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Constructing irreducible representations of discrete groups

MARC BURGER and PIERRE DE LA HARPE *
Institut de Mathématiques, Université de Lausanne, Dorigny, CH-1015 Lausanne, Suisse
e-mail: Marc.Burger@ima.unil.ch
* Section de Mathématiques, Université de Genève, C.P. 240, CH-1211 Genève 24, Suisse
e-mail: Pierre.delaHarpe@math.unige.ch

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Abstract. The decomposition of unitary representations of a discrete group obtained by induction from a subgroup involves commensurators. In particular Mackey has shown that quasi-regular representations are irreducible if and only if the corresponding subgroups are self-commensurizing. The purpose of this work is to describe general constructions of pairs of groups $\Gamma_0 < \Gamma$ with Γ_0 its own commensurator in Γ. These constructions are then applied to groups of isometries of hyperbolic spaces and to lattices in algebraic groups.

Keywords. Commensurator subgroups; unitary representations; quasi-regular representations; Gromov hyperbolic groups; arithmetic lattices.

1. Introduction

Let G be a separable locally compact group. The *unitary dual* $\hat{G}$ of G is the set of equivalence classes of irreducible representations of G, together with its Mackey Borel structure. In this paper, "representation" means "continuous unitary representation in a separable Hilbert space".

Let us recall the definition of this structure [Dix, 18.5]. For each $n \in \{1, 2, \ldots, \infty\}$, let $\mathrm{Irr}_n(G)$ denote the space of all irreducible representations of G in a given Hilbert space of dimension n. The set $\mathrm{Irr}_n(G)$ is endowed with the topology of the weak simple convergence on G (making the functions $\pi \mapsto \langle \pi(g)\xi | \eta \rangle$ continuous for all $g \in G$ and ξ, η in the Hilbert space of dimension n), and with the corresponding Borel structure. The dual $\hat{G}$ is the quotient of $\coprod_{1 \leq n \leq \infty} \mathrm{Irr}_n(G)$ by unitary equivalence, and the Mackey Borel structure on $\hat{G}$ is the quotient of the previously defined Borel structure.

In case of a countable group Γ, it follows from results of Glimm and Thoma that $\hat{\Gamma}$ is a standard Borel space if and only if Γ is virtually abelian (see [Dix], numbers 9.1, 9.5.6 and 13.11.12, or [Ped, 6.8.7]); in this case the representation theory of Γ is well understood. In all other cases there is no natural Borel coding of $\hat{\Gamma}$, i.e. $\hat{\Gamma}$ is not countably separated; for lack of a systematic procedure of constructing all irreducible representations of Γ, a natural problem is to construct large classes of irreducible representations.

Recall that two subgroups G_0 and G_1 of a group G are *commensurable* if $G_0 \cap G_1$ is of finite index in both G_0 and G_1. The *commensurator* of G_0 in G is defined to be

$$\mathrm{Com}_G(G_0) = \{g \in G \mid G_0 \text{ and } gG_0g^{-1} \text{ are commensurable}\}.$$

Let $(\Gamma_\iota)_{\iota \in I}$ be a family of pairwise non conjugate subgroups of a countable group Γ such that $\mathrm{Com}_\Gamma(\Gamma_\iota) = \Gamma_\iota$ for all $\iota \in I$. It follows from work of Mackey (see e.g. [Mac], and § 2 below) that unitary induction provides a well defined and *injective* map

$$\coprod_{\iota \in I} \widehat{\Gamma_\iota^{fd}} \subsetneq \hat{\Gamma},$$

where $\widehat{\Gamma_\iota^{fd}}$ denotes the subset of $\hat{\Gamma}_\iota$ consisting of finite dimensional representations.

Our aim in this paper is to construct actions with noncommensurable stabilizers and pairs of groups $\Gamma_0 < \Gamma$ such that $\mathrm{Com}_\Gamma(\Gamma_0) = \Gamma_0$. More generally, we construct also pairs $\Gamma_0 < \Gamma$ such that Γ_0 is a subgroup of finite index in $\mathrm{Com}_\Gamma(\Gamma_0)$; in this case, the quasiregular representation of Γ in $l^2(\Gamma/\Gamma_0)$ is a *finite* direct sum of irreducible representations.

In § 2, we recall some classical results on unitary representations. Section 3 provides elementary examples of pairs of groups $\Gamma_0 < \Gamma$ with Γ_0 its own commensurator in Γ. We consider groups of isometries of Gromov hyperbolic spaces in § 4. Then, for a lattice Γ in the group of real points of a linear algebraic group $\mathbb{G}$ defined over $\mathbb{R}$, we consider actions of Γ on appropriate sets of maximal tori in § 5 and on other sets of subgroups of $\mathbb{G}$ in § 6; in each case, we find classes of irreducible quasi-regular representations of Γ.

Note on terminology. Commensurators have been known under various names, such as quasinormalizers [Cor], commensurizers [KrR] and commensurability subgroups [Mar]. We follow the terminology of [Shi, Chapter 3] and [A' B].

2. Commensurators and induced representations

Let Γ be a discrete group, $\Gamma_0 < \Gamma$ a subgroup and $\lambda_{\Gamma/\Gamma_0}$ the left regular representation of Γ in $l^2(\Gamma/\Gamma_0)$.

A double class $\dot{x} \in \Gamma_0 \backslash \mathrm{Com}_\Gamma(\Gamma_0)/\Gamma_0$ represented by some $x \in \mathrm{Com}_\Gamma(\Gamma_0)$ corresponds to a *finite* Γ_0-orbit $\Gamma_0 x \Gamma_0$ in Γ/Γ_0, and the mapping $\Gamma_0 \to \Gamma/\Gamma_0$ applying z to $zx\Gamma_0$ induces a bijection of $\Gamma_0/(\Gamma_0 \cap x\Gamma_0 x^{-1})$ onto $\Gamma_0 x \Gamma_0$. Consequently, $\dot{x}$ gives rise to a *bounded* intertwining operator $T_{\dot{x}}$ of $\lambda_{\Gamma/\Gamma_0}$, which is defined by

$$(T_{\dot{x}} f)(y\Gamma_0) = \sum_{\zeta \in \Gamma_0/(\Gamma_0 \cap x\Gamma_0 x^{-1})} f(y\zeta x \Gamma_0)$$

for all $f \in l^2(\Gamma/\Gamma_0)$ and for all $y\Gamma_0 \in \Gamma/\Gamma_0$.

