INTERSECTION, THE MANHATTAN CURVE, AND PATTERSON-SULLIVAN THEORY IN RANK 2

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Let Γ be a nonvirtually abelian, finitely generated group and let $\mathrm{Rep}_{CC}(\Gamma)$ denote the set of injective homomorphisms $\pi: \Gamma \to G$, of Γ into the group of isometries G of a symmetric space X of rank one such that

(1) $\pi(\Gamma)$ is torsion-free

(2) the $\pi(\Gamma)$-action on X is properly discontinuous and convex cocompact.

Our aim is to describe invariants attached to pairs π_1, π_2 of convex cocompact realizations of Γ which will lead to criteria when the isomorphism

$$\pi_2\pi_1^{-1}: \Gamma_1 \to \Gamma_2$$

extends to an isomorphism $G_1 \to G_2$ of the ambient Lie groups.

Examples. (1) The free group on two generators $\Gamma = \mathbb{F}_2$ has nonhomeomorphic convex cocompact realizations in dimension 2, e.g., a thrice-punctured sphere with three expanding ends and a once-punctured torus with one expanding end.

(2) Let $\Gamma < SO(n, 1)$ be a cocompact lattice and assume that $\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$ contains at least one totally geodesic embedded codimension-one submanifold with trivial normal bundle. Then Γ admits nontrivial convex cocompact deformations into $SO(n + 1, 1)$ [JM].

At the end of §2 we indicate how our results generalize to arbitrary negative curvature.

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1. Intersection and the Manhattan curve. For notation and definitions we refer to §3.

The set $\mathscr{C}$ of Γ-conjugacy classes in $\Gamma - \{e\}$ parametrizes the set of closed geodesics of any convex cocompact realization of Γ. Therefore two convex cocompact realizations $\pi_i: \Gamma \xrightarrow{\sim} \Gamma_i \subset G_i$, $i = 1, 2$, give rise to two length functions

$$\ell_i: \mathscr{C} \to \mathbb{R}^+$$

which we now use to define the following basic invariants attached to (π_1, π_2):

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(a) Geodesic stretch.

$$\mathrm{dil}_+(\pi_1, \pi_2) := \sup_{\mathscr{C}} \frac{\ell_2(c)}{\ell_1(c)}, \qquad \mathrm{dil}_-(\pi_1, \pi_2) := \inf_{\mathscr{C}} \frac{\ell_2(c)}{\ell_1(c)}.$$

For Γ, a fundamental group of a compact surface, and $G = PSL(2, \mathbb{R})$, these invariants pertain to the minimal stretch map point of view of Teichmüller theory, as developed by W. Thurston [T].

(b) Intersection. Let μ_1 be the Patterson-Sullivan measure on the set $\Omega_1 \subset T_1(\Gamma_1 \backslash X_1)$ of recurrent points of the geodesic flow and let $(c_n)_{n \in \mathbb{N}}$ be a sequence of $\mathscr{C}$ such that the corresponding sequence of closed geodesics in Ω_1 is equidistributed with respect to μ_1. The intersection $i(\pi_1, \pi_2)$ is defined as

$$i(\pi_1, \pi_2) := \lim_{n \to \infty} \frac{\ell_2(c_n)}{\ell_1(c_n)}.$$

It measures the distortion under Morse correspondence of the length of a typical geodesic in Ω_1.

In the case of compact surfaces, W. Thurston proved, using the convexity along earthquake paths of the geodesic length function, that $i(\pi_1, \pi_2) \geqslant 1$ with equality if and only if π_1, π_2 represent the same point in Teichmüller space. The corollary of Theorem 1 below generalizes this result.

(c) The Manhattan curve. This is the continuous convex curve $\mathscr{C}_M(\pi_1, \pi_2)$ bounding the following convex subset of $\mathbb{R}^2$:

$$\left\{(a, b) \in \mathbb{R}^2 : \sum_{c \in \mathscr{C}} e^{-[a\ell_1(c) + b\ell_2(c)]} < +\infty\right\}.$$

Observe that the points $(\delta_1, 0)$ and $(0, \delta_2)$ where $\delta_i :=$ critical exponent of Γ_i, always belong to $\mathscr{C}_M(\pi_1, \pi_2)$.

Now let $\mathrm{Rep}^Z_{CC}(\Gamma)$ denote the set of $\pi \in \mathrm{Rep}_{CC}(\Gamma)$ such that $\pi(\Gamma)$ is Zariski-dense in G. Equivalently, $\pi(\Gamma)$ does not preserve any proper totally geodesic subspace of X.

