Idempotents in complex group rings: theorems of Zalesskii and Bass revisited

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Abstract. Let Γ be a group, and let $\mathbb{C}\Gamma$ be the group ring of Γ over $\mathbb{C}$. We first give a simplified and self-contained proof of Zalesskii's theorem [23] that the canonical trace on $\mathbb{C}\Gamma$ takes rational values on idempotents. Next, we contribute to the conjecture of idempotents by proving the following result: for a group Γ, denote by P_Γ the set of primes p such that Γ embeds in a finite extension of a pro-p-group; if Γ is torsion-free and P_Γ is infinite, then the only idempotents in $\mathbb{C}\Gamma$ are 0 and 1. This implies Bass' theorem [2] asserting that the conjecture of idempotents holds for torsion-free subgroups of $\mathrm{GL}_n(\mathbb{C})$.

1. Introduction

For a group Γ and a field F, we denote by $F\Gamma$ the group ring over F; evaluation at the identity $1 \in \Gamma$ defines the *canonical trace* on $F\Gamma$:

$$\tau_\Gamma : F\Gamma \to F : a \mapsto a(1)$$

($a \in F\Gamma$; most often we shall write τ for τ_Γ). In this paper we shall deal mainly, but not exclusively, with the case $F = \mathbb{C}$, the field of complex numbers. In that case, we shall also consider the *reduced C^*-algebra* $C_r^*\Gamma$ of Γ, i.e. the norm closure of $\mathbb{C}\Gamma$ acting by left convolution on the Hilbert space $\ell^2(\Gamma)$. The canonical trace on $\mathbb{C}\Gamma$ extends to $C_r^*\Gamma$ by the formula

$$\tau(T) = \langle T(\delta_1) | \delta_1 \rangle \tag{1}$$

($T \in C_r^*\Gamma$; here δ_1 denotes the characteristic function of $\{1\}$). For a unital algebra A over a field F, denote by $K_0(A)$ the Grothendieck group of projective, finite type modules over A; if A is endowed with a trace $\mathrm{Tr} : A \to F$, then Tr defines a homomorphism $\mathrm{Tr}_* : K_0(A) \to F$. The starting point of this paper was the following conjecture, due to Baum and Connes [3].

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Conjecture 1.1. For any group Γ, the range of $\tau_* : K_0(C_r^*\Gamma) \to \mathbb{C}$ is the subgroup of $\mathbb{Q}$ generated by the $\frac{1}{|H|}$'s, where H runs over finite subgroups of Γ.

Since $\tau_*(K_0(\mathbb{C}\Gamma))$ clearly contains this subgroup of $\mathbb{Q}$, we see that conjecturally $\tau_*(K_0(\mathbf{C}\Gamma))$ should *coincide* with this subgroup. The main evidence for this conjecture is:

Corollary 1.2. *$\tau_*(K_0(\mathbb{C}\Gamma))$ is a subgroup of $\mathbb{Q}$, for any group Γ.*

This is an easy consequence of the following nice result of Zalesskii [23] (see also [17], Theorem 3.5 in Chapter 2), for which we present a simplified and self-contained proof in section 3.

Theorem 1.3. *If $e \in \mathbb{C}\Gamma$ is an idempotent, then $\tau(e)$ is a rational number.*

Note that conjecture 1.1 implies the following conjecture of Farkas ([6], # 17): if $e \in \mathbb{C}\Gamma$ is an idempotent, and if some prime number p divides the denominator of $\tau(e)$ but not its numerator, then Γ should contain an element of order p.

Assume now that Γ is a torsion-free group. Then Conjecture 1.1 says that $\tau_*(K_0(C_r^*\Gamma)) = \mathbb{Z}$. By a standard argument involving positivity and faithfulness of τ on $C_r^*\Gamma$, which for completeness we recall in section 2, this implies the Kaplansky-Kadison conjecture on idempotents (see [21] for a survey):

Conjecture 1.4. If Γ is a torsion-free group, then $C_r^*\Gamma$ has no idempotent except 0 and 1.