It is then a fact (see [Bin], Theorem 2.2) that the linear space generated by

$$\{T_{\dot{x}} : l^2(\Gamma/\Gamma_0) \to l^2(\Gamma/\Gamma_0) \mid \dot{x} \in \Gamma_0 \backslash \mathrm{Com}_\Gamma(\Gamma_0)/\Gamma_0\}$$

is weakly dense in the space $\mathrm{Int}(\lambda_{\Gamma/\Gamma_0})$ of bounded intertwining operators of $\lambda_{\Gamma/\Gamma_0}$. Hence, if $\Gamma_0 \backslash \mathrm{Com}_\Gamma(\Gamma_0)$ is finite, we have

$$\dim \mathrm{Int}\,(\lambda_{\Gamma/\Gamma_0}) = \mathrm{Card}\,(\Gamma_0 \backslash \mathrm{Com}_\Gamma(\Gamma_0)/\Gamma_0)$$

and $\lambda_{\Gamma/\Gamma_0}$ is a finite direct sum of irreducible representations. In particular $\lambda_{\Gamma/\Gamma_0}$ is irreducible if and only if $\mathrm{Com}_\Gamma(\Gamma_0) = \Gamma_0$.

The above considerations then lead to the following theorem. Here and in the sequel we call two subgroups Γ_0, Γ_1 of Γ *quasiconjugate* if there exists $\gamma \in \Gamma$ such that Γ_0 and $\gamma\Gamma_1\gamma^{-1}$ are commensurable.

Theorem 2.1 [Mackey]. *Let Γ be a discrete group and let Γ_0, Γ_1 be subgroups of Γ.* (1) *The representation $\lambda_{\Gamma/\Gamma_0}$ is irreducible if and only if* $\mathrm{Com}_\Gamma(\Gamma_0) = \Gamma_0$, *in which case* $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\pi)$ *is irreducible for any $\pi \in \widehat{\Gamma_0^{fd}}$, and unitary induction*

$$\mathrm{Ind}_{\Gamma_0}^{\Gamma}: \widehat{\Gamma_0^{fd}} \longrightarrow \hat{\Gamma}$$

is an injective map.

(2) *If* $\mathrm{Com}_\Gamma(\Gamma_i) = \Gamma_i$, $i = 0, 1$, *then* $\lambda_{\Gamma/\Gamma_0}$ *and* $\lambda_{\Gamma/\Gamma_1}$ *are unitarily equivalent if and only if* Γ_0 *and* Γ_1 *are quasiconjugate in* Γ.

In case Γ_0 *and* Γ_1 *are not quasiconjugate in* Γ, *if* π_0, *respectively* π_1, *are finite dimensional irreducible unitary representations of* Γ_0, *respectively* Γ_1, *then* $\mathrm{Ind}_{\Gamma_0}^{\Gamma}(\pi_0)$ *and* $\mathrm{Ind}_{\Gamma_1}^{\Gamma}(\pi_1)$ *are not equivalent.*

Remark. We do not know whether the condition $\pi \in \widehat{\Gamma_0^{fd}}$ in (1) can be replaced by $\pi \in \hat{\Gamma}_0$.

Let us restate the previous Theorem in a slightly different way. Let Γ be a discrete group acting on a set A, and denote by

$$\mathscr{Z}_\Gamma(a) \doteqdot \{\gamma \in \Gamma \mid \gamma a = a\}$$

the stabilizer of a point $a \in A$; if more precision is needed, we write $\mathscr{Z}_{\Gamma,A}(a)$ for $\mathscr{Z}_\Gamma(a)$.

DEFINITION

The action $\Gamma \times A \longrightarrow A$ has *noncommensurable stabilizers* (N.C.S.) if any two points a_1, $a_2 \in A$ with commensurable stabilizers coincide.

The following lemma is an easy observation.

Lemma 2.2. (1) *Let* $\Gamma \times A \longrightarrow A$ *be a N.C.S. action. For* $a_1, a_2 \in A$ *and* $\gamma \in \Gamma$, *we have* $\gamma a_1 = a_2$ *if and only if* $\gamma \mathscr{Z}_\Gamma(a_1)\gamma^{-1} = \mathscr{Z}_\Gamma(a_2)$, *if and only if* $\gamma \mathscr{Z}_\Gamma(a_1)\gamma^{-1}$ *and* $\mathscr{Z}_\Gamma(a_2)$ *are commensurable.*

In particular $(\mathscr{Z}_\Gamma(a))_{a\in A}$ *is a set of self-commensurizing subgroups of* Γ, *two subgroups* $Z_\Gamma(a_1)$, $Z_\Gamma(a_2)$ *of the set being quasiconjugate if and only if* a_1, a_2 *are in the same* Γ*-orbit.*

(2) *Let* $\mathscr{G}$ *be a set of self-commensurizing subgroups of* Γ *which is stable under conjugation. Then the action of* Γ *on* $\mathscr{G}$ *by conjugation is N.C.S.*

It follows from Theorem 2.1 and Lemma 2.2. that, for a N.C.S. action $\Gamma \times A \longrightarrow A$, unitary induction

$$\mathrm{Ind}: \widehat{\coprod_{a\in\Gamma\backslash A} \mathscr{Z}_\Gamma(a)^{fd}} \longrightarrow \hat{\Gamma}$$

is an injective map.

For later use we record the following general fact. Let π, ρ be unitary representations of a group Γ. We write $\pi \prec \rho$ to express that π is *weakly contained* in ρ [Dix, 18.1.3], and $\pi \sim \rho$ to express that π and ρ are *weakly equivalent* [namely that $\pi \prec \rho$ and $\rho \prec \pi$].

Lemma 2.3. *Let* Γ_0 *be a subgroup of* Γ. *Then* $\lambda_{\Gamma/\Gamma_0} \prec \lambda_\Gamma$ *if and only if* Γ_0 *is amenable.*

Proof. If Γ_0 is amenable, $1_{\Gamma_0} \prec \lambda_{\Gamma_0}$ and hence $\lambda_{\Gamma/\Gamma_0} = \mathrm{Ind}_{\Gamma_0}^{\Gamma}(1_{\Gamma_0}) \prec \mathrm{Ind}_{\Gamma_0}^{\Gamma}(\lambda_{\Gamma_0}) = \lambda_\Gamma$.

Conversely, since 1_{Γ_0} is contained in $\mathrm{Res}_{\Gamma_0}(\lambda_{\Gamma/\Gamma_0})$ and since $\mathrm{Res}_{\Gamma_0}(\lambda_\Gamma)$ is a multiple of λ_{Γ_0}, the assumption $\lambda_{\Gamma/\Gamma_0} \prec \lambda_\Gamma$ implies

$$1_{\Gamma_0} < \mathrm{Res}_{\Gamma_0}(\lambda_{\Gamma/\Gamma_0}) \prec \mathrm{Res}_{\Gamma_0}(\lambda_\Gamma) \sim \lambda_{\Gamma_0}$$

and hence Γ_0 is amenable. □

3. Elementary examples of N.C.S. actions

Define a group action $G \times A \to A$ to be *large* if, for all $a \in A$, all $\mathscr{Z}_G(a)$-orbits in $A \setminus \{a\}$ are infinite. The next lemma is a convenient tool for constructing N.C.S. actions.

