Invent. math. 106, 1–11 (1991)

Inventiones mathematicae

Ramanujan duals II

M. Burger[1] **and P. Sarnak**[2]

[1] School of Mathematics, Institute for Advanced Study, Princeton, NJ 08540, USA
[2] Department of Mathematics, Stanford University, Stanford, Ca 94305, USA

Oblatum 5-XII-1990

1 Introduction

This is a continuation of our paper (joint with J.S. Li) "Ramanujan duals and automorphic spectrum", which we will refer to as I. For the convenience of the reader as well as making this note self contained, we review the key definitions from I. Let G be a semisimple linear algebraic group defined over $\mathbb{Q}$. Then $G(\mathbb{Z})$ is a lattice in $G(\mathbb{R})$ and we denote by $\Gamma(N)$ the principal congruence subgroup of $G(\mathbb{Z})$:

$$\Gamma(N) = \{\gamma \in G(\mathbb{Z}) \colon \gamma \equiv I \bmod N\} .$$

Let $\hat{G}(\mathbb{R})$ be the unitary dual of $G(\mathbb{R})$ endowed with the Fell topology (see [D, 18.1]) and $\hat{G}^1(\mathbb{R})$ the subset of $\hat{G}(\mathbb{R})$ consisting of all class one representations. We are interested in the spectrum $\sigma(\Gamma(N)\backslash G(\mathbb{R}))$, that is the set of all $\pi \in \hat{G}(\mathbb{R})$ occurring weakly in the regular representation of $G(\mathbb{R})$ in $L^2(\Gamma(N)\backslash G(\mathbb{R}))$ (see [D, 18.1.4]). We let $\sigma^1(\Gamma(N)\backslash G(\mathbb{R}))$ denote the class one part of $\sigma(\Gamma(N)\backslash G(\mathbb{R}))$. We recall from I the definitions of $\hat{G}_{\mathrm{Aut}}$ and $\hat{G}_{\mathrm{Raman}}$.

$$\hat{G}_{\mathrm{Aut}} = \overline{\bigcup_{N=1}^{\infty} \sigma(\Gamma(N)\backslash G(\mathbb{R}))}$$

$$\hat{G}_{\mathrm{Raman}} = \hat{G}_{\mathrm{Aut}} \cap \hat{G}^1(\mathbb{R}) . \tag{1.1}$$

The closure in (1.1) is taken w.r.t. the Fell topology of $\hat{G}(\mathbb{R})$. Identifying $\hat{G}_{\mathrm{Raman}}$ may be viewed as the general Ramanujan conjectures. The main result in I is the following: let H be a $\mathbb{Q}$-subgroup of G and $\pi \in \hat{H}_{\mathrm{Aut}}$, then any π' weakly contained in $\mathrm{Ind}_{H(\mathbb{R})}^{G(\mathbb{R})}\pi$ lies in $\hat{G}_{\mathrm{Aut}}$. Symbolically:

$$\mathrm{Ind}_{H(\mathbb{R})}^{G(\mathbb{R})} \hat{H}_{\mathrm{Aut}} \subset \hat{G}_{\mathrm{Aut}} \tag{1.2}$$

(Ind always denotes unitary induction.)

The inclusion (1.2) gives nontrivial lower bounds on $\hat{G}_{\mathrm{Aut}}$ and in particular yields a new method to construct automorphic forms and spectrum. Such applications are described in I. Our main result here is to establish two other very useful functorial properties of the sets $\hat{G}_{\mathrm{Aut}}$.

Theorem 1.1 *Let G be a semisimple linear algebraic connected group defined over $\mathbb{Q}$. Let $H < G$ be a semisimple $\mathbb{Q}$-subgroup. Then*
(a) $\mathrm{Res}_{H(\mathbb{R})}\hat{G}_{\mathrm{Aut}} \subset \hat{H}_{\mathrm{Aut}}$: *for $\pi \in \hat{G}_{\mathrm{Aut}}$, any π' weakly contained in $\mathrm{Res}_{H(\mathbb{R})}\pi$ lies in $\hat{H}_{\mathrm{Aut}}$;*
(b) $\hat{G}_{\mathrm{Aut}} \otimes \hat{G}_{\mathrm{Aut}} \subset \hat{G}_{\mathrm{Aut}}$: *for $\pi, \omega \in \hat{G}_{\mathrm{Aut}}$ any π' weakly contained in $\pi \otimes \omega$ lies in $\hat{G}_{\mathrm{Aut}}$.*

Part (a) of the Theorem may be used to prove nontrivial upper bounds on $\hat{G}_{\mathrm{Aut}}$ and $\hat{G}_{\mathrm{Raman}}$ via "lifting" such bounds from $\hat{H}_{\mathrm{Aut}}$ to $\hat{G}_{\mathrm{Aut}}$. This, of course, requires that we can find such suitable $\mathbb{Q}$-subgroups. These upper bounds give partial results towards the general Ramanujan conjectures. We illustrate this technique for certain orthogonal groups. Let $k/\mathbb{Q}$ be a totally real field, I its ring of integers and q a quadratic form over k such that:
1. q has signature $(n, 1)$ over $\mathbb{R}$
2. q^σ is definite at each archimedean place $\sigma \neq \mathrm{id}$.

