPRS].

1.4. In this section we prove Proposition 1.

Let $(\mathscr{H}, \pi)$ be a continuous unitary representation of $PSL(2, \mathbb{R})$. We assume that the Casimir operator w acts as scalar multiplication on $\mathscr{H}^\infty$: $d\pi(w) = \lambda \operatorname{Id}$ and we fix an $\alpha \in \mathbb{C}$ such that $\alpha(\alpha - 1) = \lambda$. We introduce also the subgroups of $PSL(2, \mathbb{R})$

$$K = \left\{ k(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \quad 0 \leqslant \theta \leqslant \pi \right\}$$

$$A = \left\{ a(y) = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}, \quad y > 0 \right\}$$

$$N = \left\{ n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{R} \right\}.$$

LEMMA 1.

(A) *For all* $v \in \mathscr{H}^\infty$, $Y \geqslant 1$, *and* $T > 0$ *we have*

$$\pi(m_T)v = \frac{(1-\alpha)Y^{-\alpha} - \alpha Y^{\alpha-1}}{1-2\alpha}\pi(m_T a(Y))v$$

$$-\frac{1}{2}\frac{Y^{-\alpha} - Y^{\alpha-1}}{1-2\alpha}\pi(m_T a(Y))\, d\pi(H)v$$

$$+\frac{1}{2T}\int_1^Y dy\left(\frac{y^{-\alpha} - y^{\alpha-1}}{1-2\alpha}\right)[\pi(n(-T)) - \pi(n(T))]\pi(a(y))\, d\pi(X_-)v.$$

(B) *Assume that* $\alpha \in \mathbb{R}$, $\alpha - 1 \geqslant 0$ *and that* π *has no nonzero fixed vector. Then we have for all* $T > 0$, $Y \geqslant 1$ *and* $v \in \mathscr{H}^\infty$

$$\pi(m_T)v = Y^{-\alpha}\pi(m_T a(Y))v$$

$$-\frac{1}{2T}\int_0^Y dy\, y^{\alpha-1}\left(\frac{Y^{1-2\alpha} - \max(1,y)^{1-2\alpha}}{1-2\alpha}\right)[\pi(n(-T)) - \pi(n(T))]\, d\pi(X_-)v.$$

Proof. We recall that in Iwasawa coordinates $n(x)$, $a(y)$, $k(\theta)$ the left invariant differential operators X_-, H, W are given by

$$W = \frac{\partial}{\partial\theta}$$

$$H = -2y\sin 2\theta\frac{\partial}{\partial x} + 2y\cos 2\theta\frac{\partial}{\partial y} + \sin 2\theta\frac{\partial}{\partial\theta}$$

$$X_- = y\cos 2\theta\frac{\partial}{\partial x} + y\sin 2\theta\frac{\partial}{\partial y} - \cos^2\theta\frac{\partial}{\partial\theta}$$

and the Casimir operator is

$$w = y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - y\frac{\partial^2}{\partial x\,\partial\theta}.$$

For $v \in \mathscr{H}^\infty$ the function

$$(x, y, \theta) \mapsto \pi(m_T n(x)a(y)k(\theta))v$$

is an eigenfunction of the Casimir operator of eigenvalue λ. In particular

$$y^2\frac{\partial^2}{\partial y^2}\pi(m_T a(y))v + y^2\frac{\partial^2}{\partial x^2}\bigg|_{x=0}\pi(m_T n(x)a(y))v - y\frac{\partial^2}{\partial x\,\partial\theta}\bigg|_{x=\theta=0}\pi(m_T n(x)a(y)k(\theta))v$$
$$= \lambda\pi(m_T a(y))v.$$

Now we have

$$y^2\frac{\partial^2}{\partial x^2}\bigg|_{x=0}\pi(m_T n(x)a(y))v = y^2\frac{\partial^2}{\partial x^2}\bigg|_{x=0}\frac{1}{2T}\int_{-T}^{T}\pi(n(t+x)a(y))v\,dt$$
$$= \frac{y^2}{2T}\frac{\partial}{\partial x}\bigg|_{x=0}[\pi(n(T+x)a(y))v - \pi(n(-T+x)a(y))v]$$

and

$$y\frac{\partial^2}{\partial x\,\partial\theta}\bigg|_{x=\theta=0}\pi(m_T n(x)a(y)k(\theta))v = \frac{y}{2T}\frac{\partial}{\partial\theta}\bigg|_{\theta=0}[\pi(n(T)) - \pi(n(-T))]\pi(a(y)k(\theta))v.$$

Putting everything together and using the fact that

$$X_-|_{x=\theta=0} = y\frac{\partial}{\partial x}\bigg|_{x=0} - \frac{\partial}{\partial\theta}\bigg|_{\theta=0}$$

we obtain

$$y^2\frac{\partial^2}{\partial y^2}\pi(m_T a(y))v - \lambda\pi(m_T(a(y)))v = \frac{y}{2T}[\pi(n(-T)) - \pi(n(T))]\pi(a(y))\,d\pi(X_-)v.$$

Define the following functions

$$g(y) = y^{-\alpha}\pi(m_T a(y))v$$

$$D(y) = [\pi(n(-T)) - \pi(n(T))]\pi(a(y))\,d\pi(X_-)v.$$

With this notation the above relation becomes

$$\frac{\partial}{\partial y} y^{2\alpha} \frac{\partial}{\partial y} g(y) = \frac{y^{\alpha-1}}{2T} D(y).$$

