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Kazhdan constants for SL(3, $\mathbb{Z}$)

By *M. Burger* at Stanford

Introduction

This work presents a partial answer to a question raised by J. P. Serre in [HV], Chap. 1, § 17, namely to what extent it is possible to give a quantitative version of property (T) for SL(3, $\mathbb{Z}$). We remind the reader that this property is equivalent to the following assertion:

Let S be a finite set of generators of SL(3, $\mathbb{Z}$). Then there exists an $\varepsilon>0$ such that for all unitary representations π of SL(3, $\mathbb{Z}$) in a Hilbert space $\mathscr{H}_\pi$ with no nonzero invariant vectors and for all $\xi \in \mathscr{H}_\pi$ one has the inequality:

$$\max_{\gamma \in S} \|\pi(\gamma)\xi - \xi\| \geqq \varepsilon \|\xi\|. \tag{1}$$

Here we will obtain explicit examples of pairs (S, ε) satisfying (1) for all permutation representations of SL(3, $\mathbb{Z}$). Let us now describe two corollaries of our results more precisely.

A. If π is an irreducible finite dimensional unitary representation of SL(3, $\mathbb{Z}$) we know ([Ste]) that there exists $N \in \mathbb{N}$ such that π factors through the principal congruence subgroup

$$\Gamma(N) = \{\gamma \in \mathrm{SL}(3, \mathbb{Z}) : \gamma \equiv \mathrm{Id} \bmod (N)\}.$$

We denote by N_π the smallest integer for which π has this property. Moreover we suppose that π is nontrivial, thus $N_\pi \geqq 2$.

Let

$$S = \left\{ \begin{pmatrix} 1 & 2 & j \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & j \\ 0 & 0 & 1 \end{pmatrix}, j = -1, 0, 1 \text{ and the transposed inverses of these matrices} \right\}.$$

With these notations and definitions we have:

$$\max_{\gamma \in S} \frac{\|\pi(\gamma)\xi - \xi\|}{\|\xi\|} \geqq \left[2 - 2\sqrt{1 - \left(\frac{1 - \sqrt{n_\pi}^{-1}}{4}\right)^2}\right]^{1/2} \tag{2}$$

for all $\xi \in \mathscr{H}_\pi$, where n_π is the product of all distinct primes dividing N_π. Note that the right hand side of the inequality is bigger than

$$\frac{1}{4}\left(1 - \frac{1}{\sqrt{n_\pi}}\right).$$

The inequality (2) will be a consequence of Proposition 3, § 2 and Proposition 5, § 4.

As stated before, all irreducible finite dimensional unitary representations of SL(3, $\mathbb{Z}$) occur in the action of SL(3, $\mathbb{Z}$) on $L^2(\mathrm{SL}(3,\mathbb{Z})/\Gamma)$, where Γ is some subgroup of finite index in SL(3, $\mathbb{Z}$). For the infinite index case we have the following result.

B. Let π be the permutation representation of SL(3, $\mathbb{Z}$) on $L^2(\mathrm{SL}(3,\mathbb{Z})/\Gamma)$ where Γ is a subgroup of infinite index. Then we have

$$\max_{\gamma \in S} \frac{\|\pi(\gamma)\xi - \xi\|}{\|\xi\|} \geqq \frac{1}{[8 + 2\sqrt{15}]^{1/2}} \tag{3}$$

for all $\xi \in L^2(\mathrm{SL}(3,\mathbb{Z})/\Gamma)$, and S is as before. Note that the right hand side of (3) is bigger than 1/4. This inequality will be a consequence of Proposition 3, § 2 and Proposition 4, § 4.

Example. Let ϱ be an irreducible non-trivial representation of SL(3, $\mathbb{R}$) in a real vector space V of dimension n and choose a basis of V such that $\varrho(\mathrm{SL}(3,\mathbb{Z})) \subset \mathrm{SL}(n,\mathbb{Z})$. In this way we obtain an (ergodic) action of SL(3, $\mathbb{Z}$) on the n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ with the following property:

$$\max_{\gamma \in S} \|f \circ \varrho(\gamma) - f\|_2 \geqq \frac{1}{[8 + 2\sqrt{15}]^{1/2}} \|f\|_2 \tag{4}$$

for all $f \in L^2(T^n)$ such that:

$$\int_{T^n} f(x)\, dm(x) = 0$$

where m is the Lebesgue measure on T^n. Inequality (4) follows immediately from (3) by considering the unitary representation of SL(3, $\mathbb{Z}$) in

$$\mathscr{H} = \{f \in L^2(T^n) : \int_{T^n} f(x)\, dm(x) = 0\}$$

which is equivalent via Fourier transform to a sum of representations of the above type B. Indeed, the contragredient of ϱ has no finite orbit on $\mathbb{Z}^n \setminus \{0\}$.

Outline of the paper. In the first paragraph we establish a quantitative version of the relative property (T) for the pair $(H, \mathbb{Z}^2)$ where $H = \mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ is the semi-direct product for the natural action of $\mathrm{SL}(2, \mathbb{Z})$ on $\mathbb{Z}^2$. In this case our result will include all unitary representations of H. The bounds we obtain will depend on a quantity that measures the extent to which a given set of elements in $\mathrm{SL}(2, \mathbb{Z})$ does not fix any probability measure when acting on the real projective line. This quantity will be estimated in § 4.

In the second paragraph we consider a subgroup $R < \mathrm{SL}(3, \mathbb{Z})$ isomorphic to $\mathrm{SL}(2, \mathbb{Z})$ that normalizes two subgroups A_+, A_- both isomorphic to $\mathbb{Z}^2$. We can then apply the result of § 1 to the groups RA_+ and RA_- to get a lower bound for

$$\max_{\gamma \in S} \frac{\|\pi(\gamma)\xi - \xi\|}{\|\xi\|}$$

in terms of the angle between the subspaces of A_+ and A_- invariants vectors.

In § 3 we obtain an estimate of this angle for all permutation representations of $\mathrm{SL}(3, \mathbb{Z})$.

In § 5 we will make a slight disgression on the relative property in general. Namely we are interested in the following question: Given a semidirect product $A \rtimes H$ where A is abelian, what is a sufficient condition on the action of H on the dual $\hat{A}$ such that the pair $(A \rtimes H, A)$ has the relative property (T).

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1. The relative property

Let G be a locally compact group and L a closed subgroup of G. According to Margulis ([Ma]) the pair (G, L) is said to have the relative property (T) if any continuous unitary representation that has almost invariant vectors has nonzero L-invariant vectors.

If G is a finitely generated group this means that for any finite set S of generators of G there exists $\varepsilon > 0$ such that for every unitary representation π of G with no nonzero L-invariant vectors and all $\xi \in \mathscr{H}_\pi$ one has the inequality:

$$\max_{g \in S} \|\pi(g)\xi - \xi\| \geqq \varepsilon \|\xi\|. \tag{5}$$

We will now construct examples of (S, ε) satisfying (5) for the pair

$$(\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z}), \mathbb{Z}^2).$$

To state the result we need to introduce the following invariant. Let F be a finite set of homeomorphisms of the real projective line $P^1(\mathbb{R})$. We define

$$\alpha(F) = \inf_{\mu \in M^1} \sup_{B \subset P^1(\mathbb{R})} \max_{\gamma \in F} |\mu(\gamma B) - \mu(B)|$$

where M^1 is the space of probability measures on $P^1(\mathbb{R})$ and the supremum is taken over all Borel subsets $B \subset P^1(\mathbb{R})$. Note that $\alpha(F) = 0$ if and only if there exists an F-invariant probability measure on $P^1(\mathbb{R})$. If $F \subset \mathrm{SL}(2, \mathbb{Z})$ this happens if and only if the subgroup generated by F is almost cyclic.

Example (see § 4).

$$\alpha(F) \geqq \frac{1}{4} \quad \text{if} \quad F = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\},$$

$$\alpha(F) \geqq \frac{1}{2} \quad \text{if} \quad F = \left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\}.$$

Given a subset $F \subset \mathrm{SL}(2, \mathbb{Z})$ and a measurable fundamental domain D for the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ we define the following subset of $H = \mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$:

$$P = \{(x, \gamma) \in \mathbb{Z}^2 \times F : m([x + \gamma(D)] \cap D) \neq 0\}$$

where m is the Lebesgue measure on $\mathbb{R}^2$. With these definitions and notations we want to prove the following proposition:

Proposition 1. *Let π be a unitary representation of $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ in a Hilbert space $\mathscr{H}_\pi$ and let q be the orthogonal projection of $\mathscr{H}_\pi$ on the space of $\mathbb{Z}^2$-invariant vectors.*

Let $\xi \in \mathscr{H}_\pi$ and $\delta \geqq 0$ such that:

$$\langle q(\xi) | \xi \rangle \leqq (1 - \delta) \|\xi\|^2.$$

Then:

$$\min_{\gamma \in P} \mathrm{Re} \langle \pi(\gamma) \xi | \xi \rangle \leqq \sqrt{1 - \alpha(F)^2 \delta^2} \, \|\xi\|^2.$$

Corollary. *Let π be a unitary representation of $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ with no nonzero $\mathbb{Z}^2$-invariant vectors. Then for all $\xi \in \mathscr{H}_\pi$ we have:*

$$\max_{\gamma \in P} \|\pi(\gamma) \xi - \xi\| \geqq [2 - 2\sqrt{1 - \alpha(F)^2}]^{1/2} \, \|\xi\|.$$

Remark. The proof below shows in fact that if Γ is any subgroup of $\mathrm{SL}(2, \mathbb{Z})$ which is not almost cyclic then the pair $(\mathbb{Z}^2 \rtimes \Gamma, \mathbb{Z}^2)$ has the relative property (T).

Examples. Suppose that π is a unitary representation of $\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ with no nonzero $\mathbb{Z}^2$-invariant vector. Set $n=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$, $\bar{n}=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$:

1) $F=\{n,\bar{n}\}$ and $D=(-1/2,1/2)\times(-1/2,1/2)$ then

$$P=\bigcup_{j=-1,0,1}\left\{\left(\begin{pmatrix}j\\0\end{pmatrix},n\right),\left(\begin{pmatrix}0\\j\end{pmatrix},\bar{n}\right)\right\},$$

$$\alpha(F)\geqq 1/4 \text{ (see § 4)}$$

and

$$\max_{s\in P}\|\pi(s)\xi-\xi\|\geqq\frac{1}{[8+2\sqrt{15}]^{1/2}}\|\xi\|$$

for all $\xi\in\mathscr{H}_\pi$.