Let G be the special orthogonal group of q. We consider G as a $\mathbb{Q}$-group via restriction of scalars so that

$$G(\mathbb{R}) = \mathrm{SO}(n, 1) \times \Pi\mathrm{SO}(n+1)$$

Let $G(\mathbb{R}) = KAN$ be an Iwasawa decomposition, $\mathfrak{A} = \mathrm{Lie}(A)$, $\rho = \frac{1}{2}$ – sum of positive roots and $g = ke^{H(g)}n$. The theory of spherical functions identifies $\hat{G}^1(\mathbb{R})$ with a subset of $\mathfrak{A}_{\mathbb{C}}^*/W$, where W is the Weyl group. If $\lambda \in \mathfrak{A}_{\mathbb{C}}^*/W$ corresponds to $\pi \in \hat{G}^1(\mathbb{R})$, the spherical function φ associated to π is given by:

$$\varphi(g) = \int_K e^{(\lambda - \rho)(H(gk))}\, dk\,.$$

Incidentally, this formula shows that the Fell topology on $\hat{G}^1(\mathbb{R})$ coincides with the topology on $\hat{G}^1(\mathbb{R})$ viewed as a subset of $\mathfrak{A}_{\mathbb{C}}^*/W$. Now we identify $\mathfrak{A}$ to $\mathbb{R}$ by sending ρ to $\frac{n-1}{2}$. With this normalization $\hat{G}^1(\mathbb{R})$ is identified with

$$i\mathbb{R} \cup [-\rho, \rho] \subset \mathbb{C}$$

modulo $\{\pm 1\}$. (See [Ko, Prop. 6, Th. 10]). We choose to parametrize $\hat{G}^1(\mathbb{R})$ by $s \in i\mathbb{R}^+ \cup [0, \rho]$ and denote the corresponding representation by π_s.

For these groups G we showed in I (using 1.2 with the trivial representation on $H_m \cong \mathrm{SO}(m, 1)$) that

$$\hat{G}_{\mathrm{Raman}} \supset i\mathbb{R}^+ \cup \{\rho, \rho - 1, \ldots \rho - [\rho]\}\,. \tag{1.3}$$

Recall that if π is an automorphic representation of $\mathrm{GL}(2, \mathbb{A}_F)$, F a number field, such that π_∞ is spherical and π not one dimensional it is conjectured that π_∞ is tempered. We will refer to this conjecture as the Ramanujan conjecture at ∞ for $\mathrm{GL}(2, F)$.

Theorem 1.2 *For $n \geqq 3$, and $G = \mathrm{SO}(q)$ as above*
(a) $\hat{G}_{\mathrm{Raman}} \subset i\mathbb{R}^+ \cup [0, \rho - \frac{1}{2}] \cup \{\rho\}$.

(b) *Assuming that the Ramanujan conjectures at* ∞ *for* $\mathrm{GL}(2, F)$, *where* F *is a number field, are true then*

$$\hat{G}_{\mathrm{Raman}} \subset i\mathbb{R}^+ \cup [0, \rho - 1] \cup \{\rho\} .$$

One can formulate (a) and (b) in terms of the first eigenvalue λ_1 of the Laplacian acting on $L^2(\Gamma \backslash \mathbb{H}^n)$, where $\mathbb{H}^n$ is the hyperbolic n-space.

Corollary 1.3
(a) *Let* Γ *be a congruence subgroup of* $\mathrm{SO}(q, I)$ *then*

$$\lambda_1(\Gamma \backslash \mathbb{H}^n) \geqq \frac{2n-3}{4}, \quad n \geqq 3 . \tag{1.4}$$

(b) *Assuming the Ramanujan conjectures at* ∞ *for* $\mathrm{GL}(2)$:

$$\lambda_1(\Gamma \backslash \mathbb{H}^n) \geqq n - 2, \quad n \geqq 3 . \tag{1.5}$$

Remarks 1.4 (1) The Ramanujan conjectures for GL(2) together with (1.3) show that (1.5) is the precise sharp lower bound for $\lambda_1(\Gamma \backslash \mathbb{H}^n)$ for all $n \geqq 3$.

(2) In the special case $k = \mathbb{Q}$ and $n \geqq 4$, $G(\mathbb{Q})$ is isotropic and Corollary 1.3(a) was established independently by Elstrodt-Grunewald-Mennicke [E-G-M] and Li-Piatetsky Shapiro-Sarnak [L-P-S]. The methods employed in those papers, which make use of Kloosterman sums, are restricted to the isotropic cases as well as to rank 1. (Essential use is made of the cusp in defining Poincaré series).

(3) Corollary 1.3(b) gives strong support to our conjecture that

$$\hat{G}_{\mathrm{Raman}} = i\mathbb{R}^+ \cup \{\rho, \rho - 1, \ldots\} = \hat{G}_{\mathrm{subg}}, \text{ (see I)} . \tag{1.6}$$

In this case conjecture (1.6) apparently agrees with Arthur's conjectures [A] at the infinite place. In fact, (1.3) establishes the "easier" half of these conjectures.

