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Small eigenvalues of Riemann surfaces and graphs

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Introduction

Let M be a connected surface of finite topological type (g, p, f) i.e. M is obtained by removing p points and f topological discs from a compact surface of genus $g \geqq 0$. We denote by $\mathcal{M}(g, p, f)$ the space of isometry classes of complete metrics of curvature -1 on M.

The Laplace operator Δ of a surface $S \in \mathcal{M}(g, p, f)$ acts on $C_{00}^{\infty}(S)$ the space of C^{∞}-functions with compact support and has a unique extension to an unbounded self-adjoint operator on $L^2(S)$. The essential spectrum of Δ is contained in $[1/4, +\infty)$ so that $\operatorname{Spec} \Delta \cap [0, 1/4)$ consists only of eigenvalues (see [DPRS] and 1.2). Moreover there are at most $4g+2p+3f-2$ eigenvalues of Δ in $[0, 1/4)$ and there exists a positive constant β only depending on (g, p, f) such that the number of eigenvalues in $[0, \beta]$ is at most $2g+p+f-2$.

The aim of this work is to determine the behaviour of $\operatorname{Spec} \Delta_S$ near 0 in function of $S \in \mathcal{M}(g, p, f)$. For this we cover the infinite part of $\mathcal{M}(g, p, f)$ by a finite number of "cusp neighborhoods". Each neighborhood is canonicaly associated to a finite graph. Then we show that the first order behaviour of $\operatorname{Spec} \Delta_S \cap [0, \varepsilon]$ for S in such a neighborhood is given by the spectrum of a combinatorial Laplacian (see Theor. 1.1 and Theor. 1.2). Partial results in this direction were obtained by B. Colbois [B.C.], P. Gall [P.G.] and myself [B]. Such results were used by B. Colbois and Y. Colin de Verdière [C, CdV] to construct examples of surfaces whose second eigenvalue λ_2 has large multiplicity. They obtain for all $g \geqq 2$ examples of compact surfaces with genus g and multiplicity of λ_2 of size $\sqrt{8g}/2$. Known bounds on the multiplicity of λ_2 (for small λ_2) are deduced from the fact that there are at most $2g-2+p+f$ small eigenvalues [DPRS]. It follows also from the work of G. Besson [G.B.] that if S is of signature (g, p, f) then $4g+3$ is a bound for the multiplicity of λ_2.

We will apply our result on the behaviour of small eigenvalues to reduce the problem of bounding the multiplicity of λ_2 (for λ_2 small) to the problem of bounding the multiplicity of the second eigenvalue of a weighted graph. The later problem will be discussed in part 2 of our paper.

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The output of this method is that we can bound the number of eigenvalues in very small intervals around $\lambda_2(S)$ by $\frac{2}{3}[2g-2+p+f]+2$ (see Coroll. 1.1, 1.2). In particular this gives a non-trivial bound on the multiplicity of $\lambda_2(S)$ for λ_2 smaller than a constant only depending on (g, p, f).

1 Behaviour of small eigenvalues

1.1 Statement of the results

Let S be a Riemann surface of signature (g, p, f) with $2g-2+p+f \geqq 1$. Denote by

$$LSp(S) = \{l_1 \leqq l_2 \leqq \dots\}$$

the length spectrum of S i.e. the set of lengths of closed geodesics counted according to their multiplicity. Let $r(S) = l_1(S)$. The statement of the behaviour of small eigenvalues of S depends on a description of the set of surfaces S in $\mathcal{M}(g, p, f)$ for which $r(S)$ is small. To do this we now define the cusp neighborhoods in $\mathcal{M}(g, p, f)$.

Cusp neighborhood: given a Riemann surface S we call partition of S any subset $\Lambda \subset S$ which is the union of simple closed pairwise non-intersecting geodesics. To such a partition $\Lambda \subset S$ we associate a pair $(\mathcal{G}, \omega)$ consisting of a graph $\mathcal{G} = (V, E)$ and a function $\omega: V \to \mathbb{N}^3$ defined in the following way: the set of vertices V is the set of connected components of $S \setminus \Lambda$. Each geodesic $\gamma \subset \Lambda$ is represented by an edge $e \in E$ connecting the vertices corresponding to the components of $S \setminus \Lambda$ joined by γ.

The function ω: $V \to \mathbb{N}^3$ associates to a vertex $v \in V$ the signature (g_v, p_v, f_v) of the component represented by v.

Given (g, p, f) with $2g-2+p+f \geqq 1$ it is easily verified that the pairs $(\mathcal{G}, \omega)$ arising in this way are completely characterised by the following properties:

1.) $\mathcal{G} = (V, E)$ is a connected graph
2.) $\omega: V \to \mathbb{N}^3$ is a map such that $\omega(v) = (g_v, p_v, f_v)$ verifies $2g_v - 2 + p_v + f_v \geqq 0$ with equality if and only if $(g_v, p_v, f_v) = (0, 0, 2)$.
3.) $\sum p_v = p, \quad \sum f_v = 2|E| + f$
4.) Let $d: V \to \mathbb{N}$ be the degree function of $\mathcal{G}$ where loops are counted twice. Then $d(v) \leqq f_v$ for all $v \in V$.
5.) $\sum g_v + \beta_1(\mathcal{G}) = g$ where $\beta_1(\mathcal{G})$ is the first Betti number of $\mathcal{G}$.

Two pairs $(\mathcal{G}, \omega), (\mathcal{L}, \alpha)$ are called isomorphic if the graphs $\mathcal{G} \cong \mathcal{L}$ are isomorphic and the functions ω, α correspond one to another under this isomorphism. Let us denote by $\mathcal{C}(g, p, f)$ the (finite) set of isomorphism classes of such pairs.

Given a Riemann surface S and a partition $\Lambda \subset S$ we let:

$$l(\Lambda) = \max\{l(\gamma): \gamma \text{ simple closed}, \gamma \subset \Lambda\}$$
$$L(\Lambda) = \min\{2 \operatorname{arcsh} 1, l(\eta): \eta \text{ closed geodesic } \eta \cap \Lambda = \emptyset\}$$

For $[\mathcal{G}, \omega] \in \mathcal{C}(g, p, f)$ and $\varepsilon > 0$ we define $V_\varepsilon[\mathcal{G}, \omega] \subset \mathcal{M}(g, p, f)$ as the set of Riemann surfaces S such that there exists a partition $\Lambda \subset S$ with associated pair isomorphic to $(\mathscr{G}, \omega)$ and $l(\Lambda)/L(\Lambda) < \varepsilon$, modulo the relation identifying isometric surfaces.

The fact that there are at most $3g-3+p+2f$ simple closed geodesics of length smaller than 2 arcsh 1 (see 1.2) has the following easy consequence:

$$\{S \in \mathscr{M}(g, p, f): r(S) < \varepsilon\} \subset \bigcup V_\delta[\mathscr{G}, \omega]$$

where $\varepsilon < 1$, $\delta^{3g-3+p+2f} = \varepsilon/2$ arcsh 1 and the union is taken over all cusps $[\mathscr{G}, \omega] \in \mathscr{C}(g, p, f)$. In particular if $f=0$ the complement of the union of all cusp neighborhoods is compact in $\mathscr{M}(g, p, 0)$.

Behaviour of small eigenvalues: Let $S \in \mathscr{M}(g, p, f)$. We denote by $\lambda_1 < \lambda_2 \leqq \ldots \leqq \lambda_k$ the eigenvalues of Δ_S in $[0, 1/4)$. For later purpose we also define:

$$\lambda_{k+1} = \inf\{\lambda: \lambda \in \operatorname{Spec} \Delta_S \cap (\lambda_k, \infty)\}.$$

Suppose that S is a surface representing an element in $V_\varepsilon[\mathscr{G}, \omega]$. Then S defines on the edge set E of $\mathscr{G}$ an obvious length function $l: E \to \mathbb{R}^+$ and a function $m: V \to \mathbb{N}$ defined by $m(v) = 2g_v - 2 + p_v + f_v$ if v corresponds to a component of finite volume and $m(v) = 1$ otherwise.

In this way we obtain a weighted graph $G = (\mathscr{G}, m, l)$ (see Chap. 2 for definitions) and a distinguished subset

$$P = \{v \in V: d(v) < f_v\}$$

representing the set of unbounded components. Let:

$$\lambda_1^P(G) < \lambda_2^P(G) \leqq \ldots \leqq \lambda_N^P(G), \qquad N = |V \setminus P|$$

be the spectrum of (G, P) as defined in 2.1.

Theorem 1.1. *For all $S \in V_\varepsilon[\mathscr{G}, \omega]$ and all $\varepsilon < \alpha_1$ we have:*

$$\frac{1}{2\pi^2}(1 - \alpha_2 \sqrt{\varepsilon}) \leqq \frac{\lambda_i(S)}{\lambda_i^P(G)} \leqq \frac{1}{2\pi^2}(1 + \alpha_3 \varepsilon \ln \varepsilon)$$

where G is the weighted graph attached to S, $1 \leqq i \leqq N$, $N = |V \setminus P|$ and $\alpha_1, \alpha_2, \alpha_3$ are positive constants only depending on (g, p, f).

In order to prove Theorem 1.1 we will prove a slightly stronger result whose statement needs some preliminary remarks.

It is a fundamental result due to [SWY] in the compact case and [DPRS] in the general case that the size of eigenvalues of Δ in $[0, 1/4)$ is controlled by the lengths of small closed geodesics. More precisely:

(a) There exists a positive constant $\beta = \beta(g, p, f)$ such that the number of eigenvalues of Δ in $[0, \beta]$ is at most $2g+p+f-2$.
Fix $0 < \mu \leqq 2$ arcsh 1. Let $L_j(S)$ be the minimum sum of lengths of simple closed geodesics of length $\leqq \mu$ separating S into $j+1$ components where we regard the union of all pieces of infinite volume as a single component.
(b) If $\lambda_j < 1/4$ then $\beta_1 L_j(S) \leqq \lambda_j \leqq \beta_2 L_j(S)$
(c) If $\beta_1 L_j(S) < 1/4$ then Δ has at least j eigenvalues in $[0, 1/4)$ and (b) holds.
Here β_1, β_2 are positive constants which depend only on (g, p, f, μ).

Let us draw a consequence of this Theorem. Define for $1 \leqq j \leqq 2g-2+p+f$ and $\delta > 0$:

$$\mathcal{M}_{j,\delta} = \{S \in \mathcal{M}(g,p,f): \lambda_j(S) < \beta_3 \quad \text{and} \quad \lambda_j/\lambda_{j+1} \leqq \delta\}$$

where

$$\beta_3 = \min(\beta_1 [4\beta_2(3g-3+p+2f)]^{-1}, \beta_1 \mu, 1/4)$$

and for a surface S:

$$\mathrm{Geod}(\varepsilon) = \{\gamma: \text{closed geodesic in } S \text{ of length } l(\gamma) \leqq \varepsilon\}$$

then we have:

Lemma 0.

