Rigidity Properties of Group Actions on CAT(0)-Spaces

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In this lecture we shall discuss certain aspects of the general rigidity problem of classifying isometric actions of a given group Λ on a CAT(0)-space Y. The CAT(0) property, introduced by Alexandrov [Al], [Wa], generalizes to singular metric spaces the notion of nonpositive curvature. Among such spaces one finds simply connected non-positively curved Riemannian manifolds and Euclidean buildings; in particular, geometric rigidity problems and the linear representation theory of Λ over local fields are put into the same framework. There are various types of additional structures on Λ that lead to different rigidity properties. In this lecture we shall discuss the following three situations.

1. Let Λ be a subgroup of a locally compact group G, satisfying $\Gamma < \Lambda < \mathrm{Com}_G\Gamma$ where Γ is a sufficiently large discrete subgroup of G and $\mathrm{Com}_G\Gamma = \{g \in G \ : \ g^{-1}\Gamma g$ and Γ share a subgroup of finite index$\}$ is the commensurator of Γ in G. One expects then that any isometric action of Λ on a CAT(0)-space, which satisfies a suitable geometric irreducibility property, extends continuously to the closure $\overline{\Lambda} \subset G$. Rigidity properties of this type were already established by Margulis in the early 1970s, in the case where Γ is a lattice in a semisimple Lie group G, and led to his arithmeticity criterion [Ma 1], [Ma 4].

In Section 2, we state two recent results in this direction. The first result, due to Margulis, concerns the case where Γ is a cocompact, finitely generated lattice in a locally compact group G, and is based on his theory of generalized harmonic maps. The second, based on ergodic theoretic methods introduced by Margulis in [Ma 3], treats the case where the discrete subgroup $\Gamma < G$ admits a (Γ, G)-boundary (see Section 3 for examples). In Section 5 we apply these results to the study of the commensurator of uniform tree lattices.

2. Let Λ be an irreducible lattice in a group $G = \prod_{\alpha=1}^{n} \mathbb{G}_\alpha(k_\alpha)$, where $\mathbb{G}_\alpha$ is a semisimple algebraic group defined over a local field k_α. In this lecture, a local field is a locally compact nondiscrete field. Thanks to Margulis' work, one has a fairly complete criterion for the existence of a continuous extension to G of an action by isometries of Λ on a CAT(0)-space Y when $\sum_\alpha \mathrm{rank}_{k_\alpha} \mathbb{G}_\alpha \geq 2$, and either the Λ-action comes from a linear representation over a local field [Ma 4, VII. 5,6] or Y is a tree [Ma 2]. The case of irreducible representations leads to his arithmeticity theorem [Ma 4, IX. Theorem A], whereas the case of actions on trees gives a classification of those lattices Λ that are nontrivial amalgams [Ma 2, Theorem 2].

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Thus, one would like to understand in general when an isometric action of Λ on Y extends continuously to G. This is understood when Y is a locally compact CAT(-1)-space; see Section 4.

3. Let $\Lambda = \pi_1(M)$ be the fundamental group of a compact manifold M, where M is equipped with a nontrivial action of a semisimple Lie group G of higher rank preserving some geometric structure, for instance an "H-structure" [Zi 2] or a finite measure and a connection [Zi 3], [Sp-Zi]. Following Zimmer's program one expects that the higher rank hypothesis on G implies strong restrictions on the class of CAT(0)-spaces that admit $\pi_1(M)$-actions without fixed points. We illustrate this in Section 4 by a recent result of Adams, generalizing [Sp-Zi, Theorem A].