(4) It follows from Theorem 1.1(b) that if $\{s\} \subset \hat{G}_{\mathrm{Raman}}$ for some $0 < s < \rho$ then $\{k(s - \rho) + \rho \colon k \in \mathbb{N}, k(s - \rho) + \rho > 0\} \subset \hat{G}_{\mathrm{Raman}}$.

The proof of Theorem 1.1 makes use of the equidistribution of a certain sequence of points in $\Gamma \backslash G(\mathbb{R})$. In Sect. 2 we construct these sequences and establish their equidistribution using Hecke operators. In Sect. 3 we prove Theorem 1.1 while in Sect. 4 we establish Theorem 1.2. Various comments and extensions are described in Sect. 5.

Acknowledgements. We thank M. Borovoi for clarifying discussions concerning weak approximation and J. Schwermer for general comments.

2

Let G and $\Gamma = \Gamma(N)$ be as in Theorem 1.1. Let $M \subset \Gamma \backslash G(\mathbb{R})$ be a finite subset which is invariant under right Γ-action. Then

$$T_M f(g) = \sum_{m \in M} f(mg) \tag{2.1}$$

is a bounded operator on $L^2(\Gamma \backslash G(\mathbb{R}))$ of norm

$$\| T_M \| = |M| . \tag{2.2}$$

The basic example of such an invariant set comes from $h \in \mathrm{Comm}(\Gamma)$, the commensurator of Γ. Namely, let M be the Γ-orbit of Γh in $\Gamma\backslash G(\mathbb{R})$. Then M is finite and we set $T_h := T_M$. It is easy to check that

$$\| T_h \| = |h^{-1}\Gamma h \cap \Gamma \backslash \Gamma|$$

and

$$T_h^* = T_{h^{-1}} . \tag{2.3}$$

Now let $P_f = \{2, 3, 5, \ldots\}$ denote the set of finite primes. If $S \subset P_f$ is a finite subset, $\mathbb{Z}_{(S)}$ denotes the ring of S-integers. Under the assumptions on G we have:

$$G(\mathbb{Z}_{(S)}) \text{ is finitely generated} . \tag{2.4}$$

(See [BS, Th. 6.2(i)]).

Lemma 2.1 *Under the assumptions on G, there exists $S \subset P_f$ finite such that $G(\mathbb{Z}_{(S)})$ is dense in $G(\mathbb{R})$.*

Proof. Assume first that G is simply connected. Then one is easily reduced to the case where G is $\mathbb{Q}$-almost simple in which case Lemma 2.1 follows from the strong approximation theorem. ([K, P, Th. 4.2]).

In general, let $\rho: G^{sc} \to G$ be the simply connected covering of G and let $S \subset P_f$ be such that $G^{sc}(\mathbb{Z}_{(S)})$ is dense in $G^{sc}(\mathbb{R})$. Enlarging S if necessary we may assume that $\rho(G^{sc}(\mathbb{Z}_{(S)})) \subset G(\mathbb{Z}_{(S)})$. Since $\rho(G^{sc}(\mathbb{R})) = G(\mathbb{R})^0$, we conclude that $G(\mathbb{Z}_{(S)}) \cap G(\mathbb{R})^0$ is dense in $G(\mathbb{R})^0$. On the other hand, it follows from the weak approximation theorem for G (see [San, Coroll. 3.5]) that $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$. Enlarging S if necessary we conclude that $G(\mathbb{Z}_{(S)})$ meets every connected component of $G(\mathbb{R})$ and therefore is dense in $G(\mathbb{R})$. Q.E.D.

Let $\{\varepsilon_1, \ldots, \varepsilon_r\}$ be a finite set of generators of $G(\mathbb{Z}_{(S)})$ and set

$$T = \sum_{i=1}^{r} T_{\varepsilon_i} + T_{\varepsilon_i^{-1}} \tag{2.5}$$

then T is self-adjoint by (2.3). Let

$$\| T \| = \sum_{i=1}^{r} \| T_{\varepsilon_i} \| + \| T_{\varepsilon_i^{-1}} \| = k . \tag{2.6}$$

In view of (2.2), the averaging operator T^m may be written in the following way:

$$T^m f(g) = \sum_{j=1}^{k^m} f(\delta_j^{(m)} g) \tag{2.7}$$

Here the $\delta^{(m)}$ are in $G(\mathbb{Z}_{(S)})$ and products in the ε_j's and ε_j^{-1}. Remark that any element of $G(\mathbb{Z}_{(S)})$ appears as a $\delta_j^{(m)}$ for m big enough.

Our aim is to prove that these $\delta_j^{(m)}$ become equidistributed in $\Gamma\backslash G(\mathbb{R})$ as $m \to \infty$. Precisely let

$$\tilde{T}_m = (\tilde{T}_1)^m . \tag{2.8}$$

Lemma 2.2 (a) *For all $f_1, f_2 \in L^2(\Gamma\backslash G(\mathbb{R}))$:*

$$\lim_{m \to +\infty} \langle \tilde{T}_m f_1, f_2 \rangle = \int_{\Gamma\backslash G(\mathbb{R})} f_1(g)\,dg \int_{\Gamma\backslash G(\mathbb{R})} \bar{f}_2(g)\,dg$$

where dg is the $G(\mathbb{R})$ – invariant probability measure on $\Gamma\backslash G(\mathbb{R})$.