(a) *Let $\delta < 4\beta_3$ and $S \in \mathcal{M}_{j,\delta}$. Then* $\mathrm{Geod}(\lambda_j/\beta_1)$ *cuts S into $j+1$ pieces exactly.*

(b) *Let $\varepsilon \leqq \beta$ and $\delta = (\varepsilon/\beta)^{\frac{1}{2g+p+f-2}}$ then*

$$\{S \in \mathcal{M}(g,p,f): \Delta_S \textit{ has at least } k \textit{ eigenvalues in } (0,\varepsilon]\}$$

is contained in $\bigcup_{j=k}^{2g-2+p+f} \mathcal{M}_{j,\delta}$.

Proof. (a) That $\mathrm{Geod}(\lambda_j/\beta_1)$ cuts S into at least $j+1$ pieces follows from $L_j(S) \leqq \lambda_j/\beta_1$. Suppose that there are more than $j+1$ pieces. Then $\beta_2 L_{j+1}(S) \leqq (3g+p+2f-3)\beta_2 \lambda_j/\beta_1 < 1/4$ since there are at most $3g+p+2f-3$ closed geodesics of length smaller than μ (cf. 1.2). Thus $\lambda_{j+1} < 1/4$ and $\lambda_{j+1} \leqq \beta_2 L_{j+1}(S) \leqq (3g-3+p+2f)\beta_2 \lambda_j/\beta_1$ which contradicts the assumption that $\lambda_j/\lambda_{j+1} \leqq \delta < 4\beta_3$.

(b) Let $\mathrm{Spec}\, \Delta \cap [0, 1/4) = \{\lambda_1 < \lambda_2 \leqq \ldots \leqq \lambda_k \leqq \lambda_{k+1} \leqq \ldots \leqq \lambda_r\}$. If $r \geqq 2g-1+p+f$ then:

$$\min\left\{\left(\frac{\lambda_j}{\lambda_{j+1}}\right)^{2g-2+p+f}: k \leqq j \leqq 2g-2+p+f\right\} \leqq \prod_{j=k}^{2g-2+p+f} \left(\frac{\lambda_j}{\lambda_{j+1}}\right) = \frac{\lambda_k}{\lambda_{2g-1+p+2f}} \leqq \varepsilon/\beta$$

which shows that $S \in \mathcal{M}_{j,\delta}$ for some $j \geqq k$.

If $r \leqq 2g-2+p+f$ an analogous argument shows that $S \in \mathcal{M}_{j,\delta}$ for some $j \geqq k$ and $\delta^{2g-2+p+f} \leqq 4\varepsilon \leqq \varepsilon/\beta$. Q.E.D.

This being said we will prove

Theorem 1.2. *Let j, $1 \leqq j \leqq 2g-2+p+f$ and $S \in \mathcal{M}_{j,\delta}$. Let G be the weighted graph associated to the partition* $\mathrm{Geod}(\lambda_j/\beta_1)$. *Then:*

$$\frac{1}{2\pi^2}(1-\alpha_2\sqrt{\delta}) \leqq \frac{\lambda_i(S)}{\lambda_i^P(G)} \leqq \frac{1}{2\pi^2}(1+\alpha_3 \delta \ln \delta)$$

for all $\delta \leqq \alpha_1$ and $1 \leqq i \leqq j$. Here α_1, α_2, α_3 are positive constants only depending on (g,p,f).

Remark. 1.) In general $S \in \mathcal{M}_{j,\delta}$ does not imply that S belongs to the cusp neighborhood defined by the partition $\mathrm{Geod}(\lambda_j/\beta_1)$.

2.) Let $S \in \mathcal{M}_\varepsilon[\mathcal{G}, \omega]$ and let A be the corresponding partition. Let j be the number of bounded components of $S \setminus A$. Then it is clear that for ε small $A \subset \mathrm{Geod}(\lambda_j/\beta_1)$. Moreover it is also easily checked that $\lambda_j/\lambda_{j+1} \leqq c \cdot \varepsilon$ where c is some constant depending only on (g, p, f). This shows that $\mathcal{M}_\varepsilon[\mathcal{G}, \omega] \subset \mathcal{M}_{j,\delta}$ where $\delta = c \cdot \varepsilon$. Since $\mathrm{Geod}(\lambda_j/\beta_1)$ cuts S into j bounded components as does A, both associated weighted graphs have the same spectrum. This shows that Theor. 1.2 implies Theor. 1.1.

3.) Lemma 0 b.) and Theorem 1.2 show that for ε sufficiently small the first order behaviour of $\mathrm{Spec}\, \Delta_S \cap [0, \varepsilon]$ is given by the spectrum of a weighted graph associated to some partition $A \subset S$.

In Chap. 2 we will obtain upper bounds on the multiplicity of the second eigenvalue of a weighted graph. These bounds together with Theor. 1.2. will imply the following

Corollary 1.1. *Let $\varepsilon: [0, 1/4] \to [0, 1/4]$ be any function such that $\lim_{x \to 0} \varepsilon(x) = 0$. There exists a constant $c = c(g, p, f, \varepsilon) > 0$ such that for all surfaces S for which $\lambda_2(S) < c$ we have:*

$$|\mathrm{Spec}\, \Delta_S \cap [\lambda_2, \lambda_2(1 + \varepsilon(\lambda_2))]| \leqq \tfrac{2}{3}[2g - 2 + p + f] + 2.$$

In particular the same bound holds for the multiplicity of $\lambda_2(S)$.

Example. There exists a constant $K(g) > 0$ and a sequence of compact surfaces S_n of genus $g \geqq 2$ such that $\lim_{n \to \infty} \lambda_2(S_n) = 0$ and the number of eigenvalues in $[\lambda_2, \lambda_2(1 + K\sqrt{\lambda_2})]$ is at least $g - 1$. These surfaces are modelled on a star on g vertices (see Example 2.1) and all small geodesics have the same length. This example shows that the estimate of Corollary 1.1 has the true order of magnitude in g. However for the multiplicity of $\lambda_2(S)$ it is conjectured that it does not exceed $\sqrt{g}$, at least if S is compact (see [C, CdV]).

The next corollary shows that if the eigenvalues $\lambda_i(S)$, $2 \leqq i \leqq 2g - 2 + p + f$ are all of the same size then one has a bound on the number of eigenvalues in $[\lambda_2, \lambda_2(1 + \varepsilon(\lambda_2))]$ which depends only on the genus of S.

Corollary 1.2: *Let $\varepsilon: [0, 1/4] \to [0, 1/4]$ be any function such that $\lim_{x \to 0} \varepsilon(x) = 0$ and let $K > 0$. Then there is a constant $c = c(\varepsilon, K, g, p, f) > 0$ such that if $\lambda_2(S) < c$ and $\lambda_{2g-2+p+f}(S) < K\lambda_2(S)$ we have:*

$$|\mathrm{Spec}\, \Delta_S \cap [\lambda_2, \lambda_2(1 + \varepsilon(\lambda_2))]| \leqq g + 3.$$

1.2 Preliminaries

Here we collect some well-known facts about the geometry and the spectrum of geometrically finite Riemann surfaces.

1.2.1 Let $S \in \mathcal{M}(g, p, f)$. Then according to [Bu] any closed geodesic of length $l \leqq 2$ arcsh 1 is simple and there are at most $3g-3+p+2f$ simple closed geodesics of length $\leqq 2$ arcsh 1.

Collar theorem (see [R, Bu]). *Let γ be a simple closed geodesic on S of length $l = l(\gamma)$ and let* $\mathrm{d}(p, \gamma)$ *denote the distance of a point $p \in S$ to γ. Then*:

$$C_\gamma = \left\{ p \in S : \operatorname{sh} \mathrm{d}(\gamma, p) \operatorname{sh} \frac{l}{2} \leqq 1 \right\}$$

is a topological cylinder isometric to

$$[-d_\gamma, d_\gamma] \times \mathbb{R}/\mathbb{Z} \quad \textit{with metric } \mathrm{d}x^2 + l^2 \operatorname{ch}^2 x \, \mathrm{d}\theta^2 \quad \textit{where } \operatorname{sh} d_\gamma = 1/\operatorname{sh} \frac{l}{2}.$$

Moreover if γ, η are (simple) closed geodesics of length $l \leqq 2$ arcsh 1 *then*

$$C_\gamma \cap C_\eta = \emptyset.$$

For more detailed information about the geometry of such surfaces we refer the reader to [DPRS] §. 4, [Be, Bu].

1.2.2 Let $S \in \mathcal{M}(g, p, f)$ and $S_0 \subset S$ be a connected surface with smooth compact boundary. The Laplacian Δ acts in the space of C^2-functions on S_0 which are with compact support and with vanishing normal derivative on ∂S_0. It has an extension to a self-adjoint operator Δ_n on $L^2(S_0)$. Then one proves exactly as in ([DPRS] Lemma 3.2) that the essential spectrum of Δ_n is contained in $[1/4, +\infty)$. Suppose that each boundary component $\gamma \subset \partial S_0$ has a neighborhood which is isometric to

$$[a, b] \times \mathbb{R}/\mathbb{Z} \quad \text{with metric } \mathrm{d}x^2 + l^2 \operatorname{ch}^2 x \, \mathrm{d}\theta^2$$

for some $l \leqq 2$ arcsh 1 and $b - a \geqq 1$, $b > a \geqq 0$. Then, along the same lines that in [DPRS] one can show that the small eigenvalues of Δ_n are controlled in terms of the small simple closed geodesics contained in S_0.

We introduce one further notation:

$\mu_1(S_0)$ is the infimum of the L^2-spectrum of Δ_n. If $\mathrm{Vol}(S_0) < +\infty$ then $\mu_1(S_0) = 0$ and $\mu_2(S_0)$ denotes the infimum of the L^2-spectrum of the operator Δ_n acting in the space of L^2-functions of mean zero.

1.3 Proof of Theorem 1.2: the upper bound

The upper bound of Theor. 1.2 follows essentially from work of B. Colbois and Y. Colin de Verdière. (see [C, CdV]). Our treatment differs from theirs in that it gives an improvement of a $\ln \varepsilon$-factor in the final result. We recall the main facts for the convenience of the reader.