2) $F=\{n^2,\bar{n}^2\}$ and D is as in example 1) then

$$P=\bigcup_{j=-1,0,1}\left\{\left(\begin{pmatrix}j\\0\end{pmatrix},n^2\right),\left(\begin{pmatrix}0\\j\end{pmatrix},\bar{n}^2\right)\right\},$$

$$\alpha(F)\geqq 1/2 \text{ (see § 4)}$$

and

(6) $$\max_{s\in P}\|\pi(s)\xi-\xi\|\geqq[2-\sqrt{3}]^{1/2}\|\xi\|$$

for all $\xi\in\mathscr{H}_\pi$.

Remark. $F=\{n^2,\bar{n}^2\}$, $D=(0,1)\times(0,1)$ then

$$P=\bigcup_{j=0,1,2}\left\{\left(\begin{pmatrix}j\\0\end{pmatrix},n^2\right),\left(\begin{pmatrix}0\\j\end{pmatrix},\bar{n}^2\right)\right\}.$$

For this P Gabber and Galil derived ([GG]) the inequality (6) (with the same constant) for all unitary representations of $\mathbb{Z}^2\rtimes\mathrm{SL}(2,\mathbb{Z})$ obtained by letting this group act on the subspace of functions in $l^2((\mathbb{Z}/m\mathbb{Z})^2)$ orthogonal to the constants ($m\in\mathbb{N}$).

To prove Proposition 1 we first need to establish the corresponding assertion for the pair $(\mathbb{R}^2\rtimes\mathrm{SL}(2,\mathbb{Z}),\mathbb{R}^2)$.

Proposition 2. *Let ω be a continuous unitary representation of $\mathbb{R}^2\rtimes\mathrm{SL}(2,\mathbb{Z})$ in a Hilbert space $\mathscr{H}_\omega$ and Q the orthogonal projection of $\mathscr{H}_\omega$ on the space of $\mathbb{R}^2$-invariant vectors.*

For all $\xi \in \mathcal{H}_\omega$ and $\delta \geqq 0$ with

$$\langle Q(\xi) | \xi \rangle \leqq (1-\delta) \, \|\xi\|^2$$

we have:

$$\min_{s \in F} |\mathrm{Re} \langle \omega(s) \xi | \xi \rangle| \leqq \sqrt{1-\alpha(F)^2 \delta^2} \, \|\xi\|^2.$$

Proof. Let P be the spectral measure of the restriction of ω to $\mathbb{R}^2$. This is a map which associates to each Borel set $B \subset \mathbb{R}^2$ an orthogonal projection $P(B)$ of $\mathcal{H}_\omega$ which has (among others) the following properties:

(a) $P(\mathbb{R}^2) = \mathrm{Id}$.

(b) $B \to \langle P(B) \xi | \xi \rangle$ is a positive measure on $\mathbb{R}^2$ of total mass $\|\xi\|^2$, for any $\xi \in \mathcal{H}_\omega$.

(c) $Q = P(\{0\})$.

(d) $P(sB) = \omega(s)^{-1} P(B) \, \omega(s), \; s \in \mathrm{SL}(2, \mathbb{Z})$.

Let now $\xi \in \mathcal{H}_\omega$, $\|\xi\| = 1$ and $\langle Q(\xi) | \xi \rangle \leqq 1 - \delta$.

Consider the restriction of the probability measure $B \to \langle P(B) \xi | \xi \rangle$ to $\mathbb{R}^2 - \{0\}$ and let μ be its direct image on $P^1(\mathbb{R})$ via the canonical projection:

$$p: \mathbb{R}^2 - \{0\} \to P^1(\mathbb{R}).$$

By hypothesis on ξ we have $\mu(P^1(\mathbb{R})) \geqq \delta$. For $s \in F$, $B \subset P^1(\mathbb{R})$ and $B' = p^{-1}(B)$ we have:

$$\begin{aligned} \mu(sB) - \mu(B) &= \langle P(sB') \xi | \xi \rangle - \langle P(B') \xi | \xi \rangle \\ &= \langle P(B') \, \omega(s) \xi | \omega(s) \xi \rangle - \langle P(B') \xi | \xi \rangle \\ &= \mathrm{Re} \langle P(B') \, (\omega(s) \xi + \xi) | (\omega(s) \xi - \xi) \rangle. \end{aligned}$$

Note that if A is an orthogonal projection and ζ, η vectors with $\|\zeta\| = \|\eta\|$ we have:

$$|\mathrm{Re} \langle A(\eta - \zeta) | (\eta + \zeta) \rangle| = \frac{1}{2} \, |\mathrm{Re} \langle (2A - I) \, (\eta - \zeta) | (\eta + \zeta) \rangle| \leqq \frac{1}{2} \, \|\eta - \zeta\| \, \|\eta + \zeta\|$$

using $\mathrm{Re} \langle \eta - \zeta | \eta + \zeta \rangle = 0$ and the fact that $2A - I$ is unitary.

This implies:

$$|\mu(sB) - \mu(B)| \leqq \frac{1}{2} \, \|\omega(s) \xi + \xi\| \, \|\omega(s) \xi - \xi\| = \sqrt{1 - (\mathrm{Re} \langle \omega(s) \xi | \xi \rangle)^2}.$$

Taking the supremum over s and B and the minimum over all μ with $\mu(P^1(\mathbb{R})) \geqq \delta$ one obtains the result. Q.E.D.

Remark. The idea of the proof of Proposition 2 is taken from Furstenberg's proof of the relative property of ($\mathbb{R}^2 \rtimes \mathrm{SL}(2, \mathbb{R})$, $\mathbb{R}^2$). See [HV], Chap. 2, Proposition 2.

Proof of Proposition 1. To derive Proposition 1 from Proposition 2 we relate the coefficients $\langle \pi(\gamma)\xi \mid \xi\rangle$ of the representation π of $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ to coefficients of ω, the representation of $\mathbb{R}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ induced by π.

More precisely, let E be the quotient of $\mathbb{R}^2 \times \mathscr{H}_\pi$ by the following action of $\mathbb{Z}^2$:

$$x(y, \xi) = (x + y, \pi(x, I)\xi), \qquad x \in \mathbb{Z}^2$$

and $\mathscr{H}_\omega$ the space of L^2-sections of the canonical projection:

$$E \longrightarrow \mathbb{R}^2/\mathbb{Z}^2.$$

Then $\mathbb{R}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ acts by measure preserving homeomorphisms on $\mathbb{R}^2/\mathbb{Z}^2$ and thus unitarily on $\mathscr{H}_\omega$. Denote by ω this unitary representation. Given a measurable fundamental domain D for the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$, for each vector $\xi \in \mathscr{H}_\pi$ we have an obvious section of

$$E \longrightarrow \mathbb{R}^2/\mathbb{Z}^2$$

denoted by $\mathscr{I}(\xi)$. This defines a linear map $\mathscr{I}: \mathscr{H}_\pi \to \mathscr{H}_\omega$ which has the following properties:

(a) $\mathscr{I}$ is an isometry.

(b) For all $s \in \mathrm{SL}(2, \mathbb{Z})$ and $\xi \in \mathscr{H}_\pi$ we have:

$$\langle \omega(0, s)\, \mathscr{I}(\xi) \mid \mathscr{I}(\xi)\rangle = \sum_{x \in \mathbb{Z}^2} \langle \pi(x, s)\xi \mid \xi\rangle\, m([x + s(D)] \cap D).$$

(c) Let Q (q resp.) be the orthogonal projection of $\mathscr{H}_\omega$ ($\mathscr{H}_\pi$ resp.) on the subspace of $\mathbb{R}^2$ ($\mathbb{Z}^2$ resp.) invariant vectors. Then $Q\mathscr{I} = \mathscr{I} q$.

The verification of these properties is easy and left to the reader.

Now let $\xi \in \mathscr{H}_\pi$ and $\delta \geqq 0$ such that

$$\langle q(\xi) \mid \xi\rangle \leqq (1 - \delta)\, \|\xi\|^2.$$

Then $\mathscr{I}(\xi)$ verifies:

$$\langle Q\mathscr{I}(\xi) \mid \mathscr{I}(\xi)\rangle = \langle \mathscr{I} q(\xi) \mid \mathscr{I}(\xi)\rangle = \langle q(\xi) \mid \xi\rangle \leqq (1 - \delta)\, \|\mathscr{I}(\xi)\|^2$$

and Proposition 2 applied to ω and $\mathscr{I}(\xi)$ shows that:

$$\min_{s \in F} |\mathrm{Re}\,\langle \omega(s)\, \mathscr{I}(\xi) \mid \mathscr{I}(\xi)\rangle| \leqq \sqrt{1 - \alpha(F)^2 \delta^2}\, \|\mathscr{I}(\xi)\|^2.$$

It follows from the definition of P:

$$P = \{(x, s) \in \mathbb{Z}^2 \times F : m([x + s(D)] \cap D) \neq 0\}$$

and property (b) that:

$$\min_{\gamma \in P} \operatorname{Re} \langle \pi(\gamma) \xi \mid \xi \rangle \leqq \min_{s \in F} \operatorname{Re} \langle \omega(s) \mathscr{I}(\xi) \mid \mathscr{I}(\xi) \rangle \leqq \sqrt{1 - \alpha(F)^2 \delta^2} \, \|\mathscr{I}(\xi)\|^2$$

$$= \sqrt{1 - \alpha(F)^2 \delta^2} \, \|\xi\|^2. \qquad \text{Q.E.D.}$$

2. The case of SL(3, $\mathbb{Z}$)

Let $\mathscr{H}$ be a Hilbert space and U, V two closed subspaces. We define the "cosine" of the angle between U and V to be

$$\beta(U, V) = \sup_{u \in U, v \in V} \frac{\operatorname{Re} \langle u, v \rangle}{\|u\| \, \|v\|}$$

if U and V are nonzero. If $U = \{0\}$ or $V = \{0\}$ we set $\beta(U, V) = 0$.

Remark that if P_U (resp. P_V) denotes the orthogonal projection on U (resp. V) then $\beta(U, V) = \|P_U P_V\|$ where $\| \ \|$ denotes the operator norm.

Consider the following two subgroups of SL(3, $\mathbb{Z}$):

$$A_+ = \left\{ \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{Z}^2 \right\},$$

$$A_- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x_1 & x_2 & 1 \end{pmatrix} : {}^t(x_1, x_2) \in \mathbb{Z}^2 \right\}$$

which together generate SL(3, $\mathbb{Z}$).

For a unitary representation π of SL(3, $\mathbb{Z}$) in $\mathscr{H}$ and for the subspace V_+ (V_- resp.) of vectors fixed under A_+ (A_- resp.) we set $\beta_\pi = \beta(V_+, V_-)$. In this paragraph we shall prove a lower bound on

$$\max_{\gamma \in S} \frac{\|\pi(\gamma) \xi - \xi\|}{\|\xi\|}$$

in terms of β_π, for suitable subsets $S \subset$ SL(3, $\mathbb{Z}$).