(b) *For all $f \in C_0(\Gamma\backslash G(\mathbb{R}))$*

$$\lim_{m\to\infty} \tilde{T}_m f = \int_{\Gamma\backslash G(\mathbb{R})} f(g)dg$$

uniformly on compact sets.

Proof. From (2.6) we have $\|\tilde{T}_1\| = 1$ and $\tilde{T}_1$ is selfadjoint. Thus the spectrum of $\tilde{T}_1$ is contained in $[-1, 1]$. From the spectral theorem applied to $\tilde{T}_1$ it follows easily that (a) of Lemma 2.2 will follow on showing that -1 is not an eigenvalue of $\tilde{T}_1$ and that 1 is a simple eigenvalue of $\tilde{T}_1$ (with corresponding eigenfunctions the constants.) That is we must show that if

$$\dot{v}_1 = \left\{f : \tilde{T}_1 f = f, \int_{\Gamma\backslash G(\mathbb{R})} f\,dg = 0\right\} \tag{2.9}$$

$$v_{-1} = \{f : \tilde{T}_1 f = -f\}$$

then $\dot{v}_1 = v_{-1} = \{0\}$. Now $\dot{v}_1$ and v_{-1} are invariant under $G(\mathbb{R})$-action. Hence,

$$\dot{v}_1 \cap C(\Gamma\backslash G(\mathbb{R})) \text{ and } v_{-1} \cap C(\Gamma\backslash G(\mathbb{R}))$$

are dense in $\dot{v}_1$ and v_{-1} respectively. Observe that these spaces are invariant under complex conjugation. So it suffices to show that if f is continuous, real valued,

$$\|f\|_2 = 1, \quad \int_{\Gamma\backslash G(\mathbb{R})} f(g)dg = 0$$

and $|\langle \tilde{T}_1 f, f\rangle| = 1$ then $f = 0$. Under the above clearly $\langle \tilde{T}_1 |f|, |f|\rangle = 1$ and hence $\tilde{T}_1|f| = |f|$.

Moreover $|f|(g) = 0$ for some g, since f is of mean value 0. From the definition of $\tilde{T}_1$ it follows that

$$|f|(\varepsilon_j^{\pm 1}\cdot g) = 0 \quad \text{for} \quad j = 1, 2, \ldots r\,.$$

Since $\tilde{T}_m|f| = |f|$ for all $m \geqq 1$ we get

$$|f|(\varepsilon_{j1}^{\pm 1} \ldots \varepsilon_{jk}^{\pm 1} g) = 0$$

for any choice of $k, j_1, \ldots, j_k$. It follows now from Lemma 2.1 that $|f|$ is zero on a dense set and hence $f = 0$.

To prove (b) we note that $f \in C_0(\Gamma\backslash G(\mathbb{R}))$ is uniformly continuous. Hence for $\varepsilon > 0$ and all $g_1 \in \Gamma\backslash G(\mathbb{R})$ there is a neighborhood $U_\varepsilon(g_1)$ of g_1 such that for all $m \geqq 1$ and $g \in U_\varepsilon(g_1)$.

$$|\tilde{T}_m f(g) - \tilde{T}_m f(g_1)| < \varepsilon\,. \tag{2.10}$$

Now take $\psi \in C_0(\Gamma\backslash G(\mathbb{R}))$, $\sup \psi \subset U_\varepsilon(g_1)$, $\psi \geqq 0$ and of integral 1. For m big enough (a) implies:

$$\left|\langle \tilde{T}_m f, \psi\rangle - \int_{\Gamma\backslash G(\mathbb{R})} f(g)dg\right| < \varepsilon\,. \tag{2.11}$$

On the other hand, (2.10) implies that

$$|\langle \tilde{T}_m f, \psi\rangle - \tilde{T}_m f(g_1)| < \varepsilon$$

therefore $\tilde{T}_m f$ converges pointwise to

$$\int_{\Gamma\backslash G(\mathbb{R})} f(g)dg$$

and in fact uniformly on compact sets. Q.E.D.