Let $0 < a < b$. Integrating this equality from a to b with respect to y we obtain

$$(*) \qquad b^{2\alpha} g'(b) - a^{2\alpha} g'(a) = \frac{1}{2T} \int_a^b y^{\alpha-1} D(y)\, dy.$$

Proof of (A). Multiplying $(*)$ by $a^{-2\alpha}$ and integrating from 1 to b with respect to a we obtain

$$(**) \qquad g(1) = g(b) - \left(\frac{b - b^{2\alpha}}{1 - 2\alpha}\right) g'(b) + \frac{1}{2T} \int_1^b dy \left(\frac{y^{-\alpha} - y^{\alpha-1}}{1 - 2\alpha}\right) D(y).$$

From the definition of g it follows that

$$yg'(y) = -\alpha y^{-\alpha} \pi(m_T a(y)) v + y^{-\alpha} \left(y \frac{\partial}{\partial y} \pi(m_T a(y)) v \right),$$

but $H|_{\theta=0} = 2y(\partial/\partial y)$ so that

$$(***) \qquad yg'(y) = -\alpha y^{-\alpha} \pi(m_T a(y)) v + \tfrac{1}{2} y^{-\alpha} \pi(m_T a(y))\, d\pi(H) v.$$

Substituting $(***)$ in $(**)$ and using $g(y) = y^{-\alpha}\pi(m_T a(y))v$ and setting $b = Y$ we obtain (A).

Proof of (B). We write $(***)$ in the following form

$$y^{2\alpha} g'(y) = y^{\alpha-1} [-\alpha \pi(m_T a(y)) v + \tfrac{1}{2} \pi(m_T a(y))\, d\pi(H) v].$$

(1) $\alpha - 1 > 0$: then $\lim_{y\to 0} y^{2\alpha} g'(y) = 0$ in $\mathscr{H}$.
(2) $\alpha - 1 = 0$: since π has no nonzero invariant vectors it follows from [H-M] that for every $w \in \mathscr{H}$ $\lim_{y\to 0} \pi(a(y))w = 0$ weakly in $\mathscr{H}$. In particular $\lim_{y\to 0} y^2 g'(y) = 0$ weakly in $\mathscr{H}$.

In both cases equation $(*)$ implies that

$$b^{2\alpha} g'(b) = \frac{1}{2T} \int_0^b y^{\alpha-1} D(y)\, dy.$$

Multiplying both sides with $b^{-2\alpha}$ and integrating from 1 to Y with respect to b we obtain

$$g(Y) - g(1) = \frac{1}{2T}\int_0^Y dy\, D(y) y^{\alpha-1}\left[\frac{Y^{1-2\alpha} - \max(1, y)^{1-2\alpha}}{1-2\alpha}\right]$$

which proves (B). □

Proof of Proposition 1. Let $(\mathscr{H}, \pi)$ be a continuous unitary representation of $PSL(2, \mathbb{R})$ in a separable Hilbert space $\mathscr{H}$. Let

$$(\mathscr{H}, \pi) = \bigoplus_{n=1}^{\infty} (\mathscr{L}_n, \beta_n)$$

be its decomposition in a direct sum of multiplicity free representations $(\mathscr{L}_n, \beta_n)$. For each n there is a bounded Borel measure μ_n on $\hat{G}$ such that

$$\beta_n = \int_{\hat{G}}^{\oplus} \beta\, d\mu_n(\beta)$$

(see [D, 8.6.5]).

We are going to use the following elementary fact: any bounded Borel function $F\colon \hat{G} \to \mathbb{C}$ defines via the direct integral decomposition a bounded intertwining operator of π

$$T_F\colon \mathscr{H} \to \mathscr{H}$$

whose operator norm satisfies $\|T_F\| \leqslant \sup_{\alpha \in \operatorname{supp} \pi} |F(\alpha)|$. Remark also that any intertwining operator acts in the space of C^∞ vectors $\mathscr{H}^\infty$.

We show now how to deduce Proposition 1 (1) from Proposition 1 (2). Assume that $(\mathscr{H}, \pi)$ does not contain the trivial representation. Take $\varepsilon < 0$ and let P_ε be the orthogonal projection in $\mathscr{H}$ corresponding to the characteristic function of the set

$$(-\infty, \varepsilon] \cup Z \subset \hat{G}.$$

Then it follows from the fact that π has no fixed vector that $\lim_{\varepsilon \to 0} \|P_\varepsilon v - v\| = 0$ for every $v \in \mathscr{H}$. Assume that we are given a norm N on $\mathscr{H}^\infty$ satisfying the conditions (a), (b) of §1.3. Let $-1/4 < \varepsilon < 0$, $\varepsilon = \alpha(\alpha - 1)$, $0 < \alpha < 1/2$ and define $\mathscr{H}_\varepsilon = P_\varepsilon \mathscr{H}$. Then $P_\varepsilon \mathscr{H}^\infty = \mathscr{H}_\varepsilon^\infty$ and we can apply Proposition 1 (2) to the restriction of π to $\mathscr{H}_\varepsilon$ and the restriction of N to $\mathscr{H}_\varepsilon^\infty$. Namely if $F = S \cup \{X_-, H\}$ then there is a constant $c > 0$ such that for all $v \in \mathscr{H}^\infty$

$$N(\pi(m_T)P_\varepsilon v) \leqslant cT^{-\alpha}\|P_\varepsilon v\|_F.$$

Let $v \in \mathscr{H}^\infty$, $\delta > 0$ and choose $\varepsilon > 0$ such that

$$\max_{L \in S} \|P_\varepsilon\, d\pi(L)v - d\pi(L)v\| \leqslant \delta/c.$$

Writing $v = P_\varepsilon v + (v - P_\varepsilon v)$ we have

$$N(\pi(m_T)v) \leqslant N(\pi(m_T)P_\varepsilon v) + N(\pi(m_T)(v - P_\varepsilon v))$$

and now

$$N(\pi(m_T)(v - P_\varepsilon v)) \leqslant N(v - P_\varepsilon v) \leqslant c \max_{L \in S} \|d\pi(L)(v - P_\varepsilon v)\| \leqslant \delta .$$

Hence

$$N(\pi(m_T)v) \leqslant cT^{-\alpha}\|P_\varepsilon v\|_F + \delta$$

from which it follows that $\limsup_{T\to\infty} N(\pi(m_T)v) \leqslant \delta$ for each $\delta > 0$. This proves Proposition 1 (1).