1 CAT(0)-spaces

A geodesic space is a metric space in which every pair of points x, y is joinable by a geodesic segment, i.e. a continuous curve of length $d(x, y)$. A geodesic triangle is obtained by joining pairwise three points with geodesic segments; it maps sidewise isometrically to a geodesic comparison triangle in the Euclidean plane $\mathbb{E}^2$ or in the hyperbolic plane $\mathbb{H}^2$. A geodesic space is CAT(0) (resp. CAT(-1)) if the comparison maps of its geodesic triangles into $\mathbb{E}^2$ (resp. $\mathbb{H}^2$) are not distance decreasing. Observe that a CAT(-1)-space is CAT(0). Many global geometric properties of Cartan-Hadamard manifolds (see [B-G-S]) generalize to CAT(0)-spaces and we refer the reader to [Gr 2], [Bri-Ha], [B] for general expositions. The following basic properties of the distance function of a CAT(0)-space Y hold:

P1. Convexity of distance: for any geodesic segments $c_1, c_2 : I \to Y$, $I \subset \mathbb{R}$, the function $t \to d(c_1(t), c_2(t))$ is convex;

P2. Uniform convexity of balls: for every $R, a > 0$, there exists $\varepsilon = \varepsilon(R, a) > 0$ such that given any three points $x, y_1, y_2 \in Y$ with $d(x, y_i) \leq R$ and $d(y_1, y_2) \geq a$, the midpoint m on the geodesic segment $[y_1, y_2]$ satisfies $d(x, m) \leq R - \varepsilon$.

A geodesic space Y is UC (uniformly convex) if it satisfies P1 and P2. Whereas P1 implies that Y is uniquely geodesic, P2 often serves as a substitute to local compactness. Finally we mention that a CAT(0)-space Y has a visual boundary $Y(\infty)$; when Y is locally compact and complete, the space $\overline{Y} := Y \sqcup Y(\infty)$ is an Isom(Y)-equivariant compactification of Y.

Example (1). L^p-spaces are UC for $1 < p < \infty$. Hilbert spaces are CAT(0); more generally, if S is a finite measure space and X is CAT(0), then $L^2(S, X)$ is CAT(0) [Ko-Sc].

Example (2). Simply connected Riemannian manifolds of sectional curvature $K \leq 0$ (respectively $K \leq -1$) and their convex subsets are CAT(0) (respectively CAT(-1)) [B-G-S].

Example (3). Euclidean buildings are CAT(0) [Bru-Ti], [Bro, VI. 3].

Example (4). For every Coxeter system (W, S), there exists a piecewise Euclidean cell complex Σ which is CAT(0) for the induced length metric and on which W acts properly discontinuously by isometries with compact quotient. Furthermore, Σ admits a piecewise hyperbolic CAT(-1)-structure if and only if W does not contain $\mathbb{Z}^2$. See [Mou] and the expository paper [Da].

EXAMPLE (5). Complexes of groups with a metric of nonpositive curvature are developable and their universal covering is CAT(0) [Ha], [Sp].

EXAMPLE (6). Two dimensional complexes with prescribed link L_v at every 0-cell v and whose 2-cells are regular polygons in $\mathbb{E}^2$ (resp. $\mathbb{H}^2$). Conditions for their existence and for the induced length metric to be CAT(0) (resp. CAT(-1)) are given in [Hag], [Be 1], [Be 2], [B-Br]. First examples of such polyhedra were constructed by Gromov, see [Gr 1, Section 4. C$''$].

EXAMPLE (7). Metric trees are CAT(-1)-spaces.

2 Rigidity properties of commensurators

2.1. For a finitely generated group Γ and a homomorphism $\pi : \Gamma \to \mathrm{Isom}(Y)$, where Y is a UC space, we introduce the properties:

HP1. For some (and hence every) finite generating set $S \subset \Gamma$, the sublevels of the function $d_S : Y \to \mathbb{R}_+$, $d_S(y) = \max_{\gamma \in S} d(y, \pi(\gamma)y)$, are bounded subsets of Y.

HP2. For every $y_1 \neq y_2$ in Y, there exists $\gamma \in \Gamma$ such that the geodesic segments $[y_1, \pi(\gamma)y_1]$, $[y_2, \pi(\gamma)y_2]$ are not parallel, meaning that if c parametrizes $[y_1, y_2]$, $t \to d(c(t), \pi(\gamma)c(t))$ is not constant.