1.3.1 Let $\varepsilon \leqq 2$ arcsh 1 and $G_\varepsilon = (V, E, m, l)$ be the weighted graph associated to $\mathrm{Geod}(\varepsilon)$. We identify $\mathbb{R}_P[V]$ with a subspace of

$$H^1(S) = \{ f : S \to \mathbb{R}, \| f \|_2 + \| \nabla f \|_2 < +\infty \}$$

in the following way:

Let $a>0$ be such that $\operatorname{sh} a \operatorname{sh}\frac{\varepsilon}{2}=1$ and for $\gamma\in\mathrm{Geod}(\varepsilon)$ define:

$$C_\gamma(a)=\{p\in S: d(p,\gamma)<a\}\subset C_\gamma.$$

Recall that if $S_1\sqcup\ldots\sqcup S_k$ is the decomposition of $S\setminus\mathrm{Geod}(\varepsilon)$ into connected components then:

$$V=\{S_i: 1\leqq i\leqq k\}.$$

We denote by S_i' the complement in S_i of all cylinders $C_{\gamma,a}$ meeting S_i. Given $F\in\mathbb{R}_P[V]$ we define $f\in H^1(S)$ as follows:

- $f(x)=F(S_i)\quad$ for all $x\in S_i',\quad 1\leqq i\leqq k.$

Then f is already defined on $\partial C_{\gamma,a}$ and we define f on $C_{\gamma,a}$ to be the unique harmonic extension of this function.

This defines a subspace of $H^1(S)$ denoted by $H_\varepsilon(S)$. It is associated in a canonical way to $\mathrm{Geod}(\varepsilon)$.

Using the map

$$\begin{aligned}\mathbb{R}_P[V]&\to H_\varepsilon(S)\\ F&\mapsto f\end{aligned}$$

we want to compare $Q(F)$ with $\|\nabla f\|_2^2$ and $\|F\|$ with $\|f\|_2$. In order to do this we have to establish some elementary estimates about harmonic functions on cylinders $C_\gamma(a)$.

1.3.2 Let $a>0, l>0$ and consider the cylinder $C=[-a,a]\times\mathbb{R}/\mathbb{Z}$ endowed with the metric $dx^2+l^2\operatorname{ch}^2 x\, d\theta^2$. The volume element is $dv(x,\theta)=l\operatorname{ch}x\,dx\,d\theta$ and the Laplacian $\Delta=\partial_x^2+l^{-2}\operatorname{ch}^{-2}x\,\partial_\theta^2+\operatorname{th}x\,\partial_x$.

It is easy to verify that the harmonic function f on C with boundary values c_- on $\{-a\}\times\mathbb{R}/\mathbb{Z}$ and c_+ on $\{a\}\times\mathbb{R}/\mathbb{Z}$ is given by:

$$f(x,\theta)=\frac{(c_++c_-)}{2}+\frac{(c_+-c_-)}{2}\,\frac{\operatorname{arcsinth} x}{\operatorname{arcsinth} a}.$$

Lemma 1. (compare with [C, CdV] Prop. III.3).

Let $C_+=[0,a]\times\mathbb{R}/\mathbb{Z}$ and $C_-=[-a,0]\times\mathbb{R}/\mathbb{Z}$. Then we have:

(a) $\|f\|_2^2<c_+^2\operatorname{Vol}(C_+)+c_-^2\operatorname{Vol}(C_-)$

(b)
$$\|f\|_2^2>c_+^2\operatorname{Vol}(C_+)+c_-^2\operatorname{Vol}(C_-)-\frac{la(c_+-c_-)^2}{\operatorname{arcsinth} a}.$$

(c)
$$\|\nabla f\|_2^2=\frac{(c_+-c_-)^2 l}{2\operatorname{arcsinth} a}.$$

Proof (c) is obtained by explicit integration.

We prove (a) and (b):

$$\|f\|_2^2 = 2\,\mathrm{sh}\,a\cdot l\cdot\left(\frac{c_+ - c_-}{2}\right)^2 + \frac{2l}{(\operatorname{arcsinth} a)^2}\left(\frac{c_+ - c_-}{2}\right)^2 \int_0^a \mathrm{d}x\,\mathrm{ch}\,x(\operatorname{arcsinth} x)^2$$

we have

$$\int_0^a \mathrm{d}x\,\mathrm{ch}\,x(\operatorname{arcsinth} x)^2 < \mathrm{sh}\,a(\operatorname{arcsinth} a)^2$$

which proves (a).

$$\int_0^a \mathrm{d}x\,\mathrm{ch}\,x(\operatorname{arcsinth} x)^2 = \mathrm{sh}\,a(\operatorname{arcsinth} a)^2 - 2\int_0^a \mathrm{d}x\,\mathrm{th}\,x\operatorname{arcsinth} x$$
$$> \mathrm{sh}\,a(\operatorname{arcsinth} a)^2 - 2a\operatorname{arcsinth} a$$

which proves (b). Q.E.D.

1.3.3 Let:

$$\mathbb{R}_P[V] \to H_\varepsilon(S)$$
$$F \mapsto f$$

be the map defined in 1.3.1. Then we have:

Lemma 2.

(a) $\frac{1}{\pi}Q(F) \leqq \|\nabla f\|_2^2 \leqq \frac{1}{\pi}Q(F)(1+c\cdot\varepsilon)$
(b) $\|f\|_2^2 \leqq 2\pi\|F\|^2$
(c) $\|f\|_2^2 \geqq 2\pi\|F\|^2(1-c\cdot\varepsilon\ln\varepsilon)$.

Here $c>0$ *is some universal constant.*

Proof. (a) and (b) follows immediately from Lemma 1 and the fact that $\mathrm{sh}\,a\,\mathrm{sh}\frac{\varepsilon}{2} = 1$. To prove (c) we remark that Lemma 1 implies:

$$\|f\|^2 \geqq 2\pi\|F\|^2 - Q(F)\frac{a}{\operatorname{arcsinth} a}.$$

Now we have to bound $Q(F)$:

Clearly:

$$Q(F) \leqq 2\varepsilon \sum_{x,y\in V} (F(x)^2 + F(y)^2)$$
$$= 4\varepsilon \sum_{x\in V} F(x)^2\,\mathrm{d}(x)$$

where $d(x)$ is the degree of the vertex x. Let S_x be the surface of finite volume corresponding to x. Let g_x be its genus, p_x the number of cusps and f_x the number of boundary geodesics. Then:

$$d(x) = f_x$$

and we have by Gauss-Bonnet:

$$m(x) = \frac{1}{2\pi} \operatorname{Vol}(S_x) = 2g_x - 2 + p_x + f_x.$$

This quantity is always bigger or equal to $f_x/3$ as is easily verified.

Thus:

$$Q(F) \leqq 12\varepsilon \|F\|^2.$$

On the other hand $a/\operatorname{arcsinth} a \leqq c \cdot \ln \varepsilon$ where $c > 0$ is some constant. This proves (c). Q.E.D.

1.3.4 The upper bound in Theor. 1.2 is now an immediate consequence of Lemma 2.

1.4 *Proof of Theorem 1.2: the lower bound*

1.4.1 The case of one separating geodesic. Let $S \in \mathcal{M}(g, p, f)$ and $F \subset S$ a surface with smooth compact boundary. We assume that there is a simple closed geodesic $\gamma \subset F$ of length $l \leqq 2 \operatorname{arcsh} 1$ separating F into two components F_1, F_2. We assume also that the cylinder $C_\gamma(a)$ is contained in F for some $a \leqq d_\gamma$. (cf. 1.2.1)

Using a method introduced by Y. Colin de Verdière (cf. [C, CdV] Lemma PVP) we prove the following

Lemma 3. a.) *Suppose* $\operatorname{Vol}(F) < +\infty$ *and* $\mu_2(F) < \frac{1}{4}$. *Then:*

$$\mu_2(F) \geqq \frac{1}{\pi} \frac{\operatorname{Vol}(F)\, l(\gamma)}{\operatorname{Vol}(F_1) \operatorname{Vol}(F_2)} [1 - c\, l(\gamma)(1 + \eta^{-1})]$$

where $\eta = \min(\mu_2(F_1), \mu_2(F_2))$.

b.) *Suppose* $\operatorname{Vol}(F) = +\infty$, $\operatorname{Vol}(F_1) < +\infty$ *and* $\mu_1(F) < \frac{1}{4}$. *Then:*

$$\mu_1(F) \geqq \frac{1}{\pi} \frac{l(\gamma)}{\operatorname{Vol}(F_1)} [1 - c\, l(\gamma)(1 + v^{-1})]$$

where $v = \min(\mu_2(F_1), \mu_1(F_2))$.

In both cases c is a constant only depending on a lower bound for $\operatorname{Vol}(C_\gamma)$.

Proof. We prove a.) since the proof of b.) is the same.

Let $h \in H^1(F)$ such that h is constant $= c_i$ on $F_i \setminus C_\gamma(a)$ and harmonic inside $C_\gamma(a)$. Set:

$$c_1 = [\operatorname{Vol}(F_2)/\operatorname{Vol}(F) \operatorname{Vol}(F_1)]^{\frac{1}{2}}, \qquad c_2 = -[\operatorname{Vol}(F_1)/\operatorname{Vol}(F) \operatorname{Vol}(F_2)]^{\frac{1}{2}}.$$

In particular we have using Lemma 1:

$$\int_F h(x)\,\mathrm{d}v(x)=0 \quad \text{and} \quad \|h\|_2 \leqq 1.$$

Let $h=\varphi+\varphi_\infty$ where φ is the orthogonal projection of h on the eigenspace of Δ_n (cf. 1.2.2) corresponding to the eigenvalue $\mu_2(F)$ (recall that $\mu_2(F)<\frac{1}{4}$) and $\langle\varphi_\infty, \varphi\rangle=0$.