THEOREM 1. (a) *For $\pi_1, \pi_2 \in \mathrm{Rep}^Z_{CC}(\Gamma)$, the Manhattan curve is the straight line connecting $(\delta_1, 0)$ to $(0, \delta_2)$ if and only if $\pi_2 \pi_1^{-1} : \Gamma_1 \to \Gamma_2$ extends to an isomorphism $G_1 \to G_2$ of the ambient Lie groups, in which case $\Gamma_1 \backslash X_1$, $\Gamma_2 \backslash X_2$ are isometric and $\delta_1 = \delta_2$.*

(b) *The Manhattan curve is C^1. It has two asymptotes whose normals have slope $\mathrm{dil}_-(\pi_1, \pi_2)$ at $-\infty$ and $\mathrm{dil}_+(\pi_1, \pi_2)$ at $+\infty$. Moreover, the slope of the normal to $\mathscr{C}_M(\pi_1, \pi_2)$ at $(\delta_1, 0)$ is the intersection $i(\pi_1, \pi_2)$.*

COROLLARY. *Under the hypothesis of Theorem 1, we have*

$$i(\pi_1, \pi_2) \geqslant \frac{\delta_1}{\delta_2}$$

with equality if and only if $\pi_2\pi_1^{-1}\colon \Gamma_1 \to \Gamma_2$ *extends.*

Remark. The definition of the Manhattan curve was motivated by [BS], and Theorem 1(a) generalizes Theorem 1 of [BS].

The following fact is used in the proof of Theorem 1(a).

PROPOSITION 1. *Let* G_i, $i = 1, 2$ *be rank-one connected adjoint Lie groups,* $\Gamma_i \subset G_i$ *Zariski-dense subgroups consisting of hyperbolic elements, and* $\Theta\colon \Gamma_1 \to \Gamma_2$ *an isomorphism such that* $L(\Theta(\gamma)) = c \cdot L(\gamma)$, $\forall \gamma \in \Gamma_1$, *where* $c > 0$ *is a constant and* $L(\ )$ *denotes the translation length of a hyperbolic element. Then* $c = 1$ *and* Θ *extends to an isomorphism* $\Theta_{\mathrm{ext}}\colon G_1 \to G_2$ *of the corresponding Lie groups.*

2. The Manhattan curve and Patterson theory in rank 2. The properties of the Manhattan curve, stated in §1, are intimately connected with recurrence properties of the geodesic flow on a rank-2 manifold, which we now define. To $\pi_1, \pi_2 \in \mathrm{Rep}_{CC}(\Gamma)$, we associate the diagonal action of Γ on $X := X_1 \times X_2$:

$$\gamma_*(x_1, x_2) := (\pi_1(\gamma)x_1, \pi_2(\gamma)x_2), \qquad \gamma \in \Gamma,$$

and the (infinite volume) quotient manifold $M := \Gamma\backslash X$. This manifold fibers over $\Gamma_1\backslash X_1$ with fiber X_2, and over $\Gamma_2\backslash X_2$ with fiber X_1. In order to study the set of recurrent points of the geodesic flow on $T_1(M)$, we first describe the limit set $\Lambda \subset X(\infty)$ of Γ in the ideal boundary $X(\infty)$ of X.

Recall that $X(\infty)$ is the set of equivalence classes of parametrized geodesic rays in X, two such rays being equivalent if they stay at bounded distance. We have $X(\infty) = X(\infty)_{\mathrm{sing}} \bigsqcup X(\infty)_{\mathrm{reg}}$, where $X(\infty)_{\mathrm{reg}}$ is the set of rays which are contained in a unique maximal flat subspace of X. Furthermore, for every $\lambda \in (0, \infty)$, the set $X(\infty)_\lambda$ consisting of all rays of slope λ w.r.t. the canonical splitting $X = X_1 \times X_2$ is a closed $(G_1 \times G_2)$-orbit in $X(\infty)$ and

$$X(\infty)_{\mathrm{reg}} = \bigsqcup_{\lambda \in (0,\infty)} X(\infty)_\lambda .$$

Observe that there is a $(G_1 \times G_2)$-equivariant identification $X(\infty)_\lambda \tilde{\to} X_1(\infty) \times X_2(\infty)$, and let $(\mathrm{Graph}\ \varphi)_\lambda$ be the image in $X(\infty)_\lambda$ of the graph of the Mostow map $\varphi\colon \Lambda_1 \to \Lambda_2$ under this identification (see §3.2).

