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A lower bound on λ_0 for geometrically finite hyperbolic n-manifolds

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1. Introduction

Let N be a complete hyperbolic n-manifold and $C(N)$ its convex core. The manifold N is said to be geometrically finite if the metric neighborhood $C_1(N)$ of radius one about $C(N)$ is of finite volume (see Bowditch [2] for equivalent definitions). The Laplace-Beltrami operator Δ of N acts on the space of C^∞-functions with compact support and admits a unique extension to an unbounded self-adjoint operator on $L^2(N)$. Let $\lambda_0(N)$ denote the bottom of the L^2-spectrum of $-\Delta$. (Notice that $\lambda_0(N) \geqq 0$.) When N is geometrically finite, $\lambda_0(N) = 0$ if and only if N has finite volume. Here we are interested in infinite volume geometrically finite hyperbolic manifolds and our main result is a lower bound on $\lambda_0(N)$ in terms of the volume of $C_1(N)$, provided $n \geqq 3$.

Main Theorem. *For all $n \geqq 3$, there exists a constant $K_n > 0$ (depending only on n) such that if N is an infinite volume, geometrically finite hyperbolic n-manifold, then*

$$\lambda_0(N) \geqq \frac{K_n}{\mathrm{vol}\,(C_1(N))^2}$$

where $\mathrm{vol}\,(C_1(N))$ *denotes the volume of the neighborhood $C_1(N)$ of radius one of the convex core.*

Observe that in dimension 2 the above inequality does not hold. Indeed by pinching all boundary geodesics of the convex core $C(N)$ one can make $\lambda_0(N)$ arbitrarily small, while $\mathrm{vol}\,(C_1(N))$ remains bounded. We add that in dimension 2, the dependence of $\lambda_0(N)$ on the geometry of N is well-understood (see Dodziuk-Pignataro-Randol-Sullivan [11] or Burger [4]).

Let $N = \mathbb{H}^n/\Gamma$ be a geometrically finite hyperbolic n-manifold and let D denote the Hausdorff dimension of the limit set L_Γ of Γ's action on the sphere at infinity of $\mathbb{H}^n$. Sullivan (see Theorem 2.17 in [20]) proved that either

$$\lambda_0(N) = (n-1)^2/4 \quad \text{and} \quad D \leqq (n-1)/2$$

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or

$$\lambda_0(N) = D((n-1) - D).$$

Combining Sullivan's result with our main theorem, we see that the volume of $C_1(N)$ provides an upper bound on the Hausdorff dimension of the limit set.

Corollary A. *Let N be a geometrically finite, infinite volume hyperbolic n-manifold and let D denote the Hausdorff dimension of L_Γ. Then*

$$D \leqq (n-1) - \frac{K_n}{(n-1)\operatorname{vol}(C_1(N))^2}$$

where K_n is the constant in the main theorem.

When $n = 3$, the main result should be contrasted with Theorem A from Canary [5], which asserts that there exists a constant A such that if N is a geometrically finite hyperbolic 3-manifold, then

$$\lambda_0(N) \leqq \frac{A|\chi(\partial C(N))|}{\operatorname{vol}(C(N))}$$

where $\chi(\partial C(N))$ denotes the Euler characteristic of the boundary $\partial C(N)$ of the convex core. (In fact, A may be taken to be 4π.) Thus, if N is a hyperbolic 3-manifold, the volume of the convex core provides bounds from above and below for $\lambda_0(N)$ and the Hausdorff dimension of the limit set.

Corollary B. *Let N be a geometrically finite, infinite volume hyperbolic 3-manifold and let D denote the Hausdorff dimension of L_Γ. If $\lambda_0(N) \neq 1$, then*

$$2 - \frac{4\pi|\chi(\partial C(N))|}{\operatorname{vol}(C(N))} \leqq D \leqq 2 - \frac{K}{\operatorname{vol}(C_1(N))^2}$$

where K is the constant K_3 obtained in the main theorem.

The condition $\lambda_0(N) \neq 1$ is not very restrictive. For example, if N is geometrically finite and $\lambda_0(N) = 1$, then N is either homeomorphic to the interior of a handlebody or to an $\mathbb{R}$-bundle over a closed surface (see Canary-Taylor [8], see also Sullivan [18] and Braam [3]).

Let N be a geometrically finite hyperbolic 3-manifold. We will see (Lemma 7.3) that given a, there exists L such that if $\partial C(N)$ contains no compressible curves with length $\leqq a$, then

$$\operatorname{vol}(C(N)) + 2\pi|\chi(\partial C(N))| \leqq \operatorname{vol}(C_1(N)) \leqq \operatorname{vol}(C(N)) + L|\chi(\partial C(N))|.$$

(A curve in $\partial C(N)$ is called *compressible* if it is homotopically trivial in N, but homotopically non-trivial in $\partial C(N)$.) So we obtain the following corollary of the main result:

Corollary C. *Given $a > 0$, there exists $L > 0$ such that if N is an infinite volume, geometrically finite hyperbolic 3-manifold and $\partial C(N)$ contains no compressible curves with length $\leqq a$, then*

$$\lambda_0(N) \geqq \frac{K}{(\mathrm{vol}(C(N)) + L|\chi(\partial C(N))|)^2}$$

where K is the constant K_3 obtained in the main theorem.

Corollary C may be phrased much more simply when every component of $\partial C(N)$ is incompressible. We recall that if N has no cusps, this is equivalent to Γ being freely decomposable. More generally, it is equivalent to Γ satisfying Bonahon's condition (B) (see Proposition 1.2 in Bonahon [1]). Recall that Γ is said to satisfy Bonahon's condition (B) if for every non-trivial free decomposition $A * B$ of Γ, there exists a parabolic element of Γ which is not conjugate into either A or B.

Corollary D. *There exists a constant $M > 0$ such that if $N = \mathbb{H}^3/\Gamma$ is an infinite volume, geometrically finite hyperbolic 3-manifold and Γ satisfies Bonahon's condition* (B), *then*

$$\lambda_0(N) \geqq \frac{K}{(\mathrm{vol}(C(N)) + M|\chi(\partial C(N))|)^2}$$

where K is the constant K_3 obtained in the main theorem.

Our main result is an analogue of results of Schoen [17], for closed hyperbolic n-manifolds, and Dodziuk-Randol [12], for finite volume hyperbolic n-manifolds. (In both cases $n \geqq 3$.) They proved, in these cases, that

$$\lambda_1(N) \geqq \frac{B_n}{\mathrm{vol}(N)^2}$$

for some constant B_n, depending only on n. Our proof will follow the outline of Dodziuk and Randol's proof, although it seems likely that a variant of Schoen's argument could also be made to work.

The extra element needed in our extension of Dodziuk and Randol's argument to the infinite volume setting is an analysis of the behavior of the eigenfunction corresponding to λ_0 on the complement of the convex core. The biggest technical difficulties are presented by the possibility that components of the thin part may intersect the complement of the convex core. In section 6 we note that one may prove that $K_3 > 10^{-11}$.

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2. Patterson-Sullivan measure and the spectral theory of geometrically finite hyperbolic manifolds

We recall that any complete hyperbolic n-manifold N may be written as the quotient of hyperbolic n-space by a group Γ of isometries. Let L_Γ denote the limit set for Γ's action on the sphere at infinity S_∞^{n-1} for $\mathbb{H}^n$. A hyperbolic n-manifold is said to be *elementary* if $\pi_1(N)$ contains an abelian subgroup of finite index. If N is elementary, then

$$\lambda_0(N) = \frac{(n-1)^2}{4}$$

and L_Γ contains at most two points.