Now let $H < G$ be a semisimple $\mathbb{Q}$-subgroup. If Γ is a congruence subgroup of $G(\mathbb{Z})$ then $\Delta = \Gamma \cap H(\mathbb{R})$ is a congruence subgroup of $H(\mathbb{Z})$. Identifying $\Delta\backslash H(\mathbb{R})$ with the $H(\mathbb{R})$-orbit of Γe in $\Gamma\backslash G(\mathbb{R})$ we can think of $\Delta\backslash H(\mathbb{R})$ as a "cycle" or as a positive measure μ in $\Gamma\backslash G(\mathbb{R})$ by defining

$$\langle \mu, f \rangle = \int_{\Delta\backslash H(\mathbb{R})} f(h)dh \tag{2.12}$$

for $f \in C_0(\Gamma\backslash G(\mathbb{R}))$. An averaging operator T_M as defined before, operates on cycles by

$$\langle T_M(\mu), f \rangle = \langle \mu, T_M^* f \rangle .$$

Lemma 2.3 *For $f \in C_0(\Gamma\backslash G(\mathbb{R}))$ and μ as above*

$$\lim_{m\to\infty} \langle \tilde{T}_m(\mu), f \rangle = \mathrm{Vol}(\Delta\backslash H(\mathbb{R})) \int_{\Gamma\backslash G(\mathbb{R})} f(g)dg .$$

Proof. This is immediate from Lemma 2.2 since $\tilde{T}_m f$ is uniformly bounded and converges uniformly on compacta. Q.E.D.

Before closing this section we express $\langle T_M f_1, f_2 \rangle$ and $\langle T_M(\mu), f \rangle$ in forms to be used later. We will only need to consider the case where T_M is self adjoint. Let

$$M = \bigsqcup_{i=1}^{\lambda} M_i$$

be the decomposition of M into Γ-orbits, $M_i = \Gamma h_i \Gamma$ where $h_i \in \mathrm{Comm}(\Gamma)$. Now let

$$B_i = \{\gamma \in \Gamma : \Gamma h_i \gamma = \Gamma h_i\} .$$

One checks that

$$\langle T_M f_1, f_2 \rangle = \sum_{j=1}^{\lambda} \int_{B_j\backslash G(\mathbb{R})} f_1(h_j g)\bar{f}_2(g)dg . \tag{2.13}$$

Of course if $h_j \in G(\mathbb{Q})$, which is the case for us, then B_j is a congruence subgroup of $G(\mathbb{Z})$.

Similarly, if $\Delta\backslash H(\mathbb{R})$ is a cycle as above, M decomposes into Δ-orbits:

$$M = \bigsqcup_{i=1}^{\lambda} \Gamma h_i \Delta .$$

Let $\Delta_i = \{h \in \Delta : \Gamma h_i h = \Gamma h_i\}$ then:

$$\langle T_M(\mu), f \rangle = \sum_{j=1}^{\lambda} \int_{\Delta_j\backslash H(\mathbb{R})} \bar{f}(h_j h)dh . \tag{2.14}$$

Again for our H and T_M, $\Delta_j \subset H(\mathbb{Z})$ is a congruence subgroup.

3 Proof of Theorem 1.1

We begin with part (a). Let $\Gamma < G(\mathbb{Z})$ be a congruence subgroup and $f \in C_0(\Gamma \backslash G(\mathbb{R}))$. Correspondingly, we have the diagonal matrix coefficient

$$\psi(h) = \int_{\Gamma \backslash G(\mathbb{R})} f(g)\overline{f(gh)}dg . \tag{3.1}$$

To prove (a) it suffices to show that $\psi(h)$ is a limit (uniform on compacta of $H(\mathbb{R})$) of diagonal matrix coefficients of representations of $H(\mathbb{R})$ whose spectra lie in $\hat{H}_{\text{Aut}}$ ([D, 18.1]). By Lemma 2.3 we can write

$$\psi(h) = \lim_{m \to \infty} \langle \tilde{T}_m(\mu), R \rangle \tag{3.2}$$

where

$$R(g) = \frac{1}{\operatorname{Vol}(\Delta \backslash H(\mathbb{R}))} f(g)\overline{f(gh)} .$$

From (2.14) this means that

$$\psi(h) = \lim_{m \to \infty} \frac{1}{k^m} \sum_{j=1}^{\lambda^{(m)}} \int_{\Delta_j^{(m)} \backslash H(\mathbb{R})} f(h_j^{(m)} h_1) \bar{f}(h_j^{(m)} h_1 h) dh_1 . \tag{3.3}$$

In as much as the $\Delta_j^{(m)}$ above are congruence subgroups of $H(\mathbb{Z})$, it follows that each term in the sum in (3.3) is a diagonal matrix coefficient of a representation of $H(\mathbb{R})$ whose spectrum is in $\hat{H}_{\text{Aut}}$. Thus the same is true for the sum. Therefore $\operatorname{Res}_{H(\mathbb{R})} \rho_\Gamma \subset \hat{H}_{\text{Aut}}$. Γ being an arbitrary congruence subgroup of $G(\mathbb{Z})$, part (a) is established.