In particular, there should not be any nontrivial idempotent in $\mathbb{C}\Gamma$ when Γ is torsion-free. Denote by $B\Gamma$ the classifying space of Γ, and by $RK_0(B\Gamma)$ its even K-homology with compact support. In [3], Baum and Connes define an *index map* (or *analytical assembly map*)

$$\mu_0^\Gamma : RK_0(B\Gamma) \to K_0(C_r^*\Gamma)$$

which they conjecture to be an isomorphism when Γ is torsion-free. In this case, Conjectures 1 and 2 are known to follow from the surjectivity ([1]) of μ_0^Γ. At this juncture, we mention that this conjecture of Baum and Connes was recently proved by Higson and Kasparov ([10]; see also [20]) for torsion-free amenable groups; in particular, for such an amenable torsion-free group Γ, the group ring $\mathbb{C}\Gamma$ has no non-trivial idempotent: there is no algebraic proof of this result.

Our contribution to the conjecture of idempotents is the following:

Theorem 1.5. *For a group Γ, denote by P_Γ the set of prime numbers p such that Γ embeds in a finite extension of a pro-p-group. If Γ is torsion-free and P_Γ is infinite, then there is no non-trivial idempotent in $\mathbb{C}\Gamma$.*

We shall see that Theorem 1.5 implies the following result of Bass ([2], Corollary 9.3 and Theorem 9.6):

[1]On the other hand, the injectivity of μ_0^Γ implies deep results in topology, e.g. the Novikov conjecture on homotopy invariance of higher signatures for manifolds with fundamental group Γ.

Corollary 1.6. *If Γ is torsion-free and linear in characteristic 0, then $\mathbb{C}\Gamma$ has no non-trivial idempotent.*

Actually Bass proves this for torsion-free linear groups in any characteristic, but our proof only works in characteristic 0.

2. Kaplansky's theorem

Kaplansky's theorem (see [12]) is the ancestor of all results on values of the trace on idempotents in group algebras. Existing proofs involve embedding $\mathbb{C}\Gamma$ in a suitable completion (see e.g. [16]). For completeness, we shall give a proof, by embedding $\mathbb{C}\Gamma$ in the von Neumann algebra $vN(\Gamma)$, i.e. the commutant of the right regular representation of Γ on $\ell^2(\Gamma)$ [2].

Theorem 2.1. *1. Let e be an idempotent in $vN(\Gamma)$. Then $0 \leq \tau(e) \leq 1$, with equality if and only if e is a trivial idempotent.*

2. If e is an idempotent in $\mathbb{C}\Gamma$, then $\tau(e)$ belongs to the field $\overline{\mathbb{Q}}$ of algebraic numbers.

Proof. 1. The trace τ on $vN(\Gamma)$ enjoys the following properties:

- positivity: $\tau(T^*T) \geq 0$ for $T \in C_r^*\Gamma$;
- faithfulness: $\tau(T^*T) = 0$ if and only if $T = 0$.

Fix an idempotent $e \in vN(\Gamma)$. Then the element $z = 1 + (e^* - e)^*(e^* - e)$ is self-adjoint and invertible in $vN(\Gamma)$. Set $f = ee^*z^{-1}$. Using the fact that z commutes with e, one sees that $f = f^*$. From $ee^*z = (ee^*)^2$, one deduces $f = f^2$; from $ez = ee^*e$, one deduces $fe = e$; clearly $ef = f$. So f is a self-adjoint idempotent and $\tau(f) = \tau(e)$. Since $\tau(f) = \tau(f^*f)$ and $\tau(1-f) = \tau((1-f)^*(1-f))$, it follows from $1 = \tau(f) + \tau(1-f)$ and positivity of τ that $0 \leq \tau(e) \leq 1$. If $\tau(e) = 0$, then by faithfulness $f = 0$, hence $e = 0$; replacing e by $1 - e$, one gets the other case of equality.