The *convex core* $C(N)$ of a (non-elementary) hyperbolic 3-manifold N is defined to the quotient by Γ of the convex hull $CH(L_\Gamma)$ of L_Γ. There exists a retraction $R: N \to C(N)$, called the *nearest point retraction*, such that $R(x)$ is the (unique) point on $C(N)$ nearest to x.

A (non-elementary) hyperbolic n-manifold is said to be *geometrically finite* if the neighborhood $C_1(N)$ of radius one of the convex core has finite volume.

A *conformal density* of exponent δ for N is a measure μ supported on L_Γ such that

$$(*) \qquad \mu(\gamma(E)) = \int_E |\gamma'|^\delta d\mu$$

where E is any Borel subset of the sphere and γ is any element of Γ. Given a conformal density μ of exponent δ we may define a function ϕ_μ on $\mathbb{H}^n$ where

$$\phi_\mu(x) = \int_{S_\infty^{n-1}} |\alpha_x'|^\delta d\mu$$

and α_x is a hyperbolic isometry taking x to 0. Explicitly,

$$|\alpha_x'(\xi)| = \frac{1-|x|^2}{|x-\xi|^2}.$$

ϕ_μ then descends to a function on N. Moreover, ϕ_μ is a positive eigenfunction of the Laplacian with eigenvalue $\delta(\delta-(n-1))$.

Patterson [15], [16] and Sullivan [18], [19], [20] showed how to construct a conformal density of exponent $\delta(N)$ where $\delta(N)$ is the exponent of convergence of the Poincaré series. The situation is particularly satisfactory if N is geometrically finite.

Theorem 2.1 (Patterson-Sullivan). *Let $N = \mathbb{H}^n/\Gamma$ be a geometrically finite hyperbolic n-manifold.*

a) *The exponent of convergence $\delta(N)$ of the Poincaré series equals the Hausdorff dimension D of the limit set L_Γ and there is, supported on L_Γ, a conformal density μ of exponent $\delta(N)$ which is unique up to scaling.*

b) $\lambda_0(N) < \left(\frac{n-1}{2}\right)^2$ *if and only if* $\delta > \frac{n-1}{2}$ *in which case λ_0 is an L^2-eigenvalue of $-\Delta$. The corresponding eigenspace has dimension 1 and is spanned by ϕ_μ. Otherwise,* $\lambda_0(N) = \left(\frac{n-1}{2}\right)^2$.

We will call the unique conformal density μ obtained in Theorem 2.1 the *Patterson-Sullivan measure* and refer to ϕ_μ as the Patterson-Sullivan function. If $\lambda_0(N) < \left(\frac{n-1}{2}\right)^2$, we will always normalize so that $\int_N \phi_u^2 \, dv = 1$ where dv is the volume element on N. In this normalization, $\int_N |\nabla \phi_\mu|^2 \, dv = \lambda_0(N)$.

3. The thick-thin decomposition and a key lemma of Dodziuk and Randol

In this section we will recall the thick-thin decomposition of a hyperbolic manifold. We will then recall a lemma of Dodziuk and Randol which asserts that if T is a component of the thin part such that the L^2-norm of a function f is "big" on T and the L^2-norms of f and ∇f are "small" on a neighborhood of ∂T then the L^2-norm of ∇f is "big" on T. This lemma is a relative version of the fact that the first Dirichlet eigenvalue of T is "big."

We recall that the injectivity radius $\mathrm{inj}_N(x)$ of a point $x \in N$ is defined to be half the length of the shortest homotopically non-trivial closed curve passing through x. We define

$$N_{\text{thick}(\varepsilon)} = \{x \in N \mid \mathrm{inj}_N(x) \geqq \varepsilon\}$$

and

$$N_{\text{thin}(\varepsilon)} = \{x \in N \mid \mathrm{inj}_N(x) \leqq \varepsilon\}.$$

There exists a constant $\mathscr{M}_n$, called the Margulis constant (see p. 64 in Morgan [14] or section 5.10 of Thurston [21]) which depends only on n, such that if $\varepsilon < \mathscr{M}_n$ and N is a hyperbolic n-manifold, then every component of $N_{\text{thin}(\varepsilon)}$ is either

(a) a tubular neighborhood of a closed geodesic, or

(b) homeomorphic to $F \times [0, \infty)$ where F is a, possibly noncompact, flat manifold.

Notice that if $n \geqq 3$, this guarantees that $N_{\text{thick}(\varepsilon)}$ is connected if $\varepsilon < \mathscr{M}_n$. We recall (see Bowditch [2]) that N is geometrically finite if and only if $C(N) \cap N_{\text{thick}(\varepsilon)}$ is compact for all $\varepsilon > 0$.

If T is a component of $N_{\text{thin}(\varepsilon)}$, we define T's shell to be

$$\mathscr{S}(T) = \{x \in T \mid d(x, \partial T) \leqq 1\}.$$

One immediate consequence of the thick-thin decomposition is a lower bound on the volume of $C_1(N)$. In the remainder of the paper we will use V_r^n to denote the volume of a ball of radius r in $\mathbb{H}^n$.

Lemma 3.1. *Let N be an infinite volume, geometrically finite, non-elementary hyperbolic n-manifold. Then*

$$\mathrm{vol}(C_1(N)) \geqq V_r^n$$

where $r = \min\{1, \mathscr{M}_n\}$.

Proof of 3.1. We need only show that there exists a point $x \in C(N)$ such that $\operatorname{inj}_N(x) \geqq \mathcal{M}_n$. Then the ball $B(x, r)$ of radius r about x is contained in $C_1(N)$ and isometric to a ball of radius r in $\mathbb{H}^n$.

If $\varepsilon < \mathcal{M}_n$, then either there exists a point $x_\varepsilon \in C(N)$ such that

$$\operatorname{inj}_N(x_\varepsilon) > \varepsilon \quad \text{or} \quad C(N) \subset N_{\mathrm{thin}(\varepsilon)}\,.$$

If $C(N)$ is contained in a component of $N_{\mathrm{thin}(\varepsilon)}$, then $\pi_1(N) = \pi_1(C(N))$ has an abelian subgroup of finite index, which contradicts our assumption that N is non-elementary. Therefore, for all $\varepsilon < \mathcal{M}_n$, there exists $x_\varepsilon \in C(N)$ such that $\operatorname{inj}_N(x_\varepsilon) > \varepsilon$. Since

$$C(N) \cap N_{\mathrm{thick}\left(\frac{\mathcal{M}_n}{2}\right)}$$

is compact and injectivity radius is a continuous function, we see that there must exist $x \in C(N)$ such that $\operatorname{inj}_N(x) \geqq \mathcal{M}_n$. □

It will also be useful to notice that there is a lower bound on the volume of

$$T \cap C_1(N)$$

if $T \neq \mathcal{S}(T)$.

Lemma 3.2. *Let $N = \mathbb{H}^3/\Gamma$ be a non-elementary hyperbolic n-manifold and $\varepsilon < \mathcal{M}_n$. If T is a component of $N_{\mathrm{thin}(\varepsilon)}$ such that $T \neq \mathcal{S}(T)$, then*

$$\operatorname{vol}(T \cap C_1(N)) \geqq V_s^n$$

where $s = \min\left\{\dfrac{\varepsilon}{2}, \dfrac{1}{2}\right\}$.