Lemma 3.1. (1) *A large action is N.C.S.*
(2) *Let $G \times A \to A$ be a large transitive action and let $\Gamma < G$ be a subgroup such that* $\mathrm{Com}_G \Gamma = G$. *Assume that there exists a point $a_0 \in A$ such that all $\mathscr{Z}_{\Gamma,A}(a_0)$-orbits in $A \setminus \{a_0\}$ are infinite. Then the restricted action $\Gamma \times A \to A$ is large.*

Proof. (1) For a large action $G \times A \to A$ and for two points $a_1, a_2 \in A$ with $\mathscr{Z}_G(a_1)$ and $\mathscr{Z}_G(a_2)$ commensurable, the $\mathscr{Z}_G(a_1)$-orbit of a_2 is finite and hence $a_1 = a_2$.

(2) For $a \in A$ and $g \in G$ such that $ga_0 = a$, the $\mathscr{Z}_{\Gamma,A}(a)$-orbits in $A \setminus \{a\}$ are infinite if and only if the $(g^{-1}\mathscr{Z}_{\Gamma,A}(a)g)$-orbits in $A \setminus \{a_0\}$ are infinite. Since

$$g^{-1}\mathscr{Z}_{\Gamma,A}(a)g = g^{-1}\Gamma g \cap \mathscr{Z}_{G,A}(a_0)$$

and $G = \mathrm{Com}_G \Gamma$, the subgroup

$$\Delta_0 \doteqdot \mathscr{Z}_{\Gamma,A}(a_0) \cap g^{-1}\mathscr{Z}_{\Gamma,A}(a)g = Z_{\Gamma,A}(a_0) \cap g^{-1}\Gamma g$$

is of finite index in $\mathscr{Z}_{\Gamma,A}(a_0)$. In particular all Δ_0-orbits in $A \setminus \{a_0\}$ are infinite and the same holds therefore for $g^{-1}Z_{\Gamma,A}(a)g$. □

(Claim (1) of Lemma 3.1 is a straightforward generalization of Theorem 4 in [Oba], which delas with doubly transitive actions on infinite sets.)

Example 1. Let $\mathbb{K}$ be an infinite field and let $\mathrm{Gr}_k(\mathbb{K}^n)$ denote the Grassmannian of k-dimensional subspaces of $\mathbb{K}^n$, where n,k are integers with $n \geqslant 2$ and $1 \leqslant k \leqslant n-1$.

The natural action of $GL(n,\mathbb{K})$ on $\mathrm{Gr}_k(\mathbb{K}^n)$ is N.C.S.

If $\mathbb{K}$ is a number field and if $\mathcal{O}_{\mathbb{K}}$ denotes its ring of integers, the action of $GL(n, \mathcal{O}_{\mathbb{K}})$ on $\mathrm{Gr}_k(\mathbb{K}^n)$ is N.C.S.

Proof. For two distinct points y_1, y_2 in $\mathrm{Gr}_k(\mathbb{K}^n)$, the maximal parabolic subgroup

$$P_{y_1} \doteqdot \{g \in GL(n,\mathbb{K}) \mid gy_1 = y_1\}$$

acts transitively on the infinite subset

$$\{y \in \mathrm{Gr}_k(\mathbb{K}^n) \mid \dim_{\mathbb{K}}(y \cap y_1) = \dim_{\mathbb{K}}(y_2 \cap y_1)\}$$

of the Grassmannian. Hence the transitive action of $GL(n,\mathbb{K})$ on $\mathrm{Gr}_k(\mathbb{K}^n)$ is large; in particular P_y is its own commensurator in $GL(n,\mathbb{K})$ for all $y \in G_k(\mathbb{K}^n)$.

Let $\mathbb{K}$ be now a number field. If $y_0 \in \mathrm{Gr}_k(\mathbb{K}^n)$ denote the subspace spanned by the first k vectors of the canonical basis of $\mathbb{K}^n$ and if $\Gamma = GL(n, \mathcal{O}_{\mathbb{K}})$, one has

$$\mathscr{Z}_\Gamma(y_0) = \left\{ \gamma \in \Gamma \mid \gamma \text{ of the form } \begin{pmatrix} * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * \\ 0 & \cdots & 0 & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & * & \cdots & * \end{pmatrix} \right\}$$

(with the block of zeros having $n-k$ rows and k columns). Let $y_1 \in \mathrm{Gr}_k(\mathbb{K}^n)\backslash\{y_0\}$; set $l = k - \dim_{\mathbb{K}}(y_0 \cap y_1)$. We identify $\mathbb{K}^n/y_0$ with the vector space $\mathbb{K}^{n-k}$. The actions of P_{y_0} on $\mathbb{K}^n$ and on $\{g \in \mathrm{Gr}_k(\mathbb{K}^n) | \dim(y \cap y_0) = \dim(y_1 \cap y_0)\}$ factor as actions of $GL(n-k, \mathbb{K})$ on $\mathbb{K}^{n-k}$ and $\mathrm{Gr}_l(\mathbb{K}^{n-k})$ respectively, so that the action of $\mathscr{Z}_\Gamma(y_0)$ on $\mathrm{Gr}_k(\mathbb{K}^n)\backslash\{y_0\}$ factors as an action of $GL(n-k, \mathcal{O}_{\mathbb{K}})$ on $\mathrm{Gr}_l(\mathbb{K}^{n-k})$. The latter action has clearly all its orbits infinite, since the Zariski closure of $GL(n-k, \mathcal{O}_{\mathbb{K}})$ contains that of $GL(n-k, \mathbb{Z})$ and thus contains $SL(n-k, \mathbb{C})$. It follows first that all orbits of $\mathscr{Z}_\Gamma(y_0)$ on $\mathrm{Gr}_k(\mathbb{K}^n)\backslash\{y_0\}$ are infinite, and second that $\mathscr{Z}_\Gamma(y) = \Gamma \cap P_y$ is its own commensurator in $\Gamma = GL(n, \mathcal{O}_{\mathbb{K}})$ for all $y \in \mathrm{Gr}_k(\mathbb{K}^n)$. □

We observe the following consequence of Example 1.

PROPOSITION 3.2

The unitary representation π of $SL(n, \mathbb{Z})$ in $L^2(\mathbb{R}^n/\mathbb{Z}^n)$ is an orthogonal direct sum of irreducible representations.