Proof of Proposition 1 (2). We begin by defining certain functions on $\hat{G}$.

(1) The function β

(a) on $(-\infty, 0]$: $-1/4 \leqslant \lambda \leqslant 0, \quad \lambda = \beta(\beta - 1)$ and $0 \leqslant \beta \leqslant 1/2$

$$\lambda < -1/4, \quad \lambda = \beta(\beta-1) \quad \text{and} \quad \operatorname{Im}\beta \geqslant 0$$

(b) on Z: $\beta(\pm m) = \dfrac{m}{2}$.

(2) The function f_y, $y \geqslant 1$

$$f_y(\lambda) = \frac{(1-\beta)y^{-\beta} - \beta y^{\beta-1}}{1-2\beta}, \qquad \lambda \leqslant 0$$

$$f_y(\pm m) = y^{-\beta}, \qquad m \geqslant 2.$$

(3) The function s_y, $y \geqslant 1$

$$s_y(\lambda) = \frac{y^{-\beta} - y^{\beta-1}}{1-2\beta}, \qquad \lambda \leqslant 0,$$

$$s_y(\pm m) = 0, \qquad m \geqslant 2.$$

(4) The function t_y, $Y > y > 0$, $Y \geqslant 1$

$$t_y = s_y \quad \text{on } (-\infty, 0] \quad \text{for } y \geqslant 1,$$

$$t_y = 0 \quad \text{on} \quad (-\infty, 0] \quad \text{for } 0 < y < 1,$$

$$t_y(\pm m) = y^{\beta-1}\left[\frac{\max(1, y)^{1-2\beta} - Y^{1-2\beta}}{1-2\beta}\right] \qquad \text{for } y > 0.$$

If F_y, S_y, T_y denote the corresponding intertwining operators on $\mathscr{H}$ it follows from Lemma 1 that for all $v \in \mathscr{H}^\infty$, $T > 0$, $Y > 1$

$$\pi(m_T)v = F_Y\pi(m_T a(Y))v - \frac{1}{2}S_Y\pi(m_T a(Y))\, d\pi(H)v$$

$$+\frac{1}{2T}\int_0^Y dy\; T_y[\pi(n(-T)) - \pi(n(T))]\pi(a(y))\, d\pi(X_-)v.$$

From this and the properties of N it follows that

$$N(\pi(m_T)v) \leqslant N(F_Y v) + \frac{1}{2}N(S_Y\, d\pi(H)v) + \frac{1}{T}\int_0^Y dy\; N(T_y\, d\pi(X_-)v)$$

$$\leqslant c\left[\|F_Y v\|_S + \|S_Y\, d\pi(H)v\|_S + \frac{1}{T}\int_0^Y \|T_y\, d\pi(X_-)v\|_S\, dy\right].$$

Moreover,

$$\|F_Y v\|_S \leqslant \|f_Y\|_\infty \|v\|_S, \qquad \|S_Y\, d\pi(H)v\|_S \leqslant \|s_Y\|_\infty \|d\pi(H)v\|_S,$$

$$\|T_y\, d\pi(X_-)v\|_S \leqslant \|t_y\|_\infty \|d\pi(X_-)v\|_S,$$

where the supremum $\| \;\|_\infty$ is taken over $(-\infty, \alpha(\alpha-1)] \cup Z$.

It is now easy to verify from the definitions that

$$\|f_y\|_\infty \leqslant C\frac{(1-\alpha)y^{-\alpha} - \alpha y^{\alpha-1}}{1-2\alpha} \qquad y \geqslant 1,$$

$$\|s_y\|_\infty \leqslant C\frac{y^{-\alpha} - y^{\alpha-1}}{1-2\alpha} \qquad y \geqslant 1,$$

$$\|t_y\|_\infty \leqslant C\frac{y^{-\alpha} - y^{\alpha-1}}{1-2\alpha} \qquad y \geqslant 1,$$

$$\|t_y\|_\infty \leqslant 1, \qquad 0 < y < 1,$$

where $C > 0$ is some absolute constant. From this it follows that

$$\|F_Y v\|_S \leqslant C\frac{(1-\alpha)Y^{-\alpha} - \alpha Y^{\alpha-1}}{1-2\alpha}\|v\|_S$$

$$\|S_Y\, d\pi(H)v\|_S \leqslant C\frac{Y^{-\alpha} - Y^{\alpha-1}}{1-2\alpha}\|d\pi(H)v\|_S$$

$$\frac{1}{T}\int_0^Y dy\|T_y\, d\pi(X_-)v\|_S \leqslant \frac{1}{T}\|d\pi(X_-)v\|_S + \frac{1}{T}\int_1^Y dy\left(\frac{y^{-\alpha} - y^{\alpha-1}}{1-2\alpha}\right)\|d\pi(X_-)v\|_S$$