2. The group of all automorphisms of $\mathbb{C}$ acts on $\mathbb{C}\Gamma$. If $e = e^2 \in \mathbb{C}\Gamma$, then $\tau(\sigma(e)) = \sigma(\tau(e))$ for every $\sigma \in \operatorname{Aut}\mathbb{C}$, so that $0 \leq \sigma(\tau(e)) \leq 1$ by the first part of the theorem. Since $\operatorname{Aut}\mathbb{C}$ acts transitively on transcendental numbers, this implies $\tau(e) \in \overline{\mathbb{Q}}$.

■

Remark 2.2. In the beginning of the proof of Theorem 2.1, the argument (taken from [7], 3.2.1) really shows that, in a unital C*-algebra A, any idempotent is equivalent to a self-adjoint idempotent. What is needed is the fact that every element of A of the form $1 + x^*x$ is invertible in A.

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[2] The double commutant theorem shows that $vN(\Gamma)$ is the weak closure of $\mathbb{C}\Gamma$ acting in the left regular representation; the canonical trace extends to $vN(\Gamma)$ by formula (1).

Remark 2.3. The theorems of Kaplansky and Zalesskii are trivial for finite groups. Indeed, if Γ is a finite group of order n, denote by Tr the standard trace on $M_n(\mathbb{C})$, and by $\lambda : \mathbb{C}\Gamma \to M_n(\mathbb{C})$ the left regular representation. Then

$$\tau(a) = \frac{\mathrm{Tr}\ \lambda(a)}{n}\ (a \in \mathbb{C}\Gamma).$$

In particular, if e is an idempotent in $\mathbb{C}\Gamma$, we get

$$\tau(e) = \frac{\mathrm{Rank}\ \lambda(e)}{n},$$

a rational number between 0 and 1. A similar argument appears in lemma 1.2 of Chapter 2 of [17].

Remark 2.4. Say that a group is *locally residually finite* if every finitely generated subgroup is residually finite. For example, abelian groups are locally residually finite, and so are linear groups (in any characteristic!), by a theorem of Mal'cev [14] (see [1] for a recent proof). We observe that the theorems of Kaplansky and Zalesskii are basically obvious for a locally residually finite group Γ. Indeed, let $e \in \mathbb{C}\Gamma$ be a non-zero idempotent, and denote by H the subgroup generated by supp e. Since H is residually finite, we can find in H a normal subgroup N of finite index, such that $N \cap (\mathrm{supp}\ e) = 1$. Let $\pi : \mathbb{C}H \to \mathbb{C}(H/N)$ be the homomorphism induced by the quotient map $H \to H/N$. Denote by $\tau_{H/N}$ the canonical trace on $\mathbb{C}(H/N)$, so that

$$\tau_{H/N}(\pi(a)) = \sum_{n \in N} a(n)\ (a \in \mathbb{C}H).$$

Because of the assumption on N, we have

$$\tau(e) = \tau_{H/N}(\pi(e));$$

by the case of finite groups, we deduce that $\tau(e)$ is a rational number in $[0, 1]$.

3. Zalesskii's theorem

We follow Zalesskii's original strategy, i.e. we first prove a result in positive characteristic, and then lift it to characteristic 0. Thus we shall prove the following extension of Theorem 1.3:

Theorem 3.1. *Let F be a field. Let $e \in F\Gamma$ be an idempotent. Then $\tau(e)$ belongs to the prime field of F.*

Proof. char $F = p$. This part of the proof is basically Zalesskii's beautiful argument. Start with the remark that, if A is an algebra over F endowed with a trace $\mathrm{Tr} : A \to F$, then one enjoys "Frobenius under the trace": for every $x, y \in A$:

$$\mathrm{Tr}((x+y)^p) = \mathrm{Tr}(x^p) + \mathrm{Tr}(y^p). \qquad (2)$$

To see it, expand $(x+y)^p$ in 2^p monomials, and let the cyclic group of order p act by cyclic permutations on this set of monomials. The trace Tr is constant along orbits, so the traces along orbits with p elements sum up to 0; therefore only the two monomials x^p and y^p contribute to $\mathrm{Tr}((x+y)^p)$.