Proof of 3.2. Let T be a component of $N_{\mathrm{thin}(\varepsilon)}$ and let

$$T_s = \{x \in T \mid d(x, \partial T) \geqq s\}\,.$$

We first prove that $T_s \cap C(N)$ is non-empty. If T is a compact component of $N_{\mathrm{thin}(\varepsilon)}$, then T_s contains a closed geodesic and this closed geodesic is contained entirely within the convex core. If T is non-compact, and $\tilde{T}_s$ is a lift of T_s, then $\tilde{T}_s$ is a horoball based at some point $p \in S_\infty^{n-1}$ left invariant by some parabolic subgroup Γ_p of Γ. Let Z be a geodesic ray joining some point $x \in CH(L_\Gamma)$ to p, then Z is contained within $CH(L_\Gamma)$. Thus, in either case, $C(N) \cap T_s$ is non-empty.

Since $T_s \cap C(N)$ is non-empty and $C(N)$ is not contained entirely within T_s, we see that there exists a point $x \in \partial T_s \cap C(N)$. One may check that $\operatorname{inj}_N(x) \geqq \varepsilon - s \geqq \dfrac{\varepsilon}{2}$. Thus $B(x, s)$ is contained entirely within $T \cap C_1(N)$ and isometric to a ball of radius s in $\mathbb{H}^n$. □

We will make key use of Lemma 2 from Dodziuk and Randol's paper [12]:

Lemma 3.3 (Dodziuk-Randol). *Let N be a hyperbolic n-manifold and let $\varepsilon < \mathcal{M}_n$. There exists a constant $d_0 > 0$ (depending only on n) such that if T is a component of $N_{\mathrm{thin}(\varepsilon)}$ with $T \neq \mathcal{S}(T)$, and $f \in C^1(N) \cap L^2(T)$ such that*

1. $\int_T f^2 dv \geqq c,$

2. $\int_{S(T)} |\nabla f|^2 dv \leqq d_0 c,$ *and*

3. $\int_{S(T)} f^2 dv \leqq d_0 c.$

Then

$$\int_T |\nabla f|^2 dv \geqq \frac{c}{2}\left(\frac{n-1}{2}\right)^2 .$$

The proof in Dodziuk-Randol [12] does not explicitly deal with the case where F is non-compact, however the argument carries over directly. One may notice from their proof that if $n \geqq 3$ then one may choose $d_0 = \frac{1}{16}$.

4. Exponential decay on the complement of the convex core

In this section we will explore the behavior of the Patterson-Sullivan function on the complement of the convex core.

Proposition 4.1. *Let $N = \mathbb{H}^n/\Gamma$ be a hyperbolic n-manifold and let μ be a Γ-invariant conformal density with exponent δ. Then there exists a constant $d_1 > 0$, such that if $x \in N - C_1(N)$, then*

$$|\nabla \phi_\mu(x)| \geqq d_1 \delta |\phi_\mu(x)| .$$

Here d_1 may be taken to be $(e^2 - 1)/(e^2 + 1)$.

Proof of 4.1. Again we will be working in the ball model for $\mathbb{H}^n$. We may normalize so that $x = 0$ and that $R(x)$ lies on the positive portion of the x_n-axis. (Here $R: N \to C(N)$ denotes the nearest point retraction defined in section 2.) In this normalization $R(z) = (0, \ldots, R_n(x))$ where

$$R_n(x) \geqq \frac{e-1}{e+1} .$$

This implies that every point in L_Γ is contained in the portion of S^{n-1} enclosed by the geodesic hemisphere passing through $R(x)$ and perpendicular to the x_n-axis. Let

$$X = \{(x_1, \ldots, x_n) \in S^{n-1} | x_n \geqq (e^2 - 1)/(e^2 + 1)\} .$$

In particular, we see that $L_\Gamma \subset X$.

Let S be the cone of vectors v in $T_0(\mathbb{H}^n)$ whose associated geodesics γ_v have endpoints in X. Then there exists a constant d_1, such that if v is a vector in $T_0(\mathbb{H}^n)$ and u is a unit vector in the direction of the positive x_n-axis, then $u \cdot v \geqq d_1 |v|$. Here d_1 may be taken to be $(e^2-1)/(e^2+1)$.

Let $f_\xi = \dfrac{1-|x|^2}{|x-\xi|^2}$, then $\phi(x) = \int_X f_\xi^\delta(x)\,d\mu$. We now notice that $\nabla f_\xi^\delta \in S$ if $\xi \in X$ and that $|\nabla f_\xi^\delta| = \delta f_\xi^\delta$. We then see, by elementary calculus, that

$$|\nabla\phi_\mu(0)| = \left|\int_X \nabla(f_\xi^\delta)(0)\,d\mu\right| \geqq d_1 \int_X |\nabla f_\xi^\delta(0)|\,d\mu = d_1\,\delta \int_X f_\xi^\delta d\mu = d_1\,\delta\,\phi_\mu(0)\,. \quad \square$$

We will not actually make use of this fact, but it is interesting to observe that ϕ_μ decays exponentially as one moves away from $C(N)$. Let $S: N \to C_1(N)$ denote the nearest-point retraction from N to $C_1(N)$. Notice that $S(x)$ is the intersection of the geodesic joining x and $R(x)$ with $\partial C_1(N)$.

Proposition 4.2. *Let N be a hyperbolic n-manifold and μ be a conformal density of exponent δ. If $x \in N - C_1(N)$, then*

$$\phi_\mu(x) \leqq e^{-d_1\delta d(x,S(x))}\,\phi_\mu(S(x))$$

where d_1 is the constant in Proposition 4.1.

Proof of 4.2. Let γ be a unit-speed geodesic arc joining x to $S(x)$. Then the proof of Proposition 4.1 implies that $\nabla\phi_\mu \cdot \gamma' \geqq d_1\,\delta\,\phi_\mu$. One may now integrate along γ to obtain the result. $\square$

We now observe that if N is geometrically finite and $\lambda_0(N)$ is "small" then $\int_{C_1(N)} \phi_\mu^2 dv$ is close to 1. (Recall that we have normalized so that $\int_N \phi_\mu^2 dv = 1$.)

Lemma 4.3. *Let N be a geometrically finite hyperbolic n-manifold such that*

$$\lambda_0(N) \neq \frac{(n-1)^2}{4}$$

and let ϕ_μ be the Patterson-Sullivan function for N. Then,

$$\int_{C_1(N)} \phi_\mu^2 dv > 1 - \frac{2\lambda_0(N)}{(n-1)\,d_1}$$

where d_1 is the constant obtained in Proposition 4.1.

Proof of 4.3. We first notice that

$$\int_{N - C_1(N)} |\nabla\phi_\mu|^2 dv \leqq \lambda_0(N)\,.$$

We recall, from Lemma 4.1, that $|\nabla \phi_\mu(x)| \geqq d_1 \delta \phi_\mu(x)$ if $x \in N - C_1(N)$. Combining these two facts we see that

$$\lambda_0(N) \geqq \int_{N - C_1(N)} |\nabla \phi_\mu|^2 dv \geqq d_1 \delta \int_{N - C_1(N)} \phi_\mu^2 dv .$$

The proof is completed by simply noticing that $\delta > (n-1)/2$ and that

$$\int_{C_1(N)} \phi_\mu^2 dv = 1 - \int_{N - C_1(N)} \phi_\mu^2 dv . \quad \square$$

We will need to make use of the following consequence of Yau's Harnack inequality for positive eigenfunctions of the Laplacian (see [22]). We will give a quick proof which uses the definition of ϕ_μ directly.