Let $C=C_\gamma(a)$. Then

$$\int_C |\nabla h(x)|^2\,\mathrm{d}v(x)=\|\nabla h\|_2^2=\mu_2\,\|\varphi\|_2^2+\int_F |\nabla\varphi_\infty(x)|^2\,\mathrm{d}v(x).$$

Now:

$$\begin{aligned}\int_F |\nabla\varphi_\infty(x)|^2\,\mathrm{d}v(x)&=\int_F \langle\nabla\varphi_\infty(x), \nabla h(x)\rangle\,\mathrm{d}v(x)\\ &=\int_C \langle\nabla\varphi_\infty(x), \nabla h(x)\rangle\,\mathrm{d}v(x)\\ &=\frac{l(c_2-c_1)}{2\operatorname{arcsinth} a}\int_0^1 \mathrm{d}\theta\int_{-a}^{a}\mathrm{d}x\,\partial_x\varphi_\infty(x,\theta) \quad \text{where } l=l(\gamma)\\ &=\frac{l(c_2-c_1)}{2\operatorname{arcsinth} a}\int_0^1 \mathrm{d}\theta\{\varphi_\infty(a,\theta)-\varphi_\infty(-a,\theta)\}.\end{aligned}$$

An integration by parts of $\int_0^a \partial_x f(x,\theta)\,dx$ where f is any C^1-function gives the formula:

$$l\,\mathrm{sh}\,a\int_0^1 f(a,\theta)\,\mathrm{d}\theta=l\int_0^1\mathrm{d}\theta\int_0^a \mathrm{ch}\,x\,f(x,\theta)\,\mathrm{d}x+l\int_0^1\mathrm{d}\theta\int_0^a \mathrm{sh}\,x\,\partial_x f(x,\theta)\,\mathrm{d}x$$

using that $\mathrm{sh}\,x<\mathrm{ch}\,x$ and applying Cauchy-Schwarz we obtain:

$$\left|l\,\mathrm{sh}\,a\int_0^1 f(a,\theta)\,\mathrm{d}\theta\right|\leqq(l\,\mathrm{sh}\,a)^{1/2}\{\|f\|_{L^2(C\cap F_2)}+\|\nabla f\|_{L^2(C\cap F_2)}\}.$$

Similarly

$$\left|l\,\mathrm{sh}\,a\int_0^1 f(-a,\theta)\,\mathrm{d}\theta\right|\leqq(l\,\mathrm{sh}\,a)^{1/2}\{\|f\|_{L^2(C\cap F_1)}+\|\nabla f\|_{L^2(C\cap F_1)}\}.$$

Applying this to $f=\varphi_\infty$ we obtain:

$$\int_F |\nabla\varphi_\infty(x)|^2\,\mathrm{d}v(x)\leqq\frac{l|c_2-c_1|}{\operatorname{arcsinth} a}\,\frac{1}{(l\,\mathrm{sh}\,a)^{1/2}}\{\|\varphi_\infty\|_2+\|\nabla\varphi_\infty\|_2\}.$$

Now: $\|\nabla\varphi_\infty\|_2^2 \geqq \mu_3(F)\|\varphi_\infty\|_2^2$ and $\mu_3(F) \geqq \min(\mu_2(F_1), \mu_2(F_2)) = \eta$. Thus:

$$\|\nabla\varphi_\infty\|_2^2 \leqq \frac{l|c_2-c_1|\sqrt{2}}{\operatorname{arcsinth} a \operatorname{Vol}(C)^{1/2}} \|\nabla\varphi_\infty\|_2 (1+\eta^{-1/2})$$

or

$$\|\nabla\varphi_\infty\|_2^2 \leqq \frac{2l^2(c_2-c_1)^2}{(\operatorname{arcsinth} a)^2 \operatorname{Vol}(C)} (1+\eta^{-1/2})^2.$$

From this follows that:

$$\int_C |\nabla h(x)|^2 dv(x) \leqq \mu_2(F) + \frac{2l^2(c_2-c_1)^2}{(\operatorname{arcsinth} a)^2 \operatorname{Vol}(C)} (1+\eta^{-1/2})^2.$$

On the other hand, Lemma 1 shows that:

$$\int_C |\nabla h(x)|^2 \mathrm{d}v(x) = \frac{(c_2-c_1)^2 l}{2 \operatorname{arcsinth} a}.$$

Putting everything together we obtain a.). Q.E.D.

Remark. For later applications it is crucial that the error term in Lemma 3 is of the form $l(\gamma)/\eta$. This means that the estimate is optimal as long as $\mu_2(F)$ is small when compared to $\min(\mu_2(F_1), \mu_2(F_2))$. A similar statement holds for b.).

1.4.2 A modified graph. In order to prove Theorem 1.2 it is convenient to modify the graph G_ε associated to $\operatorname{Geod}(\varepsilon)$, keeping its spectrum fixed. This is done in the following way:

Let $\operatorname{Geod}'(\varepsilon) \subset \operatorname{Geod}(\varepsilon)$ be the subset of those geodesics which connect two distinct components of $S \setminus \operatorname{Geod}(\varepsilon)$ one of which at least is of finite volume. Let

$$\{S_v : v \in V'\}$$

be the set of connected components of $S \setminus \operatorname{Geod}'(\varepsilon)$. Then V' is the vertex set of our new graph. We have a distinguished subset

$$P' = \{v \in V' : \operatorname{Vol}(S_v) = +\infty\}$$

and a weight function $m \colon V' \to \mathbb{N}$ defined by:

$$m(v) = \frac{1}{2\pi} \operatorname{Vol}(S_v) \quad \text{if } \operatorname{Vol}(S_v) < +\infty$$

$$m(v) = 1 \quad \text{if } v \in P'.$$

The edge set E' is identified with $\operatorname{Geod}'(\varepsilon)$ and we get an obvious length function l' on E'.

Let $G'_\varepsilon = (V', E', m, l')$. It is clear that the spectrum of (G'_ε, P') is the same than the spectrum of the pair (G_ε, P).

Let $\delta \leqq 4\beta_3$ and $\varepsilon = \lambda_j/\beta_1$, $1 \leqq j \leqq 2g-2+p+f$ and let $S \in \mathcal{M}_{j,\delta}$. Then we know by Lemma 0 that Geod(ε) cuts S into $j+1$ pieces exactly where the union of all components of infinite volume is seen as one piece.

Let T be a connected component of $S \backslash \mathrm{Geod}'(\varepsilon)$. About each boundary geodesic $\gamma \subset \partial T$ there is a half cylinder

$$C_\gamma^+(a) = \{p \in T: d(p,\gamma) \leqq a\} \qquad 0 < a \leqq d_\gamma.$$

Lemma 4. *There are positive constants α, α' depending only on (g, p, f) such that if $\delta \leqq \alpha$ and T' is the surface obtained from T by removing half-cylinders $C_\gamma^+(a)$, $0 \leqq a \leqq d_\gamma - 1$ then:*

(a) *If* $\mathrm{Vol}(T') = +\infty: \mu_1(T') \geqq \alpha' \lambda_{j+1}(S)$

(b) *If* $\mathrm{Vol}(T') < +\infty: \mu_2(T') \geqq \alpha' \lambda_{j+1}(S)$.

Proof. (a) $\mathrm{Vol}(T') = +\infty$. From the discussion in 1.2.2 it follows that

$$\mu_1(T') \geqq \beta_1 L_1(T').$$

But:

$$L_1(T') + L_j(S) \geqq L_{j+1}(S) \geqq \beta_2^{-1} \lambda_{j+1}(S)$$

in virtue of ([DPRS]).

On the other hand $L_j(S) \leqq \beta_1^{-1} \lambda_j(S) \leqq \beta_1^{-1} \delta \lambda_{j+1}(S)$. Thus:

$$\mu_1(T') \geqq \beta_1 L_1(T') \geqq \beta_1 (\beta_2^{-1} - \beta_1^{-1} \delta) \lambda_{j+1}(S)$$

which proves (a) for sufficiently small δ.

(b) same proof. Q.E.D.

1.4.3 Fix j, $1 \leqq j \leqq 2g-2+p+f$. Let $S \in \mathcal{M}_{j,\delta}$ where $\delta \leqq 4\beta_3$ and consider the graph G'_{λ_j/β_1} defined in 1.4.2.

We define a map

$$H^1(S) \to \mathbb{R}_{P'}[V']$$
$$f \mapsto F$$

by

$$F(v) = \frac{1}{\mathrm{Vol}(S_v)} \int_{S_v} f(x)\, dv(x) \quad \text{if } \mathrm{Vol}(S_v) < +\infty$$
$$F(v) = 0 \quad \text{if } v \in P'.$$

Let E_j be the subspace of $H^1(S)$ spanned by all eigenfunctions of Δ_S of eigenvalue $\lambda \leqq \lambda_j$.

Lemma 5. *There are constants $\alpha, \alpha' > 0$ only depending on (g, p, f) such that if $S \in \mathcal{M}_{j,\delta}$ and $0 < \delta \leqq \alpha$ we have:*

$$2\pi \|F\|^2 (1 + \alpha' \delta) \geqq \|f\|_2^2 \geqq 2\pi \|F\|^2$$

for all $f \in E_j$.

Proof. Let $v \in V'$, and $\mathrm{Vol}(S_v) < +\infty$. Then we have:

$$\int_{S_v} |\nabla f(x)|^2\, dv(x) \geqq \mu_2(S_v) \int_{S_v} [f(x) - F(v)]^2\, dv(x)$$

and if $\mathrm{Vol}(S_v)=+\infty$:

$$\int_{S_v} |\nabla f(x)|^2 \,\mathrm{d}v(x) \geqq \mu_1(S_v) \int_{S_v} f(x)^2 \,\mathrm{d}v(x).$$

By Lemma 4:

$$\mu_2(S_v) \geqq \alpha' \lambda_{j+1}(S) \quad \text{and} \quad \mu_1(S_v) \geqq \alpha' \lambda_{j+1}(S).$$

Summing over $v \in V'$ we obtain:

$$\lambda_j \| f \|_2^2 \geqq \int_S |\nabla f(x)|^2 \,\mathrm{d}v(x) \geqq \alpha' \lambda_{j+1} \{ \| f \|_2^2 - 2\pi \| F \|^2 \}$$

which proves the upper bound for $\| f \|_2^2$. The lower bound follows from Cauchy-Schwarz. Q.E.D.