Construction of S. The following subgroup of SL(3, $\mathbb{Z}$) normalizes A_+ and A_-:

$$R=\left\{\begin{pmatrix} & g & & 0\\ & & & 0\\ 0 & & 0 & 1\end{pmatrix} : g\in \mathrm{SL}(2,\mathbb{Z})\right\}$$

and the subgroups RA_+ and RA_- are isomorphic to $\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ via the isomorphisms

$$\varphi_+\begin{pmatrix} g & & x\\ 0 & 0 & 1\end{pmatrix}=(x,g),$$

$$\varphi_-\begin{pmatrix} & g & 0\\ & & 0\\ & {}^tX & 1\end{pmatrix}=(-{}^tg^{-1}x, {}^tg^{-1}).$$

Given a subset $F\subset \mathrm{SL}(2,\mathbb{Z})$, a measurable fundamental domain D for the action of $\mathbb{Z}^2$ on $\mathbb{R}^2$ and P the associated subset of $\mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ (see § 1)

$$P=\{(x,\gamma)\in \mathbb{Z}^2\times F : m([x+\gamma(D)]\cap D)\neq 0\}$$

we define

$$S=\varphi_+^{-1}(P)\cup \varphi_-^{-1}(P).$$

Examples. 1) If $F=\{n,\bar n\}$ and $D=(-1/2,1/2)^2$ then S is the set of matrices

$$\begin{pmatrix}1&1&j\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0\\1&1&j\\0&0&1\end{pmatrix},\qquad j=-1,0,1$$

and the transposed inverses of these matrices ($|S|=12$).

2) If $F=\{n^2,\bar n^2\}$ and $D=\left(-\frac{1}{2},\frac{1}{2}\right)^2$ then S is the set of matrices

$$\begin{pmatrix}1&2&j\\0&1&0\\0&0&1\end{pmatrix},\ \begin{pmatrix}1&0&0\\2&1&j\\0&0&1\end{pmatrix},\qquad j=-1,0,1$$

and the transposed inverses of these matrices ($|S|=12$).

Given these definitions we have:

Proposition 3. *Let π be a unitary representation of* SL(3, $\mathbb{Z}$) *in a Hilbert space $\mathscr{H}_\pi$.*

1) *If $V_+ \neq \{0\}$ and $V_- \neq \{0\}$ we have for all $\xi \in \mathscr{H}_\pi$:*

$$\max_{s \in S} \|\pi(s)\xi - \xi\| \geqq \left\{2 - 2\sqrt{1 - \left[\frac{\alpha(F)(1-\beta_\pi)}{2}\right]^2}\right\}^{1/2} \|\xi\|.$$

2) *If $V_+ = \{0\}$ or $V_- = \{0\}$ we have for all $\xi \in \mathscr{H}_\pi$:*

$$\max_{s \in S} \|\pi(s)\xi - \xi\| \geqq \{2 - 2\sqrt{1 - \alpha(F)^2}\}^{1/2} \|\xi\|.$$

Remarks. 1) It is easy to see that property (T) for SL(3, $\mathbb{Z}$) implies the existence of a constant $c < 1$ such that $\beta_\pi \leqq c$ for all unitary representations π with no nonzero fixed vectors. Proposition 3 shows that the converse is also true.

2) The results A and B stated in the introduction are now direct consequences of Proposition 3 applied to the set S of example 2), together with the bounds on β_π obtained in § 3.

Proof. Let π be a unitary representation of SL(3, $\mathbb{Z}$) in a Hilbert space $\mathscr{H}_\pi$ and q_+ (q_- resp.) the orthogonal projection of $\mathscr{H}_\pi$ on the subspace V_+ (V_- resp.). Let $\xi \in \mathscr{H}_\pi$ with $\|\xi\| = 1$ and define δ_+, δ_- by:

$$\langle q_+(\xi) | \xi \rangle = 1 - \delta_+, \quad \langle q_-(\xi) | \xi \rangle = 1 - \delta_-.$$

Setting $S_+ = \varphi_+^{-1}(P)$, $S_- = \varphi_-^{-1}(P)$, Proposition 1, § 1 implies:

(a) $\min_{g \in S_+} \operatorname{Re} \langle \pi(g)\xi | \xi \rangle \leqq \sqrt{1 - \alpha(F)^2 \delta_+^2}$,

(b) $\min_{g \in S_-} \operatorname{Re} \langle \pi(g)\xi | \xi \rangle \leqq \sqrt{1 - \alpha(F)^2 \delta_-^2}$.

If the left hand side of (a) or (b) is negative then

$$\max_{g \in S} \|\pi(g)\xi - \xi\| > \sqrt{2}\, \|\xi\|$$

and the proposition is proved. If not, then the inequalities (a) and (b) are equivalent to:

$$m_\pm := \sqrt{1 - \min_{g \in S_\pm} (\operatorname{Re} \langle \pi(g)\xi | \xi \rangle)^2} \geqq \alpha(F)\delta_\pm.$$

Thus:

$$\max(m_+, m_-) \geqq \frac{m_+ + m_-}{2} \geqq \alpha(F)\left\{1 - \frac{1}{2}\langle (q_+ + q_-)\xi | \xi \rangle\right\} \geqq \alpha(F)\left(1 - \frac{1}{2}\|q_+ + q_-\|\right).$$

In other words:

$$\min_{g \in S} (\operatorname{Re} \langle \pi(g) \xi | \xi \rangle)^2 \leqq 1 - \alpha(F)^2 \{1 - \frac{1}{2} \|q_+ + q_-\|\}^2.$$

This and the equality $\|q_+ + q_-\| = 1 + \beta_\pi$ (see Lemma 1 below) proves Proposition 3. 1). Proposition 3. 2) is an immediate consequence of the corollary of Proposition 1, § 1.

Q.E.D.

The equality $\|q_+ + q_-\| = 1 + \beta_\pi$ depends only on the Euclidean structure on $\mathscr{H}_\pi$ so that we have to prove the following:

Lemma 1. *Let E be an Euclidean space and q (p resp.) be a nonzero orthogonal projection on a subspace U (V resp.). Then we have:*

$$\|q + p\| = 1 + \beta(U, V).$$

Proof. First a general remark: let $u, v \in E$ $\|u\| = \|v\| = 1$, $w = u + v$, and let P_u (resp. P_v) be the orthogonal projection of E on $\mathbb{R}u$ (resp. $\mathbb{R}v$). Then we have by direct computation:

$$\frac{\langle (P_u + P_v) w | w \rangle}{\|w\|^2} = 1 + \langle u, v \rangle. \tag{7}$$

Suppose now $u \in U$ and $v \in V$ then:

$$\|w\|^2 [1 + \langle u, v \rangle] = \langle (P_u + P_v) w | w \rangle \leqq \langle (p+q) w | w \rangle.$$

This proves that $1 + \beta(U, V) \leqq \|p + q\|$. Now we prove the opposite inequality. If $\dim E = 1$ there is nothing to prove. If $\dim E = 2$ we can suppose that U and V are two distinct lines in E. The two bisectors of U, V are eigenvectors for $p + q$ and the equality (7) shows that the largest eigenvalue is $1 + \beta(U, V)$. Thus $\|p + q\| = 1 + \beta(U, V)$. Now let E be arbitrary. Taking $\xi \in E$, $\|\xi\| = 1$ we consider P (Q resp.) the orthogonal projection of E on $\mathbb{R}p(\xi)$ ($\mathbb{R}q(\xi)$ resp.). Then we have:

$$\langle (p+q) \xi | \xi \rangle = \langle (P+Q) \xi | \xi \rangle \leqq \|P + Q\| = \|PT + QT\|$$

where T is the orthogonal projection on the subspace spanned by $p(\xi)$ and $q(\xi)$. By the preceeding discussion we have:

$$\|PT + QT\| = 1 + \frac{|\langle q(\xi) | p(\xi) \rangle|}{\|q(\xi)\| \, \|p(\xi)\|} \leqq 1 + \beta(U, V)$$

which implies:

$$\|p + q\| \leqq 1 + \beta(U, V).$$

Q.E.D.

Remark (referee). Let $u = 1 - 2p$, $v = 1 - 2q$, u and v are involutions on $\mathscr{H}$ and hence define a unitary representation of the infinite dihedral group D_∞. Via direct integral decomposition one is reduced to prove Lemma 1 for $\dim \mathscr{H} = 1, 2$.

3. The angle estimate

In this paragraph we will estimate β_π in the case where π is a permutation representation of $SL(3, \mathbb{Z})$. We first treat the case of the action of $SL(3, \mathbb{Z})$ on $L^2(SL(3, \mathbb{Z})/\Gamma)$ where Γ is of infinite index in $SL(3, \mathbb{Z})$.

Proposition 4. *Let Γ be a subgroup of infinite index in $SL(3, \mathbb{Z})$ and V_+ (resp. V_-) be the space of A_+ (resp. A_-) invariant functions in $L^2(SL(3, \mathbb{Z})/\Gamma)$. Then V_+ and V_- are orthogonal.*

In other words, if π denotes the representation of $SL(3, \mathbb{Z})$ on $L^2(SL(3, \mathbb{Z})/\Gamma)$ then $\beta_\pi = 0$.

Proof. We will need the following result of J. Tits ([T]): if N is a natural number and Ω is a subgroup of $SL(3, \mathbb{Z})$ containing the N'th power of the 6 elementary matrices $\mathrm{Id} + E_{ij}$ $(i \neq j)$ then Ω contains $\Gamma(N^2)$, in particular Ω is of finite index in $SL(3, \mathbb{Z})$.

Suppose that V_+ and V_- are not orthogonal. Then there is a finite orbit O_+ (O_- resp.) of A_+ (A_- resp.) on $SL(3, \mathbb{Z})/\Gamma$ such that $O_+ \cap O_- \neq \emptyset$. Let Γ' be the stabilizer of $x \in O_+ \cap O_-$. Then Γ' is conjugate to Γ and $\Gamma' \cap A_+$ ($\Gamma' \cap A_-$ resp.) is of finite index in A_+ (A_- resp.). In particular there is an $m \in \mathbb{N}$ such that Γ' contains

$$\begin{pmatrix} 1 & 0 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & m & 1 \end{pmatrix}$$

and thus contains also

$$\begin{pmatrix} 1 & m^2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ m^2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

These two matrices are commutators of some of the preceding ones.

Now Tits' result implies that Γ' (and thus Γ) is of finite index in $SL(3, \mathbb{Z})$. This contradicts the assumption on Γ. Q.E.D.