The limit set $\Lambda \subset X(\infty)$ of Γ has the following description:

PROPOSITION 2.

$$\Lambda = \bigsqcup_{\lambda \in F} (\mathrm{Graph}\ \varphi)_\lambda, \qquad \textit{where } F := [\mathrm{dil}_-(\pi_1, \pi_2), \mathrm{dil}_+(\pi_1, \pi_2)].$$

For every $\lambda \in [0, \infty]$, the subbundle $T_1^\lambda(M) \subset T_1(M)$ consisting of all unit vectors of slope λ is a closed subset of $T_1(M)$, invariant under the action of the geodesic flow. Denote by $R_\lambda \subset (\text{Graph } \varphi)_\lambda$ the set of geodesic rays of slope λ which project to recurrent rays in $T_1^\lambda(M)$.

THEOREM 2. (a) *The map* $\mathscr{C}_M(\pi_1, \pi_2) \to (\text{dil}_-(\pi_1, \pi_2), \text{dil}_+(\pi_1, \pi_2))$ *which to each* $(a, b) \in \mathscr{C}_M$ *associates the slope* λ *of the normal at* (a, b) *is a homeomorphism.*

(b) *For* $\lambda \in (\text{dil}_-(\pi_1, \pi_2), \text{dil}_+(\pi_1, \pi_2))$ *the Hausdorff dimension of* R_λ *is given by*

$$(a + \lambda b)\cdot \max(1, \lambda^{-1})$$

where $(a, b) \in \mathscr{C}_M$ *corresponds to* λ *under the above homeomorphism.*

The proofs of Theorems 1 and 2 use mainly the following version of Patterson-Sullivan theory:

$$\text{let } \mathscr{C}_R := \Bigg\{(\alpha, \beta) \in \mathbb{R}^2\text{: such that the critical exponent of the Poincaré series}$$

$$Q_{\alpha,\beta}(s) := \sum_{\gamma\in\Gamma} e^{-s\cdot\sqrt{\alpha d^2(\pi_1(\gamma)x_1, x_1)+\beta d^2(\pi_2(\gamma)x_2, x_2)}} \text{ is at } s = 1\Bigg\}.$$

For every $(\alpha, \beta) \in \mathscr{C}_R$ we get from $Q_{\alpha,\beta}(s)$, using Patterson's construction, a positive measure $\Pi_{\alpha,\beta}$ supported on $\Lambda = \bigsqcup_{\lambda\in F} (\text{Graph } \varphi)_\lambda$.

THEOREM 3. (a) *For every* $(\alpha, \beta) \in \mathscr{C}_R$ *there is a unique* $\lambda \in F$ *such that* $\Pi_{\alpha,\beta}$ *has* $(\text{Graph } \varphi)_\lambda$ *as its support. This* λ *is characterized as the unique one for which*

$$\left(\frac{\alpha}{\sqrt{\alpha + \beta\lambda^2}}, \frac{\beta\lambda}{\sqrt{\alpha + \beta\lambda^2}}\right) \in \mathscr{C}_M(\pi_1, \pi_2).$$

(b) $\Pi_{\alpha,\beta}$ *gives full measure to* R_λ.

Remark. The assertions of Theorems 2 and 3 hold in pinched-variable negative curvature. In this setting the conclusion of Theorem 1 and the corollary is that the length functions ℓ_1, ℓ_2 are proportional. This generalizes Theorem 1 of [La].

3. Preliminaries.

3.1. Let X be a symmetric space of rank one, G its group of isometries, and $\Pi \subset G$ a torsion-free nonelementary convex cocompact subgroup. Let $\Lambda \subset X(\infty)$ be the limit set of Π and $C(\Lambda) \subset X$ its convex hull. In particular $\Pi\backslash C(\Lambda)$ is compact. Fix $o \in X$ a base point and define $d(\xi, \eta) := e^{-(\xi\cdot\eta)_o}$, $\xi, \eta \in X(\infty)$, where $(\xi\cdot\eta)_0$ is the Gromov scalar product relative to the base point o ([Gr], [Gh, H]). Although d is not in general a distance, there exists $C > 0$, depending only on X, such that $d(\xi, \eta) \leqslant C\cdot\max(d(\xi, \alpha), d(\alpha, \eta))\,\forall \xi, \eta, \alpha \in X(\infty)$. Set $B(\xi, r) := \{\eta \in X(\infty)\text{: } d(\xi, \eta) \leqslant$