1.4.4 A lower bound for $\| \nabla f \|_2^2$. Let S be a geometrically finite surface and $f \in L^1(S)$. For each subset $A \subset S$ of positive volume we define:

$$f(A) = \frac{1}{\mathrm{Vol}(A)} \int_A f(x) \,\mathrm{d}v(x),$$

in particular $f(A)=0$ if $\mathrm{Vol}(A)=+\infty$. Then we have:

Lemma 6. *Let A, B be surfaces with smooth boundary, $A, B \subset S$ such that* $\mathrm{Vol}(A \cap B)=0$. *Set $D = A \cup B$*

a.) *If* $\mathrm{Vol}(D) < +\infty$ *then we have for all $f \in H^1(S)$:*

$$\int_D |\nabla f(x)|^2 \,\mathrm{d}v(x) \geqq \mu_2(D) \frac{\mathrm{Vol}(A)\,\mathrm{Vol}(B)}{\mathrm{Vol}(D)} (f(A) - f(B))^2$$

b.) *If* $\mathrm{Vol}(A) < +\infty$ *and* $\mathrm{Vol}(B) = +\infty$ *then we have for all $f \in H^1(S)$:*

$$\int_D |\nabla f(x)|^2 \,\mathrm{d}v(x) \geqq \mu_1(D)\,\mathrm{Vol}(A) f(A)^2.$$

Proof. a.) By definition of $\mu_2(D)$ we have:

$$\begin{aligned}
\int_D |\nabla f(x)|^2 \,\mathrm{d}v(x) &\geqq \mu_2(D) \int_D [f(x) - f(D)]^2 \,\mathrm{d}v(x) \\
&= \mu_2(D) \Big\{ \int_A f(x)^2 \,\mathrm{d}v(x) + \int_B f(x)^2 \,\mathrm{d}x - \mathrm{Vol}(D) f(D)^2 \Big\} \\
&\geqq \mu_2(D) \{ \mathrm{Vol}(A) f(A)^2 + \mathrm{Vol}(B) f(B)^2 - \mathrm{Vol}(D) f(D)^2 \} \\
&= \frac{\mu_2(D)\,\mathrm{Vol}(A)\,\mathrm{Vol}(B)}{\mathrm{Vol}(D)} (f(A) - f(B))^2.
\end{aligned}$$

b.) Is obvious. Q.E.D.

1.4.5 A combinatorial Lemma. Let $S \in \mathscr{M}_{j,\delta}$, δ small and $G'_{\lambda_j/\beta_1} = (V', E', m, l')$.

In order to apply Lemma 6 we want to cover S using surfaces S_e, $e \in E'$ such that

(1) $\mathrm{Vol}(S_e \cap S_{e'}) = 0$ if $e \neq e'$

(2) Let $\gamma_e \in \mathrm{Geod}'(\lambda_j/\beta_1)$ be the geodesic labelled by $e \in E'$. Then γ_e cuts S_e into two pieces exactly and the cylinder $C_\gamma(a)$ is contained in S_e where $a = \operatorname{arcsh}\left(1/\operatorname{sh}\frac{\lambda_j/\beta_1}{2}\right) - 1$.

To do this we need the following Lemma:

Lemma 7. *Let $\mathscr{G} = (W, E)$ be a finite connected graph and $v_0 \in W$ a fixed vertex. Then there exists an injective map*

$$\phi: W - \{v_0\} \to E$$

such that for all $v \neq v_0$, v is an extremity of $\phi(v)$.

Proof. Straightforward induction on the number of vertices of $\mathscr{G}$. Q.E.D.

For $v \in V'$ we let K_v be the complement in S_v of the union of all cylinders $C_\gamma(a)$ meeting S_v where $\gamma \in \mathrm{Geod}'(\lambda_j/\beta_1\}$.

We fix $v_0 \in V'$ and let $\phi: V' - \{v_0\} \to E'$ be the map given by Lemma 7. In order to define the surfaces S_e we have to distinguish two cases:

1.) Im ϕ does not contain any edge whose extremity is v_0. Then we extend ϕ to V' by $\phi(v_0) = e$ where e is some edge issued from v_0.

- if $e \notin \mathrm{Im}\,\phi$ we define $S_e = C_\gamma(a)$ where $\gamma \in \mathrm{Geod}'(\lambda_j/\beta_1)$ corresponds to e.
- *if* $e \in \mathrm{Im}\,\phi$, then $e = \phi(v)$ for a unique $v \in V'$ and we set $S_e = K_v \cup C_\gamma(a)$ where γ corresponds to e.

2.) Im ϕ contains edges issued from v_0. Let $e_1 = \phi(v_1)$ be one of these edges.

- if $e \notin \mathrm{Im}\,\phi$ we set $S_e = C_\gamma(a)$ as before.
- if $e = \phi(v)$ and $v \neq v_1$ we set $S_e = K_v \cup C_\gamma(a)$.
- if $e = e_1$ we set $S_e = K_{v_1} \cup C_\gamma(a) \cup K_{v_0}$.

In each case we obtain a family of surfaces $\{S_e : e \in E'\}$ satisfying properties 1.) and 2.).

1.4.6 End of the proof. Let $S = \bigcup_{v \in V'} S_v = \bigcup_{e \in E'} S_e$.

According to Lemma 5 it suffices to prove that if $\varphi \in H^1(S)$ then:

$$\int_S |\nabla \varphi(x)|^2 \,\mathrm{d}v(x) \geqq \frac{1}{\pi}\left(1 - \alpha\sqrt{\frac{\lambda_j}{\lambda_{j+1}}}\right) \sum l(\gamma)(\varphi(S_V) - \varphi(S_W))^2$$

this last sum being over all $\gamma \in \mathrm{Geod}'(\lambda_j/\beta_1)$, $\gamma \subset S_V \cap S_W$.

Each surface S_e is cut by $\gamma = \gamma_e$ into two surfaces A_e and B_e. We apply Lemma 6 to $S_e = A_e \cup B_e$ and obtain:

a.) If $\mathrm{Vol}(S_e) < +\infty$:

$$\int_{S_e} |\nabla \varphi(x)|^2 \,\mathrm{d}v(x) \geqq \mu_2(S_e) \frac{\mathrm{Vol}(A_e)\,\mathrm{Vol}(B_e)}{\mathrm{Vol}(S_e)} (\varphi(A_e) - \varphi(B_e))^2.$$

b.) If $\mathrm{Vol}(A_e) < +\infty$ and $\mathrm{Vol}(S_e) = +\infty$:

$$\int_{S_e} |\nabla \varphi(x)|^2 \, \mathrm{d}v(x) \geqq \mu_1(S_e) \, \mathrm{Vol}(A_e)(\varphi(A_e) - \varphi(B_e))^2$$

since $\varphi(B_e) = 0$.

From Lemma 3 it follows that

$$\int_{S_e} |\nabla \varphi(x)|^2 \, \mathrm{d}v(x) \geqq \frac{l(\gamma)}{\pi} (\varphi(A_e) - \varphi(B_e))^2 \left(1 - c \frac{l}{\eta}\right)$$

where $\eta \geqq \alpha' \lambda_{j+1}(S)$ using Lemma 4 and $l \leqq \lambda_j / \beta_1$. Here c, α' constants which only depend on (g, p, f). This shows that:

$$\int_{S_e} |\nabla \varphi(x)|^2 \, \mathrm{d}v(x) \geqq \frac{l(\gamma)}{\pi} \left(1 - c' \frac{\lambda_j}{\lambda_{j+1}}\right) (\varphi(A_e) - \varphi(B_e))^2. \tag{1}$$

Let v, $w \in V'$ such that $A_e \subset S_v$ and $B_e \subset S_w$. We can assume that $\mathrm{Vol}(A_e) < +\infty$. Now we estimate:

$$|(\varphi(A_e) - \varphi(B_e))^2 - (\varphi(S_v) - \varphi(S_w))^2| \leqq [|\varphi(A_e) - \varphi(S_v)| + |\varphi(B_e) - \varphi(S_w)|] \cdot [|\varphi(A_e) - \varphi(S_w)| + |\varphi(B_e) - \varphi(S_v)|].$$

Let $A' = S_v \setminus A_e$, then a simple computation shows that:

$$\varphi(A_e) - \varphi(S_v) = \frac{\mathrm{Vol}(A')}{\mathrm{Vol}(S_v)} (\varphi(A_e) - \varphi(A')).$$

Now Lemma 4 and 6 imply:

$$(\varphi(A_e) - \varphi(S_v))^2 \leqq \frac{\alpha}{\lambda_{j+1}} \int_{S_v} |\nabla \varphi(x)|^2 \, \mathrm{d}v(x) \tag{2}$$

where $\alpha = \alpha(g, p, f)$. Remark that the inequality is trivialy satisfied if $\mathrm{Vol}(A_e) = \infty$.

Consider the surfaces $A_e \cup S_w$ and $B_e \cup S_v$. Then the same arguments as in the proof of inequality (1) show that:

$$|\varphi(A_e) - \varphi(S_w)| \leqq \alpha \, l(\gamma)^{-1/2} \Big[\int_{S_v \cup S_w} |\nabla \varphi(x)|^2 \, \mathrm{d}v(x) \Big]^{1/2} \tag{3}$$

$$|\varphi(B_e) - \varphi(S_v)| \leqq \alpha \, l(\gamma)^{-1/2} \Big[\int_{S_v \cup S_w} |\nabla \varphi(x)|^2 \, \mathrm{d}v(x) \Big]^{1/2}. \tag{4}$$

Putting the inequalities (2), (3) and (4) together we obtain that

$$l(\gamma) |(\varphi(A_e) - \varphi(B_e))^2 - (\varphi(S_v) - \varphi(S_w))^2|$$

is bounded by:

$$\alpha \cdot \left(\frac{l(\gamma)}{\lambda_{j+1}}\right)^{1/2} \cdot \int_{S_v \cup S_w} |\nabla \varphi(x)|^2 \, \mathrm{d}v(x) \leqq \alpha' \cdot \left(\frac{\lambda_j}{\lambda_{j+1}}\right)^{1/2} \cdot \int_{S_v \cup S_w} |\nabla \varphi(x)|^2 \, \mathrm{d}v(x).$$

This together with inequality (1) shows that:

$$\int_{S_e} |\nabla \varphi(x)|^2 \, \mathrm{d}v(x) \geqq \frac{l(\gamma)}{\pi}\left(1 - c' \frac{\lambda_j}{\lambda_{j+1}}\right)(\varphi(S_v) - \varphi(S_w))^2$$

$$-\alpha \sqrt{\frac{\lambda_j}{\lambda_{j+1}}} \int_{S_v \cup S_w} |\nabla \varphi(x)|^2 \, \mathrm{d}v(x)$$

here $\alpha = \alpha(g, p, f)$ always denotes some constant depending only on (g, p, f). Summing over $e \in E'$ we obtain the desired estimate. Q.E.D.

1.5 Proof of the corollaries

We prove Corollary 1.1. The proof of Corollary 1.2 is completely analogous, and uses Coroll. 2 of Theor. 2.1.

Suppose that the corollary is false. Then there exists a sequence of Riemann surfaces $\{S_n\}_{n=1}^{\infty}$ such that:

(a) $\lim_{n \to \infty} \lambda_2(S_n) = 0$

(b) $|\mathrm{Spec}\, \Delta_{S_n} \cap [\lambda_2, \lambda_2(1 + \varepsilon(\lambda_2))]| \geqq q$ where $q = \frac{2}{3}[2g - 2 + p + f] + 3$.