The rest of this paragraph is devoted to the estimate of β_π for finite dimensional unitary representations of $SL(3, \mathbb{Z})$. Since the discussion will involve various rings, we introduce the following notation: Let A be a commutative ring with identity. Then:

$$A_+(A) = \left\{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} x \\ y \end{pmatrix} \in A \times A \right\},$$

$$A_-(A) = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix} : {}^t(x, y) \in A \times A \right\}.$$

The following is our main result:

Proposition 5. *Let $\mathbb{Z}_p$ be the ring of p-adic integers and π a continuous unitary representation of* SL$(3, \mathbb{Z}_p)$ *with no nonzero fixed vector. Let V_+ (V_- resp.) be the subspace of $A_+(\mathbb{Z}_p)$ ($A_-(\mathbb{Z}_p)$ resp.) invariant vectors. Then*:

$$\beta(V_+, V_-) \leqq \frac{1}{\sqrt{p}}.$$

Corollary. *Let π be an irreducible nontrivial finite dimensional unitary representation of* SL$(3, \mathbb{Z})$. *Then* $\beta_\pi \leqq \frac{1}{\sqrt{n_\pi}}$.

We show first how to deduce the corollary from Proposition 5. For this we need the following lemma:

Lemma 2. *Let E_1, E_2 be finite dimensional Euclidean spaces and V_1, W_1 (resp. V_2, W_2) subspaces of E_1 (resp. E_2). Then*:

$$\beta(V_1 \otimes V_2, W_1 \otimes W_2) = \beta(V_1, W_1)\, \beta(V_2, W_2).$$

Proof.

$$\beta(V_1 \otimes V_2, W_1 \otimes W_2) = \|P_{V_1 \otimes V_2} P_{W_1 \otimes W_2}\|$$

$$= \|(P_{V_1} \otimes P_{V_2})(P_{W_1} \otimes P_{W_2})\| = \|P_{V_1} P_{W_1} \otimes P_{V_2} P_{W_2}\| = \|P_{V_1} P_{W_1}\|\, \|P_{V_2} P_{W_2}\|$$

$$= \beta(V_1, W_1)\, \beta(V_2, W_2).$$ Q.E.D.

Proof of the corollary. We can interpret π as a representation of

$$\mathrm{SL}(3, \mathbb{Z}/N_\pi \mathbb{Z}) \cong \prod_{i=1}^{k} \mathrm{SL}(3, \mathbb{Z}/p_i^{n_i} \mathbb{Z})$$

where $N_\pi = \prod_{i=1}^{k} p_i^{n_i}$ is written as a product of distinct primes. Due to this representation as a product we have the decomposition:

$$\pi \cong \bigotimes_{i=1}^{k} \pi_i, \quad \mathscr{H}_\pi \cong \bigotimes_{i=1}^{k} \mathscr{H}_i \tag{8}$$

π_i being a nontrivial irreducible representation of SL$(3, \mathbb{Z}/p_i^{n_i}\mathbb{Z})$ in $\mathscr{H}_i$.

Because

$$A_{\pm}(\mathbb{Z}/N_\pi \mathbb{Z}) \cong \prod_{i=1}^{k} A_{\pm}(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$$

the decomposition (8) induces an isomorphism

$$V_{\pm} \cong \bigotimes_{i=1}^{k} V_{\pm}(\pi_i)$$

where $V_{\pm}(\pi_i)$ is the subspace of $\mathscr{H}_i$ consisting of $A_{\pm}(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$-fixed vectors. Proposition 5 shows that $\beta(V_+(\pi_i), V_-(\pi_i)) \leqq 1/\sqrt{p_i}$ and by Lemma 2 we are done. Q.E.D.

To show Proposition 5 we first need to show an analogous result for SL$(2, \mathbb{Z}_p)$.

If A is a commutative ring with identity we consider the following subgroups of SL$(2, A)$:

$$N_+(A) = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in A \right\},$$

$$N_-(A) = \left\{ \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} : b \in A \right\}.$$

Then we have:

Proposition 6. *Let π be a continuous unitary representation of* SL$(2, \mathbb{Z}_p)$ *in a Hilbert space $\mathscr{H}$ with no nonzero fixed vectors and W_+ (resp. W_-) the subspace of $N_+(\mathbb{Z}_p)$ (resp. $N_-(\mathbb{Z}_p)$) fixed vectors. Then*:

$$\beta(W_+, W_-) \leqq \frac{1}{\sqrt{p}}.$$

Proof. We can suppose that π is an irreducible representation of SL$(2, \mathbb{Z}/p^n\mathbb{Z})$ for some $n \geqq 1$ and $W_+ \neq \{0\}$. Thus π occurs in the regular representation of SL$(2, \mathbb{Z}/p^n\mathbb{Z})$ on SL$(2, \mathbb{Z}/p^n\mathbb{Z})/N_+(\mathbb{Z}/p^n\mathbb{Z})$. We identify this homogeneous space with V_n, the SL$(2, \mathbb{Z}/p^n\mathbb{Z})$-orbit of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in $(\mathbb{Z}/p^n\mathbb{Z})^2$:

$$V_n = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a \text{ or } b \text{ is a unit in } \mathbb{Z}/p^n\mathbb{Z} \right\}.$$

Let $W_+(n)$ (resp. $W_-(n)$) be the subspace of $l^2(V_n)$ consisting of $N_+(\mathbb{Z}/p^n\mathbb{Z})$ (resp. $N_-(\mathbb{Z}/p^n\mathbb{Z})$) invariant functions orthogonal to the constant ones. We prove by induction on n that:

$$|\langle u_+|u_-\rangle| \leqq \frac{1}{\sqrt{p}} \|u_+\| \, \|u_-\|$$

for all $u_+ \in W_+(n)$, $u_- \in W_-(n)$.

Case $n=1$: $V_1 = \left\{\begin{pmatrix} x \\ y \end{pmatrix} \in (\mathbb{Z}/p\mathbb{Z})^2 : \begin{pmatrix} x \\ y \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\}$. Let F_+ (F_- resp.) be the set of fixed points under N_+ (N_- resp.) in V_1. For $f \in F = F_+ \cup F_-$ we denote ψ_f^+ (resp. ψ_f^-) the characteristic function of the N_+ (resp. N_-) orbit of f.

$$\{\psi_f^+ : f \in F\} \quad \text{resp.} \quad \{\psi_f^- : f \in F\}$$

is an orthogonal basis of the space of N_+ (resp. N_-) invariant functions in $l^2(V_1)$. Let

$$u_+ = \sum_{f \in F} x_f \psi_f^+, \quad u_- = \sum_{f \in F} y_f \psi_f^-$$

then

$$\|u_+\|^2 = \sum_{f \in F_+} |x_f|^2 + p \sum_{f \in F_-} |x_f|^2,$$

$$\|u_-\|^2 = p \sum_{f \in F_+} |y_f|^2 + \sum_{f \in F_-} |y_f|^2,$$

$$\langle u_+|u_-\rangle = \sum_{f \in F} x_f \bar{y}_f + \Big(\sum_{f \in F_-} x_f\Big)\Big(\sum_{f' \in F_+} \bar{y}_{f'}\Big).$$

Set $X_\pm = \sum_{f \in F_\pm} x_f$. Then:

$$|\langle u_+|u_-\rangle| = \left|\sum_{f \in F_-} x_f \bar{y}_f + \sum_{f \in F_+} (x_f + X_-)\bar{y}_f\right| \leqq \|u_-\|_2 \left\{\sum_{f \in F_-} |x_f|^2 + \frac{1}{p} \sum_{f \in F_+} |x_f + X_-|^2\right\}^{1/2}.$$

If $u_+ \in W_+(1)$ then $pX_- + X_+ = 0$ and as the right hand side of the inequality we get:

$$\frac{1}{\sqrt{p}} \|u_-\|_2 \{\|u_+\|_2^2 - (1+p)|X_-|^2\}^{1/2} \leqq \frac{\|u_-\|_2 \, \|u_+\|_2}{\sqrt{p}}.$$

Case $n \geqq 2$: Let $r_n : V_n \to V_{n-1}$, $q_n : \mathrm{SL}(2, \mathbb{Z}/p^n\mathbb{Z}) \to \mathrm{SL}(2, \mathbb{Z}/p^{n-1}\mathbb{Z})$ be the maps of reduction mod p^{n-1}. Define:

$$S_n : l^2(V_n) \to l^2(V_{n-1})$$

by $S_n f(x) = \sum_{y : r_n(y) = x} f(y)$.

Then $\frac{1}{p} \cdot S_n$ is an q_n-equivariant isometry between $(\operatorname{Ker} S_n)^\perp$ and $l^2(V_{n-1})$. By induction hypothesis it suffices to estimate the angle between the subspaces of $N_+(\mathbb{Z}/p^n\mathbb{Z})$ and $N_-(\mathbb{Z}/p^n\mathbb{Z})$ invariant functions in $\operatorname{Ker} S_n$.

We have an orthogonal decomposition of $\operatorname{Ker} S_n$ given by restricting the functions on V_n to the fibers of r_n:

$$\operatorname{Ker} S_n = \bigoplus_{x \in V_{n-1}} l^2(r_n^{-1}(x))^\circ$$

where $l^2(r_n^{-1}(x))^\circ$ is the subspace of functions in $l^2(r_n^{-1}(x))$ orthogonal to the constants. Let

$$N_\pm(x) = \{g \in N_\pm(\mathbb{Z}/p^n\mathbb{Z}) : g \text{ preserves } r_n^{-1}(x)\}$$

and $V_\pm(x)$ the subspace of $l^2(r_n^{-1}(x))^\circ$ of $N_\pm(x)$-invariant functions. The subspace of $N_\pm(\mathbb{Z}/p^n\mathbb{Z})$-invariant functions of $\operatorname{Ker} S_n$ is then contained in

$$\bigoplus_{x \in V_{n-1}} V_\pm(x)$$

so that it suffices to estimate the angle between $V_+(x)$ and $V_-(x)$.

If $x = \begin{pmatrix} a \\ b \end{pmatrix}$ with $a \cdot b \neq 0$ (in $\mathbb{Z}/p^{n-1}\mathbb{Z}$) one verifies easily that all $N_+(x)$ and $N_-(x)$ orbits in $r_n^{-1}(x)$ have the same number of points namely p, and that every $N_+(x)$ orbit meets every $N_-(x)$ orbit in exactly one point. From this it follows easily that $V_+(x)$ and $V_-(x)$ are orthogonal.