$r\}$. Using these balls, we have the notion of Hausdorff dimension and Hausdorff measure on $X(\infty)$. For instance, $HD(X(\infty)) = 2\rho$ where ρ = half-sum of positive roots of G. The Patterson measure m is the unique (up to scaling) positive bounded measure on Λ such that $d(\pi_* m)(\xi) = e^{-\delta\beta_\xi(\pi \cdot 0)}\, dm(\xi)\ \forall \pi \in \Pi$, where δ = the critical exponent of Π and $\beta_\xi(x)$ is the Busemann functon on $X(\infty) \times X$ normalized by $\beta_\xi(o) = 0$. The Patterson measure class coincides with the Hausdorff measure class of dimension δ on Λ, and $HD(\Lambda) = \delta$. Let $\Omega \subset T_1(\Pi\backslash X)$ be the set of points which are recurrent, for positive and negative time, under the action of the geodesic flow. This set is compact and carries a unique invariant ergodic probability measure μ (Patterson-Sullivan measure) of maximal entropy δ. This measure μ is gotten from the Π-invariant measure

$$\frac{dm(\xi)\, dm(\eta)}{e^{2\delta(\xi \cdot \eta)_0}} \qquad \text{on } \Lambda \times \Lambda.$$

3.2. Given $\pi_i\colon \Gamma \tilde{\rightarrow} \Gamma_i \subset G_i$, $i = 1, 2$, convex cocompact realizations of Γ, and $\Lambda_i \subset X_i(\infty)$ their respective limit sets, we have the Mostow map $\varphi\colon \Lambda_1 \to \Lambda_2$ which is the unique $\pi_2\pi_1^{-1}\colon \Gamma_1 \to \Gamma_2$ equivariant homeomorphism from Λ_1 to Λ_2. It has the following property:

LEMMA 1. *φ is quasi-conformal: for all $\xi \in \Lambda_1$ and $r > 0$ there is $r' > 0$ such that*

$$B_2(\varphi(\xi), C^{-1}r') \subseteq \varphi(B_1(\xi, r)) \subset B_2(\varphi(\xi), Cr')$$

where $C > 0$ is an absolute constant and $B_i(\xi, r) := B(\xi, r) \cap \Lambda_i$.

3.3. Let $f\colon \Gamma_1\backslash C(\Lambda_1) \to \Gamma_2\backslash C(\Lambda_2)$ be a homotopy equivalence inducing $\pi_2\pi_1^{-1}$: $\Gamma_1 \to \Gamma_2$. For $t \geqslant 0$ and $v \in \Omega_1$, let $C_{t,v} \subset \Gamma_2\backslash C(\Lambda_2)$ be the curve which is the image by f of the geodesic starting at v and of length t. Let $\varphi_t(v)$ be the length of the unique geodesic arc homotopic to $C_{t,v}$. Then $\varphi_t(v)$ is a subadditive cocycle (triangle inequality), and there exists $C > 0$ such that $\varphi_t - C$ is superadditive (quasi-geodesic lemma). It is easy to see that the formula

$$i(\pi_1, \pi_2) = \lim_{t\to\infty} \frac{1}{t} \int_{\Omega_1} \varphi_t(v)\, d\mu_1(v)$$

holds.

4. Sketch of proofs.

4.1. Choose base points $x_i \in X_i$. A slight modification of an argument of Knieper [Kn] shows that the Manhattan curve $\mathscr{C}_M$ coincides with the set of $(a, b) \in \mathbb{R}^2$ for which the Poincaré series

$$P_s(a, b) := \sum_{\gamma\in\Gamma} e^{-s[ad(\pi_1(\gamma)x_1, x_1) + bd(\pi_2(\gamma)x_2, x_2)]}$$

has critical exponent $s = 1$. We compactify $X_1 \times X_2$ by $\overline{X}_1 \times \overline{X}_2$, where $\overline{X}_i := X_i \cup X_i(\infty)$, and observe that in this compactification the limit set of the diagonal Γ-action on $X_1 \times X_2$ is Graph $\varphi \subset \Lambda_1 \times \Lambda_2$ where φ is the Mostow map. We call a measure μ on $\Lambda_1 \times \Lambda_2$ (a, b)-dimensional if

(1) supp $\mu \subset$ Graph φ;
(2) $d(\gamma_* \mu)(\xi_1, \xi_2) = e^{-a\beta_{\xi_1}(\pi(\gamma_1)x_1) - b\beta_{\xi_2}(\pi(\gamma_2)x_2)}\, d\mu(\xi_1, \xi_2)$ for all $\gamma \in \Gamma$.