Write now $|\gamma|$ for the order of an element γ in Γ. Define a family of traces on $F\Gamma$ by

$$\mathrm{Tr}_k(a) = \sum_{|\gamma|=p^k} a(\gamma) \quad (k \in \mathbb{N}; a \in F\Gamma);$$

notice that $\mathrm{Tr}_0 = \tau$. Write $e = \sum_{\gamma\in\Gamma} e(\gamma).\gamma$; since $e = e^p$, formula (2) yields

$$\mathrm{Tr}_k(e) = \sum_{|\gamma|=p^k} e(\gamma)^p \, \mathrm{Tr}_k(\gamma^p). \tag{3}$$

But, for $k \geq 1$:

$$\mathrm{Tr}_k(\gamma^p) = \begin{cases} 1 & \text{if } |\gamma| = p^{k+1} \\ 0 & \text{otherwise;} \end{cases}$$

while, for $k = 0$:

$$\tau(\gamma^p) = \begin{cases} 1 & \text{if either } \gamma = 1 \text{ or } |\gamma| = p \\ 0 & \text{otherwise.} \end{cases}$$

For $k \geq 1$, we get from (3):

$$\mathrm{Tr}_k(e) = \sum_{|\gamma|=p^{k+1}} e(\gamma)^p = (\mathrm{Tr}_{k+1}(e))^p.$$

Since e has finite support, we clearly have $\mathrm{Tr}_k(e) = 0$ for k large enough. Going backwards, we get:

$$\mathrm{Tr}_1(e) = \mathrm{Tr}_2(e) = \ldots = 0.$$

For $k = 0$, we get from (3):

$$\tau(e) = e(1)^p + \sum_{|\gamma|=p} e(\gamma)^p = (\tau(e))^p + (\mathrm{Tr}_1(e))^p = (\tau(e))^p,$$

so that $\tau(e)$ lies in the prime field of F.

This concludes the proof of Theorem 3.1 in positive characteristic. ■

We now want to lift this proof to characteristic 0.

Lemma 3.2. *If e is an idempotent in $\mathbb{C}\Gamma$, there exists an idempotent f in $\overline{\mathbb{Q}}\Gamma$ such that* $\mathrm{supp}\; e \supset \mathrm{supp}\; f$ *and* $\tau(e) = \tau(f)$.

Proof. Set $S = \{st : s, t \in \mathrm{supp}\; e\}$ and consider the affine algebraic variety in $\mathbb{C}^S$ defined by the following set of equations:

$$x_\gamma = \sum_{s,t\in \mathrm{supp}\; e \,:\, st=\gamma} x_s x_t, \quad \gamma \in S \tag{4}$$

$$x_\gamma = 0, \quad \gamma \in S - \mathrm{supp}\; e \tag{5}$$

$$x_1 = \tau(e). \tag{6}$$

This variety has to be understood as follows: suppose that $x \in \mathbb{C}\Gamma$ is defined by this set of equations inside S, and by 0 outside S. Then (4) says that x is an idempotent, (5) prescribes the support, and (6) prescribes the trace. By Kaplansky's theorem, this variety is defined over $\overline{\mathbb{Q}}$, and it has a point over $\mathbb{C}$ (namely e); by the Nullstellensatz, it has points over $\overline{\mathbb{Q}}$. ■

We shall need a particular case of the Frobenius density theorem [9]; see [19] for interesting historical comments on this not so well-known result.