If $x = \begin{pmatrix} a \\ 0 \end{pmatrix}$, a a unit in $\mathbb{Z}/p^{n-1}\mathbb{Z}$ then

$$r_n^{-1}\begin{pmatrix} a \\ 0 \end{pmatrix} = \left\{\begin{pmatrix} a' \\ 0 \end{pmatrix} + p^{n-1}\begin{pmatrix} c \\ d \end{pmatrix} : c, d \bmod p\right\}$$

a' being a fixed unit such that $a' \equiv a \bmod p^{n-1}$. Then:

$$N_+(x) = \left\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{Z}/p^n\mathbb{Z}\right\}$$

and

$$N_-(x) = \left\{\begin{pmatrix} 1 & 0 \\ yp^{n-1} & 1 \end{pmatrix} : y \bmod p\right\}.$$

If we identify $r_n^{-1}\begin{pmatrix}a\\0\end{pmatrix}$ with $(\mathbb{Z}/p\mathbb{Z})^2$ in an obvious way, the action of $N_+(x)$ and $N_-(x)$ becomes

$$\begin{pmatrix}1 & z\\0 & 1\end{pmatrix}\times\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}c+zd\\d\end{pmatrix},$$

$$\begin{pmatrix}1 & 0\\yp^{n-1} & 1\end{pmatrix}\times\begin{pmatrix}c\\d\end{pmatrix}=\begin{pmatrix}c\\ya'+d\end{pmatrix}.$$

For $t\in\mathbb{Z}/p\mathbb{Z}$ let φ_t^+ be the characteristic function of $\left\{\begin{pmatrix}t\\0\end{pmatrix}\right\}=N_+(x)\begin{pmatrix}t\\0\end{pmatrix}$ and φ_t^- the characteristic function of $N_-(x)\begin{pmatrix}t\\0\end{pmatrix}=\left\{\begin{pmatrix}t\\y\end{pmatrix}: y\neq 0\right\}$. For $y\neq 0$, $y\in\mathbb{Z}/p\mathbb{Z}$, ψ_y is the characteristic function of

$$N_+(x)\begin{pmatrix}0\\y\end{pmatrix}=\left\{\begin{pmatrix}s\\y\end{pmatrix}: s\in\mathbb{Z}/p\mathbb{Z}\right\}.$$

Then $\{\varphi_t^+, \psi_y : t,\ y\in\mathbb{Z}/p\mathbb{Z},\ y\neq 0\}$ is an orthogonal basis of the $N_+(x)$-invariant functions and

$$\{\varphi_t^- : t\in\mathbb{Z}/p\mathbb{Z}\}$$

is an orthogonal basis for the $N_-(x)$-invariant functions.

Let:

$$u_+=\sum x_t\varphi_t^+ + \sum_{y\neq 0} f_y\psi_y,$$

$$u_-=\sum z_t\varphi_t^-.$$

Then:

$$\|u_+\|^2=\sum |x_t|^2+p\sum_{y\neq 0}|f_y|^2,$$

$$\|u_-\|^2=p\sum |z_t|^2,$$

$$\langle u_+|u_-\rangle=\sum_t x_t\bar z_t+\Big(\sum_{y\neq 0} f_y\Big)\Big(\sum_t \bar z_t\Big)$$

if u_- is orthogonal to the constants: $\sum_t z_t=0$ and

$$|\langle u_+|u_-\rangle|\leqq\Big(\sum |x_t|^2\Big)^{1/2}\Big(\sum |z_t|^2\Big)^{1/2}\leqq\frac{1}{\sqrt{p}}\|u_+\|\,\|u_-\|.$$

The case $x=\begin{pmatrix}0\\b\end{pmatrix}$ is completely analogous. Q.E.D.

Remark. The proof shows that the inequality of Proposition 6 is sharp.

Let us define the following subgroups of SL(3, A):

$$P(A)=\left\{\begin{pmatrix}1 & x & y\\ 0 & a & b\\ 0 & c & d\end{pmatrix} : ad-bc=1;\ x, y, a, b, c, d \in A\right\},$$

$$S(A)=\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & a & b\\ 0 & c & d\end{pmatrix} : ad-bc=1;\ a, b, c, d \in A\right\}.$$

Lemma 3. *Let $V_\pm$ be the subspace of $A_\pm(\mathbb{Z}/p^n\mathbb{Z})$ invariant functions in*

$$l^2(\mathrm{SL}(3, \mathbb{Z}/p^n\mathbb{Z})/P(\mathbb{Z}/p^n\mathbb{Z}))$$

which are orthogonal to the constants. Then we have for all $u_+ \in V_+$, $u_- \in V_-$:

$$|\langle u_+ | u_-\rangle| \leqq \frac{1}{\sqrt{p}}\, \|u_+\|\, \|u_-\|.$$

The proof will be given further on.

We identify SL(2, A) with $S(A)$ so that N_+ and N_- becomes:

$$N_+(A)=\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & a\\ 0 & 0 & 1\end{pmatrix} : a \in A\right\},$$

$$N_-(A)=\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & a & 1\end{pmatrix} : a \in A\right\}.$$

Lemma 4. *Let W_+ (E_- resp.) be the subspace of $A_+(\mathbb{Z}/p^n\mathbb{Z})$ ($N_-(\mathbb{Z}/p^n\mathbb{Z})$ resp.) invariant functions in*

$$l^2(P(\mathbb{Z}/p^n\mathbb{Z})/S(\mathbb{Z}/p^n\mathbb{Z}))$$

which are orthogonal to the constants. Then we have for all $e_- \in E_-$ and $w_+ \in W_+$:

$$|\langle e_- | w_+\rangle| \leqq \frac{1}{\sqrt{p}}\, \|e_-\|\, \|w_+\|.$$

Proof of Proposition 5. We can suppose π is an irreducible nontrivial representation of SL(3, $\mathbb{Z}/p^n\mathbb{Z}$) for some $n \geqq 1$ and $V_+ \neq \{0\}$, $V_- \neq \{0\}$.

If π has a nonzero $P(\mathbb{Z}/p^n\mathbb{Z})$-invariant vector π is contained in the regular representation of SL(3, $\mathbb{Z}/p^n\mathbb{Z}$) on $l^2(\mathrm{SL}(3, \mathbb{Z}/p^n\mathbb{Z})/P)$ and we can apply Lemma 3.

Suppose π has no nonzero $P(\mathbb{Z}/p^n\mathbb{Z})$-invariant vector. Note that V_- is contained in the space of $N_-(\mathbb{Z}/p^n\mathbb{Z})$-invariant vectors.

Decomposing $\pi|_P$ into irreducible components we are left with the following problem: For a nontrivial irreducible representation ω of $P(\mathbb{Z}/p^n\mathbb{Z})$ estimate the angle between the subspace of $N_-(\mathbb{Z}/p^n\mathbb{Z})$ and $A_+(\mathbb{Z}/p^n\mathbb{Z})$-invariants. If ω has a nonzero $S(\mathbb{Z}/p^n\mathbb{Z})$-invariant vector, ω is contained in $l^2(P/S)$ and we apply Lemma 4. If ω has no nonzero $S(\mathbb{Z}/p^n\mathbb{Z})$-invariant vector we apply Proposition 6 to the irreducible components of $\omega|_S$. Q.E.D.

Proof of Lemma 3. Let us introduce the following notations:

$$U_m = (\mathbb{Z}/p^m\mathbb{Z})^x, \quad |U_m| = p^m(1-p^{-1}).$$

$V_m \subset (\mathbb{Z}/p^m\mathbb{Z})^2$ is the SL(2, $\mathbb{Z}/p^m\mathbb{Z}$) orbit of $\begin{pmatrix}1\\0\end{pmatrix}$. It has $|V_m| = p^{2m}(1-p^{-2})$ elements. $E_m \subset (\mathbb{Z}/p^n\mathbb{Z})^2$, $(m \leqq n-1)$ is the SL(2, $\mathbb{Z}/p^n\mathbb{Z}$) orbit of $\begin{pmatrix}p^m\\0\end{pmatrix}$; $|E_m| = p^{2(n-m)}(1-p^{-2})$. Finally R_m denotes a set of representatives of U_m in U_n $(n \geqq m)$.

We identify SL(3, $\mathbb{Z}/p^n\mathbb{Z}$)/$P(\mathbb{Z}/p^n\mathbb{Z})$ with Ω, the SL(3, $\mathbb{Z}/p^n\mathbb{Z}$) orbit of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ in $(\mathbb{Z}/p^n\mathbb{Z})^3$:

$$\Omega = \left\{\begin{pmatrix}a\\b\\c\end{pmatrix} : \text{at least one coordinate is a unit}\right\}.$$

Recall that the action of $A_+(\mathbb{Z}/p^n\mathbb{Z})$ on $(\mathbb{Z}/p^n\mathbb{Z})^3$ is given by

$$\begin{pmatrix}1 & 0 & x\\0 & 1 & y\\0 & 0 & 1\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = \begin{pmatrix}a+xc\\b+yc\\c\end{pmatrix}.$$

From this one deduces that the orbits of $A_+(\mathbb{Z}/p^n\mathbb{Z})$ on Ω are:

(a$_+$) $$c \in U_n : W_c^+ = \left\{\begin{pmatrix}w\\c\end{pmatrix} : w \in (\mathbb{Z}/p^n\mathbb{Z})^2\right\}.$$

Then $|W_c^+| = p^{2n}$.

(b$_+$) $$1 \leqq k \leqq n-1, \quad d \in R_{n-k}, \quad v \in V_k:$$

$$W_{k,d,v}^+ = \left\{ \begin{pmatrix} v' \\ d p^k \end{pmatrix} : v' \equiv v \bmod p^k \right\},$$

$$|W_{k,d,v}^+| = p^{2(n-k)}.$$

(c$_+$) $$w \in V_n : W_w^+ = \left\{ \begin{pmatrix} w \\ 0 \end{pmatrix} \right\}, \quad |W_w^+| = 1.$$

The action of $A_-(\mathbb{Z}/p^n\mathbb{Z})$ is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c + xa + yb \end{pmatrix}.$$

The orbits of $A_-(\mathbb{Z}/p^n\mathbb{Z})$ on Ω are the following:

(a$_-$) $$w \in V_n : W_w^- = \left\{ \begin{pmatrix} w \\ x \end{pmatrix} : x \in \mathbb{Z}/p^n\mathbb{Z} \right\},$$

$$|W_w^-| = p^n.$$

(b$_-$) $$1 \leqq m \leqq n-1, \quad d \in U_m, \quad v \in E_m:$$

$$W_{m,d,v}^- = \left\{ \begin{pmatrix} v \\ d' \end{pmatrix} : d' \equiv d \bmod p^m \right\},$$

$$|W_{m,d,v}^-| = p^{n-m}.$$

(c$_-$) $$c \in U_n : W_c^- = \left\{ \begin{pmatrix} 0 \\ 0 \\ c \end{pmatrix} \right\}, \quad |W_c^-| = 1.$$

Denote by F_c^+, $F_{k,d,v}^+$, F_w^+, F_w^-, $F_{m,d,v}^-$, F_c^- the characteristic functions of the orbits of type resp. (a$_+$), (b$_+$), (c$_+$), (a$_-$), (b$_-$), (c$_-$).