Lemma 4.4. *Let $N = \mathbb{H}^n/\Gamma$ be a hyperbolic n-manifold and let ϕ_μ be its Patterson-Sullivan function. Then*

$$\int_{B(x,\varepsilon)} \phi_\mu^2 dv \geqq e^{-2\varepsilon(n-1)} \phi_\mu^2(x) \operatorname{vol}(B(x,\varepsilon)) .$$

In particular, there exists a constant d_2, depending only on ε and n, such that whenever $x \in N_{\mathrm{thick}(\varepsilon)}$, then

$$\int_{B(x,\varepsilon)} \phi_\mu^2 dv \geqq d_2 \phi_\mu^2(x) .$$

d_2 may be taken to be $e^{-2\varepsilon(n-1)} V_\varepsilon^n$ where V_ε^n denotes the volume of a ball of radius ε in $\mathbb{H}^n$.

Proof of 4.4. It follows from the explicit formula for ϕ_μ that

$$|\nabla \phi_\mu(y)| \leqq (n-1) \phi_\mu(y)$$

for all $y \in N$. Therefore, if $y \in B(x,\varepsilon)$, then $\phi_\mu(y) \geqq e^{-\varepsilon(n-1)} \phi_\mu(x)$. Thus we see that

$$\int_{B(x,\varepsilon)} \phi_\mu^2 dv \geqq e^{-2\varepsilon(n-1)} \phi_\mu^2(x) \operatorname{vol}(B(x,\varepsilon)) . \quad \square$$

We next observe that if N is geometrically finite and $\lambda_0(N)$ is "small", then ϕ_μ is "small" on the thick part of the complement of the convex core.

Lemma 4.5. *Let N be a geometrically finite hyperbolic n-manifold with*

$$\lambda_0(N) \neq \frac{(n-1)^2}{4}$$

and let μ be its Patterson-Sullivan measure. There exists a constant d_3, depending only on ε and n, such that if $x \in N_{\mathrm{thick}(\varepsilon)} - C(N)$, then

$$\phi_\mu(x) \leqq d_3 \sqrt{\lambda_0(N)} .$$

Here d_3 may be taken to be $e^{(n-1)(1+\frac{\varepsilon}{2})}/d_1\sqrt{d_2}$.

Proof of 4.5. First suppose that $x \in N_{\mathrm{thick}(\varepsilon)}$ and $d(x, C(N)) \geqq 1 + \frac{\varepsilon}{2}$. Lemma 4.4 implies that

$$\int_{B(x,\varepsilon/2)} \phi_\mu^2 \, dv \geqq d_2 \phi_\mu^2(x).$$

We may then apply Lemma 4.1 to see that $\|\nabla \phi_\mu\|_{2,B(x,\varepsilon/2)} \geqq d_1 \sqrt{d_2}\, \phi_\mu(x)$. But

$$\|\nabla \phi_\mu\|_{2,B(x,\varepsilon/2)} \leqq \sqrt{\lambda_0(N)}$$

which implies that

$$\phi_\mu(x) \leqq \frac{\sqrt{\lambda_0(N)}}{d_1\sqrt{d_2}}.$$

Now let x be a point in $N_{\mathrm{thick}(\varepsilon)} - C(N)$, such that $d(x, C(N)) = d < 1 + \frac{\varepsilon}{2}$. Let Z be the geodesic ray beginning at $R(x)$ and passing through x. Notice that Z is orthogonal to $\partial C(N)$. Let y be the (unique) point on Z such that $d(y, C(N)) = 1 + \frac{\varepsilon}{2}$.

We recall that the neighborhood of radius d, $\mathcal{N}_d(C(N))$, of $C(N)$ is strictly convex (see Corollary 2.4.11 in [7]). Thus, the nearest-point projection $R_d : N \to \mathcal{N}_d(C(N))$ is distance-decreasing. Since $R_d(y) = x$, we see that $\mathrm{inj}_N(y) > \mathrm{inj}_N(x) \geqq \varepsilon$. By the above argument, $\phi_\mu(y) \leqq \sqrt{\lambda_0(N)}/d_1\sqrt{d_2}$. But since $|\nabla \phi_\mu| \leqq (n-1)|\phi_\mu|$ on all of N we may conclude, by integrating along Z, that

$$\phi_\mu(x) \leqq \frac{e^{(n-1)(1+\frac{\varepsilon}{2})}\sqrt{\lambda_0(N)}}{d_1\sqrt{d_2}}. \quad \square$$

Remark. Another, perhaps slightly more general, way to obtain the information needed about the behavior of ϕ_μ on the complement of the convex core, is to prove that there is a lower bound for the first Neumann eigenvalue of each component of $N - C_1(N)$, depending only on n.

5. The thick part of the convex core

In this section we will obtain pointwise bounds on ϕ_μ on the thick part of the convex core. We will first need the following result from elliptic theory. This result may be obtained as a direct consequence of Gärding's inequality. However, in order to obtain explicit constants we will give a more concrete proof in an appendix.

Lemma 5.1. *Let N be a hyperbolic n-manifold and let ϕ be an eigenfunction of $-\Delta$ with eigenvalue $(n-1)^2/4 \geqq \lambda \geqq 0$. If $\varepsilon \leqq 2\,\mathrm{arcsinh}\left(\sqrt{\frac{2}{n-1}}\right)$, then there exists a constant d_4, depending only on ε and n, such that if $x \in N_{\mathrm{thick}(\varepsilon)}$, then*

$$\|\nabla\phi\|_{\infty, B(x,\varepsilon/2)} \leqq d_4 \|\nabla\phi\|_{2, B(x,\varepsilon)}.$$

If $n \geqq 3$, then d_4 may be taken to be

$$\frac{2^n(3n+1)\sqrt{n}\sqrt{V^n_{\frac{\varepsilon}{2}}}}{\sqrt{v_{n-1}}\,\varepsilon^n}$$

where v_{n-1} denotes the volume of the Euclidean $(n-1)$-sphere.

We now see that $\mathrm{vol}(C_1(N))$ and $\lambda_0(N)$ provide a bound on the size of ϕ_μ in the thick part.

Lemma 5.2. *Let N be an infinite volume, geometrically finite hyperbolic n-manifold with $n \geqq 3$ and $\lambda_0(N) \neq \frac{(n-1)^2}{4}$ and let μ be its Patterson-Sullivan measure. Suppose that $\varepsilon < \min\left\{1, \mathscr{M}_n, 2\operatorname{arcsinh}\left(\sqrt{\frac{2}{n-1}}\right)\right\}$. There exists d_5, depending only on ε and n, such that if $x \in N_{\mathrm{thick}(\varepsilon)}$, then*

$$\phi_\mu(x) \leqq d_5 \sqrt{\mathrm{vol}(C_1(N))}\sqrt{\lambda_0(N)}.$$

Here d_5 may be taken to be $(d_3/\sqrt{V^n_r}) + (\varepsilon d_4\sqrt{V^n_\varepsilon}/V^n_{\frac{\varepsilon}{4}})$.