Proof. By Fourier transform, π is equivalent to the permutation representation of $SL(n, \mathbb{Z})$ in $l^2(\mathbb{Z}^n)$; the latter is a direct sum of quasi-regular representations $\pi_k \doteqdot \lambda_{SL(n,\mathbb{Z})/\Gamma_k}$, where Γ_k denotes the stabilizer of $(k, 0, \dots, 0) \in \mathbb{Z}^n$ in $SL(n, \mathbb{Z})$, for all $k \geqslant 0$. The one-dimensional representation π_0 is irreducible. For $k \geqslant 1$, and Γ'_k the stabilizer of $(k:0:\cdots:0) \in \mathbb{P}^{n-1}(\mathbb{Q})$, Mackey's result and Example 1 imply that $\lambda_{SL(n,\mathbb{Z})/\Gamma'_k}$ is irreducible. As Γ_k is of index 2 in Γ'_k, the representation π_k is either irreducible or sum of 2 irreducibles. □

For a group action $G \times A \to A$ and subsets $B \subset A$, $S \subset G$ we set

$$\mathscr{Z}_{G,A}(B) \doteqdot \bigcap_{b \in B} \mathscr{Z}_{G,A}(b)$$

$$\mathscr{N}_{G,A}(B) \doteqdot \{g \in G | g(B) = B\}$$

and $\mathscr{F}_A(S)$ the set of common fixed points of elements in S. Observe that

$$\mathscr{N}_{G,A}(B) = \mathscr{Z}_{G,\mathscr{P}(A)}(B),$$

where $\mathscr{P}(A)$ denotes the power set of A.

Lemma 3.3. *Let $G \times A \to A$ be an action and let $S \subset G$ be a union of conjugacy classes of G such that*

$$\mathscr{F}_A(g) = \mathscr{F}_A(g^n) \quad and \quad |\mathscr{F}_A(g)| < \infty$$

for all $g \in S$ and for all $n > 1$. Then the action of G on the set

$$\{F \in \mathscr{P}(A) | F = \mathscr{F}_A(g) \text{ for some } g \in S\}$$

is N.C.S.

Proof. Let $g, h \in S$ be such that the subgroups $\mathscr{N}_{G,A}(\mathscr{F}_A(g))$ and $\mathscr{N}_{G,A}(\mathscr{F}_A(h))$ are commensurable in G. Since $\mathscr{F}_A(g)$ and $\mathscr{F}_A(h)$ are both finite subsets of A, the subgroup

$$K \doteqdot \mathscr{Z}_{G,A}(\mathscr{F}_A(g)) \cap \mathscr{Z}_{G,A}(\mathscr{F}_A(h))$$

is of finite index in $\mathscr{Z}_{G,A}(\mathscr{F}_A(g))$ and $\mathscr{Z}_{G,A}(\mathscr{F}_A(h))$.

Hence there exists an integer $N \geqslant 1$ such that g^N and h^N are in K. One has

$$\mathscr{F}_A(g) = \mathscr{F}_A(g^N) \supset \mathscr{F}_A(K) \supset \mathscr{F}_A(\mathscr{Z}_{G,A}(\mathscr{F}_A(h))) = \mathscr{F}_A(h)$$

and similarly $\mathscr{F}_A(h) \supset \mathscr{F}_A(g)$, so that $\mathscr{F}_A(h) = \mathscr{F}_A(g)$. □

Example 2. Consider a subgroup Γ of $SL(n, \mathbb{C})$ and an element $\gamma \in \Gamma$ which is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$ and which is regular in the following sense: one has $\lambda_j^N \neq \lambda_k^N$ for each integer $N \geqslant 1$ whenever j, k are distinct in $\{1, \ldots, n\}$; in other words, the fixed point set $\mathscr{F}(\gamma)$ of γ in $\mathbb{P}^{n-1}(\mathbb{C})$ has cardinality n and $\mathscr{F}(\gamma^N) = \mathscr{F}(\gamma)$ for all integers $N \in \mathbb{Z}$, $N \neq 0$. Then the subgroup

$$\mathscr{N}_{\Gamma, \mathbb{P}^{n-1}(\mathbb{C})}(\mathscr{F}(\gamma)) = \{\gamma' \in \Gamma \mid \gamma' \text{ permutes the eigen-directions of } \gamma\}$$

of Γ is its own commensurator in Γ by Lemma 3.3. (This subgroup of Γ is distinct from Γ itself as soon as Γ is not virtually abelian.)

Observe that the group

$$\mathbb{T} \doteqdot \mathscr{Z}_{SL(n,\mathbb{C}), \mathbb{P}^{n-1}(\mathbb{C})}(\mathscr{F}(\gamma))$$

is a maximal torus in $SL(n, \mathbb{C})$ and that $\mathscr{N}_{\Gamma, \mathbb{P}^{n-1}(\mathbb{C})}(\mathscr{F}(\gamma))$ is the intersection with Γ of the normalizer of $\mathbb{T}$ in $SL(n, \mathbb{C})$. More on this in § 5 below.

Example 3. Consider an integer $n \geq 2$, the group $\Gamma = SL(n, \mathbb{Z})$ and the subgroup Γ_0 of upper triangular matrices in Γ (with diagonal entries ± 1).

Then Γ_0 is its own commensurator in Γ.

Proof. Let Flag$(\mathbb{C}^n)$ be the set of complete flags in $\mathbb{C}^n$. Let S be the subset of Γ consisting of matrices which have precisely one Jordan block. Then, for the action of Γ on Flag$(\mathbb{C}^n)$, one has $\mathscr{F}(\gamma) = \mathscr{F}(\gamma^n)$ and $|\mathscr{F}(\gamma)| = 1$ for all $\gamma \in S$. This ends the proof because Γ_0 is the stabilizer of the flag $\mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^{n-1}$ associated to the canonical basis of $\mathbb{C}^n$. □

Consider the group $\Gamma = SL(3, \mathbb{Z})$. For a subgroup $\Gamma_0 = \Gamma \cap P_y$ as in Example 1, it follows from Lemma 2.3 that the irreducible representation $\lambda_{\Gamma/\Gamma_0}$ is not weakly contained in λ_Γ. But for a subgroup $\Gamma_0 = \mathscr{N}_{\Gamma, \mathbb{P}^{n-1}(\mathbb{C})}(\mathscr{F}(\gamma))$ as in Example 2 or for the triangular subgroup Γ_0 of Example 3, one has $\lambda_{\Gamma/\Gamma_0} \prec \lambda_\Gamma$ by Lemma 2.3, and consequently $\lambda_{\Gamma/\Gamma_0} \sim \lambda_\Gamma$ by [BCH].

There are examples of self-commensurizing subgroups of braid groups and of related groups in [FRZ] and in [Par].

4. Groups of isometries of hyperbolic spaces

4.1. Let X be a Gromov hyperbolic space; let $X(\infty)$ be its Gromov boundary and Is(X) its group of isometries. Then Is(X) acts on $X(\infty)$ and on $S^2X(\infty)$, the set of unordered pairs of points in $X(\infty)$.

Let Γ be a subgroup of Is(X). Denote by $X(\infty)_p \subset X(\infty)$ the set of fixed points of parabolic elements in Γ and by $S^2X(\infty)_h \subset S^2X(\infty)$ the set of fixed point sets of hyperbolic elements in Γ.

PROPOSITION 4.1

The action of Γ on

$$X(\infty)_p \coprod S^2 X(\infty)_h$$

has noncommensurable stabilizers.