Take i minimal such that $q \leqq i \leqq 2g - 2 + p + f$ and $\lim_{n \to \infty} \lambda_i(S_n)/\lambda_{i+1}(S_n) = 0$.

By passing to a subsequence of $\{S_n\}_{n=1}^{\infty}$ we can assume that the graph with weight function associated to $\mathrm{Geod}(\lambda_i(S_n)/\beta_1)$ is isomorphic to a fixed one $(\mathscr{G}, m)$, $\mathscr{G} = (V, E)$. From the definition of i it follows that there exists $c > 0$ such that $\lambda_2(S_n) \geqq c\,\lambda_i(S_n)$ for all $n \geqq 1$. If l_n is the length function on E defined by S_n then we have for all $e \in E$:

$$l_n(e) \leqq \lambda_i(S_n)/\beta_1 \leqq \lambda_2(S_n)/\beta_1 c.$$

Thus we can assume that the sequence $l_n/\lambda_2(S_n)$, converges to a function $l: E \to \mathbb{R}^+ \cup \{0\}$. Let $E' = \{e \in E: l(e) \neq 0\}$. Then it follows from Theor. 1.2 and the hypotheses of Corollary 1.1 that the second eigenvalue $\lambda_2^g(G')$ of the weighted graph:

$$G' = (V, E', m, l)$$

is equal to $2\pi^2$ and has multiplicity at least q. Moreover this graph satisfies the hypothesis of Theor. 2.2. Indeed let S_v be the component corresponding to $v \in V \setminus P$ and let $\mathrm{d}(v)$ be the degree of the vertex v. The $\mathrm{d}(v) = f_v$ and:

$$\mathrm{d}(v) - 2 = f_v - 2 \leqq 2g_v - 2 + p_v + f_v = m(v).$$

In this way we obtain a contradiction with Theor. 2.2. Q.E.D.

2 Weighted graphs

A weighted graph $G = (V, E, m, l)$ is a graph $\mathscr{G} = (V, E)$ together with a weight function

$$m: V \to \mathbb{R}^+$$

defined on the set of vertices and a length function

$$l: E \to \mathbb{R}^+$$

defined on the set of edges. We assume that these two functions take strictly positive values. Given a distinguished subset $P \subset V$ we define the spectrum of the pair (G, P) in the following way:

On $\mathbb{R}[V]$ we have a quadratic form

$$Q(F) = \sum_{e \in E} \partial F(e)^2 \, l(e), \qquad F \in \mathbb{R}[V]$$

where

$$\partial F(e) = F(v) - F(w) \in \mathbb{R}/\{\pm 1\}, \qquad e = \{v, w\}$$

and a scalar product:

$$\langle F_1, F_2 \rangle = \sum_{v \in V} F_1(v) \, F_2(v), \qquad F_1, F_2 \in \mathbb{R}[V].$$

Let $\mathbb{R}_P[V]$ be the subspace of functions $f \in \mathbb{R}[V]$ that vanishes on P.

The restriction of $F \to Q(F)$ to $\mathbb{R}_P[V]$ defines a symmetric operator M_P:

$$\langle M_P F, F \rangle = Q(F), \qquad F \in \mathbb{R}_P[V].$$

The spectrum of (G, P) is the set of eigenvalues of M_P listed according to their multiplicities:

$$\lambda_1^P(G) \leqq \lambda_2^P(G) \leqq \ldots \leqq \lambda_N^P(G)$$

where $N = |V| - |P|$. If $P = \emptyset$ this is simply the spectrum of G.

Let $\mathscr{G}' = (V, E')$ be the graph obtained from $\mathscr{G}$ by replacing all multiple edges by one edge and by deleting all loops. Up to a obvious modification $l': E' \to \mathbb{R}^+$ of our length function l we obtain a weighted graph $G' = (V, E', m, l')$ such that the spectrum of (G', P) is identical with the spectrum of (G, P).

This being said we will assume throughout this chapter that the graph $\mathscr{G} = (V, E)$ is finite without loops and without multiple edges.

We will see that if $V \setminus P$ generates a connected graph $\lambda_1^P(G)$ is of multiplicity one. Our first estimate of the multiplicity of $\lambda_2^P(G)$ involves the following invariant of a graph $\mathscr{G} = (V, E)$:

Let $d: V \to \mathbb{N}$ be the degree function of $\mathscr{G}$ and $d_{\mathscr{G}} = \max_{v \in V} \mathrm{d}(v)$ the maximal degree of a vertex $v \in V$. If $A \subset V$ we denote by $G(A)$ the graph generated by A. Then we define:

$$\alpha(\mathscr{G}) = \min_{Q \subset V} (d_\tau + |Q|)$$

where the minimum is taken over all subsets $Q \subset V$ such that $\mathscr{T} = G(V \setminus Q)$ is a connected tree.

Theorem 2.1. *Let $G = (V, E, m, l)$ be a weighted graph and $P \subset V$ a distinguished subset such that $G(V \setminus P)$ is connected. Then the multiplicity of $\lambda_2^P(G)$ is at most $\alpha(\mathscr{G})$ where $\mathscr{G} = G(V \setminus P)$.*

Examples. 1. K_n is the complete graph on n vertices. We set the edge and length function to be identically 1. Then $\lambda_2(K_n)=n$ is of multiplicity $n-1$ and it is easily seen that $\alpha(K_n)=n-1$.

2. S_n is the star with n vertices. As before edge and length function are identically 1. Then $\lambda_2(S_n)=1$ is of multiplicity $n-2$ and $\alpha(S_n)=n-1$.

Corollary 1. *Let $T=(\mathscr{T},m,l)$ be a weighted connected tree. Then*:

$$\text{mult}\,\lambda_2(T)\leqq d_{\mathscr{T}}.$$

In 2.2 we will prove that if $\mathscr{G}$ is a connected graph and $\beta_1(\mathscr{G})$ its first Betti number then:

$$\alpha(\mathscr{G})\leqq d_{\mathscr{G}}+\beta_1(\mathscr{G}).$$

From this follows:

Corollary 2. *Let $G=(V,E,m,l)$ be a weighted graph and $P\subset V$ a distinguished subset such that $\mathscr{X}=G(V\setminus P)$ is connected. Suppose that $d_{\mathscr{X}}\leqq 3$ then*:

$$\text{mult}\,\lambda_2^P(G)\leqq \tfrac{1}{2}|V\setminus P|+4.$$

In 2.3 we will show another approach to the problem of bounding the multiplicity of λ_2. This will lead us to the following result which is well suited for applications to Riemann surfaces:

Theorem 2.2. *Let $G=(V,E,m,l)$ be a weighted graph and $P\subset V$ such that $G(V\setminus P)$ is connected. Assume that m is integer valued and that $d(v)-2\leqq m(v)$ for all $v\in V\setminus P$. Then*:

$$\text{mult}\,\lambda_2^P(G)\leqq \tfrac{2}{3}m(V\setminus P)+2.$$

Here we set for $A\subset V, m(A)=\sum_{v\in A} m(v)$. Essential use of Theor. 2.2 will be made in the proof of Corollary 1.2.

2.1 The first method

Let $G=(V,E,m,l)$ be a weighted graph and $P\subset V$ a subset such that $G(V\setminus P)$ is connected. One verifies that M_P acts on functions $f\in\mathbb{R}_P[V]$ as follows:

$$x\in V\setminus P \qquad M_P f(x)=\frac{1}{m(x)}\sum_{x\equiv y} l(\{x,y\})(f(x)-f(y))$$

$$x\in P \qquad M_P f(x)=0.$$

Here $\{x,y\}$ is the edge joining x and y, the symbol $x\equiv y$ means that x,y are adjacent vertices.

We have the following easy

Lemma 1. *If $G(V\setminus P)$ is connected $\lambda_1^P(G)$ is of multiplicity one and any nonzero eigenfunction of M_P of eigenvalue $\lambda_1^P(G)$ is everywhere nonzero on $V\setminus P$.*

Proof. For all $f \in \mathbb{R}[V]$ we have $Q(|f|) \leqq Q(f)$ with equality if and only if for all x, y such that $x \equiv y$ we have $f(x) \cdot f(y) \geqq 0$. Since

$$\lambda_1^P(G) = \min \frac{Q(f)}{\|f\|^2}$$

where the minimum is taken over $f \in \mathbb{R}_P[V]$, the inequality $Q(|f|) \leqq Q(f)$ shows that if f is an eigenfunction of eigenvalue $\lambda_1^P(G)$ then $|f|$ has also this property. Let f be an eigenfunction such that for some $x \in V \setminus P$ $f(x) = 0$. Since $\varphi = |f|$ is also eigenfunction we have:

$$\sum_{y \equiv x} l(\{x, y\}) \, \varphi(y) = 0$$

from which follows $\varphi(y) = 0$ for all $y \equiv x$. Since $G(V \setminus P)$ is connected this implies $\varphi = 0$ and hence $f = 0$. This shows that $\lambda_1^P(G)$ is of multiplicity one. Q.E.D.

Let $\mathscr{G} = (V, E)$ be a graph. For $A \subset V$ we define the boundary of A:

$$\partial A = \{a \in A : a \text{ is adjacent to some point in } V \setminus A\}.$$

Given a weighted graph $G = (V, E, m, l)$ and a subset $P \subset V$ we take two pairs of subsets (P_1, V_1), (P_2, V_2) with the following properties:

(1) $P_i \subset V_i \subset V \quad i = 1, 2$
(2) $G_i = G(V_i \setminus P_i)$ is a nonvoid connected graph for $i = 1, 2$
(3) $V_1 \cap V_2 \subset P_1 \cap P_2$
(4) $P_1 \cap P_2 \supset P$
(5) $P_i \supset \partial V_i$ for $i = 1, 2$
(6) $V_i \setminus P_i \not\subset P$ for $i = 1, 2$.

Lemma 2. *Assume that $G(V \setminus P)$ is connected, then*

$$\max(\lambda_1^{P_1}(G_1), \lambda_1^{P_2}(G_2)) \geqq \lambda_2^P(G)$$

with equality if and only if:

$$\lambda_1^{P_1}(G_1) = \lambda_1^{P_2}(G_2) = \lambda_2^P(G).$$

Proof. Let $F_i \in \mathbb{R}_{P_i}[V_i]$ be a positive eigenfunction corresponding to the eigenvalue $\lambda_1^{P_i}(G_i)$. We extend F_i to V by setting $F_i = 0$ on $V \setminus V_i$. Let $F \neq 0$ be an eigenfunction of M_P of eigenvalue $\lambda_1^P(G)$. Property (6) and Lemma 1 implies that $\langle F, F_i \rangle \neq 0$ for $i = 1, 2$. From this follows that there exist $c_1 \neq 0$, $c_2 \neq 0$ such that $f = c_1 F_1 + c_2 F_2 \in \mathbb{R}_P[V]$ is orthogonal to F.