Proof of 5.2. Let $\{x_i\}$ be a maximal collection of points in $C(N) \cap N_{\mathrm{thick}(\varepsilon)}$ such that $d(x_i, x_j) \geqq \frac{\varepsilon}{2}$. There are at most $\mathrm{vol}(C_1(N))/V^n_{\frac{\varepsilon}{4}}$ such points, and $N_{\mathrm{thick}(\varepsilon)} \cap C(N)$ is covered by the collection of balls $\{B(x_i, \varepsilon/2)\}_{i=1}^k$ of radius $\frac{\varepsilon}{2}$ centered at $\{x_i\}$. Moreover, at most $V^n_{\varepsilon+\frac{\varepsilon}{4}}/V^n_{\frac{\varepsilon}{4}}$ of the associated ε-balls intersect at any point.

Thus,

$$(V^n_{\frac{5\varepsilon}{4}}/V^n_{\frac{\varepsilon}{4}})\,\lambda_0(N) \geqq \sum_i \int_{B(x_i,\varepsilon)} |\nabla\phi_\mu|^2\, dv.$$

We may apply Cauchy-Schwartz and Lemma 5.1 to see that

$$\begin{aligned}\sqrt{\mathrm{vol}(C_1(N))/V^n_{\varepsilon/4}}\Big(\sum_i \int_{B(x_i,\varepsilon)} |\nabla\phi_\mu|^2 dv\Big)^{\frac{1}{2}} &\geqq \sum_i \|\nabla\phi_\mu\|_{2, B(x_i,\varepsilon)} \\ &\geqq \frac{1}{d_4}\sum_i \|\nabla\phi_\mu\|_{\infty, B(x_i,\varepsilon/2)}.\end{aligned}$$

Therefore,

$$\sum_i \|\nabla\phi_\mu\|_{\infty, B(x_i,\varepsilon/2)} \leqq d_4 \sqrt{(V^n_{\frac{5\varepsilon}{4}}/V^n_{\frac{\varepsilon}{4}})\,\lambda_0(N)}\sqrt{\mathrm{vol}(C_1(N))/V^n_{\frac{\varepsilon}{4}}}.$$

So if x and y are in the same component of $N_{\mathrm{thick}(\varepsilon)} \cap C(N)$ we see that

$$|\phi_\mu(x) - \phi_\mu(y)| \leqq d_4\,\varepsilon \sqrt{(V^n_{\frac{5\varepsilon}{4}}/V^n_{\frac{\varepsilon}{4}})\,\lambda_0(N)}\sqrt{\mathrm{vol}(C_1(N))/V^n_{\frac{\varepsilon}{4}}}.$$

Recall that if $y \in \partial C(N) \cap N_{\mathrm{thick}(\varepsilon)}$ then, by Lemmas 4.5 and 3.1

$$\phi_\mu(y) \leqq d_3 \sqrt{\lambda_0(N)} \leqq \frac{d_3}{\sqrt{V_r^n}} \sqrt{\lambda_0(N) \operatorname{vol}(C_1(N))}\,.$$

We also notice that every component of $C(N) \cap N_{\mathrm{thick}(\varepsilon)}$ contains a point of $\partial C(N)$, since, if $n \geqq 3$ and $\varepsilon < \mathcal{M}_n$, then $N_{\mathrm{thick}(\varepsilon)}$ is connected and contains points of $N - C(N)$.

Combining these two observations we obtain

$$\Phi_\mu(x) \leqq \left(\frac{d_3 \sqrt{\operatorname{vol}(C_1(N))}}{\sqrt{V_r^n}} + d_4 \varepsilon \sqrt{V_{\frac{5\varepsilon}{4}}^n / V_{\frac{\varepsilon}{4}}^n} \sqrt{\operatorname{vol}(C_1(N)) / V_{\frac{\varepsilon}{4}}^n} \right) \sqrt{\lambda_0(N)}\,. \quad \square$$

It will also be useful to have an extension of Lemma 5.2 to the shell of a component of $N_{\mathrm{thin}(\varepsilon)}$.

Lemma 5.3. *Let N be an infinite volume, geometrically finite hyperbolic n-manifold with $\lambda_0(N) \neq \frac{(n-1)^2}{4}$, and let μ be its Patterson-Sullivan measure. Let*

$$\varepsilon < \min \left\{ 1, \mathcal{M}_n, 2 \operatorname{arcsinh} \left(\sqrt{\frac{2}{n-1}} \right) \right\}$$

and T be a component of $N_{\mathrm{thin}(\varepsilon)}$. If $x \in \mathcal{S}(T)$, then

$$\phi_\mu(x) \leqq d_6 \sqrt{\operatorname{vol}(C_1(N))} \sqrt{\lambda_0(N)}$$

where $d_6 = e^{(n-1)} d_5$.

Proof of 5.3. This follows immediately from Lemma 5.2 and the fact that

$$|\nabla \phi_\mu'| \leqq (n-1)\, \phi_\mu\,.$$

Remark. Lemmas 5.2 and 5.3 are false if $n = 2$. This is a result of the fact that $N_{\mathrm{thick}(\varepsilon)}$ may be disconnected if N is a hyperbolic surface.

6. Proof of Main Theorem

In this section we will give the proof of our main theorem.

Main Theorem. *For all $n \geqq 3$ there exists a constant $K_n > 0$ such that if N is an infinite volume, geometrically finite hyperbolic n-manifold, then*

$$\lambda_0(N) \geqq \frac{K_n}{\operatorname{vol}(C_1(N))^2}\,.$$

Proof of Main Theorem. We first recall that $\operatorname{vol}(C_1(N)) \geqq V_r^n$ (see Lemma 3.1). So if we take $K_n \leqq ((n-1)^2/4)(V_r^n)^2$, then we may assume that $\lambda_0(N) \neq (n-1)^2/4$.

We will use the shorthand $\lambda = \lambda_0(N)$. Let ϕ denote the Patterson-Sullivan function for N. Then $\phi \in L^2(N)$, $\Delta\phi = -\lambda\phi$ and $\int\limits_N \phi^2\,dv = 1$ (see Theorem 2.1). We will also choose a fixed $\varepsilon < \min\left\{1, \mathcal{M}_n, 2\operatorname{arcsinh}\left(\sqrt{\frac{2}{n-1}}\right)\right\}$. So $V_r^n \geqq V_\varepsilon^n$ and $V_s^n \geqq V_{\frac{\varepsilon}{2}}^n$.

We will give a short outline of the proof, as the presence of actual constants can obscure the relatively simple line of argument. If λ is small enough, in comparison with $\operatorname{vol}(C_1(N))^2$, then we see, using Lemmas 5.2 and 5.3, that ϕ has at least half its support on the thin part. We may then find a component T of the thin part, such that the L^2 norm of ϕ on T is large with respect to $\operatorname{vol}(C_1(N))$. However, both the L^2 norms of ϕ and $\nabla\phi$ are small, in comparison to $\operatorname{vol}(C_1(N))$, on the shell of T. Therefore Lemma 3.3 implies that the L^2-norm of $\nabla\phi$ on T is large with respect to $\operatorname{vol}(C_1(N))$. Combining these observations, we see that λ cannot be too small, in comparison with $\operatorname{vol}(C_1(N))^2$.