Proof. Let Γ_{ne} denote the set of non elliptic elements in Γ. For the Γ-action on $X(\infty)$ and for each $\gamma \in \Gamma_{ne}$, one has

$$\mathscr{F}_{X(\infty)}(\gamma) = \mathscr{F}_{X(\infty)}(\gamma^n) \text{ for all } n \geq 1$$

and $\mathscr{F}_{X(\infty)}(\gamma)$ is of cardinality 1 or 2 depending on whether γ is parabolic or hyperbolic. Thus Proposition 4.1 follows from Lemma 3.3. □

Remark. For each hyperbolic element $\gamma \in \Gamma$, recall that the cyclic group $\gamma^{\mathbb{Z}}$ is of finite index in the group $\mathscr{Z} = \mathscr{Z}_{\Gamma, S^2 X(\infty)}(\mathscr{F}_{X(\infty)}(\gamma))$; see e.g. [GhH, chap. 8, n⁰ 33]; in particular, the group $\mathscr{Z}$ is amenable. By Lemma 2.3, the quasi-regular representation $\lambda_{\Gamma/\mathscr{Z}}$ is weakly contained in the regular representation λ_Γ.

Assume moreover that X is a discrete space which has at most exponential growth and that $\Gamma \subset \mathrm{Is}(X)$ is a discrete subgroup. For each parabolic element $\gamma \in \Gamma$, the group $\mathscr{Z} = \mathscr{Z}_{\Gamma, X(\infty)}(\mathscr{F}_{X(\infty)}(\gamma))$ is amenable (see Proposition 1.6 in [BuM]), so that one has also $\lambda_{\Gamma/\mathscr{Z}} \prec \lambda_\Gamma$. Indeed, the set

$$\{\mathscr{Z}_{\Gamma, X(\infty)}(\omega) \mid \omega \in X(\infty)_p \coprod S^2 X(\infty)_h\}$$

coincides with the set of all maximal amenable infinite subgroups of Γ [Ada].

In case Γ is a Gromov hyperbolic group, the set $X(\infty)_p$ is empty because there is no parabolic. If Γ is moreover torsion free, then $\mathscr{Z}_\Gamma(\omega)$ is infinite cyclic for all $\omega \in S^2 X(\infty)_h$.

It is known that the reduced C^*-algebra of a torsion free Gromov hyperbolic group Γ is simple [Har]. From this and Lemma 2.3, it follows that the quasi-regular representation $\lambda_{\Gamma/\mathscr{Z}_\Gamma(\omega)}$ is quasi-equivalent to the regular representation λ_Γ for each $\omega \in S^2 X(\infty)_h$.

For a nonabelian free group, this is Proposition 1 of [Boz], itself a paper strongly motivated by [Yos].

4.2. Let now X be a proper CAT(-1)-space and let

$$\mathscr{G}X = \{c : \mathbb{R} \longrightarrow X \mid c \text{ is isometric}\}$$

be the space of parametrized geodesics in X with the topology of uniform convergence on compactas. The action of $\mathbb{R}$ on $\mathscr{G}X$ via reparametrizations

$$g_t c(s) = c(s+t), \quad c \in \mathscr{G}X, \quad s, t \in \mathbb{R}$$

commutes with that of $\mathrm{Is}(X)$ and defines for any discrete subgroup $\Gamma < \mathrm{Is}(X)$ a flow on $\Gamma \backslash \mathscr{G}X$, called the *geodesic flow*. We recall that, for a discrete divergence group $\Gamma < \mathrm{Is}(X)$, there is a canonical *Patterson–Sullivan measure* m_{PS} on $\Gamma \backslash \mathscr{G}X$ which is invariant and ergodic for the geodesic flow. The notion of a divergence group is borrowed from Patterson–Sullivan theory of Kleinian groups ([Pat], [Sul]; see also [Bou], [Coo], [CoP] which is generalized to CAT(-1)-spaces in [BuM]).

PROPOSITION 4.2

Let $\Lambda < \mathrm{Is}(X)$ be a discrete subgroup. Let

$$\mathscr{S}(\Lambda) = \{\Gamma < \Lambda \mid \Gamma \text{ is a divergence group with } m_{PS}(\Gamma\backslash\mathscr{G}X) < \infty\}$$

be endowed with the ordering given by inclusion and let $\mathscr{C} \subset \mathscr{S}(\Lambda)$ be a commensurability class.

Then $\mathscr{C}$ has a unique maximal element $\Gamma_{\mathscr{C}}$, and this subgroup $\Gamma_{\mathscr{C}}$ satisfies $\mathrm{Com}_\Lambda \Gamma_{\mathscr{C}} = \Gamma_{\mathscr{C}}$. Moreover, if $\sim$ denotes the relation of commensurability on $\mathscr{S}(\Lambda)$, the action of Λ on $\mathscr{S}(\Lambda)/\sim$ by conjugation is N.C.S.

In particular, for each $\Gamma < \mathscr{S}(\Lambda)$, the quasi-regular representation $\lambda_{\Lambda/\Gamma}$ is a finite sum of irreducible representations; if $\Gamma_+ = \mathrm{Com}_\Lambda(\Gamma)$, then Γ is of finite index in Γ_+ and $\lambda_{\Lambda/\Gamma_+}$ is irreducible.

Remarks. (i) Let $\Gamma < \mathrm{Is}(X)$ be a non-elementary discrete subgroup, $\mathscr{L}_\Gamma \subset X(\infty)$ its limit set and $Q_\Gamma = \mathrm{Co}(\mathscr{L}_\Gamma) \subset X$ the convex hull of the latter. If $\Gamma\backslash Q_\Gamma$ is compact (that is, if Γ is convex-cocompact) then Γ is a divergence group with $m_{PS}(\Gamma\backslash\mathscr{G}X) < \infty$; see [Bou].

(ii) Let X be a symmetric space of rank 1 and $\Gamma < \mathrm{Is}(X)$ a geometrically finite subgroup (see [Bow]). Then Γ is a divergence group with $m_{PS}(\Gamma\backslash\mathscr{G}X) < \infty$.

Example. Let $\Lambda < PSL(2, \mathbb{R})$ be a discrete subgroup. Then $\mathscr{S}(\Lambda)$ contains all finitely generated non virtually cyclic subgroups of Λ. Indeed, such subgroups are non-elementary and geometrically finite.

Thus, for a finitely generated infinite subgroup Γ of Λ, the quasi-regular representation $\lambda_{\Lambda/\Gamma}$ is a *finite* sum of irreducible representations: this follows from Proposition 4.1 if Γ is virtually cyclic, in which case $\lambda_{\Lambda/\Gamma} \prec \lambda_\Lambda$, and from Proposition 4.2 in other cases, for which $\lambda_{\Lambda/\Gamma} \not\prec \lambda_\Lambda$.