Property (3) implies that $\|f\|^2 = c_1^2 \|F_1\|^2 + c_2^2 \|F_2\|^2$.
Properties (1) to (6) imply $Q(f) = c_1^2 Q(F_1) + c_2^2 Q(F_2)$.
Thus

$$\begin{aligned} Q(f) &= c_1^2 \lambda_1^{P_1}(G_1) \|F_1\|^2 + c_2^2 \lambda_1^{P_2}(G_2) \|F_2\|^2 \\ &\geqq \lambda_2^P(G) \|f\|^2 = \lambda_2^P(G)(c_1^2 \|F_1\|^2 + c_2^2 \|F_2\|^2). \end{aligned}$$

From this follows $\max(\lambda_1^{P_1}(G_1), \lambda_1^{P_2}(G_2)) \geqq \lambda_2^P(G)$. Q.E.D.

Here is a immediate consequence of Lemma 2.

Corollary. *Let $(P_1, V_1), \ldots, (P_n, V_n)$ be pairs of subsets of V such that for all $i \neq j$, $(P_i, V_i), (P_j, V_j)$ verify properties* (1) *to* (6) *above.*

Then either (1.) *there is a unique i, $1 \leqq i \leqq n$ such that $\lambda_1^{P_i} < \lambda_2^P$ and $\lambda_1^{P_j} > \lambda_2^P$ for all $j \neq i$ or*

(2.) *$\lambda_1^{P_j} \geqq \lambda_2^P$ for all j, $1 \leqq j \leqq n$.*

Proof of Theorem 2.1. Let $Q \subset V \backslash P$ such that $G(V') = T$ is a connected tree where $V' = V \backslash (P \cup Q)$.

1. If $d_T(p) = 1$ for all $p \in V'$ then $|V'| = 2$. Choose $p \in V'$ and let f be an eigenfunction of M_P such that $f = 0$ on $Q \cup P \cup \{p\}$. Then it is easily seen that $f = 0$ on V.

2. $\mathbf{T} = \{p \in V' : d_T(p) \geqq 2\}$ is non void.

To each $p \in \mathbf{T}$ we associate $d_T(p) = l$ pairs of subsets $(P_1, V_1), \ldots, (P_l, V_l)$ in the following way:

Let $V' - \{p\} = W_1 \sqcup \ldots \sqcup W_l$ where $G(W_i)$ are the connected components of $G(V' - \{p\})$.

Define $V_i = \{p \in V: p \text{ is adjacent to a point of } W_i\} \cup P$ and $P_i = V_i \backslash W_i$.

We claim that for all $i \neq j (P_i, V_i), (P_j, V_i)$ verify properties (1) to (6). Let us verify (3) and (5):

(3) $i \neq j$, $V_i \cap V_j \subset P_i \cap P_j$: Let $x \in V_i \cap V_j$. If $x \in P$ then $x \in P_i \cap P_j$ by construction. If $x \notin P$ there exist $x_i \in W_i$ and $x_j \in W_j$ such that $x \equiv x_i$ and $x \equiv x_j$. If x were in W_i or W_j then $G(W_i \cup W_j)$ would be connected. This is a contradiction. Thus $x \in P_i \cap P_j$.

(5) $P_i \supset \partial V_i$: by definition $V_i \backslash \partial V_i \supset W_i$.

Remark that a similar argument than in (3) shows that for all i: $\{p\} = P_i \cap V'$. Let us say that a point $p \in \mathbf{T}$ has property (*) if there exists $1 \leqq i \leqq l$ such that:

$$\lambda_1^{P_i}(G_i) < \lambda_2^P(G), \qquad G_i = G(V_i),$$

and thus $\lambda_1^{P_j}(G_j) > \lambda_2^P(G)$ for all $j \neq i$.

[Recall that P_i, V_i, l depend on p].

We distinguish two cases:

1. There is a point $p \in \mathbf{T}$ not having property (*).
2. All points in $\mathbf{T}$ have property (*).

First case. Take $p \in \mathbf{T}$ not having property (*). The corollary of Lemma 2 implies that

$$\lambda_1^{P_i}(G_i) \geqq \lambda_2^P(G) \qquad \text{for all } i, \ 1 \leqq i \leqq l.$$

Let $A_p = \{q \in V', q \equiv p\}$ and choose a point $q \in A_p$. Suppose $f \in \mathbb{R}_P[V]$ is an eigenfunction of M_P of eigenvalue $\lambda_2^P(G)$ and f is zero on $Q \cup \{p\} \cup (A_p - \{q\})$. Then we have $0 = \lambda_2 f(p) = \sum_{x \equiv p} l(\{p, x\})(f(p) - f(x))$ which implies that $f(q) = 0$.

Remark also that since $P_i \cap V' = \{p\}$, $f = 0$ on P_i and $f|_{V_i} \in \mathbb{R}_{P_i}[V_i]$ is an eigenfunction of M_{P_i} of eigenvalue $\lambda_2^P(G) \leqq \lambda_1^{P_i}(G_i)$. Moreover $A_p \cap (V_i \backslash P_i) \neq \emptyset$ and $f|_{A_p} = 0$ together with Lemma 1 implies that $f = 0$ on V_i for all i. Thus f is identically 0. This shows that

$$\operatorname{mult} \lambda_2^P(G) \leqq |Q| + d_T(p).$$

Second case. All points in $\mathbf{T}$ have property (*). To each $p \in \mathbf{T}$ we associate a unique point $p' \in V'$ in the following way: since p has property (*) there is a unique component $G(W_i)$ of $G(V'-\{p\})$ such that $\lambda_1^{P_i}(G_i) < \lambda_2^P(G)$. In W_i there is a unique point $p' \equiv p$. Now given $p \in \mathbf{T}$ we construct a maximal path in V':

$$p_0 = p, p_1, p_2, \ldots, p_r$$

with the following properties: $p_i = p'_{i-1}$, $1 \leqq i \leqq r$ all p_i are distinct and $p_i \in \mathbf{T}$ for all i, $0 \leqq i \leqq r$.

Now we make the following remark: let $p \in \mathbf{T}$ and f be an eigenfunction of M_P of eigenvalue $\lambda_2^P(G)$. If f is zero on $Q \cup \{p\}$ then $f|_{P_j} = 0$ for all j and f is eigenfunction of M_{P_j}. Thus $f|_{V_j} = 0$ for all j for which $\lambda_1^{P_j}(G_j) > \lambda_2^P(G)$. By definition of p' this implies that f is zero on all components of $G(V'-\{p\})$ which do not contain p'. In particular $f(q) = 0$ for all $q \equiv p$, $q \neq p'$. Since $f(p) = 0$ this implies that $f(p') = 0$.

The maximality of the path has the following consequence. There are two cases:

1. $p'_r \notin \mathbf{T}$: This means that the component of $V'-\{p_r\}$ containing p'_r is $\{p'_r\}$. From the preceeding discussion it follows that if f is zero on $Q \cup \{p_r\}$ then f is identically zero.

2. $p'_r = p_{r-1}$: If f is zero on $Q \cup \{p_{r-1}\}$ then f is zero on all components of $G(V'-\{p_{r-1}\})$ not containing p_r and $f(p_r) = 0$. But then f is zero on all components of $G(V \setminus \{p_r\})$ not containing p_{r-1}. Since $G(V')$ is a tree this implies that f is zero on V' hence f is identically zero.

In both cases we obtain mult $\lambda_2^P(G) \leqq |Q| + 1$. Q.E.D.

Let $\mathscr{G} = (V, E)$ be a connected graph. We show now how to construct a set $Q \subset V$ such that $G(V \setminus Q)$ is a connected tree. Doing this we prove:

Lemma 3. $\alpha(\mathscr{G}) \leqq d_{\mathscr{G}} + \beta_1(\mathscr{G})$.

Proof. Let $T = (V, E')$ be a maximal tree in $\mathscr{G}$ and $d: V \to \mathbb{N}$ the degree function of $\mathscr{G}$. Let $W_i = \{v \in V: d_T(v) = 1,\ d(v) > 1\}$ and W'_1 a maximal subset of points in W_1 that are pairwise non-adjacent in $\mathscr{G}$. Let $V_1 = V \setminus W'_1$, $\mathscr{G}_1 = G(V_1)$ the subgraph of $\mathscr{G}$ generated by V_1 and T_1 the subtree of T generated by V_1. Then T_1 is connected and is a maximal tree in $\mathscr{G}_1$. Let E_1 resp. E'_1 be the edge set of $\mathscr{G}_1$ resp. T_1. Then:

$$E_1 = E \setminus \{\text{all edges issued from points in } W'_1\}$$
$$E'_1 = E' \setminus \{\text{all edges in } E' \text{ issued from } W'_1\}.$$

Thus

$$\beta_1(\mathscr{G}_1) = |E_1| - |E'_1| = \beta_1(\mathscr{G}) - \sum_{v \in W'_1} (\mathrm{d}(v) - 1).$$

In particular $\beta_1(\mathscr{G}_1) \leqq \beta_1(\mathscr{G}) - |W'_1|$.

We can apply the same procedure to $(\mathscr{G}_1, T_1)$ and so on to get after a finite number of steps a pair $(\mathscr{G}_n, T_n)$ such that $\mathscr{G}_n = T_n$ and such that if $V_n \subset V$ is the vertex set of $\mathscr{G}_n$ we have:

$$0 = \beta_1(\mathscr{G}_n) \leqq \beta_1(\mathscr{G}) - |V \setminus V_n|.$$

Then $Q = V \setminus V_n$ satisfies obviously the property that $G(V \setminus Q)$ is a connected tree. Moreover $|Q| \leqq \beta_1(\mathscr{G})$. Q.E.D.

2.2 The second method

Let $G=(V, E, m, l)$ be a weighted graph and $P\subset V$ such that $G(V\backslash P)$ is connected. Let $f\in\mathbb{R}_P[V]$ be an eigenfunction of M_P of eigenvalue λ. For $q\in V$ we denote by A_q the set of $p\in V$ adjacent to q. If f is zero on A_q then:

$$\lambda f(q)=\frac{1}{m(q)} f(q) \sum_{q\equiv p} l(\{p, q\})$$

which implies $f(q)=0$ unless:

$$\lambda=\frac{1}{m(q)} \sum_{p\equiv q} l(\{p, q\}).$$

From this follows that if

$$B=\left\{q\in V\backslash P: \lambda_2^P(G)=\frac{1}{m(q)} \sum_{p\equiv q} l(\{p, q\})\right\}$$

and if S is a subset of $V\backslash(P\cup B)$ of pairwise non-adjacent points then

$$\text{mult } \lambda_2^P(G)\leqq |V\backslash P|-|S|+|B|.$$

Here is an application of this inequality in the case of ordinary graphs.