Let $\{T_1, \ldots, T_n\}$ denote the components of $N_{\text{thin}(\varepsilon)}$. Let N_ε^S denote $N_{\text{thick}(\varepsilon)} \cup_i \mathcal{S}(T_i)$. One immediate consequence of Lemma 5.2 and 5.3 is that

$$\int\limits_{C_1(N)\cap N_\varepsilon^S} \phi^2\,dv \leqq d_6^2\,\lambda \operatorname{vol}(C_1(N))^2 .$$

Combining this with Lemma 4.3 we see that

$$\int\limits_{(N - N_\varepsilon^S)\cap C_1(N)} \phi^2\,dv \geqq 1 - \left(\frac{2}{(n-1)\,d_1} + d_6^2 \operatorname{vol}(C_1(N))^2\right)\lambda .$$

So if

$$\lambda \leqq \frac{1}{\dfrac{4}{(n-1)\,d_1} + 2d_6^2 \operatorname{vol}(C_1(N))^2},$$

we have

$$\int\limits_{(N - N_\varepsilon^S)\cap C_1(N)} \phi^2\,dv \geqq \frac{1}{2} .$$

Therefore, there exists a component T of $N_{\text{thin}(\varepsilon)}$ such that $T \neq \mathcal{S}(T)$ and

$$\int\limits_{T\cap C_1(N)} \phi^2 \geqq \frac{\operatorname{vol}(T\cap C_1(N))}{2\operatorname{vol}(C_1(N))} .$$

If

$$\lambda \leqq \frac{d_0 \operatorname{vol}(\dot{T}\cap C_1(N))}{2\operatorname{vol}(C_1(N))}$$

then,

$$\int\limits_{\mathcal{S}(T)} |\nabla\phi|^2\,dv \leqq \lambda \leqq \frac{d_0}{2}\,\frac{\operatorname{vol}(T\cap C_1(N))}{\operatorname{vol}(C_1(N))} \leqq d_0 \int\limits_{T\cap C_1(N)} \phi^2\,dv .$$

To be able to invoke Lemma 3.3 it only remains to guarantee that

$$\int_{\mathscr{S}(T)} \phi^2 dv \leqq d_0 \int_T \phi^2 dv .$$

Unfortunately, this must be done in two steps. First we handle the portion of $\mathscr{S}(T)$ which is "near" $C(N)$ and then we handle the portion which is far from $C(N)$.

Lemma 5.3 guarantees that

$$\begin{aligned}\int_{\mathscr{S}(T)\cap C_1(N)} \phi^2 dv &\leqq d_6^2 \lambda \operatorname{vol}(C_1(N)) \operatorname{vol}(\mathscr{S}(T)\cap C_1(N))\\ &\leqq d_6^2 \lambda \operatorname{vol}(C_1(N)) \operatorname{vol}(T\cap C_1(N))\\ &\leqq 2 d_6^2 \lambda \operatorname{vol}(C_1(N))^2 \int_{T\cap C_1(N)} \phi^2 dv .\end{aligned}$$

Therefore, if $\lambda \leqq d_0/(4 d_6^2 \operatorname{vol}(C_1(N))^2)$, then

$$\int_{\mathscr{S}(T)\cap C_1(N)} \phi^2 dv \leqq \frac{d_0}{2} \int_T \phi^2 dv .$$

Moreover, by Lemma 4.3,

$$\int_{\mathscr{S}(T) - C_1(N)} \phi^2 dv \leqq \frac{2\lambda}{(n-1) d_1} .$$

$$\lambda \leqq \frac{d_0 d_1 (n-1) \operatorname{vol}(T\cap C_1(N))}{8 \operatorname{vol}(C_1(N))} ,$$

$$\int_{\mathscr{S}(T) - C_1(N)} \phi^2 dv \leqq \frac{d_0}{2} \int_T \phi^2 dv .$$

Hence if

$$\lambda \leqq \min \left\{ \frac{d_0}{4 d_6^2 \operatorname{vol}(C_1(N))^2}, \frac{d_0 d_1 (n-1) \operatorname{vol}(T\cap C_1(N))}{8 \operatorname{vol}(C_1(N))}, \frac{d_0 \operatorname{vol}(T\cap C_1(N))}{2 \operatorname{vol}(C_1(N))}, \frac{1}{\frac{4}{(n-1) d_1} + 2 d_6^2 \operatorname{vol}(C_1(N))^2} \right\},$$

then, by Lemma 3.3,

$$\lambda \geqq \int_T |\nabla \phi|^2 dv > \left(\frac{n-1}{2}\right)^2 \frac{\operatorname{vol}(T\cap C_1(N))}{2 \operatorname{vol}(C_1(N))} .$$

Thus, if we let $A = \operatorname{vol}(T \cap C_1(N)) \operatorname{vol}(C_1(N))$, then

$$\lambda \geqq \frac{1}{\operatorname{vol}(C_1(N))^2} \min \left\{ \frac{d_0}{4d_6^2}, \frac{d_0 d_1 (n-1) A}{8}, \frac{d_0 A}{2}, \frac{1}{\left(\frac{4}{d_1(n-1)\operatorname{vol}(C_1(N))^2}\right) + 2d_6^2}, \frac{(n-1)^2 A}{8} \right\}.$$

We now recall, from Lemma 3.2, that $\operatorname{vol}(T \cap C_1(N)) \geqq V^n_{\frac{\varepsilon}{2}}$ and, from Lemma 3.1, that $\operatorname{vol}(C_1(N)) \geqq V^n_\varepsilon$. We also recall that we assumed at the beginning of the proof that $K_n \leqq ((n-1)^2/4)(V^n_r)^2$. Therefore, if we set

$$K_n = \min \left\{ \frac{d_0}{4d_6^2}, \frac{d_0 d_1 (n-1)}{8} (V^n_{\frac{\varepsilon}{2}} V^n_\varepsilon), \frac{d_0}{2} (V^n_{\frac{\varepsilon}{2}} V^n_\varepsilon), \frac{1}{\left(\frac{4}{d_1(n-1)(V^n_\varepsilon)^2} + 2d_6^2\right)}, \frac{V^n_\varepsilon V^n_{\frac{\varepsilon}{2}}(n-1)^2}{8}, \frac{(n-1)^2 (V^n_\varepsilon)^2}{4} \right\}$$

then

$$\lambda_0(N) \geqq \frac{K_n}{\operatorname{vol}(C_1(N))^2}. \quad \square$$

Remark. In order to find a lower bound for K_n it is only necessary to choose $\varepsilon \leqq \min \left\{ 1, \mathcal{M}_n, 2 \operatorname{arcsinh}\left(\sqrt{\frac{2}{n-1}}\right) \right\}$ and then explicitly bound all the constants used in the paper. It is a consequence of the work of Culler and Shalen [10] that if $n = 3$, then we may choose $\varepsilon = \frac{\log 3}{2}$. In this case, $.75 > V^3_\varepsilon > .7$. We may then obtain the following estimates, when $n = 3$, in a straightforward manner $d_0 = \frac{1}{16}$, $d_1 > .75$, $d_2 > .04$, $d_3 < 86$, $d_4 < 72$, $d_5 < 4528$, and $d_6 < 33{,}458$. Thus, plugging into the above formula for K_3, we see that $K_3 > 10^{-11}$. Given a choice of ε for $n \geqq 4$, similar bounds can be obtained for all K_n.

7. The boundary of the convex core of a hyperbolic 3-manifold

In this section we will derive the inequality (Lemma 7.3) used in the derivation of Corollaries C and D. Much of the necessary information about $\partial C(N)$ is summarized by the fact that the boundary of the convex core of a hyperbolic 3-manifold is (the image of) a pleated surface (see Theorem 1.12.1 in Epstein-Marden [13]).