THEOREM 4. *$\mathscr{C}_M$ is the set of points in $\mathbb{R}^2$ for which there exists an (a, b)-dimensional measure. Furthermore, for every $(a, b) \in \mathscr{C}_M$ this measure is unique up to scaling and is therefore ergodic.*

The existence follows from Patterson's construction using the Poincaré series $P_s(a, b)$. The uniqueness follows in a standard way from the local behavior of (a, b)-dimensional measures:

LEMMA 2. *Let μ be an (a, b)-dimensional measure. Then there are constants $0 < c_1 \leqslant c_2 < +\infty$ such that*

$$c_1 r^a r'^b \leqslant \mu_{a,b}(B(\xi, r) \times \varphi(B(\xi, r))) \leqslant c_2 r^a r'^b$$

$\forall \xi \in \Lambda_1$, $r > 0$, and r' is given by Lemma 1.

Proof. Use Sullivan's shadowing technique. □

Denote by $\mu_{a,b}$ the unique (a, b)-dimensional probability measure.

Proof of Theorem 1(a). Assume that $\mathscr{C}_M$ is a straight line and pick $(a, b) = (\delta_1/2, \delta_2/2) \in \mathscr{C}_M$. Let ν_i be the projection of $\mu_{a,b}$ on Λ_i, and hence $\varphi_* \nu_1 = \nu_2$. Vitali's covering lemma, Cauchy Schwarz, and Lemma 2 imply then that ν_i is equivalent to Patterson measure m_i on Λ_i. Hence if $d\alpha_i := dm_i(\xi)\, dm_i(\eta)/e^{2\delta_i(\xi,\eta)_{x_i}}$, we have $(\varphi \times \varphi)_* \alpha_1 = c\alpha_2$ for some constant $c > 0$. From this it follows easily that there are constants $0 < c_1 \leqslant c_2 < +\infty$ such that

$$c_1 \leqslant \frac{d(\varphi(\xi), \varphi(\eta))^{\delta_2}}{d(\xi, \eta)^{\delta_1}} \leqslant c_2 \qquad \forall \xi, \eta \in \Lambda_1,$$

and hence

$$\frac{\ell_2(c)\delta_2}{\ell_1(c)\delta_1} = 1 \qquad \forall c \in \mathscr{C}.$$

Theorem 1(a) follows then from Proposition 1. □

4.2. Proof of Theorem 3(a). For every $(\alpha, \beta) \in \mathscr{C}_R$, Patterson's construction using the Poincaré series $Q_{\alpha,\beta}(s)$ produces a measure $\Pi_{\alpha,\beta}$ supported on $\Lambda = \bigsqcup_{\lambda \in F} (\text{Graph } \varphi)_\lambda$. Desintegrating $\Pi_{\alpha,\beta}$ along F, we get measures $\mu_{\alpha,\beta}(\lambda)$ on Graph φ and a computation shows that they are $(a_\lambda, b_\lambda) := (\alpha/\sqrt{\alpha + \beta\lambda^2}, \beta\lambda/\sqrt{\alpha + \beta\lambda^2})$-dimensional. Hence (a_λ, b_λ) lies on the intersection of the conic $x^2/\alpha + y^2/\beta = 1$ and the Manhattan curve $\mathscr{C}_M$ (Theorem 4).

Assume for simplicity $\alpha > 0$, $\beta > 0$. Then the above ellipse cannot meet the interior of

$$\left\{(a, b) \in \mathbb{R}^2 \colon \sum_{\gamma \in \Gamma} e^{-[ad(\pi_1(\gamma)x_1, x_1) + bd(\pi_2(\gamma)x_2, x_2)]} < +\infty\right\}.$$

Indeed, $(\alpha, \beta) \in \mathscr{C}_R$ and for any point (a, b) on this ellipse, we have $ad_1 + bd_2 \leqslant \sqrt{\alpha d_1^2 + \beta d_2^2}$, where $d_i = d(\pi_i(\gamma)x_i, x_i)$. This proves Theorem 3(a). □