Let F be an $A_+(\mathbb{Z}/p^n\mathbb{Z})$-invariant function on Ω and H an $A_-(\mathbb{Z}/p^n\mathbb{Z})$-invariant function. Then:

$$F=\sum_{U_n} x_c F_c^+ + \sum_{k,d,v} x_{k,d,v} F_{k,d,v}^+ + \sum_{V_n} x_w F_w^+,$$

$$H=\sum_{V_n} y_w F_w^- + \sum_{m,d,v} y_{m,d,v} F_{m,d,v}^- + \sum_{U_n} y_c F_c^-.$$

Their norms and inner product are:

$$\|F\|^2 = p^{2n}\sum_{U_n}|x_c|^2 + \sum_{k,d,v} p^{2(n-k)}|x_{k,d,v}|^2 + \sum_{V_n}|x_w|^2, \tag{9}$$

$$\|H\|^2 = p^n\sum_{V_n}|y_w|^2 + \sum_{m,d,v} p^{n-m}|y_{m,d,v}|^2 + \sum_{U_n}|y_c|^2,$$

$$\langle F|H\rangle = \left(\sum_{U_n} x_c\right)\left(\sum_{V_n}\bar{y}_v\right) + \sum_{m,d,v}\bar{y}_{m,d,v}\sum_{d'\equiv d(p^m)} x_{d'} + \sum_{U_n} x_c\bar{y}_c + \sum_{k,d,v} x_{k,d,v}\sum_{v'\equiv v(p^k)}\bar{y}_{c'} + \sum_{V_n} x_v\bar{y}_v.$$

Denote by $h_m: U_n \longrightarrow U_m$ the canonical homomorphism. We can write the second term of the sum (9) in the following way:

$$\sum_{m,d,v}\bar{y}_{m,d,v}\sum_{d'\equiv d(p^m)} x_{d'} = \sum_{d\in U_n} x_d\bar{t}_d$$

where

$$t_d = \sum_{m=1}^{n-1}\sum_{v\in E_m} y_{m,h_m(d),v}.$$

Substituting this into (9) and writing $Y=\sum_{v\in V_n} y_v$ we obtain:

$$\langle F|H\rangle = \sum_{c\in U_n} x_c[\bar{Y}+\bar{t}_c+\bar{y}_c] + \sum_{k,d,v} x_{k,d,v}\sum_{v'\equiv v(p^k)}\bar{y}_{v'} + \sum_{v\in V_n} x_v\bar{y}_v.$$

By Cauchy-Schwarz: $|\langle F|H\rangle| \leq \|F\|\, T^{1/2}$ where

$$T = p^{-2n}\sum_{c\in U_n}|Y+t_c+y_c|^2 + (1-p^{-1})\sum_{k=1}^{n-1} p^{k-n}\sum_{v\in V_k}\Big|\sum_{v'\equiv v(p^k)} y_{c'}\Big|^2 + \sum_{v\in V_n}|y_v|^2. \tag{10}$$

We want to estimate T under the assumption that H is orthogonal to the constants.

In order to estimate the second term of the sum (10) we introduce an orthogonal decomposition of $l^2(V_n)$ defined as follows.

Let $r_j: V_n \longrightarrow V_j$ be the map given by reduction mod p^j. Consider the linear map

$$S_j: l^2(V_n) \longrightarrow l^2(V_j)$$

defined by $S_j h(v) = \sum_{v' \equiv v(p^j)} h(v')$, $h \in l^2(V_n)$. For $j = -1$ we put $S_{-1} = 0$. Then:

$$\{0\} = \operatorname{Ker} S_n \subset \operatorname{Ker} S_{n-1} \subset \ldots \subset \operatorname{Ker} S_0 \subset \operatorname{Ker} S_{-1} = l^2(V_n).$$

Define K_j to be the orthogonal complement of $\operatorname{Ker} S_j$ in $\operatorname{Ker} S_{j-1}$, $j = 0, \ldots, n$. In this way we obtain an orthogonal decomposition of $l^2(V_n) = \bigoplus_{j=0}^{n} K_j$ with the following property:

$$h = h_n + \ldots + h_0, \quad h_i \in K_i, \quad h \in l^2(V_n),$$

then

$$S_j h = S_j h_j + \ldots + S_j h_0, \quad 0 \leqq j \leqq n.$$

To compute the norm of $S_j h$ we make the following remarks:

(i) The adjoint S_j^* of S_j is given by $S_j^* f = f \circ r_j$, $f \in l^2(V_j)$, $1 \leqq j \leqq n$.

(ii) $n \geqq j \geqq 1$, $0 \leqq l \leqq j: (S_j h_l) \circ r_j = p^{2(n-j)} h_l$ (because h_l is constant on the fibers of r_j). From this we deduce:

$$\|S_j h\|^2 = \langle (S_j h) \circ r_j | h \rangle = p^{2(n-j)} \sum_{l=0}^{j} \|h_l\|^2. \tag{11}$$

We apply the preceding discussion to the function $h(v) = y_v$ and obtain:

$$\begin{aligned}(1-p^{-1}) \sum_{k=1}^{n-1} p^{k-n} \sum_{v \in V_k} \Big| \sum_{v' \equiv v(p^k)} y_{v'} \Big|^2 &= (1-p^{-1}) \sum_{k=1}^{n-1} p^{k-n} \|S_k h\|^2 \\ &= (p^{n-1}-1) \|h_0\|^2 + \sum_{k=1}^{n-1} (p^{n-k}-1) \|h_k\|^2 \\ &\qquad \text{(use (11))} \\ &= p^{n-1} \|h_0\|^2 + \sum_{k=1}^{n} p^{n-k} \|h_k\|^2 - \|h\|^2.\end{aligned}$$

Substituting this into (10) we obtain:

$$\begin{aligned} T &= p^{-2n} \sum_{c \in U_n} |Y + t_c + y_c|^2 + p^{n-1} \|h_0\|^2 + \sum_{k=1}^{n} p^{n-k} \|h_k\|^2 \\ &\leqq p^{-2n} \sum_{c \in U_n} |Y + t_c + y_c|^2 + p^{n-1} \sum_{v \in V_n} |y_v|^2. \end{aligned} \tag{12}$$

Now we deal with the first term of the sum (12):

$$p^{-2n}\sum_{c\in U_n}|Y+t_c+y_c|^2=p^{-n}(1-p^{-1})\,|Y|^2+p^{-2n}\,2\,\mathrm{Re}\big[\bar{Y}\sum_{c\in U_n}(t_c+y_c)\big]+p^{-2n}\sum_{c\in U_n}|t_c+y_c|^2.$$

The definition of t_c and the hypothesis that H is orthogonal to the constants imply:

$$\sum_{c\in U_n}(t_c+y_c)=\sum_{m=1}^{n-1}\sum_{v\in E_m}\sum_{d\in U_m}p^{n-m}y_{m,d,v}+\sum_{c\in U_n}y_c=-p^n\sum_{v\in V_n}y_v=-p^nY.$$

Thus:

$$p^{-2n}\sum_{c\in U_n}|Y+t_c+y_c|^2=-p^{-n}(1+p^{-1})\,|Y|^2+p^{-2n}\sum_{c\in U_n}|t_c+y_c|^2.$$

Now we apply Cauchy-Schwarz:

$$|t_c+y_c|^2=\left|y_c+\sum_{m=1}^{n-1}\sum_{v\in E_m}y_{m,h_m(c),v}\right|^2\leqq\left[1+\sum_{m=1}^{n-1}|E_m|\right]\left[|y_c|^2+\sum_{m=1}^{n-1}\sum_{v\in E_m}|y_{m,h_m(c),v}|^2\right].$$

Recall that $|E_m|=p^{2(n-m)}(1-p^{-2})$.

We obtain:

$$p^{-2n}\sum_{c\in U_n}|Y+t_c+y_c|^2\leqq p^{-2n}\sum_{c\in U_n}|t_c+y_c|^2\leqq p^{-2}\left\{\sum_{c\in U_n}|y_c|^2+\sum_{m=1}^{n-1}\sum_{v\in E_m}\sum_{d\in U_m}p^{n-m}|y_{m,d,v}|^2\right\}.$$

Substituting this into (12) we obtain $T\leqq p^{-1}\|H\|^2$ and thus

$$|\langle F|H\rangle|\leqq\frac{1}{\sqrt{p}}\,\|F\|\,\|H\|. \qquad \text{Q.E.D.}$$

Proof of Lemma 4. We parametrize a set of representatives of

$$P(\mathbb{Z}/p^n\mathbb{Z})/S(\mathbb{Z}/p^n\mathbb{Z})$$

by $(\mathbb{Z}/p^n\mathbb{Z})^2$:

$$(a,b)\to\begin{pmatrix}1&a&b\\0&1&0\\0&0&1\end{pmatrix}.$$

In this parametrization $A_+(\mathbb{Z}/p^n\mathbb{Z})$ acts as follows:

$$\begin{pmatrix}1&0&x\\0&1&y\\0&0&1\end{pmatrix}\times(a,b)=(a,b+x-ay)$$

and $N_-(\mathbb{Z}/p^n\mathbb{Z})$ like:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{pmatrix} \times (a, b) = (a - bc, b).$$

Orbits of $A_+(\mathbb{Z}/p^n\mathbb{Z})$:

$$a \in \mathbb{Z}/p^n\mathbb{Z}, \quad V_a^+ = \{(a, x) : x \in \mathbb{Z}/p^n\mathbb{Z}\}.$$

Orbits of $N_-(\mathbb{Z}/p^n\mathbb{Z})$:

(1) $b \in U_n: \quad V_b^- = \{(x, b) : x \in \mathbb{Z}/p^n\mathbb{Z}\}$,

(2) $1 \leqq j \leqq n-1, \ u \in R_{n-j}, \ x \in \mathbb{Z}/p^j\mathbb{Z}: \quad V_{j,u,x}^- = \{(x', p^j u) : x' \equiv x \bmod p^j\}$,

(3) $x \in \mathbb{Z}/p^n\mathbb{Z}: \quad V_{0,x}^- = \{(x, 0)\}$.

The inner product of an A_+-invariant function with an N_--invariant function leads to the same type of sums as the ones in Lemma 3 (in fact they are somewhat simpler). And the method of estimating them is exactly the same, so we leave the proof as an exercise to the reader. Q.E.D.