A *geodesic lamination* on a finite area hyperbolic surface S is a closed, disjoint union of simple geodesics. Any geodesic lamination on a surface of finite area has measure zero (see Theorem 4.9 in Casson-Bleiler [9]). A *pleated surface* $f: S \to N$ is a proper pathwise isometry of a finite area hyperbolic surface S into N such that there exists a geodesic lamination α, called the *pleating locus*, on S such that f is totally geodesic on $S-\alpha$ and f takes each geodesic in α to a geodesic in N. Moreover, if $g \in \pi_1(S)$, then $f_*(g)$ is a parabolic element of $\pi_1(N)$ if g is a parabolic element of $\pi_1(S)$. (See [7] for an extensive discussion of pleated surfaces.)

Let α denote the pleating locus of $\partial C(N)$. Since $\partial C(N)-\alpha$ is totally geodesic, we observe that if A is any subset of $\partial C(N)-\alpha$, then

$$\operatorname{vol}(C_1(N) \cap R^{-1}(A)) \geqq \operatorname{area}(A).$$

Since α has measure zero we may conclude that:

Lemma 7.1. *Let N be a hyperbolic 3-manifold. If A is a measurable subset of $\partial C(N)$, then*

$$\operatorname{vol}(C_1(N) \cap R^{-1}(A)) \geqq \operatorname{area}(A).$$

In particular,

$$\operatorname{vol}(C_1(N) - C(N)) \geqq \operatorname{area}(\partial C(N)) \geqq 2\pi.$$

We also need a bound from above on the volume of $C_1(N)-C(N)$. This may be obtained from Lemma 8.12.1 in [21], whose proof is the same as that of Lemma 8.2 in [6]. Recall that if $f\colon S \to N$ is a pleated surface, then a homotopically non-trivial curve γ on S is said to be compressible if $f(\gamma)$ is homotopically trivial in N.

Lemma 7.2 (Thurston). *Let $f\colon S \to N$ be a pleated surface such that every compressible curve in S has length $\geqq a$. Then there exists a constant L (depending only on a) such that the volume of $\mathcal{N}_1(f(S))$ is less than $L|\chi(S)|$ (where $\mathcal{N}_1(f(S))$ denotes the neighborhood of radius one of $f(S)$).*

We may now combine Lemma 7.1 and 7.2 to obtain the inequality which was used in the introduction to derive Corollaries C and D.

Lemma 7.3. *Given $a>0$, there exists $L>0$ such that if N is an infinite volume, geometrically finite hyperbolic 3-manifold such that $\partial C(N)$ contains no compressible curves with length $\leqq a$, then*

$$\operatorname{vol}(C(N)) + 2\pi|\chi(\partial C(N))| \leqq \operatorname{vol}(C_1(N)) \leqq \operatorname{vol}(C(N)) + L|\chi(\partial C(N))|.$$

8. Appendix: Proof of Lemma 5.1

Lemma 5.1 is easily seen to follow from the following pointwise result in $\mathbb{H}^n$.

Lemma 8.1. *Let $F\colon \mathbb{H}^n \to \mathbb{R}$ be an eigenfunction of $-\Delta$ with eigenvalue*

$$(n-1)^2/4 \geqq \lambda \geqq 0.$$

If $s \leqq \operatorname{arcsinh}\left(\sqrt{\frac{2}{n-1}}\right)$ *and* $x \in \mathbb{H}^n$, *then*

$$\|\nabla_x F\|^2 \leqq \frac{c(n)\, V_s^n}{s^{2n}} \int_{B(x,s)} \|\nabla F\|^2\, dv$$

where $c(n) = \frac{(3n+1)^2 n}{v_{n-1}}$ *if* $n \geqq 3$, $c(2) = \frac{147}{\pi}$ *and* v_{n-1} *denotes the volume of the Euclidean* $(n-1)$*-sphere.*

Proof of 8.1. We first pull back the Riemannian metric on $\mathbb{H}^n$ via the exponential map to obtain polar coordinates. In these coordinates the metric takes the form

$$ds^2 = d\varrho^2 + \sinh^2(\varrho)\, d\sigma^2$$

where $d\sigma^2$ is the standard metric on S^{n-1}. Here the volume form takes the form

$$dv = \sinh^{n-1}(\varrho)\, d\varrho\, d\xi$$

where $d\xi$ is the volume form on S^{n-1}. For a function F we have

$$\nabla F = \left(F_\varrho, \frac{1}{\sinh^2 \varrho} \nabla_\sigma F\right)$$

so $\|F_\varrho\| \leqq \|\nabla F\|$. Finally the Laplacian is

$$\Delta F = F_{\varrho\varrho} + (n-1)\coth(\varrho)\, F_\varrho + \frac{1}{\sinh^2 \varrho} \Delta_{S^{n-1}} F .$$

Now, let P be an eigenfunction of $\Delta_{S^{n-1}}$ on S^{n-1} and consider

$$c_P(\varrho) = \int_{S^{n-1}} F(\varrho, \sigma)\, P(\sigma)\, d\xi(\sigma) . \tag{1}$$

If $\Delta F = -\lambda F$ and $\Delta_{S^{n-1}} P = -\mu P$, $\mu > 0$, then c_P satisfies the ordinary differential equation:

$$c_P''(\varrho) + (n-1)\coth(\varrho)\, c_P'(\varrho) + \left(\lambda - \frac{\mu}{\sinh^2 \varrho}\right) c_p(\varrho) = 0 \tag{2}$$

and

$$c_P(0) = \int_{S^{n-1}} F(0)\, P(\sigma)\, d\xi(\sigma) = 0 .$$

Taking the Taylor expansion to second order at 0 of F:

$$F(\varrho, \sigma) = F(0) + \varrho \sum \sigma_i \partial_i F(0) + O(\varrho^2)$$

(where $\sigma = (\sigma_1, \ldots, \sigma_n)$) and plugging into (1), we get:

$$c'_P(0) = \sum_{i=1}^{n} \partial_i F(0) \int_{S^{n-1}} \sigma_i P(\sigma)\, d\xi(\sigma)\,.$$

Set

$$v_P = \Big(\int_{S^{n-1}} \sigma_1 P(\sigma)\, d\xi(\sigma), \ldots, \int_{S^{n-1}} \sigma_n P(\sigma)\, d\xi(\sigma)\Big)\,.$$

Hence if y denotes the unique solution of the ordinary differential equation (2) with initial conditions

$$y(0) = 0, \quad y'(0) = 1$$

we get $c_P(\varrho) = \langle \nabla_0 F, v_P \rangle y(\varrho)$ and therefore

$$\langle \nabla_0 F, v_P \rangle y'(\varrho) = c'_P(\varrho) = \int_{S^{n-1}} F_\varrho(\varrho, \sigma) P(\sigma)\, d\xi(\sigma)\,. \tag{3}$$

Multiplying by $\sinh^{n-1}(\varrho)$, integrating from 0 to r, and expanding using equation (3) we see that,

$$\langle \nabla_0 F, v_P \rangle \int_0^r \sinh^{n-1}(\varrho)\, y'(\varrho)\, d\varrho = \int_0^r \int_{S^{n-1}} F_\varrho(\varrho, \sigma) P(\sigma) \sinh^{n-1}(\varrho)\, d\varrho\, d\xi(\sigma)\,.$$