Putting $T = Y$ we obtain Proposition 1 (2). □

2. Geometrically finite surfaces

2.1. Let $S = \Gamma\backslash D^2$ be any hyperbolic surface. A positive measure λ on $\Gamma\backslash PSL(2, \mathbb{R})$ is $P = AN$ quasi invariant if

$$p_*\lambda = \chi(p)\lambda \qquad \text{for all } p \in P,$$

where $\chi\colon P \to \mathbb{R}^+$ is a character of P and $g_*\lambda$ denotes the action of $g \in PSL(2, \mathbb{R})$ on measures. Let $d\theta$ be the Lebesgue measure on $S^1 = \{e^{i\theta}\colon 0 \leqslant \theta < 2\pi\}$ viewed as the boundary of D^2. A finite measure ν on S^1 is α conformal for Γ if

$$\gamma_*\nu = j(\gamma)^\alpha\nu \qquad \text{for all } \gamma \in \Gamma,$$

where $j(\gamma)$ is the Radon-Nikodym derivative of $\gamma_*\, d\theta$ with respect to $d\theta$. Here α is a real number (see [S1] for an intrinsic definition).

We show now that there is a natural bijection between the set of P-quasi invariant positive measures on $\Gamma\backslash PSL(2, \mathbb{R})$ and the set of positive Γ conformal measures on S^1. Let λ be a P-quasi invariant positive measure on $\Gamma\backslash PSL(2, \mathbb{R})$ and consider its lift $\tilde{\lambda}$ to $PSL(2, \mathbb{R})$. This measure is left Γ invariant and satisfies

$$(*) \qquad \begin{cases} a(y)_*\tilde{\lambda} = y^\beta\tilde{\lambda} & \text{for all } y > 0 \\ n(x)_*\tilde{\lambda} = \tilde{\lambda} & \text{for all } x \in \mathbb{R}, \end{cases}$$

where g_* denotes the right action of $g \in PSL(2, \mathbb{R})$ on measures on $PSL(2, \mathbb{R})$. Using Iwasawa coordinates on $PSL(2, \mathbb{R})$

$$PSL(2, \mathbb{R}) = K \times A \times N,$$

$$g = k(\theta)a(y)n(x)$$

we obtain a projection $PSL(2, \mathbb{R}) \to A \times N$ with compact fibers. It follows from properties $(*)$ that the direct image of $\tilde{\lambda}$ on $A \times N$ via this projection is the measure

$$c\, dy\, y^{-\beta}\, dx,$$

where $c > 0$ is some constant. Therefore there exists for almost all $(y, x) \in \mathbb{R}^+ \times \mathbb{R}$ a probability measure

$$d\mu_{(y,x)}(\theta)$$

on K such that for all continuous functions f on $PSL(2, \mathbb{R})$ with compact support we have

$$\tilde{\lambda}(f) = c \int_0^\infty dy\, y^{-\beta} \int_{-\infty}^\infty dx \int_0^\pi d\mu_{(y,x)}(\theta) f(k(\theta)a(y)n(x)).$$

Using properties (*) again we see that the map

$$(y, x) \to \mu_{(y,x)}$$

is essentially constant. Let μ be its essential value. It is a probability measure supported on K. In Iwasawa coordinates the visual map is given by

$$\text{Vis: } PSL(2, \mathbb{R}) = T_1 D^2 \to S^1$$

$$k(\theta)a(y)n(x) \mapsto e^{2\pi i\theta}.$$

Denote again by μ the direct image of μ on S^1 via Vis. It follows from the left Γ invariance of $\tilde{\lambda}$ that μ is $1 - \beta$ conformal, i.e.,

$$\gamma_* \mu = j(\gamma)^{1-\beta} \mu \qquad \text{for all } \gamma \in \Gamma.$$

The inverse of the map $\lambda \to \mu$ was already considered in the Introduction. It follows also from our description that

$$\text{supp } \lambda = \text{projection on } T_1 S \text{ of } \text{Vis}^{-1} (\text{supp } \mu).$$

2.2. We remark now that in order to show Theorem 1 it suffices to prove

PROPOSITION 2. *Let S be a geometrically finite surface without cusps and assume $\delta > 1/2$. If λ is a positive N-invariant ergodic measure supported on $\mathcal{F}_c$ then λ is P-quasi invariant.*

Indeed, assume that Proposition 2 is true. Then λ is obtained from an α conformal probability measure on S^1 supported on the limit set $\Lambda \subset S^1$. Then it follows from Sullivan's characterization of Patterson's measure that $\alpha = \delta$ and $\nu = \mu_P$ [S1, Theorem 8]. In other words, λ is a multiple of the measure μ constructed in the Introduction (C).

2.3. The rest of §2 is devoted to the proof of Proposition 2. We assume from now on that S is geometrically finite without cusps and $\delta > 1/2$. Let $\lambda_k \leqslant \lambda_{k-1} \leqslant \cdots \leqslant \lambda_1 < \lambda_0 = \delta(\delta-1)$ be the eigenvalues of the Laplacian of S in $(-1/4, 0]$. Consider the unitary representation π of $PSL(2, \mathbb{R})$ in $\mathscr{H} = L^2(\Gamma \backslash PSL(2, \mathbb{R}))$. Then we have the direct sum decomposition

$$(\mathscr{H}, \pi) = \bigoplus_{i=0}^{k} (\mathscr{H}_{\lambda_i}, \pi_{\lambda_i}) \oplus (\mathscr{H}', \pi')$$

(cf. §1.2 for the definition of $\mathscr{H}_\lambda$, π_λ) and

$$\operatorname{supp} \pi' \subset (-\infty, -1/4] \cup Z.$$

In particular, the function

$$\varphi_0(h) = \int_{S^1} d\mu_P(\zeta) \left(\frac{1 - |h \cdot o|^2}{|h \cdot o - \zeta|^2} \right)^{\delta}, \qquad o \text{ being the origin of } D^2,$$

is up to scalar multiple the unique K invariant vector in $\mathscr{H}_{\lambda_0}$. Viewed as a function on $T_1 S$, φ_0 is also in $L^1(T_1 S, \mu)$ where μ is the P-quasi invariant measure associated to the Patterson measure μ_P.