4. The value of $\alpha(F)$ in two examples

Lemma 5. (a) $\alpha(F) \geqq \frac{1}{4}$ *if* $F = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$,

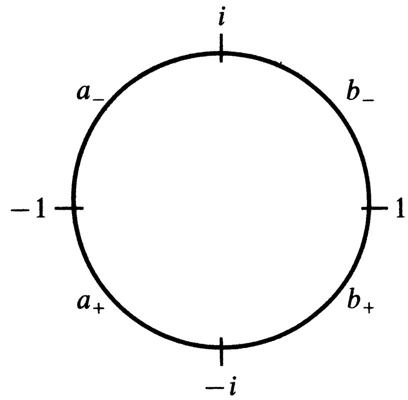
(b) $\alpha(F) = \frac{1}{2}$ *if* $F = \left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\}$.

Proof. Let us denote as usual $n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\bar{n} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. We identify $P^1(\mathbb{R})$ with $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ in such a way that:

$$n(z) = \frac{(2i-1)z+1}{-z+(2i+1)},$$

$$\bar{n}(z) = \frac{(2i+1)z+1}{-z+(2i-1)}.$$

We will use the following partition of S^1:



$$b_- = \left\{e^{i\theta} : 0 \leqq \theta < \frac{\pi}{2}\right\}, \quad a_- = \left\{e^{i\theta} : \frac{\pi}{2} \leqq \theta < \pi\right\},$$

$$a_+ = \left\{e^{i\theta} : \pi \leqq \theta < \frac{3\pi}{2}\right\}, \quad b_+ = \left\{e^{i\theta} : \frac{3\pi}{2} \leqq \theta < 2\pi\right\},$$

$$n(1) = 1, \quad \bar{n}(-1) = -1.$$

Let μ be a probability measure on S^1.

(a) Suppose that for all Borel sets $E \subset S^1$ we have

$$|\mu(nE) - \mu(E)| \leqq \frac{1}{4}.$$

We apply this to $E = a^+ \cup b^+$ and use that

$$n(a^+ \cup b^+) = b^+$$

to conclude that $\mu(a^+) \leqq 1/4$. In the same way one shows that $\mu(a^-) \leqq 1/4$. On the other hand we have: $\bar{n}(a^+ \cup b^+) = a^+$ and $\bar{n}(a^-) = a^- \cup b^-$. Finally, with $E = a^+ \cup b^+$:

$$\mu(E) - \mu(\bar{n}E) + \mu(\bar{n}a^-) - \mu(a^-) = \mu(b^+) + \mu(b^-) = 1 - \mu(a^-) - \mu(a^+) \geqq \frac{1}{2}.$$

From this follows: $\sup\limits_{B \subset S^1} |\mu(\bar{n}B) - \mu(B)| \geqq \frac{1}{4}$.

(b) Suppose that $|\mu(n^2 E) - \mu(E)| \leqq \frac{1}{2}$ for all Borel sets $E \subset S^1$.

Take $B = a^- \cup a^+ \cup b^+$, then $n^2 B = b^+$ and $\mu(a^- \cup a^+) \leqq 1/2$ by hypothesis. On the other hand, if $A = b^+ \cup b^- \cup a^+$, then $\bar{n}^2 A = a^+$ and $\mu(A) - \mu(\bar{n}^2 A) = \mu(b^+ \cup b^-) \geqq 1/2$. This proves that $\alpha(F) \geqq 1/2$. By taking $\mu = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$ one sees that $\alpha(F) = 1/2$. Q.E.D.

5. The relative property revisited

In this paragraph we want to find the relevant property of the action of SL(2, $\mathbb{Z}$) on the 2-torus T^2 which implies the relative property (T) for the pair

$$(\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z}), \mathbb{Z}^2).$$

In particular we want to give a proof of this relative property without referring to another group (e.g. $\mathbb{R}^2 \rtimes \mathrm{SL}(2, \mathbb{R})$) as done usually.

For the sake of clarity we will formulate our problem in a more general context. Namely, let A be a locally compact abelian group and H a topological group acting continuously on A by (continuous) automorphisms. The semidirect product $A \rtimes H$ is as usual the topological space $A \times H$ endowed with the group structure

$$(a, h)(a', h') = (a h(a'), h h').$$

The group Aut A (and thus H) acts on the dual group $\hat{A}$ and we extend this to an action on $\hat{A}_\infty$ the Alexandroff compactification of $\hat{A}$. Then Aut A has at least two fixed points in $\hat{A}_\infty$, namely e_* the trivial character and ∞ the point at infinity.

Proposition 7. *Let X be a compact space on which H acts by homeomorphisms. Let $p: X \to \hat{A}_\infty$ be a continuous H-equivariant map such that:*

(i) *p is a homeomorphism from the complement of $p^{-1}(e_*) \cup p^{-1}(\infty)$ onto $\hat{A}_\infty - \{e_*, \infty\}$.*

(ii) *There is no H-invariant probability measure on $p^{-1}(e_*) \cup p^{-1}(\infty)$.*

Then $(A \rtimes H, A)$ has the relative property (T).

Note. If A is discrete i.e. $\hat{A}$ compact we set $\hat{A}_\infty = \hat{A}$. Then the conditions (i) and (ii) have to be replaced by

(i′) p is an homeomorphism from the complement of $p^{-1}(e_*)$ onto $\hat{A} - \{e_*\}$.

(ii′) There is no H-invariant probability measure on $p^{-1}(e_*)$.

Examples. 1) Let V be a finite dimensional vector space over a locally compact non discrete field k. Let H be a subgroup of $GL(V)$; H acts on the dual vector space V^* and on the projective space $P(V^*)$. We assume that there is no H-invariant probability measure on $P(V^*)$. Then the pair $(V \rtimes H, V)$ has the relative property (T). Let us construct the space X and the map $p: X \to \hat{V}_\infty$. Choose a continuous nontrivial character of the additive group k and identify $\hat{V}$ with V^* in the usual way. Let Y be the space obtained by adding the hyperplane at infinity $P(V^*)$ to V^* and let $i: V^* \to Y$ be the canonical injection. The space X is obtained by blowing up the point $i(O^*) \in Y$. The map p is the composition of the canonical $GL(V)$-equivariant maps $X \to Y$ and $Y \to V^*_\infty$. By construction this map $p: X \to V^*_\infty$ satisfies the hypothesis of Proposition 7.

We remind the reader that a subgroup H of $GL(V)$ stabilizes a probability measure on $P(V^*)$ if and only if it is contained in a compact extension of some group stabilizing a line in V^* (cf. [Z], Chap. 3).

2) Let $V = \mathbb{R}^n$ and let $\Gamma \subset SL(n, \mathbb{Z})$ be a subgroup such that there is no Γ-invariant probability measure on $P(V^*)$. Then $(\mathbb{Z}^n \rtimes \Gamma, \mathbb{Z}^n)$ has the relative property (T). The dual space of $\mathbb{Z}^n$ is T^n. The space X is obtained by blowing up the point $O^* \in T^n$. Then $p^{-1}(O^*) = P(V^*)$ and the hypotheses of Proposition 7 are satisfied. This implies that if Γ is any subgroup of $SL(2, \mathbb{Z})$ which is not almost cyclic, then $(\mathbb{Z}^n \rtimes \Gamma, \mathbb{Z}^2)$ has the relative property (T).

Proof. Let π be a continuous unitary representation of $A \rtimes H$ in a Hilbert space $\mathscr{H}$. Let P be the spectral measure of $\pi|_A$. Then we have for each vector $\xi \in \mathscr{H}$ a positive bounded measure μ_ξ on $\hat{A}$ defined by:

$$\mu_\xi(B) = \langle P(B)\xi \mid \xi \rangle,$$

B Borel subset of $\hat{A}$. We have the following properties:

(a) For $h \in H: \mu_{\pi(h)\xi} = \mu_\xi \circ h$.

(b) For $f \in L^1(A): \mu_{\pi(f)\xi} = |\hat{f}|^2 \mu_\xi$

where $\hat{f}$ is the Fourier transform of f.

(c) For $\xi, \eta \in \mathscr{H}$, $\|\xi\| = \|\eta\| = 1$ one has: $\|\mu_\xi - \mu_\eta\| \leqq 2\|\xi - \eta\|$.

Now we consider μ_ξ as a measure on $\hat{A}_\infty$. If $\mathscr{H}_1 = \{\xi \in \mathscr{H} : \|\xi\| = 1\}$ we obtain in this way a map:

$$M: \mathscr{H}_1 \to M^1(\hat{A}_\infty)$$

from $\mathscr{H}_1$ into the space of probability measures on $\hat{A}_\infty$ which is compact for the weak topology. Let now $\{\xi_n\}_n \subset \mathscr{H}_1$ be a sequence of almost invariant vectors. In particular for each function $f \in L^1(A)$:

$$\lim_{n \to \infty} \|\pi(f)\xi_n - (\int_A f)\, \xi_n\| = 0.$$

Then it follows from (b) and (c) that

$$\lim_{n\to\infty} \|(|\hat{f}|^2 - |\hat{f}(e_*)|^2)\, \mu_{\xi_n}\| = 0.$$

Now if v is any positive bounded measure on $\hat{A}$ which is a weak accumulation point of the sequence μ_{ξ_n} then v has to verify:

$$|\hat{f}|^2 \cdot v = |\hat{f}(e_*)|^2 \cdot v \quad \text{for all} \quad f \in L^1(A).$$

In other words $\operatorname{supp} v = \{e_*\}$. From this discussion we deduce that any $\mu \in M^1(\hat{A}_\infty)$ which is accumulation point of the sequence $\{M(\xi_n)\}_n$ has its support in $\{e_*, \infty\}$. Suppose now that π has no nonzero A-invariant vector, then $P(e_*) = 0$ and for $\xi \in \mathscr{H}_1$, we can consider $M(\xi)$ as a probability measure on X whose direct image under $p: X \to \hat{A}_\infty$ is $M(\xi)$. But then any accumulation point μ of the sequence $M(\xi_n)$, viewed as measures on X would be an H-invariant probability measure on $p^{-1}(e_*) \cup p^{-1}(\infty)$. This is a contradiction. Q.E.D.