We turn to the proof of part (b). From Lemma 2.2(a) and (2.14) we have that for $u, v \in L^2(\Gamma \backslash G(\mathbb{R}))$

$$\lim_{m \to \infty} \frac{1}{k^m} \sum_{j=1}^{\lambda(m)} \int_{B_j^{(m)} \backslash G(\mathbb{R})} v(h_j^{(m)} g) u(g) dg = \int_{\Gamma \backslash G(\mathbb{R})} \int_{\Gamma \backslash G(\mathbb{R})} v(g_1) u(g_2) dg_1 dg_2 .$$

Hence it follows that for $h \in C_0(\Gamma \backslash G(\mathbb{R}) \times \Gamma \backslash G(\mathbb{R}))$

$$\lim_{m \to \infty} \frac{1}{k^m} \sum_{j=1}^{\lambda(m)} \int_{B_j^{(m)} \backslash G(\mathbb{R})} h(h_j^{(m)} g, g) dg = \int_{\Gamma \backslash G(\mathbb{R}) \times \Gamma \backslash G(\mathbb{R})} h(g_1, g_2) dg_1 dg_2 .$$

Let

$$\psi(g) = \int_{\Gamma \backslash G(\mathbb{R})} \int_{\Gamma \backslash G(\mathbb{R})} F(g_1, g_2) \overline{F(g_1 g, g_2 g)} dg_1 dg_2$$

with $F \in C_0(\Gamma \backslash G(\mathbb{R}) \times \Gamma \backslash G(\mathbb{R}))$, be a diagonal matrix coefficient of $\rho_\Gamma \otimes \rho_\Gamma$.

From the above

$$\psi(g) = \lim_{m \to \infty} \frac{1}{k^m} \sum_{j=1}^{\lambda(m)} \int_{B_j^{(m)} \backslash G(\mathbb{R})} F(h_j^{(m)} h, h) \overline{F(h_j^{(m)} hg, hg)} dh .$$

The terms of this sum are diagonal matrix coefficients of $\rho_{\Gamma'}$ for suitable congruence subgroups Γ' of $G(\mathbb{Z})$. Part (b) now follows as before. Q.E.D.

4

For notational simplicity we begin by assuming $k = \mathbb{Q}$. Let $e_1, \ldots, e_n$ be an orthogonal basis of $\mathbb{Q}^n$ such that

$$\begin{aligned} q(e_i) > 0 \qquad 1 \leqq i \leqq n-1 \\ q(e_n) < 0 \end{aligned} \tag{4.1}$$

and define

$$H = \{g \in \mathrm{SO}(q) \colon g(e_i) = e_i, 1 \leqq i \leqq n-4\} . \tag{4.2}$$

H is a $\mathbb{Q}$-subgroup of $G = \mathrm{SO}(q)$ and $H(\mathbb{Z})$ is a lattice in $H(\mathbb{R})$. Note that $H(\mathbb{R}) \cong \mathrm{SO}(3, 1)$.

Lemma 4.1 *Let H be the special orthogonal group of a quadratic form over $\mathbb{Q}$ in 4 variables such that $H(\mathbb{R}) \cong \mathrm{SO}(3, 1)$. Then*
(a) $\hat{H}_{\mathrm{Raman}} \subset i\mathbb{R}^+ \cup [0, \frac{1}{2}] \cup \{1\}$.
(b) *If the Ramanujan conjectures at ∞ hold for* $\mathrm{GL}(2, E)$ *where E is an imaginary quadratic extension of $\mathbb{Q}$ then*

$$\hat{H}_{\mathrm{Raman}} \subset i\mathbb{R}^+ \cup \{1\} .$$

Proof. This may be deduced from well known results as follows: Let $\mathrm{Spin}(q)$ be the spinor group associated to the quadratic form q, (see [Ca, p. 181]). Under the assumptions, $\mathrm{Spin}(q)$ may be identified (over $\mathbb{Q}$) with the elements of reduced norm 1 in a quaternion algebra A over E, where E is an imaginary quadratic extension of $\mathbb{Q}$. Moreover, the inverse image under the covering $\mathrm{Spin}(q) \to \mathrm{SO}(q)$ of a congruence subgroup of $\mathrm{SO}(q)(\mathbb{Z})$ is a congruence subgroup of $\mathrm{Spin}(q)$. (See for example [E-G-M, Prop. 3.1]). Thus to establish (a) and (b), it suffices to do so for the group of reduced norm 1 elements in a quaternion algebra over E. By using the Jacquet-Langlands correspondence [J-L], see also [V], we may reduce the problem to showing that (a) and (b) hold for $\Gamma \backslash \mathrm{SL}(2, \mathbb{C})$ where Γ is a congruence subgroup of $\mathrm{SL}(2, J)$ and J is the ring of integers of E. That (b) holds is now a tautology since this is precisely the assumption made. On the other hand, (a) has been established by Gelbart-Jacquet ([G-J, Th. 9.3(4)]), using the GL(2)–GL(3) lifting and by Sarnak [S] using Kloosterman sums. Q.E.D.