2.4. Before we go into the proof of Proposition 2 we make a preliminary remark. If $\delta = 1$ then φ_0 is an L^2 harmonic function on S which is not identically zero. It follows from [Y] that φ_0 is constant and hence $\operatorname{Vol}(S) < +\infty$. Since S is without cusps this implies that S is compact. In this case Theorem 2 (C) shows that all N orbits in $T_1 S$ are uniformly distributed with respect to the $PSL(2, \mathbb{R})$ invariant measure on $T_1 S$. This implies that the action of N on $T_1 S$ is uniquely ergodic, a result due to H. Furstenberg [F]. We therefore make the further assumption that $\delta < 1$ throughout the rest of the paper.

2.5. We first need to show a certain conservativity property of the action of N on $\mathscr{F}_c$.

LEMMA 2. *Let $F \subset \mathscr{F}_c$ be any compact set. There is a constant $c = c_F > 0$ such that for all $g \in F$ and $\tau \geqslant 2$*

$$\int_1^{\tau} dx\, x^{-\delta-1} \int_{-x}^{x} dt\, \varphi_0(gn(t)) \geqslant c(1 + \ln \tau).$$

Proof. (a) It follows from Proposition 1 (2) that for all $x > 0$ and all $g \in T_1 S$

$$\int_{-x}^{x} \varphi_0(gn(t))\, dt \leqslant cx^{\delta},$$

where $c > 0$ is some positive constant. In particular

$$\int_1^\infty dx\, x^{s-1} \int_{-x}^{x} \varphi_0(gn(t))\, dt$$

converges for all $s < -\delta$.

(b) We show now that there is a constant $c = c_F > 0$ such that for all $g \in F$ and $-1 < s < -\delta$

$$\int_1^\infty x^s[\varphi_0(gn(x)) + \varphi_0(gn(-x))]\, dx \geqslant \frac{c}{|s+\delta|}.$$

We use the following representation of φ_0:

$$\varphi_0(h) = \int_{S^1} d\mu_P(\zeta) j(h, \zeta)^\delta$$

where

$$j(h, \zeta) = \frac{1 - |h\cdot o|^2}{|h\cdot o - \zeta|^2}$$

hence

$$\varphi_0(gn(x)) = \int_{S^1} d\mu_P(\zeta) j(g, \zeta)^\delta j(n(x), g^{-1}\zeta)^\delta.$$

We can assume g to be in a fixed compact set $\tilde{F}$ in $PSL(2, \mathbb{R})$. Then $j(g, \zeta)^\delta$, $\zeta \in S^1$, is between two positive constants so that we are reduced to consider

$$\int_{S^1} d\mu_P(\zeta) \int_1^\infty dx\, x^s\{ j(n(x), g^{-1}\zeta)^\delta + j(n(-x), g^{-1}\zeta)^\delta\}$$

$$= \int_{S^1} d\mu_P(\zeta) \{u(g^{-1}\zeta) + u(\overline{g^{-1}\zeta})\}$$

where

$$u(e^{i\theta}) = 4^\delta \int_1^\infty \frac{x^s\, dx}{[2x^2(1 - \cos\theta) - 4x\sin\theta + 4]^\delta}$$

$$= (\sin^2(\theta/2))^{-\delta} \int_1^\infty \frac{x^s\, dx}{[(x - \operatorname{ctg}(\theta/2))^2 + 1]^\delta}$$

(recall that $n(x)\cdot o = x/(2i + x)$).

A few computations show that there is a constant $c > 0$ such that

$$(*) \qquad u(e^{i\theta}) + u(e^{-i\theta}) \geqslant c|\theta|^{-s-2\delta}, \qquad |\theta| \leqslant \pi.$$

Let d be the K invariant metric on S^1. There is a constant $c > 0$ such that for all $g \in \tilde{F}$ and $\zeta, \zeta' \in S^1$

$$c^{-1}\, d(\zeta, \zeta') \leqslant d(g\zeta, g\zeta') \leqslant cd(\zeta, \zeta').$$

Therefore, it follows from $(*)$ that for all $\zeta \in S^1$

$$u(g^{-1}\zeta) + u(\overline{g^{-1}\zeta}) \geqslant cd(\zeta, \xi)^{-s-2\delta}, \qquad \xi = g \cdot 1.$$

From this it follows that there is a constant $c > 0$ such that for all $g \in F$ and $-1 < s < -\delta$

$$\int_1^\infty x^s[\varphi_0(gn(x)) + \varphi_0(gn(-x))]\, dx \geqslant c \int_{S^1} d\mu_P(\zeta)\, d(\zeta, \xi)^{-s-2\delta}.$$

Now if $I(\xi, r)$ is the interval of radius r about ξ an integration by parts shows that

$$(***) \qquad \int_{S^1} d\mu_P(\zeta)\, d(\zeta, \xi)^{-s-2\delta} \geqslant (s + 2\delta) \int_0^\pi t^{-s-2\delta-1} \mu_P(I(\xi, t))\, dt.$$

Note that $g \in \mathscr{F}_c$ is equivalent to $g \cdot 1 = \xi \in \Lambda$. From [S2, §7] and the fact that S is convex cocompact we deduce that there is a constant $c > 0$ such that for all $\xi \in \Lambda$ and $0 \leqslant r \leqslant \pi$

$$c^{-1} r^\delta \leqslant \mu_P(I(\xi, r)) \leqslant cr^\delta.$$

Putting this into $(***)$ and using $(**)$ we obtain the claim (b).