4.3. Recall that for $(a, b) \in \mathscr{C}_M$, $\mu_{a,b}$ denotes the unique (a, b)-dimensional probability measure on Graph φ. The uniqueness statement in Theorem 4 implies that the map

$$\mathscr{C}_M \to M^1(\text{Graph } \varphi)$$

$$(a, b) \mapsto \mu_{a,b}$$

is continuous. Let $\mu_{a,b}^{(1)}$ be the projection of $\mu_{a,b}$ on Λ_1. From the Γ_1-invariant measure on $\Lambda_1 \times \Lambda_1$

$$\frac{d\mu_{a,b}^{(1)}(\xi)\, d\mu_{a,b}^{(1)}(\eta)}{e^{2a(\xi \cdot \eta)_{x_1}} e^{2b(\varphi(\xi) \cdot \varphi(\eta))_{x_2}}},$$

we deduce a probability measure $\nu_{a,b}$ on $\Omega_1 \subset T_1(\Gamma_1 \backslash X_1)$ which is invariant under the geodesic flow and, by Hopf's argument, ergodic since $\mu_{a,b}$ is. Moreover

$$\mathscr{C}_M \to M^1(\Omega_1)$$

$$(a, b) \mapsto \nu_{a,b}$$

is continuous. Let

$$\lambda_{a,b} := \lim_{t \to \infty} \frac{1}{t} \int_{\Omega_1} \varphi_t(v)\, d\nu_{a,b}(v),$$

where $\varphi_t(v)$ is the cocycle defined in §3.3. From the properties of $\varphi_t(v)$ stated in §3.3, we deduce easily that the map $(a, b) \to \lambda_{a,b}$ is continuous as well. Observe that for $(a, b) = (\delta_1, 0)$ we have $\mu_{a,b} = \mu_1$ and $\lambda_{a,b} = i(\pi_1, \pi_2)$.

The following lemma was provided by S. Mozes:

LEMMA 3. *Let α be a geodesic flow-invariant ergodic probability measure on Ω_1 and*

$$\lambda := \lim_{t \to +\infty} \frac{1}{t} \int_{\Omega_1} \varphi_t(v)\, d\alpha(v).$$

For almost every $v \in \Omega_1$, there exists a sequence $t_n \to +\infty$ and a constant $K > 0$ such that

$$|\varphi_{t_n}(v) - \lambda t_n| \leqslant K \qquad \forall n \in \mathbb{N}.$$

Proof of Theorem 1(b). In the proof of Theorem 3(a) we constructed a continuous map $\mathscr{C}_R \to \mathscr{C}_M$. Theorem 1(b) amounts essentially to showing that this map has a continuous inverse.

Let $(a, b) \in \mathscr{C}_M$ and assume for simplicity that $a > 0$, $b > 0$. From the local behavior of $\mu_{a,b}^{(1)}$ (Lemma 2) and Lemma 3, it follows that there is a constant $c > 0$ such that

$$\sum_{\gamma \in \mathscr{S}} e^{-[ad(\pi_1(\gamma)x_1, x_1) + bd(\pi_2(\gamma)x_2, x_2)]} = +\infty$$

where

$$\mathscr{S} = \{\gamma \in \Gamma : |d(\pi_2(\gamma)x_2, x_2) - \lambda_{a,b} d(\pi_1(\gamma)x_1, x_1)| \leqslant c\}.$$

Let $\alpha > 0$, $\beta > 0$ be the unique solution of

$$a = \frac{\alpha}{\sqrt{\alpha + \beta\lambda^2}}, \qquad b = \frac{\beta\lambda}{\sqrt{\alpha + \beta\lambda^2}}, \qquad \text{where } \lambda = \lambda_{a,b}.$$

Then there is $K > 0$ such that for all $\gamma \in \mathscr{S}$

$$\sqrt{\alpha d_1^2 + \beta d_2^2} \leqslant ad_1 + bd_2 + K$$

where $d_i = d(\pi_i(\gamma)x_i, x_i)$, whereas the inequality

$$ad_1 + bd_2 \leqslant \sqrt{\alpha d_1^2 + \beta d_2^2}$$

always holds. This shows that $(\alpha, \beta) \in \mathscr{C}_R$, and $\lambda = \lambda_{a,b}$ is the number associated to (α, β) by Theorem 3(a). In particular, at smooth points of $\mathscr{C}_M$, $\lambda_{a,b}$ is the slope of the normal at (a, b). Since $(a, b) \to \lambda_{a,b}$ is continuous, $\mathscr{C}_M$ is C^1. We already observed that $\lambda_{\delta_1, 0} = i(\pi_1, \pi_2)$.

The remaining assertions in Theorem 1(b) are easy. □

Details of the proofs of Theorem 2 and Theorem 3(b) will be published later on.

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