To complete the proof of Theorem 1.2, we use Theorem 1.1(a) and Lemma 4.1. Let $G = \mathrm{SO}(q)$, H as in (4.2) and $\pi \in \hat{G}_{\mathrm{Raman}}$, $\pi \neq 1$. Then:

$$\mathrm{Res}_{H(\mathbb{R})}\pi \subset \hat{H}_{\mathrm{Aut}} . \tag{4.3}$$

Let K be a maximal compact subgroup of $G(\mathbb{R})$ such that $K_0 = K \cap H(\mathbb{R})$ is maximal compact in $H(\mathbb{R})$. We may choose $A \subset H(\mathbb{R})$ a maximal $\mathbb{R}$-split torus such that $G(\mathbb{R}) = KAK$, $H(\mathbb{R}) = K_0 A K_0$. For the proof we may assume that $\pi = \pi_s$ with $0 < s < \rho$. Let φ_s be the associated spherical function. Then φ_s is bi-K_0-invariant of positive type and therefore

$$\varphi_s(h) = \int \varphi'_r(h) d\mu(r) \tag{4.4}$$

where μ is a probability measure on $\hat{H}(\mathbb{R})^1$ and φ'_r is the spherical function of $H(\mathbb{R})$ corresponding to the parameter $r \in i\mathbb{R}^+ \cup [0, 1]$. It follows from (4.3) and Lemma 4.1(a) that:

$$\text{support}\, \mu \subset \hat{H}_{\text{Raman}} \subset i\mathbb{R}^+ \cup [0, \tfrac{1}{2}] \cup \{1\} .$$

Notice that $\mu(\{1\}) = 0$ since π has no $H(\mathbb{R})$-invariant vectors. Therefore

$$\text{support}\, \mu \subset i\mathbb{R}^+ \cup [0, \tfrac{1}{2}] . \tag{4.5}$$

Let $X \in \text{Lie}(A)$, X of norm 1 w.r.t. the Killing form of Lie $G(\mathbb{R})$. It is well known (see [G-V, 5.1]) that
(a) $|\varphi'_r(\exp tX)| \leqq \varphi'_0(\exp tX)$, $r \in i\mathbb{R}^+$, $t \geqq 0$.
(b) $\varphi'_r(\exp tX) \leqq C(1 + t)e^{(r-1)t}$, $0 \leqq r \leqq 1$, $t \geqq 0$, and C is an absolute constant.
From this follows that for all $t \geqq 0$:

$$\varphi_s(\exp tX) = \left| \int \varphi'_r(\exp tX)\, d\mu(r) \right| \leqq C(1 + t)e^{-t/2} . \tag{4.6}$$

Observe that if $s \in [0, \rho]$, φ_s is positive. On the other hand, φ_s as a spherical function on $G(\mathbb{R})$ has the following behavior as $t \to +\infty$ (see [G-V, 5.1]):

$$\varphi_s(\exp tX) \sim C_s e^{(s-\rho)t}, \quad 0 < s < \rho .$$

This together with (4.6) imply $s \leqq \rho - \frac{1}{2}$ which proves (a). In the same way, if the Ramanujan conjecture holds for GL(2, E), then we conclude $s \leqq \rho - 1$.

The proof of Theorem 1.2 for q a quadratic form over an arbitrary number field k is similar. The only comment that need be made is that Spin(q) will be identified with a quaternion algebra over a quadratic extension E/k. The Gelbart-Jacquet result mentioned earlier then may be used to deduce (a) in this case. For (b), assuming the Ramanujan conjectures for GL(2, E), E an arbitrary number field, will suffice.

5

This section is devoted to a discussion of Hecke operators and uniform distribution of Hecke points in the light of recent results of M. Ratner, ([R]). It is also meant to illustrate the connection between the methods of Sect. 2 and certain questions of ergodic theory. For this reason we did not include proofs, therefore we are writing "Theorem 5.2" instead of Theorem 5.2.

First, we fix some notations. If X is a locally compact topological space with a continuous group action $X \times G \to X$ we let $M(X)$ denote the space of bounded measures with weak topology, $M^1(X)$ the space of probability measures, $C_0(X)$ the space of continuous functions vanishing at infinity, $M(X)^G$, $M^1(X)^G$ the space of G-invariant vectors in $M(X)$ resp $M^1(X)$. Let G be a simple connected Lie group. We are interested in the classification of Γ-invariant ergodic probability measures on $\Gamma \backslash G$. To relate this problem to the classification theorem of M. Ratner [R] we make the following observation:

Let $v \in M^1(\Gamma \backslash G)^G$, for $f \in C_0(\Gamma \backslash G \times \Gamma \backslash G)$ define

$$\tilde{v}(f) = \int_{\Gamma \backslash G} dg \int_{\Gamma \backslash G} dv(h) f(g, hg)$$

where dg is the G-invariant probability measure on $\Gamma\backslash G$. Clearly $\tilde{\nu}$ is a $\Delta(G)$ invariant probability measure on $\Gamma\backslash G\times\Gamma\backslash G$. Here $\Delta(G)$ denotes the diagonal subgroup of $G\times G$.

The following lemma is straightforward.

Lemma 5.1

$$M^1(\Gamma\backslash G)^\Gamma \to M^1(\Gamma\backslash G\times\Gamma\backslash G)^{\Delta(G)}$$

$$\nu \to \tilde{\nu}$$

is a homeomorphism.

Now it follows from Ratner's classification theorem and the fact that $\Delta(G)$ is maximal connected in $G\times G$ that any $\Delta(G)$-invariant probability measure is of the form

(1) $dg\times dg$
or
(2) $\Delta(G)$-invariant probability measure supported on the closed orbit $(z,e)\Delta(G)$ where $z\in\mathrm{Comm}(\Gamma)$.