(c) Now we prove the Lemma. Let $-1 < s < -\delta$ and $\tau \geqslant 2$:

$$\begin{aligned} &\int_1^\tau dx\, x^{-\delta-1} \int_{-x}^x \varphi_0(gn(t))\, dt \\ &\quad\geqslant \int_1^\tau dx\, x^{s-1} \int_{-x}^x \varphi_0(gn(t))\, dt \\ &\quad= \int_1^\infty dx\, x^{s-1} \int_{-x}^x \varphi_0(gn(t))\, dt - \int_\tau^\infty dx\, x^{s-1} \int_{-x}^x \varphi_0(gn(t))\, dt \end{aligned}$$

using (a) we obtain

$$\int_{\tau}^{\infty} dx\, x^{s-1} \int_{-x}^{x} \varphi_0(gn(t))\, dt \leqslant \frac{c\tau^{s+\delta}}{|s+\delta|}.$$

On the other hand

$$\int_{1}^{\infty} dx\, x^{s-1} \int_{-x}^{x} \varphi_0(gn(t))\, dt = (-s^{-1}) \int_{-1}^{1} \varphi_0(gn(t))\, dt$$

$$+ (-s^{-1}) \int_{1}^{\infty} x^{s}[\varphi_0(gn(x)) + \varphi_0(gn(-x))]\, dx$$

$$\geqslant \frac{c|s|}{|s+\delta|} \qquad \text{using (b)}.$$

Hence

$$\int_{1}^{\tau} dx\, x^{-\delta-1} \int_{-x}^{x} \varphi_0(gn(t))\, dt \geqslant \frac{c_1 - c_2\tau^{s+\delta}}{|s+\delta|}.$$

Choosing $|s+\delta|$ of size $1/\ln \tau$, we obtain the Lemma. □

LEMMA 3. *Let $\varphi \in C_K^{\infty}(S)$ and consider it as a function on $T_1 S$. Then we have*

$$\lim_{\tau\to\infty} \frac{\displaystyle\int_{1}^{\tau} dx\, x^{-\delta-1} \int_{-x}^{x} \varphi(gn(t))\, dt}{\displaystyle\int_{1}^{\tau} dx\, x^{-\delta-1} \int_{-x}^{x} \varphi_0(gn(t))\, dt} = \frac{\langle \varphi, \varphi_0 \rangle}{\|\varphi_0\|^2}$$

uniformly on compact sets in $\mathscr{F}_c$.

Proof. Let $\varphi = \langle \varphi, \varphi_0 \rangle (\varphi_0/\|\varphi_0\|_2^2) + \varphi_\perp$, where $\varphi_\perp$ is orthogonal to $\mathscr{H}_{\lambda_0}$. It follows from Proposition 1 (2) applied to $\varphi_\perp$ and the orthogonal of $\mathscr{H}_{\lambda_0}$ in $L^2(\Gamma\backslash PSL(2, \mathbb{R}))$ that

$$\sup_{g\in T_1 S} \left| \frac{1}{x} \int_{-x}^{x} \varphi_\perp(gn(t))\, dt \right| \leqslant c(S, \varphi_\perp) \frac{x^{-\alpha_1} - x^{1-\alpha_1}}{1 - 2\alpha_1},$$

where $0 < \alpha_1 \leqslant 1/2$, $\alpha_1(\alpha_1 - 1) = \lambda_1(S)$ if $\lambda_1(S) > -1/4$ and $\alpha_1 = 1/2$ if $\operatorname{Spec} \Delta_S \cap (-1/4, 0] = \{\lambda_0\}$. In any case we have $\delta + \alpha_1 > 1$ and hence for $\tau \to \infty$

$$\int_{1}^{\tau} dx\, x^{-\delta-1} \int_{-x}^{x} \varphi(gn(t))\, dt = \frac{\langle \varphi, \varphi_0 \rangle}{\|\varphi_0\|^2} \int_{1}^{\tau} dx\, x^{-\delta-1} \int_{-x}^{x} \varphi_0(gn(t))\, dt + O(1).$$

Dividing by $\int_1^\tau dx\, x^{-\delta-1} \int_{-x}^{x} \varphi_0(gn(t))\, dt$ and using Lemma 2 enable us to conclude the proof.

Remark. It follows from the proof of Lemma 3 that if $\varphi \in C_K^\infty(S)$, $\varphi \geqslant 0$, $\varphi \not\equiv 0$, then there is a constant $c > 0$ such that for all $\tau \geqslant 2$ and $g \in \mathscr{F}_c$

$$\int_1^\tau dx\, x^{-\delta-1} \int_{-x}^{x} \varphi(gn(t))\, dt \geqslant c(1 + \ln \tau)$$

from which it follows easily that

$$\limsup_{\tau \to \infty} \frac{1}{\tau^\delta} \int_{-\tau}^{\tau} \varphi(gn(t))\, dt > 0.$$

On the other hand we know from Proposition 1 (2) that this last quantity is bounded, so that one may ask if

$$\frac{1}{\tau^\delta} \int_{-\tau}^{\tau} \varphi(gn(t))\, dt > 0$$

has a limit as $\tau \to \infty$. The following example shows that this is not always the case.