Then after applying Cauchy-Schwartz and recalling that $\|F_\varrho\| \leqq \|\nabla F\|$, we see that

$$|\langle \nabla_0 F, v_P \rangle| \left| \int_0^r \sinh^{n-1}(\varrho)\, y'(\varrho)\, d\varrho \right| \leqq \|P\|_{L^2(S^{n-1})} \sqrt{\frac{V_r^n}{v_{n-1}}} \left(\int_{B(x,r)} \|\nabla F\|^2\, dv \right)^{\frac{1}{2}}. \tag{4}$$

We now specialize to $P_j(\sigma) = \sigma_j$, which satisfies $\Delta_{S^{n-1}} P_j = -(n-1) P_j$ and we observe that

$$\int_{S^{n-1}} \sigma_i \sigma_j\, d\xi(\sigma) = \delta_{ij} \frac{v_{n-1}}{n}\,. \tag{5}$$

Also notice that since y depends only of F and μ it is unaffected by the choice of j. Using (5), squaring (4) and summing over j, we get:

$$\|\nabla_x F\|^2 \left| \int_0^r \sinh^{n-1}(\varrho)\, y'(\varrho)\, d\varrho \right|^2 \leqq \frac{n}{v_{n-1}} V_r^n \int_{B(x,r)} \|\nabla F\|^2\, dv\,. \tag{6}$$

We now give a lower bound for $\int_0^r \sinh^{n-1}(\varrho)\, y'(p)\, d\varrho$. Recall that y satisfies

$$y'' + (n-1)\coth(\varrho)\, y' + y\left(\lambda - \frac{n-1}{\sinh^2 \varrho}\right) = 0$$

and that $y(0) = 0$ and $y'(0) = 1$. This can also be written as

$$\sinh^{n-1}(r)\, y'(r) = \int_0^r \sinh^{n-1}(\varrho)\, y(\varrho) \left(\frac{n-1}{\sinh^2 \varrho} - \lambda \right) d\varrho \,. \tag{7}$$

Let r_0 be the smallest zero of y', if there is any. Hence $y' \geqq 0$ on $[0, r_0]$ and therefore $y \geqq 0$ and is increasing on $[0, r_0]$. By definition of r_0 we have

$$\int_0^{r_0} \sinh^{n-1}(\varrho)\, y(\varrho) \left(\frac{n-1}{\sinh^2 \varrho} - \lambda \right) d\varrho = 0$$

and therefore, $\dfrac{n-1}{\sinh^2(r_0)} - \lambda \leqq 0$, since $\sinh^{n-1}(\varrho)\, y(\varrho) > 0$ on $(0, r_0]$. Observing that $\lambda \leqq \left(\dfrac{n-1}{2} \right)^2$, we get the bound

$$r_0 \geqq \operatorname{arcsinh}\left(\frac{2}{\sqrt{n-1}} \right).$$

Now write the differential equation in the following manner:

$$y'' + \lambda y + (n-1)\left(\coth(\varrho)\, y' - \frac{y}{\sinh^2 \varrho} \right) = 0 \,.$$

Notice that $\left(\coth(\varrho)\, y' - \dfrac{y}{\sinh^2 \varrho} \right) = (\coth(\varrho)\, y)'$ and integrate from 0 to r, to obtain

$$y'(r) - 1 + \lambda \left(\int_0^r y(\varrho)\, d\varrho \right) + (n-1)\,[\coth(r)\, y(r) - 1] = 0$$

or

$$y'(r) + \lambda \left(\int_0^r y(\varrho)\, d\varrho \right) + (n-1) \coth(r)\, y(r) = n \,.$$

For $r \leqq r_0$, y is positive and increasing. Hence,

$$y'(r) + \lambda r y(r) + (n-1) \coth(r)\, y(r) \geqq n \,,$$

i.e.

$$y'(r) + y(r)\big(\lambda r + (n-1) \coth(r)\big) \geqq n \,. \tag{8}$$

Observe that for all $r \leqq \operatorname{arcsinh}\left(\sqrt{\dfrac{2}{n-1}} \right)$ one has

$$\lambda r + (n-1) \coth(r) \leqq \left(\frac{n-1}{2} \right)^2 r + (n-1) \coth(r) \leqq \frac{3n-1}{2r} \,.$$

The second inequality is established by first observing that the function

$$f(r) = \left(\frac{n-1}{2}\right)^2 r^2 + (n-1) r \coth(r)$$

is increasing on the interval $\left(0, \operatorname{arcsinh}\left(\sqrt{\frac{2}{n-1}}\right)\right]$, and then noting that $f(r) \leqq \frac{3n-1}{2}$ when $r = \operatorname{arcsinh}\left(\sqrt{\frac{2}{n-1}}\right)$.

Now, since $\operatorname{arcsinh}\left(\sqrt{\frac{2}{n-1}}\right) < \operatorname{arcsinh}\left(\frac{2}{\sqrt{n-1}}\right) \leqq r_0$, we have from (8) that

$$y'(r) + \left(\frac{3n-1}{2r}\right) y(r) \geqq n,$$

for all $r \leqq \operatorname{arcsinh}\left(\sqrt{\frac{2}{n-1}}\right)$. Multiplying by $r^{\frac{3n-1}{2}}$, we get

$$\left(r^{\frac{3n-1}{2}} y\right)' \geqq n r^{\frac{3n-1}{2}}$$

and hence by integrating that

$$y(r) \geqq \frac{2n}{3n+1} r \tag{9}$$

for all $r \leqq \operatorname{arcsinh}\left(\sqrt{\frac{2}{n-1}}\right)$. From equation (7) we see that

$$\int_0^s \sinh^{n-1}(r) y'(r)\, dr = \int_0^s (s-\varrho) \sinh^{n-1}(\varrho) y(\varrho) \left(\frac{n-1}{\sinh^2 \varrho} - \lambda\right) d\varrho. \tag{10}$$

Now observe that for $\varrho \leqq \operatorname{arcsinh}\left(\sqrt{\frac{2}{n-1}}\right)$,

$$\frac{n-1}{\sinh^2 \varrho} - \lambda \geqq \frac{n-1}{\sinh^2 \varrho} - \left(\frac{n-1}{2}\right)^2 \geqq \frac{n-1}{2 \sinh^2 \varrho}. \tag{11}$$

Combining (9), (10) and (11), we see that if $s \leqq \operatorname{arcsinh}\left(\sqrt{\frac{2}{n-1}}\right)$, then

$$\int_0^s \sinh^{n-1}(r) y'(r)\, dr \geqq \int_0^s (s-\varrho) \varrho \sinh^{n-3}(\varrho) \left(\frac{n(n-1)}{3n+1}\right) d\varrho.$$

When $n \geqq 3$ we may use the fact that $\sinh \varrho \geqq \varrho$ to observe that

$$\int_0^s \sinh^{n-1}(r) y'(r)\, dr \geqq \int_0^s (s-\varrho) \varrho^{n-2} \left(\frac{n(n-1)}{3n+1}\right) d\varrho = \left(\frac{1}{3n+1}\right) s^n.$$

One may use this inequality and (6) to complete the proof of the theorem when $n \geqq 3$. To complete the proof when $n = 2$, use the fact that $\sinh(\varrho) \leqq \sqrt{3}\varrho$ if $\varrho \leqq \operatorname{arcsinh}(\sqrt{2})$. □

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