Observe that if Y is CAT(0), locally compact and complete, HP1 is equivalent to the property that $\pi(\Gamma)$ does not have a fixed point in $Y(\infty)$.

THEOREM 1. [Ma 5] *Let Γ be a finitely generated, cocompact lattice in a locally compact group G, $\Lambda < G$ with $\Gamma < \Lambda < \mathrm{Com}_G\Gamma$ and $\rho : \Lambda \to \mathrm{Isom}(Y)$ a homomorphism into the group of isometries of a complete UC space Y, such that*

(1) *Λ acts c-minimally on Y,*

(2) *any subgroup of finite index in Γ satisfies HP1 and HP2.*

Then ρ extends continuously to $\overline{\Lambda}$.

REMARKS: (1) A group action by isometries on a geodesic space is c-minimal if it admits no nonvoid proper closed convex invariant subspace.

(2) Theorem 1 applies to the case where Y is the Euclidean building associated to a connected k-simple group $\mathbb{H}$ defined over a local field k, the image $\rho(\Lambda) \subset \mathbb{H}(k)$ is Zariski dense in $\mathbb{H}$, and $\rho(\Gamma) \subset \mathbb{H}(k)$ is not relatively compact.

The proof of this theorem relies on uniqueness properties of generalized harmonic maps [Ma 5]. By definition such maps are critical points, in the space of Γ-equivariant measurable maps $\varphi : G \to Y$, of an "energy " functional

$$E(\varphi) := \int_{\Delta(\Gamma)\backslash G \times G} d(\varphi(g_1), \varphi(g_2))^p h(g_1^{-1} g_2) dg_1 dg_2$$

where $p \geq 1$, $\Delta(\Gamma) := \{(\gamma, \gamma) : \gamma \in \Gamma\}$ and h is a suitable positive continuous function. The basic problem for applying this method to discrete subgroups Γ of locally compact groups G is to find a positive continuous function h and an exponent $p > 1$ such that there is at least one equivariant map $\varphi : G \to Y$ of finite energy. When Γ is a finitely generated cocompact lattice in G, $p > 1$, and π satisfies HP1, HP2, there is a unique harmonic map and it is continuous; Theorem 1 follows then from the fact that such a map is automatically Λ-equivariant. These results apply notably when Γ is a cocompact lattice in a connected Lie group, or in $\prod_{\alpha=1}^{n} \mathbb{G}_\alpha(k_\alpha)$ where $\mathbb{G}_\alpha$ is a semisimple connected group defined over a local field k_α, or in the automorphism group Aut$T$ of a uniform tree (see Section 5). Indeed, in all these cases Γ is finitely generated. The above method applies also to irreducible lattices in semisimple connected Lie groups all of whose simple factors are noncompact and not locally isomorphic to $SL(2, \mathbb{R})$. Notice that lattices need not be finitely generated. For example, noncocompact lattices in $G = \mathbb{G}(k)$, where $\mathbb{G}$ is a simple k-group of k-rank one defined over a local field k of positive characteristic, are never finitely generated (see [Lu] for the structure of lattices in $\mathbb{G}(k)$). Also when G is not compactly generated, cocompact lattices are not finitely generated, see 3.3 (b) for an example with dense commensurator.

2.2. Let G be a locally compact group, $H < G$ a closed subgroup, and B a standard Borel space endowed with a Borel action $G \times B \to B$ preserving a σ-finite measure class μ. We call (B, μ) an (H, G)-boundary if it satisfies the properties:

BP1: the H-action on (B, μ) is amenable;

BP2: the diagonal H-action on $(B \times B, \mu \times \mu)$ is ergodic.