Proposition 2.1. *Let $\mathscr{G}=(V, E)$ be a connected regular graph of degree r on N vertices and*

$$0=\lambda_1<\lambda_2\leqq \dots \leqq\lambda_N$$

be the eigenvalues of $rI-A$ where A is the adjacency matrix of $\mathscr{G}$. Then

$$\text{mult } \lambda_2\leqq N\left(1-\frac{1}{r}\right)+\frac{1}{r}.$$

Proof. If $B\neq\emptyset$ then $\lambda_2=r$. But $\sum_{i=0}^{N}\lambda_i=N\cdot r$ thus $\lambda_i=r$ for all $1\leqq i\leqq N$. From this follows that $\mathscr{G}$ is the complete graph on $r+1=N$ vertices and the inequality $\text{mult } \lambda_2\leqq N\left(1-\frac{1}{r}\right)+\frac{1}{r}$ holds.

If $B=\emptyset$ then it follows from the discussion above and Lemma 6 (see below) that:

$$\text{mult } \lambda_2\leqq |V|-|S|\leqq |V|\left(1-\frac{1}{r}\right)+\frac{1}{r}. \quad \text{Q.E.D.}$$

To obtain a better estimate we study the structure of B:

Lemma 4. *Let B be as defined above. B is of diameter at most 2 and there are two cases*: (1) *$G(B)$ is connected. Fix $x_1, x_2 \in B$ with $x_1 \equiv x_2$. Then there are 3 types of vertices*:

$$A_1 = \{x \in B: x \equiv x_1, x \not\equiv x_2, x \neq x_2\}$$
$$A_2 = \{x \in B: x \equiv x_2, x \not\equiv x_1, x \neq x_1\}$$
$$A_3 = \{x \in B: x \equiv x_1 \text{ and } x \equiv x_2\}$$

and $A_1 \sqcup A_2 \sqcup A_3 \sqcup \{x_1, x_2\} = B$, for all $x \in A_1, y \in A_2$ we have $x \equiv y$, for all $x \in A_3$ and $y \in A_1 \sqcup A_2$ we have $x \equiv y$.

(2) *$G(B)$ is not connected: for all $x, y \in B$ $\operatorname{dist}(x, y) = 2$ and any $z \in V \setminus P$ adjacent to some point in B is adjacent to all of them.*

Proof. 1.) If for all, $x, y \in B\, x \equiv y$ then B is of the form (1) with $A_1 = A_2 = \emptyset$.

2.) Suppose that there exist $x, y \in B$ such that $\operatorname{dist}(x, y) \geqq 2$. Let F be a nonzero eigenfunction of M_P of eigenvalue $\lambda_1^P(G)$ and let $f = c_1 \delta_x + c_2 \delta_y\, c_1 \neq 0$, $c_2 \neq 0$ such that f is orthogonal to F. Then $Q(f) = \lambda_2^P(G) \|f\|^2$ as one verifies easily. From this follows that f is an eigenfunction of M_P of eigenvalue $\lambda_2^P(G)$. In particular if $z \in V \setminus P$ is adjacent to x then

$$0 = \lambda_2^P f(z) = -\frac{1}{m(z)} \sum_{t \equiv z} l(t, z\}) f(t).$$

This forces z to be adjacent to y and thus $\operatorname{dist}(x, y) = 2$.

This shows that for $x, y \in B$ either $x \equiv y$ or $\operatorname{dist}(x, y) = 2$ in which case any $z \in V \setminus P$ adjacent to x is also adjacent to y. (1) and (2) are then immediate consequences of this property. Q.E.D.

Let $\mathscr{G} = (V, E)$ be a connected graph. We want to choose a subset $S \subset V$ of pairwise nonadjacent points in an optimal way. Let $S \subset V$ be any such subset. On S we put the following graph structure: for $s, t \in S$ there is a edge between s and t if and only if $A_s \cap A_t \neq \emptyset$. We denote $\operatorname{st}(S)$ the graph obtained in this way. A straightforward induction on the number of vertices in $\mathscr{G}$ shows the following:

Lemma 5. *Let $\mathscr{G} = (V, E)$ be a connected graph and $p \in V$. Then there exists a maximal set of pairwise non-adjacent vertices $S \subset V$ such that* $\operatorname{st}(S)$ *is connected.*

The following consequence is important for us:

Lemma 6. *Let $\mathscr{G} = (V, E)$ be a connected graph and $p \in V$. Then there exists a maximal set of pairwise non-adjacent vertices $S \subset V$ such that*:

$$\sum_{p \in S} d(p) \geqq |V| - 1.$$

Proof. Let S be the set given by Lemma 5 and e the number of edges of $\operatorname{st}(S)$. Since $\operatorname{st}(S)$ is connected we have $e \geqq |S| - 1$. Thus

$$\sum_{p \in S} d(p) \geqq e + |V| - |S| \geqq |V| - 1$$

since each point $q \in V \setminus S$ is adjacent to at least one point in S and is counted twice when it is in $A_s \cap A_t$ for some $s, t \in S$. Q.E.D.

Proof of Theorem 2.2. Let $G=(V,E,m,l)$, $P\subset V$, $d\colon V\to\mathbb{N}$ satisfy the hypotheses of Theorem 2.2. Let $V_0=V\setminus P$ and $B\subset V_0$ such that

$$B=\left\{x\in V_0 : \lambda_2^P(G)=\frac{1}{m(x)}\sum_{y\equiv x} l(x,y\})\right\}.$$

Given $A\subset V$ we set $m(A)=\sum_{x\in A} m(x)$.

Let $V_0\setminus B=V_1\sqcup\ldots\sqcup V_n$ such that the graphs $G(V_i)$, $1\leqq i\leqq n$, are the connected components of $G(V_0\setminus B)$.

Let $|V_i|\geqq 2$ for $1\leqq i\leqq k$ and $|V_i|=1$ for $k+1\leqq i\leqq n$. For each i, $1\leqq i\leqq k$ we choose one point $p_i\in V_i$ such that p_i is adjacent to some point in B. We let $S_i\ni p_i$ be the subset of V_i given by Lemma 6.

Let now $f\in\mathbb{R}_P[V]$ be an eigenfunction of M_P of eigenvalue $\lambda_2^P(G)$ and suppose that f is zero on

$$B\cup\bigcup_{i=1}^{k}(V_i\setminus S_i)$$

then it is easily seen that f is identically zero. From this follows that:

$$(*)\qquad \operatorname{mult}\lambda_2\leqq|B|+\sum_{i=1}^{k}(|V_i|-|S_i|)$$

we first estimate $|V_i|-|S_i|$ for $1\leqq i\leqq k$:

let d_i be the degree function of the graph $G(V_i)$ and r_i the number of points in B adjacent to p_i. Then it follows from Lemma 6 that:

$$\sum_{p\in S_i} d(p)\geqq r_i+\sum_{p\in S_i} d_i(p)\geqq|V_i|+r_i-1.$$

From this and the inequality $m(x)\geqq d(x)-2$ we obtain:

$$m(S_i)\geqq|V_i|-2|S_i|+r_i-1.$$

On the other hand we have the obvious inequality:

$$|V_i|-|S_i|+m(S_i)\leqq m(V_i).$$

This and the preceeding inequality imply:

$$(**)\qquad 2(|V_i|-|S_i|)\leqq m(V_i)+|S_i|-(r_i-1).$$

Now we estimate $|B|$. According to Lemma 4 we distinguish two cases:

1.) B is connected: $B=A_1\sqcup A_2\sqcup A_3\sqcup\{x_1,x_2\}$
then we have:

$$\begin{aligned} m(B)&\geqq \mathrm{d}(x_1)-2+\mathrm{d}(x_2)-2+(|B|-2)\\ &\geqq(|A_1|+|A_3|-1)+(|A_2|+|A_3|-1)+(|B|-2)\\ &\geqq 2|B|-6\end{aligned}$$

hence:

$$|B| \leqq \tfrac{1}{2} m(B) + 3.$$

2.) B is not connected: in particular $|B| \geqq 2$. Because of Lemma 4 any point adjacent to one point in B is adjacent to all of them and this implies that $r_1 = |B|$ for $1 \leqq i \leqq k$ and that $\mathrm{d}(x) \geqq k-2$ for all $x \in B$. In particular $m(B) \geqq |B|(k-2)$.

Now we put (**) into the inequality (*) and obtain:

$$\text{mult}\, \lambda_2 \leqq |B| + \frac{1}{2} \sum_{i=1}^{k} m(V_i) + \frac{1}{2} \sum_{i=1}^{k} |S_i| - \sum_{i=1}^{k} \left(\frac{r_i - 1}{2} \right).$$

In case 1.) we use that $r_i \geqq 1$ and $|B| \leqq \frac{1}{2} m(B) + 3$ to obtain:

$$\text{mult}\, \lambda_2 \leqq \frac{1}{2} m(V_0) + 3 + \frac{1}{2} \sum_{i=1}^{k} |S_i|.$$

In case 2.):

$$|B| - \sum_{i=1}^{k} \left(\frac{r_i - 1}{2} \right) = |B| \left(1 - \frac{k}{2} \right) + \frac{k}{2}.$$

If $k \geqq 2$ this is smaller than $\frac{1}{2}|B|(k-2) + 1 \leqq \frac{1}{2} m(B) + 1$. If $k = 1$ this equals $\frac{|B|}{2} + \frac{1}{2} \leqq \frac{1}{2} m(B) + \frac{1}{2}$.

Thus:

$$\text{mult}\, \lambda_2 \leqq \frac{1}{2} m(V_0) + 1 + \frac{1}{2} \sum_{i=1}^{k} |S_i|.$$

In any case we obtain:

$$\text{mult}\, \lambda_2 \leqq \frac{1}{2} m(V_0) + 3 + \frac{1}{2} \sum_{i=1}^{k} |S_i|.$$

On the other hand it follows immediately from (*) that:

$$\frac{1}{2} \text{mult}\, \lambda_2 \leqq \frac{1}{2} m(V_0) - \frac{1}{2} \sum_{i=1}^{k} |S_i|.$$

Adding this to the preceeding inequality we obtain:

$$\text{mult}\, \lambda_2 \leqq \tfrac{2}{3} m(V_0) + 2. \quad \text{Q.E.D.}$$

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