Let S be geometrically finite with one expanding end and without cusps. Let $g \subset S$ be a closed geodesic distinct from the closed geodesic bounding the expanding end. We represent $S = \Gamma \backslash \mathbb{H}^2$ as the quotient of the upper half plane $\mathbb{H}^2$ in such a way that the geodesic $x = 0$ is a lift of g. Let $\Lambda \subset \mathbb{R} \cup \{\infty\}$ be the limit set of Γ. By construction $\infty \in \Lambda$. Let $C(\Lambda)$ be the convex hull of Λ and $S_0 = \Gamma \backslash C(\Lambda)$. In our example we take $g = e$, $\varphi \in C_K^\infty(S)$, φ nonnegative and with support in S_0. We can assume that

$$\limsup_{\tau \to \infty} \frac{1}{\tau^\delta} \int_0^\tau \varphi(gn(t))\, dt > 0.$$

Consider $t \to \Gamma\, en(t)$, the N orbit of $\Gamma\, e$ in $\Gamma \backslash PSL(2, \mathbb{R})$ and let $c(t)$ be its projection on S. We denote by $t_1 < t_1' < t_2 < t_2' < \cdots$ the sequence of times t, $t > 0$, at which $c(t)$ crosses the boundary of S_0, so that $c(t)$ leaves S_0 at t_n for all $n \geqslant 1$. By construction we have

$$(*) \qquad \frac{1}{t_n'^\delta} \int_0^{t_n'} \varphi(gn(t))\, dt = \left(\frac{t_n}{t_n'}\right)^\delta \frac{1}{t_n^\delta} \int_0^{t_n} \varphi(gn(t))\, dt.$$

Let h be the geodesic bounding S_0 and let $\tilde{h}$ be some lift of h contained in $\{z \in \mathbb{H}^2 : x > 0\}$. Let $\langle \gamma \rangle$ be the subgroup of Γ of elements with axis $x = 0$. Then $\gamma^n(\tilde{h})$ is a sequence of lifts of h and for $n \geqslant n_0$, $\gamma^n(\tilde{h})$ intersects the horocycle

$$\{i + t : t > 0\} \subset \mathbb{H}^2.$$

Let $i + s_n$, $i + s_n'$, $s_n < s_n'$ be the two intersection points. An explicit computation shows that $\lim_{n\to\infty}(s_n'|s_n) = b|a$ where $a < b$ are the end points of $\tilde{h}$. On the other hand, $(s_n, s_n')_{n=1}^{\infty}$ is a subsequence of $(t_n, t_n')_1^{\infty}$. Using this and (*) we conclude that

$$\liminf_{\tau\to\infty} \frac{1}{\tau^\delta}\int_0^\tau \varphi(gn(t))\,dt < \limsup_{\tau\to\infty}\frac{1}{\tau^\delta}\int_0^\tau \varphi(gn(t))\,dt.$$

In the sequel we will need the following version of Hopf's ergodic theorem: Given a locally compact, σ-compact topological space X with a continuous $\mathbb{R}$ action

$$\mathbb{R} \times X \to X$$

$$(t, x) \mapsto xn(t),$$

let ν be a positive N invariant ergodic Radon measure on X and assume that there exists an everywhere positive function $g \in L^1(X, \nu)$ such that for ν almost all $x \in X$

$$\int_{-\infty}^{\infty} g(xn(t))\,dt = +\infty,$$

then:

THEOREM. (Hopf [H]) *For all $f \in L^1(X, \nu)$ we have for ν almost all $x \in X$*

$$\lim_{\tau\to\infty}\frac{\int_{-\tau}^{\tau} f(xn(t))\,dt}{\int_{-\tau}^{\tau} g(xn(t))\,dt} = \frac{\int f(x)\,d\nu(x)}{\int g(x)\,d\nu(x)}.$$

Using this ratio ergodic theorem we can prove

LEMMA 4. *Let ν be an N invariant positive ergodic measure on $\mathscr{F}_c$. Then ν is an eigenmeasure of the Casimir operator of eigenvalue λ_0: for all $f \in C_K^\infty(T_1S)$ we have*

$$\int_{T_1S} d\pi(w)f(g)\,d\nu(g) = \lambda_0\int_{T_1S} f(g)\,d\nu(g).$$

Proof. It is sufficient to show that if $f \in C_K^\infty(T_1S)$ is orthogonal to $\mathscr{H}_{\lambda_0}$ then

$$\int_{T_1S} f(g)\,d\nu(g) = 0.$$

Choose an everywhere positive continuous function $\psi \in L^1(T_1 S, \nu)$. Fix some non-negative function $\varphi \in C_K^\infty(S)$, $\varphi \not\equiv 0$ and consider it as a function on $T_1 S$. Then $\psi \geqslant c\varphi$ for some positive constant c. It follows from Lemma 3 that for all $g \in \mathscr{F}_c$

$$(*) \qquad \int_1^\infty dx\, x^{-\delta-1} \int_{-x}^{x} \psi(gn(t))\, dt = +\infty .$$

In particular $\int_{-\infty}^{\infty} \psi(gn(t))\, dt = +\infty$.