THEOREM 2. *Let Γ be a discrete subgroup of a locally compact, second countable group G and $\Lambda < G$ with $\Gamma < \Lambda < \mathrm{Com}_G\Gamma$. We assume that there exists (B, μ), which is a (Γ', G)-boundary for any subgroup of finite index $\Gamma' < \Gamma$.*

A. *Let $\mathbb{H}$ be a connected adjoint k-simple group, where k is a local field, and $\pi : \Lambda \to \mathbb{H}(k)$ a homomorphism such that $\pi(\Lambda)$ is Zariski dense in $\mathbb{H}$ and $\pi(\Gamma) \subset \mathbb{H}(k)$ is not relatively compact. Then π extends continuously to $\overline{\Lambda}$.*

B. *[Bu-Mo] Let Y be a locally compact, complete $CAT(-1)$-space and $\pi : \Lambda \to$ Isom(Y) a homomorphism such that $\pi(\Lambda)$ acts c-minimally on Y and $\pi(\Gamma)$ is not elementary. Then π extends continuously to $\overline{\Lambda}$.*

REMARKS: (1) A group of isometries of a CAT(-1)-space Y is called elementary, if it has an invariant subset $\Delta \subset \overline{Y}$ consisting of one or two points.

(2) In the case where Γ is a lattice in the automorphism group of a d-regular tree T_d ($d \geq 3$) and Y is a locally finite tree, Theorem 2B is due to Lubotzky-Mozes-Zimmer [L-M-Z].

(3) Gao has recently (October 94) proved Theorem 2B in the case where Γ is a divergence group (see Section 3.3) in $G =$ Isom(X), X is locally compact complete CAT(-1), and Y is separable complete CAT(-1).

The proof of Theorem 2 relies on uniqueness properties of suitable "boundary" maps. For 2A one shows that, if (B, μ) is a (Γ', G)-boundary for every subgroup of finite index $\Gamma' < \Gamma$, and $\pi : \Gamma \to \mathbb{H}(k)$ is a homomorphism with Zariski dense and unbounded image, there is a proper k-subgroup $\mathbb{L} < \mathbb{H}$ such that for any $\Gamma' < \Gamma$, of finite index, there exists a unique Γ'-equivariant measurable map $\varphi : B \to \mathbb{H}(k)/\mathbb{L}(k)$ (see [Zi 1], [A'C-Bu] for different constructions of boundary maps). In case 2B one shows that, if (B, μ) is a (Γ, G)-boundary and $\pi : \Gamma \to \mathrm{Isom}(Y)$ a homomorphism such that $\pi(\Gamma)$ is not elementary, there is a unique Γ-equivariant measurable map $\varphi : B \to Y(\infty)$ (see [Bu-Mo]).

3 (Γ, G)-boundaries and commensurators

Theorem 2 is of interest when (Γ, G) admits a boundary and $\mathrm{Com}_G\Gamma$ is not discrete. We have the following sources for such groups:

3.1. Arithmetic lattices. Let us consider $G = \prod_{\alpha=1}^n \mathbb{G}_\alpha(k_\alpha)$ and $P = \prod \mathbb{P}_\alpha(k_\alpha)$, where $\mathbb{G}_\alpha$ is a connected, simply connected k_α-almost simple, k_α-isotropic group and $\mathbb{P}_\alpha < \mathbb{G}_\alpha$ a minimal k_α-parabolic subgroup of $\mathbb{G}_\alpha$. The homogeneous space $B := G/P$ with its G-invariant measure class is a (Γ, G)-boundary for any lattice $\Gamma < G$. When the lattice Γ is arithmetic, its commensurator is dense in G. By Margulis' arithmeticity theorem this is always the case when $\sum \mathrm{rank}_{k_\alpha}\mathbb{G}_\alpha \geq 2$ and Γ is irreducible.

3.2. Regular tree lattices. Consider $G = \mathrm{Aut}\ T_d$, $d \geq 3$, and $P < G$ the stabilizer of a point in $T_d(\infty)$. Again the homogeneous space $B := G/P$ with its G-invariant measure class is a (Γ, G)-boundary for any lattice $\Gamma < G$. Here BP2 follows from the Howe-Moore property of Aut T_d [Lu-Mo]. We remark that cocompact lattices in Aut T_d have dense commensurators, see Section 5.