Via Lemma 5.1 we deduce that any Γ-invariant ergodic probability measure on $\Gamma\backslash G$ is:
(1) dg
or
(2) μ_M, $\mu_M(f)=\frac{1}{|M|}\sum_{m\in M} f(m)$, where $M\subset\Gamma\backslash G$ is a finite Γ-orbit.

This classification has an interesting consequence for intertwining operators. Namely, let ρ be the regular representation of G in $C_0(\Gamma\backslash G)$ and $\mathrm{Int}\,C_0(\Gamma\backslash G)$ the space of continuous intertwining operators with strong topology. It is plain that the map

$$M(\Gamma\backslash G)^\Gamma \to \mathrm{Int}\,C_0(\Gamma\backslash G)$$

$$\mu \to T_\mu$$

defined by $T_\mu(f)(g)=\mu(\rho(g)f)$ is a homeomorphism of topological vector spaces. Remark that if $\mu=dg$ then $T_\mu=P$ the projection onto the space of constant functions. Also, if $\mu=\mu_M$ where $M=\Gamma$ orbit of $\Gamma\gamma$, $\gamma\in\mathrm{Comm}\,\Gamma$ then $T_\mu=\tilde{T}_\gamma$ the normalized Hecke operator. It follows now from the above classification theorem that any intertwining operator is limit of linear combinations of Hecke operators and the projection onto the constants.

Concerning the uniform distribution of Hecke points one may obtain the following theorem using the results and methods of M. Ratner:

"Theorem 5.2" *Let $(x_n)_{n=1}^\infty\subset\Gamma\backslash\mathrm{Comm}\,\Gamma$ and assume that*

$$\lim_{n\to\infty} x_n = x\notin\Gamma\backslash\mathrm{Comm}\,\Gamma\,.$$

Then for $f\in C_0(\Gamma\backslash G)$, $\tilde{T}_{x_n}f\to\int_{\Gamma\backslash G} f(g)dg$ uniformly on compact sets.

References

[A] Arthur, J.: On Some Problems Suggested by the Trace Formula. (Lect. Notes Math., vol. 1041, pp. 1–50) Berlin Heidelberg New York: Springer 1983

[BS] Borel, A., Serre, J.P.: Cohomologie d'immeubles et de Groupes S-arithmétiques. Topology **15**, 211–232 (1976)

[BLSI] Burger, M., Li, J.S., Sarnak, P.: Ramanujan Duals and automorphic Spectrum. (Preprint)

[Ca] Cassels, J.W.S.: Rational Quadratic Forms. Reading, MA: Academic Press 1978

[Dd] Dixmier, J.: C*-Algebras. Amsterdam: North Holland 1982

[E-G-M] Elstrodt, J., Grunewald, F., Mennicke, J.: Kloosterman Sums for Clifford Algebras and a Lower Bound for the Positive Eigenvalues of the Laplacian for Congruence Subgroups Acting on Hyperbolic Spaces. Invent. Math. **101**, 641–685 (1990)

[G-J] Gelbart, S.S., Jacquet, H.: A Relation Between Automorphic Representations of GL(2) and GL(3). Ann. Sci. Éc. Norm. Supér., IV. Sér. II., 471–542 (1978)

[G-V] Gangolli, R., Varadarajan, V.S.: Harmonic Analysis of Spherical Functions on Real Reductive Groups. (Ergeb. Math. Grenzgeb., vol. 101) Berlin Heidelberg New York: Springer 1988

[J-L] Jacquet, H., Langlands, R.P.: Automorphic Forms on GL(2). (Lect. Notes Math., vol. 114) Berlin Heidelberg New York: Springer 1970

[K] Kneser, M.: Strong Approximation, in Algebraic Groups and Discontinuous Subgroups. Proc. Symp. Pure Math. IX., 187–196 (1966)

[Ko] Kostant, B.: On the Existence and Irreducibility of a Certain Series of Representations. Bull. Am. Math. Soc. **75**, 627–642 (1969)

[L-P-S] Li, J.S., Piatetskii-Shapiro, I.I., Sarnak, P.: Poincaré Series for $SO(n, 1)$. Proc. Indian Acad. Sci., Math. Sci. **97**, 231–237 (1987)

[P] Platonov, V.P.: Arithmetic Theory of Algebraic Groups. Russ. Math. Surv. **37**, 1–62 (1982)

[R] Ratner, M.: On Measure Rigidity of Unipotent Subgroups of Semisimple Groups. (Preprint)

[S] Sarnak, P.: The Arithmetic and Geometry of Some Hyperbolic 3-manifolds. Acta Math. **151**, 253–295 (1983)

[San] Sansuc, J.J.: Groupe de Brauer et Arithmétique des Groupes Algébriques Linéaires sur un Corps de Nombres. J. Reine Angew. Math. **327**, 12–80 (1981)

[V] Vigneras, M.F.: Quelques Remarques sur la Conjecture $\lambda_1 \geqq \frac{1}{4}$. Séminaire de Théorie des Nombres, Paris 1981–82. (Prog. Math.) Boston Basel Stuttgart: Birkhäuser