Proof of Proposition 4.2. It suffices to show that, given a discrete divergence group $\Gamma_0 < \mathrm{Is}(X)$ with $m_{PS}(\Gamma_0\backslash\mathscr{G}X) < \infty$ and a discrete subgroup $\Gamma < \mathrm{Is}(X)$ with $\Gamma_0 < \Gamma < \mathrm{Com}_{\mathrm{Is}(X)}(\Gamma_0)$, the subgroup Γ_0 is of finite index in Γ.

Indeed, assuming this is true, consider the commensurability class $\mathscr{C}$ of a subgroup Γ_0 of Λ which is in $\mathscr{S}(\Lambda)$. Setting $\Gamma_{\mathscr{C}} = \mathrm{Com}_\Lambda(\Gamma_0)$ one has Γ_0 of finite index in $\Gamma_{\mathscr{C}}$; one has therefore $\Gamma_{\mathscr{C}} \in \mathscr{S}(\Lambda)$ and $\mathrm{Com}_\Lambda \Gamma_{\mathscr{C}} = \Gamma_{\mathscr{C}}$. As any group commensurable with Γ_0 is in $\Gamma_{\mathscr{C}}$, the latter group is clearly the *unique* maximal element of $\mathscr{C}$. The last claim of the proposition is now obvious.

For the convenience of the reader we recall the construction of m_{PS} (see § 1.3 in [BuM]). Let δ be the critical exponent of Γ_0, let $\mu: X \to M^+(X(\infty))$ be the δ-dimensional Patterson–Sullivan density for Γ_0 and let $(\xi|\eta)_x$ denote the Gromov scalar product of ξ, $\eta \in X(\infty)$. Using the Γ-invariant measure

$$\frac{\mathrm{d}\mu_x(\xi) \times \mathrm{d}\mu_y(\xi)}{\mathrm{e}^{-2\delta(\xi|\eta)_x}}$$

on $X(\infty) \times X(\infty)\backslash\{\text{diagonal}\}$, one obtains a Γ-invariant and geodesic-flow invariant measure $\tilde{m}_\mu$ on $\mathscr{G}X$; the Patterson–Sullivan measure m_{PS} is then the corresponding geodesic-flow invariant measure on $\Gamma\backslash\mathscr{G}X$.

We recall furthermore that $\gamma_* \mu_x = \mu_{\gamma x}$ for all $\gamma \in \Gamma_0$, $x \in X$, and that there exists a homomorphism χ: $\mathrm{Com}_{\mathrm{Is}(X)}(\Gamma_0) \to \mathbb{R}_+^*$ such that $\gamma_* \mu_x = \chi(\gamma)\mu_x$ for all $\gamma \in \mathrm{Com}_{\mathrm{Is}(X)}(\Gamma_0)$, $x \in X$. From this follows $\gamma_* \tilde{m}_\mu = \chi(\gamma)^2 \tilde{m}_\mu$ for all $\gamma \in \mathrm{Com}_{\mathrm{Is}(X)}(\Gamma_0)$ (see [BuM], Corollary 6.5.3).

Since Γ acts properly discontinuously on $\mathscr{G}X$, there exists a compact set $K \subset \mathscr{G}X$ of positive $\tilde{m}_\mu$-measure such that $\gamma K \cap K = \emptyset$ for all $\gamma \in \Gamma$ with $\gamma \neq e$. (We argue as if Γ was acting effectively on $\mathscr{G}X$; when it is not the case, we leave the minor appropriate changes to the reader.) For a set $\mathscr{T} \subset \Gamma$ of representatives of $\Gamma_0 \backslash \Gamma$, the set $\coprod_{\tau \in \mathscr{T}} \tau K$ injects into $\Gamma_0 \backslash \mathscr{G}X$ and therefore

$$\left(\sum_{\tau \in \mathscr{T}} \chi(\tau)^2\right) \tilde{m}_\mu(K) = \tilde{m}_\mu\left(\coprod_{\tau \in \mathscr{T}} \tau K\right) \leqslant m_{\mathrm{PS}}(\Gamma_0 \backslash \mathscr{G}X) < \infty.$$

Hence, since $\chi | \Gamma_0 = 1$, we obtain

$$\sum_{\tau \in \Gamma_0 \backslash \Gamma} \chi(\tau)^2 < \infty.$$

For every $\gamma \in \Gamma$, we have thus

$$\left(\sum_{\tau \in \Gamma_0 \backslash \Gamma} \chi(\tau)^2\right) \chi(\gamma)^2 = \sum_{\sigma \in \Gamma_0 \backslash \Gamma} \chi(\sigma)^2$$

which shows first that $\chi(\gamma)^2 = 1$ for all $\gamma \in \Gamma$ and second that $|\Gamma_0 \backslash \Gamma| < \infty$. □

5. Maximal tori and actions of lattices with noncommensurable stabilizers

Let $\mathbb{G}$ be a linear algebraic group defined over $\mathbb{R}$, let $\Gamma < \mathbb{G}(\mathbb{R})$ be a discrete subgroup and set

$$\mathscr{T}(\Gamma) = \{\mathbb{T} \subset \mathbb{G} | \mathbb{T} \text{ is a maximal } \mathbb{R}\text{-split torus such that } \mathbb{T}(\mathbb{R})/(\mathbb{T}(\mathbb{R}) \cap \Gamma) \text{ is compact}\}.$$

PROPOSITION 5.1

The Γ-action by conjugation on $\mathscr{T}(\Gamma)$ is N.C.S.

Here and in the sequel, we will use the following simple lemma.

Lemma 5.2. *Let $\mathbb{G}$ be a linear algebraic group and let A_0, A_1 be two commensurable subgroups of $\mathbb{G}$. Then $(\overline{A_0})^0 = (\overline{A_1})^0$.*

Proof of Proposition 5.1. We have to show that, given $\mathbb{T}, \mathbb{T}' \in \mathscr{T}(\Gamma)$ such that $\mathscr{N}_{\mathbb{G}}(\mathbb{T}) \cap \Gamma$ and $\mathscr{N}_{\mathbb{G}}(\mathbb{T}') \cap \Gamma$ are quasiconjugate in Γ, then $\mathbb{T}$ and $\mathbb{T}'$ are Γ-conjugate.

First we observe that, for $\mathbb{T} \in \mathscr{T}(\Gamma)$, the group $(\mathscr{N}_{\mathbb{G}}(\mathbb{T})(\mathbb{R}) \cap \Gamma)/(\mathbb{T}(\mathbb{R}) \cap \Gamma)$ is finite. Indeed, since $\mathbb{T}(\mathbb{R})/(\mathbb{T}(\mathbb{R}) \cap \Gamma)$ is compact, the canonical map

$$\mathscr{N}_{\mathbb{G}}(\mathbb{T})(\mathbb{R})/(\mathbb{T}(\mathbb{R}) \cap \Gamma) \longrightarrow \mathscr{N}_{\mathbb{G}}(\mathbb{T})(\mathbb{R})/\mathbb{T}(\mathbb{R})$$

is proper and therefore $(\mathscr{N}_{\mathbb{G}}(\mathbb{T})(\mathbb{R}) \cap \Gamma)/(\mathbb{T}(\mathbb{R}) \cap \Gamma)$ is a discrete subgroup of the compact group $\mathscr{N}_{\mathbb{G}}(\mathbb{T})(\mathbb{R})/\mathbb{T}(\mathbb{R})$.