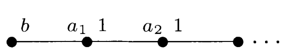
3.3. Divergence groups. Let X be a locally compact and complete CAT(-1)-space, $H < \mathrm{Isom}(X)$ a closed subgroup, and

$$\delta_H = \inf\{s > 0 : \ P_x(s) = \int_H e^{-sd(hx,x)}dh < +\infty\}$$

its critical exponent, which does not depend on the choice of $x \in X$. A nonelementary, discrete subgroup $\Gamma < \mathrm{Isom}(X)$ is called a divergence group if $\delta_\Gamma < +\infty$ and $P_x(\delta_\Gamma) = +\infty$. Generalizing the Patterson-Sullivan theory of Kleinian groups (see [Pa], [Su], and [Bo] for compact quotients of CAT(-1)-spaces), one constructs a canonical measure class μ_Γ on $X(\infty)$ that is invariant under $G := \overline{\mathrm{Com}_{\mathrm{Isom}(X)}\Gamma} < \mathrm{Isom}(X)$ and such that $(X(\infty), \mu_\Gamma)$ is a (Γ', G)-boundary for any subgroup Γ' of finite index in Γ. Hence when Γ is a divergence group, Theorem 2 may be applied to any subgroup $\Lambda < \mathrm{Isom}(X)$ with $\Gamma < \Lambda < \mathrm{Com}_{\mathrm{Isom}(X)}\Gamma$. Concerning property BP1, Adams [Ad] has shown that when X is at most of exponential growth, the action on $X(\infty)$ of any closed subgroup of $\mathrm{Isom}(X)$ is universally amenable.
We mention the following examples of divergence groups (see [Bu-Mo]):

(a) any lattice in $\mathrm{Isom}(X)$ with $\mathrm{Isom}(X)\backslash X$ compact, is a divergence group;

(b) let $\mathbb{A}$ be a tree of finite groups with edge indexed graph (see [Ba-Ku] for definitions),



where $b \geq 2$ and not all a_i's are 1. Let R be the radius of convergence of the power series $P(x) := \sum_{k=1}^{\infty}(a_k - 1)a_{k-1} \dots a_1 x^k$, T the universal covering tree, and Γ the fundamental group of $\mathbb{A}$. Then Γ is a divergence group if and only if $P(R) \geq \frac{1}{b-1}$. This happens for example if P is rational. When the edge indexed graphs associated respectively to $\mathrm{Aut}T \backslash T$ and $\mathbb{A}$ coincide (e.g. $b \neq a_i + 1$, $\forall i$) then Γ is cocompact in $\mathrm{Aut}T$ and $\mathrm{Aut}T$ is not compactly generated. One can show that no $(\Gamma, \mathrm{Aut}\ T)$-boundary is a homogeneous space of $\mathrm{Aut}T$. Finally, one can choose the vertex and edge groups of $\mathbb{A}$ in such a way that $\mathrm{Com}_{\mathrm{Aut}\ T}\Gamma$ is dense in $\mathrm{Aut}T$.

(c) Let $\Gamma <$ Aut T_4 be the fundamental group of the Cayley graph of the free abelian group on two generators. Then Γ is a divergence group whose commensurator is dense in Aut T_4.

4 Lattices in higher rank groups and CAT(-1)-spaces

THEOREM 3. [Ad], [Bu-Mo] *Let Γ be an irreducible lattice in $G = \prod_{\alpha=1}^{n} \mathbb{G}_\alpha(k_\alpha)$, where $\mathbb{G}_\alpha$ is a simply connected, k_α-almost simple, k_α-isotropic group and $\sum_{\alpha=1}^{n}$* $\mathrm{rank}_{k_\alpha}\mathbb{G}_\alpha \geq 2$. *Let Y be a locally compact, complete CAT(-1)-space and $\pi : \Gamma \to$* $\mathrm{Isom}(Y)$ *a homomorphism with nonelementary image. Let $X \subset Y$ be the closed convex hull of the limit set of $\pi(\Gamma)$. Then $\pi : \Gamma \to$* $\mathrm{Isom}(X)$ *extends to a continuous homomorphism of G, factoring through a proper homomorphism from a rank one factor of G into* $\mathrm{Isom}(X)$.