We want to close this paragraph by discussing the following point: let $V \rtimes H$ be the semidirect product of example 1) above. Then Furstenberg's proof of the relative property of $(V \rtimes H, V)$ (see Proposition 2, § 1) shows that any continuous unitary representation of $V \rtimes H$ that almost has H-invariant vectors has nonzero V-invariant vectors. This stronger property does not hold for the pair $(\mathbb{Z}^n \rtimes \mathrm{SL}(n, \mathbb{Z}), \mathbb{Z}^n)$. Indeed, consider the translation representation of $\mathbb{Z}^n \rtimes \mathrm{SL}(n, \mathbb{Z})$ on $l_0^2((\mathbb{Z}/m\mathbb{Z})^n)$, the space of functions on $(\mathbb{Z}/m\mathbb{Z})^n$ which are orthogonal to the constants. In this space there is an obvious $\mathrm{SL}(n, \mathbb{Z})$-invariant vector but no $\mathbb{Z}^n$-invariant one. However we have the following

Proposition 8. *Let π be a unitary representation of $\mathbb{Z}^n \rtimes \mathrm{SL}(n, \mathbb{Z})$ having almost invariant vectors for $\mathrm{SL}(n, \mathbb{Z})$.*

Then either there exists a nonzero vector which is invariant under a subgroup of finite index of $\mathbb{Z}^n$ or:

(i) *if $n \geqq 3$, π contains the natural representation of $\mathbb{Z}^n \rtimes \mathrm{SL}(n, \mathbb{Z})$ on $l^2(\mathbb{Z}^n)$;*

(ii) *if $n = 2$, π contains weakly the natural representation of $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ on $l^2(\mathbb{Z}^2)$.*

In order to prove Proposition 8 we need to know the classification of $\mathrm{SL}(n, \mathbb{Z})$-invariant probability measures on the n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. This follows from the following result.

Proposition 9. *Let μ be a $\mathrm{SL}(n, \mathbb{Z})$-invariant ergodic probability measure on the n-torus T^n. Then either μ is concentrated on a finite $\mathrm{SL}(n, \mathbb{Z})$-orbit or μ is the Lebesgue measure of T^n.*

In order to prove this proposition we recall the following theorem due to N. Wiener ([B], p. 54, 1.3.1).

Theorem (Wiener). *Let μ be a bounded measure on T^n. Then μ has no atoms if and only if*

$$\lim_{N\to\infty} \frac{1}{N^n} \sum_{\|X\|\leq N} |\hat{\mu}(X)|^2 = 0.$$

Proof of Proposition 9. Let μ be a SL$(n, \mathbb{Z})$-invariant ergodic probability measure and suppose that μ is not the Lebesgue measure. Then there exists $X \in \mathbb{Z}^n$, $X \neq 0$ such that $\hat{\mu}(X) \neq 0$. If m is the g.c.d. of the components of X, the SL$(n, \mathbb{Z})$-orbit of X is:

$$\{mY : Y \in \mathbb{Z}^n,\ Y \text{ primitive}\}.$$

From this follows that:

$$\sum_{\|Z\|\leq N} |\hat{\mu}(Z)|^2 \geq |\hat{\mu}(X)|^2 \cdot \mathrm{card}\,\{Y \in \mathbb{Z}^n : Y \text{ primitive}, \|Y\| \leq N/m\}$$

$$\geq |\hat{\mu}(X)|^2\, c(n)\, N^n/m^n$$

where $c(n)$ is a positive constant which depends only on n. From Wiener's theorem it follows that μ has atoms. Since μ is ergodic it has to be supported on one finite SL$(n, \mathbb{Z})$-orbit. Q.E.D.

To prove Proposition 8 we need one more result on almost SL$(n, \mathbb{Z})$-invariant measures on T^n:

Lemma 6. *Let $(\mu_m)_{m=1}^{\infty}$ be a sequence of probability measures on the n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ such that*:

(i) $$\lim_{m\to\infty} \|\mu_m \circ \gamma - \mu_m\| = 0 \quad \textit{for all} \quad \gamma \in \mathrm{SL}(n, \mathbb{Z}).$$

(ii) $$\mu_m(\mathbb{Q}^n/\mathbb{Z}^n) = 0 \quad \textit{for all} \quad m \geq 1.$$

Then $(\mu_m)_{m=1}^{\infty}$ converges weakly to the Lebesgue measure.

To prove this lemma we need the following construction:

For each finite SL$(n, \mathbb{Z})$-invariant subset $F \subset T^n$ (hence $F \subset \mathbb{Q}^n/\mathbb{Z}^n$) let T_F^n be the topological space obtained by blowing up each point $p \in F$. Then SL$(n, \mathbb{Z})$ acts by homeomorphisms on T_F^n and if $F' \subset F$ we have continuous SL$(n, \mathbb{Z})$-invariant maps

$$p_{FF'} : T_F^n \to T_{F'}^n, \quad p_F : T_F^n \to T^n.$$

Let $\tilde{T}^n$ be the projective limit of this system. This is a compact topological space on which SL$(n, \mathbb{Z})$ acts by homeomorphisms. If μ is a probability measure on T^n such that $\mu(\mathbb{Q}^n/\mathbb{Z}^n) = 0$ then using the homeomorphisms

$$p_F : T_F^n - p_F^{-1}(F) \to T^n - F$$

we obtain on each T_F^n a probability measure μ_F such that $p_{FF'}(\mu_F)=\mu_{F'}$. This defines a unique probability measure $\tilde{\mu}$ on $\tilde{T}^n$ such that $p_F(\tilde{\mu})=\mu_F$ ([Bo], chap. 5, § 8, ex. 19). In particular the Lebesgue measure on T^n gives a SL$(n, \mathbb{Z})$-invariant probability measure on $\tilde{T}^n$ denoted by L.

The point of our construction is that L is the unique SL$(n, \mathbb{Z})$-invariant probability measure on $\tilde{T}^n$. Let us show this: Let μ be an ergodic SL$(n, \mathbb{Z})$-invariant probability measure on $\tilde{T}^n$ and μ_F its direct image on T_F^n for all finite SL$(n, \mathbb{Z})$-invariant sets $F \subset T^n$. In particular μ_ϕ is a SL$(n, \mathbb{Z})$-invariant ergodic probability measure on T^n. If μ_ϕ where supported on a finite SL$(n, \mathbb{Z})$-orbit $F \subset \mathbb{Q}^n/\mathbb{Z}^n$ then μ_F would be supported on:

$$p_F^{-1}(F) = \bigcup_{f \in F} P^{n-1}(\mathbb{R})_f .$$

Let Γ_F be the subgroup of elements of SL$(n, \mathbb{Z})$ stabilizing each point $f \in F$. Then Γ_F is of finite index in SL$(n, \mathbb{Z})$ and the restriction of μ_F to each component $P^{n-1}(\mathbb{R})_f$ is Γ_F-invariant. This would produce a probability measure on $P^{n-1}(\mathbb{R})$ which is invariant under a subgroup of finite index of SL$(n, \mathbb{Z})$. This contradiction and Proposition 9 show that μ_ϕ is the Lebesgue measure of T^n, hence $\mu = L$.

Proof of Lemma 6. The preceding discussion shows that we have an isometric SL$(n, \mathbb{Z})$-equivariant map:

$$\{\mu \in M^1(T^n) : \mu(\mathbb{Q}^n/\mathbb{Z}^n) = 0\} \longrightarrow M^1(\tilde{T}^n).$$

If $(\mu_m)_{m=1}^\infty$ is a sequence in $M^1(\tilde{T}^n)$ satisfying the hypotheses of Lemma 6 any weak accumulation point of the sequence $(\tilde{\mu}_m)_{m=1}^\infty$ is SL$(n, \mathbb{Z})$-invariant. Hence $(\tilde{\mu}_m)_{m=1}^\infty$ converges weakly to L. From this it follows that $(\mu_m)_{m=1}^\infty$ converges weakly to the Lebesgue measure. Q.E.D.

Proof of Proposition 8. Let $(\xi_m)_{m=1}^\infty$ be the sequence of almost SL$(n, \mathbb{Z})$-invariant vectors with $\|\xi_m\| = 1$, $1 \leqq m \leqq \infty$. Let P denote the spectral measure of $\pi|_{\mathbb{Z}^n}$.

If $n \geqq 3$ then there exists $m \geqq 1$ such that ξ_m is SL$(n, \mathbb{Z})$-invariant, this because SL$(n, \mathbb{Z})$ has property (T). Then

$$\mu_\xi(B) = \langle \pi(B)\xi \,|\, \xi \rangle \quad \text{where} \quad \xi = \xi_m$$

is a SL$(n, \mathbb{Z})$-invariant probability measure on T^n. If μ_ξ has atoms then there exists a nonzero vector invariant under some subgroup of finite index in $\mathbb{Z}^n$. If μ_ξ has no atoms it is the Lebesgue measure. In this case we associate to each function f on $\mathbb{Z}^n$ with finite support the vector:

$$\eta = \sum_{x \in \mathbb{Z}^n} f(x)\, \pi(x)\, \xi .$$

Using the fact that μ_ξ is the Lebesgue measure one verifies that for all

$$(y, \gamma) \in \mathbb{Z}^n \rtimes \mathrm{SL}(n, \mathbb{Z}): \langle \pi(y, \gamma)\eta \mid \eta\rangle = \sum_{z \in \mathbb{Z}^n} f(\gamma^{-1}(z-y))\,\overline{f(z)}.$$

This shows that the map $f \longrightarrow \eta$ realizes the natural representation of $\mathbb{Z}^n \rtimes \mathrm{SL}(n, \mathbb{Z})$ on $l^2(\mathbb{Z}^n)$ as subrepresentation of π.

If $n=2$ there are also two cases. If for some $m \geqq 1$:

$$\mu_{\xi_m}(\mathbb{Q}^2/\mathbb{Z}^2) > 0$$

then there exists a nonzero vector which is fixed by some subgroup of finite index in $\mathbb{Z}^2$. If

$$\mu_{\xi_m}(\mathbb{Q}^2/\mathbb{Z}^2) = 0 \quad \text{for all} \quad m \geqq 1$$

we are in the hypotheses of Lemma 6. Hence the sequence $(\mu_{\xi_m})_{m=1}^{\infty}$ converges weakly to the Lebesgue measure of T^2. From this one deduces that for each $\gamma \in \mathrm{SL}(2, \mathbb{Z})$ the sequences of measures

$$\nu_m(B) = \langle P(B)\,\pi(\gamma)\,\xi_m \mid \xi_m\rangle$$

also converges weakly to the Lebesgue measure.

Now let f be a function on $\mathbb{Z}^2$ with finite support and consider the sequence of vectors

$$\eta_m = \sum_{x \in \mathbb{Z}^2} f(x)\,\pi(x)\,\xi_m.$$

Then we have:

$$\langle \pi(y, \gamma)\eta_m \mid \eta_m\rangle = \sum_{x, z} f(x)\,\overline{f(z)}\,\langle \pi(-z+y+\gamma(x), e)\,\pi(\gamma)\,\xi_m \mid \xi_m\rangle$$

as $m \longrightarrow \infty$ this expression has the following limit:

$$\sum_{y+\gamma(x)=z} f(x)\,\overline{f(z)} = \sum_{z \in \mathbb{Z}^2} f(\gamma^{-1}(z-y))\,\overline{f(z)}.$$

This shows that π contains weakly the natural representation of $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ on $l^2(\mathbb{Z}^2)$. Q.E.D.

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Department of Mathematics, Stanford University, 94305 Stanford, Ca, U.S.A.

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