If now $\mathscr{N}_{\mathbb{G}}(\mathbb{T}) \cap \Gamma$ and $\mathscr{N}_{\mathbb{G}}(\mathbb{T}') \cap \Gamma$ are quasiconjugate in Γ, there exist $\Delta < \mathbb{T}(\mathbb{R}) \cap \Gamma$ of finite index and $\gamma \in \Gamma$ such that $\gamma \Delta \gamma^{-1}$ is of finite index in $\Gamma \cap \mathbb{T}'(\mathbb{R})$. Passing to Zariski closure, we obtain $\mathbb{T}' = \gamma \overline{\Delta} \gamma^{-1} = \gamma \mathbb{T} \gamma^{-1}$. □

Examples. (1) Let $\mathbb{G}$ be a semisimple $\mathbb{R}$-group and $\Gamma < \mathbb{G}(\mathbb{R})$ a lattice. Then $\mathcal{T}(\Gamma) \neq \emptyset$; this follows from the existence of $\mathbb{R}$-hyper-regular elements in Γ [PrR]. Indeed, for such a $\gamma \in \Gamma$, the centralizer $\mathcal{Z}_{\mathbb{G}}(\gamma)$ contains an $\mathbb{R}$-split torus $\mathbb{T}$ which is maximal in $\mathbb{G}$ and such that $\mathbb{T}(\mathbb{R})/(\Gamma \cap \mathbb{T}(\mathbb{R}))$ is compact.
(2) Let $\mathcal{P}$ be the set of primitive indefinite integral binary forms

$$Q(X,Y) = aX^2 + bXY + cY^2$$

with $a > 0$. Then the map which to every $Q \in \mathcal{P}$ associates $SO(Q)^0$ gives a bijection between $\mathcal{P}$ and the set of $\mathbb{R}$-split tori $\mathbb{T} \subset SL(2)$ for which $SL(2,\mathbb{Z}) \cap \mathbb{T}(\mathbb{R})$ is a lattice in $\mathbb{T}(\mathbb{R})$:

$$\mathcal{P} \cong \mathcal{T}(SL(2,\mathbb{Z})).$$

(3) It is a general fact due to Ono [Ono] that, for a $\mathbb{Q}$-torus $\mathbb{T}$ with $X_{\mathbb{Q}}(\mathbb{T}) = 1$, the group $\mathbb{T}(\mathbb{R})/\mathbb{T}(\mathbb{Z})$ is compact. Hence, given a semisimple $\mathbb{Q}$-group $\mathbb{G}$, the set $\mathcal{T}(\mathbb{G}(\mathbb{Z}))$ contains all $\mathbb{Q}$-torii $\mathbb{T}$ which are maximal $\mathbb{R}$-split and such that $X_{\mathbb{Q}}(\mathbb{T}) = 1$. As examples of such torii in $SL(n)$, let $\mathbb{K}/\mathbb{Q}$ be a totally real number field or degree n, let $\mathbb{H} \doteqdot \mathrm{Res}_{\mathbb{K}/\mathbb{Q}} \mathbb{G}L_1 \subset \mathbb{G}L_n$ and $\mathbb{T} \doteqdot \mathbb{H} \cap \mathbb{S}L(n)$. The group $\mathcal{U}_{\mathbb{K}}$ of units of $\mathbb{K}$ is abelian of rank $n-1$ and isomorphic to $\mathbb{H}(\mathbb{Z})$. As $\mathbb{T}(\mathbb{Z})$ is of index at most two in $\mathbb{H}(\mathbb{Z})$, the torus $\mathbb{T}(\mathbb{Z})$ is of rank $n-1$ and hence $\mathbb{T}(\mathbb{R})/\mathbb{T}(\mathbb{Z})$ is compact.

6. Algebraic subgroups and actions of arithmetic lattices with noncommensurable stabilizers

In this section $\mathbb{G}$ denotes a connected linear algebraic $\mathbb{Q}$-group; let

$$\mathcal{S}_{\mathbb{G}} = \{\mathbb{H} \mid \mathbb{H} \text{ is a connected } \mathbb{Q}\text{-subgroup of } \mathbb{G}, \text{ of finite index in } \mathcal{N}_{\mathbb{G}}(\overline{\mathbb{H}(\mathbb{Z})^0})\}.$$

We will show below that if $\mathbb{H}$ is a connected $\mathbb{Q}$-subgroup of $\mathbb{G}$, one always has the inclusion

$$\mathbb{H} < \mathcal{N}_{\mathbb{G}}(\overline{\mathbb{H}(\mathbb{Z})}^0).$$

PROPOSITION 6.1

The action by conjugation of $\mathbb{G}(\mathbb{Z})$ on $\mathcal{S}_{\mathbb{G}}$ is N.C.S. and $\mathcal{S}_{\mathbb{G}}$ contains all parabolic $\mathbb{Q}$-subgroups of $\mathbb{G}$.

Lemma 6.2. Let $\mathbb{H}$ be a $\mathbb{Q}$-subgroup of $\mathbb{G}$.

(1) $\mathcal{N}_{\mathbb{G}}(\mathbb{H})(\mathbb{Q}) < \mathrm{Com}_{\mathbb{G}}(\mathbb{H}(\mathbb{Z}))$

(2) $\mathcal{N}_{\mathbb{G}}(\mathbb{H})^0 < \mathcal{N}_{\mathbb{G}}(\overline{\mathbb{H}(\mathbb{Z})}^0)$.

Proof of Lemma 6.2. Let us first show the implication (1) $\Longrightarrow$ (2). As $\mathcal{N}_{\mathbb{G}}(\mathbb{H})$ is defined over $\mathbb{Q}$, one has

$$\mathcal{N}_{\mathbb{G}}(\mathbb{H})^0 < \overline{\mathcal{N}_{\mathbb{G}}(\mathbb{H})(\mathbb{Q})}$$

by a theorem of Rosenlicht [Bor, 18.3]. On the other hand Lemma 5.2 implies

$$\overline{\mathrm{Com}_{\mathbb{G}}(\mathbb{H}(\mathbb{Z}))} < \mathcal{N}_{\mathbb{G}}(\overline{\mathbb{H}(\mathbb{Z})}^0)$$

and hence (1) implies (2).