When all the almost simple factors of G have rank at least 2, $\pi(\Gamma)$ must be elementary; when $\mathrm{Isom}(Y)$ has finite critical exponent, stabilizers of boundary points are amenable and, as Γ has property T, $\pi(\Gamma)$ is relatively compact and hence fixes a point in Y.

See [Ad, Theorem 11.2] in the case where $G = \mathbb{G}(\mathbb{R})$ $(n = 1)$ and [Bu-Mo] in the general case.

THEOREM 4. [Ad] *Let M be a connected, compact real analytic manifold equipped with a nontrivial real analytic action of a simple Lie group G of real rank ≥ 2. Assume that G preserves a probability measure and a real analytic connection. Then $\pi_1(M)$ cannot act properly discontinuously on a locally compact CAT(-1) space that is at most of exponential growth.*

5 Uniform tree lattices

A uniform tree is a locally finite tree X whose automorphism group contains a discrete subgroup Γ such that $\Gamma \backslash X$ is finite. In particular, such a subgroup Γ is a cocompact lattice in Aut X. Uniform trees are characterized by the property that Aut $X \backslash X$ is finite and Aut X is unimodular [Ba-Ku]. It is a remarkable fact that for a uniform tree X, the commensurators of any two cocompact lattices in Aut X are conjugate [Le], [Ba-Ku]. Denoting by $C(X)$ a representative of this conjugacy class of subgroups, we have

THEOREM 5. [Li] *$C(X)$ is dense in* Aut X.

This was conjectured by Bass and Kulkarni who proved it for regular trees, see [Ba-Ku].

Theorem 1 and 2B are useful in studying the problem of whether $C(X)$ determines X. The analogous problem of whether Aut X determines X was solved by Bass and Lubotzky, see [Ba-Lu]. Concerning the former, we have a complete answer only in the case of regular trees.

COROLLARY 1. [L-M-Z] *If* $\rho : C(T_d) \to C(T_m)$ *is an isomorphism, then* $d = m$ *and* ρ *is conjugation by an element of* Aut T_d.

In [L-M-Z] it is also shown that $C(T_d)$, $d \geq 3$, is not linear over any field. The proof of this fact used a description of the commensurator of certain cocompact lattices in Aut T_d in terms of "recolorings" of d-regular graphs. This was also used there to give a proof, via length functions, of Theorem 2B for the case where $\Lambda = C(T_d)$, Y is an abitrary tree, and the lattice Γ is cocompact.

Applying Theorem 2 one has

COROLLARY 2. [Bu-Mo] *Let* X *be a uniform tree all of whose vertices have degree* ≥ 3. *Assume that* Aut X *acts c-minimally on* X *and is not discrete. Then* $C(X)$ *is not linear over any field.*

Observe that if Aut X is discrete, the group $C(X) =$ Aut X is virtually free and finitely generated, hence linear. In connection with Corollary 2, the subgroup $\mathrm{Aut}^+ X$ generated by all oriented-edge stabilizers plays an important role. It follows from a theorem of Tits [Ti] that if X is uniform and Aut X acts c-minimally, then $\mathrm{Aut}^+ X$ is simple. In the proof of Corollary 2 one shows that if π is a linear representation of $C(X)$, the subgroup Ker $\pi \cap \mathrm{Aut}^+ X$ is nonamenable. It would therefore be interesting to know whether $C(X) \cap \mathrm{Aut}^+ X$ is simple.

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