Now it follows from Hopf's ergodic theorem and $(*)$ that for ν almost all $g \in \mathscr{F}_c$

$$\lim_{\tau\to\infty} \frac{\displaystyle\int_1^\tau dx\, x^{-\delta-1} \int_{-x}^{x} f(gn(t))\, dt}{\displaystyle\int_1^\tau dx\, x^{-\delta-1} \int_{-x}^{x} \psi(gn(t))\, dt} = \lim_{\tau\to\infty} \frac{\displaystyle\int_1^\tau dx\, x^{-\delta-1} h(x) \int_{-x}^{x} \psi(gn(t))\, dt}{\displaystyle\int_1^\tau dx\, x^{-\delta-1} \int_{-x}^{x} \psi(gn(t))\, dt},$$

where

$$h(x) = \frac{\displaystyle\int_{-x}^{x} f(gn(t))\, dt}{\displaystyle\int_{-x}^{x} \psi(gn(t))\, dt},$$

and this last limit equals

$$(**) \qquad \lim_{x\to\infty} h(x) = \frac{\displaystyle\int f(g)\, d\nu(g)}{\displaystyle\int \psi(g)\, d\nu(g)} .$$

If $f \in C_K^\infty(T_1 S)$ and is orthogonal to $\mathscr{H}_{\lambda_0}$ we apply Proposition 1 (2) to find that for all $g \in T_1 S$

$$\left| x^{-\delta-1} \int_{-x}^{x} f(gn(t))\, dt \right| \leqslant c x^{-(\delta+\alpha_1)},$$

where $\delta + \alpha_1 > 1$ and hence

$$\int_1^\infty dx\, x^{-\delta-1} \int_{-x}^{x} f(gn(t))\, dt < +\infty .$$

It follows now from $(**)$ that $\int f(g)\, d\nu(g) = 0$. □

2.5.

Proof of Proposition 2. Let ν be an N invariant positive ergodic measure on $\mathscr{F}_c$. Let $f \in C_K^\infty(T_1 S)$ and consider

$$u(n(x)a(y)k(\theta)) = \int_{T_1 S} \pi(n(x)a(y)k(\theta))f(g)\, d\nu(g).$$

It follows from Lemma 4 that u satisfies

$$y^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - y\frac{\partial^2 u}{\partial x\, \partial\theta} = \lambda_0 u.$$

But u is also left N invariant hence

$$y^2 \frac{\partial^2 u}{\partial y^2} = \lambda_0 u$$

in particular there are constant $c_1(f)$, $c_2(f)$ such that for all $y > 0$

$$(*) \qquad \int_{T_1 S} \pi(a(y))f(g)\, d\nu(g) = c_1(f)y^\delta + c_2(f)y^{1-\delta}.$$

From this equality we deduce that $f \to c_1(f)$, $f \to c_2(f)$ are positive N invariant Radon measures and $\nu = c_1 + c_2$. Since ν is ergodic there are numbers α, $\beta \geqslant 0$, $|\alpha| + |\beta| > 0$ such that

$$(**) \qquad \alpha c_1(f) = \beta c_2(f) \qquad \text{for all } f \in C_K(T_1 S).$$

On the other hand it follows from $(*)$ that

$$c_1(\pi(a(y))f) = y^\delta c_1(f)$$

$$c_2(\pi(a(y))f) = y^{1-\delta} c_2(f).$$

Hence $(**)$ is only possible if $\alpha = 0$ or $\beta = 0$, so $c_1 = 0$ or $c_2 = 0$. This proves Proposition 2. □

2.6.

Proof of the Corollary. Consider the following family of measures

$$V_{g,T}(\varphi) = \frac{\int_1^T dx\, x^{-\delta-1} \int_{-x}^{x} \varphi(gn(t))\, dt}{\int_1^T dx\, x^{-\delta-1} \int_{-x}^{x} \varphi_0(gn(t))\, dt}, \qquad g \in \mathscr{F}_c, \qquad T \geqslant 2.$$

Since φ_0 has a positive lower bound on each compact set of $T_1 S$, it follows that the set

$$\{V_{g,T}: g \in \mathscr{F}_c, T \geqslant 2\}$$

is relatively compact in the vague topology of $\mathscr{M}(T_1 S)$. Suppose that the Corollary is false. Then there exists a compact set $F \subset \mathscr{F}_c$, sequences $(g_n)_{n=0}^{\infty} \subset F$, $T_n \to \infty$, a function $\varphi \in C_K^{\infty}(T_1 S)$ and $\varepsilon > 0$ such that for all $n \geqslant 0$

$$(*) \qquad |V_{g_n, T_n}(\varphi) - \mu(\varphi)/\|\varphi_0\|^2| \geqslant \varepsilon.$$

Let v be an accumulation point of the sequence $(V_{g_n, T_n})_{n=1}^{\infty}$. From the fact that

$$\lim_{\tau \to \infty} \int_1^{\tau} dx\, x^{-\delta-1} \int_{-x}^{x} \varphi_0(gn(t))\, dt = +\infty$$

uniformly on compact sets in $\mathscr{F}_c$ (Lemma 2) it follows easily that v is N invariant and supported on $\mathscr{F}_c$. Hence $v = \lambda\mu$, where $\lambda \geqslant 0$ is some constant. From Lemma 3 it follows that for all $\psi \in C_K^{\infty}(S)$

$$\int \psi(g)\, dv(g) = \frac{\langle \psi, \varphi_0 \rangle}{\|\varphi_0\|^2} = \frac{\mu(\varphi)}{\|\varphi_0\|^2}.$$

Hence $\lambda = 1/\|\varphi_0\|^2$ which contradicts $(*)$. ∎

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