In order to prove (1) we may assume that $\mathbb{H}$ is connected. Let $X_{\mathbb{Q}}(\mathbb{H})$ be the set of $\mathbb{Q}$-characters of $\mathbb{H}$ and set

$$\mathbb{H}_0 \doteqdot \bigcap_{\chi \in X_{\mathbb{Q}}(\mathbb{H})} \operatorname{Ker} \chi.$$

Clearly, $\mathbb{H}_0(\mathbb{Z})$ is a subgroup of finite index in $\mathbb{H}(\mathbb{Z})$ and it follows from [BHC] that $\mathbb{H}_0(\mathbb{Z})$ is a lattice in $\mathbb{H}_0(\mathbb{R})$. Observe also that $\mathcal{N}_{\mathbb{G}}(\mathbb{H})(\mathbb{Q})$ acts on $X_{\mathbb{Q}}(\mathbb{H})$ and hence normalizes $\mathbb{H}_0$.

Let $\mathbb{G} < GL(n, \mathbb{C})$ for some n, fix $g \in \mathcal{N}_{\mathbb{G}}(\mathbb{H})(\mathbb{Q})$ and choose an integer $m \geq 1$ such that mg and mg^{-1} are in $M_n(\mathbb{Z})$. For the subgroup

$$\Gamma \doteqdot \{\gamma \in \mathbb{H}_0(\mathbb{Z}) \mid \gamma \equiv \mathrm{id} \bmod m^2\},$$

we have $g\Gamma g^{-1} \subset M_n(\mathbb{Z})$ and $\det(g\Gamma g^{-1}) \subset \{1, -1\}$; hence $g\Gamma g^{-1} < \mathbb{H}_0(\mathbb{Z})$. Furthermore, Γ is of finite index in $\mathbb{H}_0(\mathbb{Z})$ and since $\mathbb{H}_0(\mathbb{Z})$ is a lattice in $\mathbb{H}_0(\mathbb{R})$, the conjugate $g\Gamma g^{-1}$ is of finite index in $\mathbb{H}_0(\mathbb{Z})$ as well. Hence

$$g \in \operatorname{Com}_{\mathbb{G}}(\mathbb{H}_0(\mathbb{Z})) = \operatorname{Com}_{\mathbb{G}}(\mathbb{H}(\mathbb{Z})). \qquad \square$$

Proof of Proposition 6.1. For the first assertion, take $\mathbb{H}_1, \mathbb{H}_2 \in \mathcal{S}_{\mathbb{G}}$ such that $\mathcal{N}_{\mathbb{G}}(\mathbb{H}_1)(\mathbb{Z})$ and $\mathcal{N}_{\mathbb{G}}(\mathbb{H}_2)(\mathbb{Z})$ are commensurable, hence $\mathcal{N}_{\mathbb{G}}(\mathbb{H}_1)^0(\mathbb{Z})$ and $\mathcal{N}_{\mathbb{G}}(\mathbb{H}_2)^0(\mathbb{Z})$ are also commensurable. Since $\mathbb{H}_i$ is connected, we have $\mathbb{H}_i < \mathcal{N}_{\mathbb{G}}(\mathbb{H}_i)^0$ and since $\mathbb{H}_i \in \mathcal{S}_{\mathbb{G}}$, Lemma 6.2.2 implies that $\mathbb{H}_i$ is of finite index in $\mathcal{N}_{\mathbb{G}}(\mathbb{H}_i)^0$, in particular $\mathbb{H}_1(\mathbb{Z})$ and $\mathbb{H}_2(\mathbb{Z})$ are commensurable. This implies $\overline{\mathbb{H}_1(\mathbb{Z})}^0 = \overline{\mathbb{H}_2(\mathbb{Z})}^0$, and hence

$$\mathbb{H}_1 = \mathcal{N}_{\mathbb{G}}((\overline{\mathbb{H}_1(\mathbb{Z})}^0)^0 = \mathcal{N}_{\mathbb{G}}((\overline{\mathbb{H}_2(\mathbb{Z})}^0)^0 = \mathbb{H}_2.$$

For the second assertion, let $\mathbb{P}$ be a parabolic $\mathbb{Q}$-subgroup of $\mathbb{G}$. Since $\mathbb{P} \subset \mathcal{N}_{\mathbb{G}}(\overline{\mathbb{P}(\mathbb{Z})}^0)$, the subgroup $\mathbb{P}' \doteqdot \mathcal{N}_{\mathbb{G}}(\overline{\mathbb{P}(\mathbb{Z})}^0)$ is $\mathbb{Q}$-parabolic and hence $\mathcal{R}_u(\mathbb{P}') \subset \mathcal{R}_u(\mathbb{P})$. Since $\overline{\mathbb{P}(\mathbb{Z})}^0$ is normal in $\mathbb{P}'$ we have

$$\mathcal{R}_u(\overline{\mathbb{P}(\mathbb{Z})}^0) \subset \mathcal{R}_u(\mathbb{P}').$$

On the other hand, $\overline{\mathcal{R}_u(\mathbb{P})(\mathbb{Z})} = \mathcal{R}_u(\mathbb{P})$ and hence $\mathcal{R}_u(\mathbb{P})$ is a (normal) subgroup of $\overline{\mathbb{P}(\mathbb{Z})}^0$, which implies $\mathcal{R}_u(\overline{\mathbb{P}(\mathbb{Z})}^0) \supset \mathcal{R}_u(\mathbb{P})$. This finally shows that $\mathcal{R}_u(\mathbb{P}) = \mathcal{R}_u(\mathbb{P}')$ and hence $\mathbb{P} = \mathbb{P}'$. $\square$

Examples. Assume $\mathbb{G}$ to be a semi-simple, defined over $\mathbb{Q}$ and $\mathbb{Q}$-simple. Let $\mathbb{H}$ be a connected semi-simple $\mathbb{Q}$-subgroup of $\mathbb{G}$ which is maximal as a $\mathbb{Q}$-subgroup. Then $\mathbb{H} = \mathcal{N}_{\mathbb{G}}(\mathbb{H})$, and hence $\mathbb{H} = \operatorname{Com}_{\mathbb{G}}(\mathbb{H})$ by Lemma 5.2. Observe that $\mathbb{G}(\mathbb{Z})$ is a lattice in $\mathbb{G}(\mathbb{R})$ and that $\mathbb{H}(\mathbb{Z})$ is a lattice in $\mathbb{H}(\mathbb{R})$, by [BHC].

Maximal subgroups of the classical groups have been classified by Dynkin [Dyn]. In case $\mathbb{G}$ is $SL(n, \mathbb{C})$ with its standard $\mathbb{Q}$-structure, examples of subgroups $\mathbb{H}$ as above include (to quote but a few):

(i) orthogonal groups $SO(q) \subset SL(n, \mathbb{C})$ for a non degenerate quadratic form q over $\mathbb{Q}$.
(ii) the symplectic group $Sp(n, \mathbb{C}) \subset SL(n, \mathbb{C})$ (n even),
(iii) the images of the fundamental representations $SL(m, \mathbb{C}) \to SL(\binom{m}{p}, \mathbb{C})$.

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