Lemma 3.3. *Let $f \in \mathbb{Z}[X]$ be an irreducible, monic polynomial; denote by $\mathrm{Gal}(f/\mathbb{Q})$ the Galois group of f over $\mathbb{Q}$. The set of prime numbers p such that f is a product of linear factors over $\mathbb{F}_p$, has an analytical density of $\frac{1}{|\mathrm{Gal}(f/\mathbb{Q})|}$.*

Proof. Let K be the splitting field of f over $\mathbb{Q}$, denote by

$$\zeta_K(s) = \prod_{\wp}(1 - \frac{1}{N(\wp)^s})^{-1} \quad (s > 1)$$

the Dedekind ζ-function of K, where the product is over prime ideals $\wp$ in the ring of integers $\Re$ of K. We shall use the fact that

$$\lim_{s\to 1^+} \frac{\ln \zeta_K(s)}{\ln \frac{1}{s-1}} = 1,$$

which follows easily from the fact that $\zeta_K(s)$ has a simple pole at $s = 1$ (see 1(2) and 1(4) in Chapter V of [4]; note that we do *not* need the exact value of the residue at $s = 1$). But

$$\ln \zeta_K(s) = \sum_{\wp}\sum_{k=1}^{\infty} \frac{N(\wp)^{-ks}}{k} = \sum_{\wp} N(\wp)^{-s} + \psi(s),$$

where ψ is a continuous function on $[1, \infty[$. For an ordinary prime p, denote by $\wp_1, \ldots, \wp_{g_p}$ the prime ideals in $\Re$ lying above p, so that

$$p\Re = (\wp_1 \ldots \wp_{g_p})^{e_p},$$

all $\wp_i$'s have the same norm $N(\wp_i) = p^{f_p}$ $(1 \le i \le g_p)$, and

$$e_p f_p g_p = [K : \mathbb{Q}] = |\,\mathrm{Gal}(f/\mathbb{Q})|$$

(see [18], Proposition 1 in Chapter VI). Then

$$\sum_{\wp} N(\wp)^{-s} = \sum_{p} g_p . p^{-f_p s}$$

$$= |\,\mathrm{Gal}(f/\mathbb{Q})| \sum_{p: f_p=1, e_p=1} p^{-s} + \sum_{p: f_p=1, e_p>1} g_p . p^{-s} + \sum_{p: f_p>1} g_p . p^{-f_p s}.$$

The first sum is exactly over primes p such that f is a product of linear factors over $\mathbb{F}_p$; the second sum is over some primes which are ramified in K, so that it is a finite sum (see [4], 5(4) of Chapter III); the third sum converges at $s = 1$. Finally, recalling that

$$\lim_{s \to 1^+} \frac{\sum_p \frac{1}{p^s}}{\ln \frac{1}{s-1}} = 1,$$

we get

$$1 = \lim_{s \to 1^+} \frac{\sum_\wp N(\wp)^{-s}}{\sum_p p^{-s}} = |\mathrm{Gal}(f/\mathbb{Q})| \lim_{s \to 1^+} \frac{\sum_{p: f_p = 1, e_p = 1} p^{-s}}{\sum_p p^{-s}};$$

this concludes the proof. ∎

Note that this proof shows that the primes p for which f is a product of linear factors over $\mathbb{F}_p$, are responsible for the pole of ζ_K at $s = 1$.

Proof of Theorem 3.1: char $F = 0$. Clearly we may assume that F is a subfield of $\mathbb{C}$. By lemma 3.2, we may assume that F is a finite algebraic extension of $\mathbb{Q}$. Enlarging F if necessary, we may assume this extension to be Galois. Let $\Re$ be the ring of integers of F. For a prime ideal $\wp$ of $\Re$ not dividing denominators of coefficients of e, we may reduce modulo $\wp$ and get an idempotent $\overline{e} \in (\Re/\wp)\Gamma$. By the first part of the proof, $\tau(\overline{e})$ is an element of the prime field of $\Re/\wp$; the same holds with e replaced by $\sigma(e)$, for every $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$. Write $\tau(e) = \frac{\alpha}{d}$, where $\alpha \in \Re$ and $d \in \mathbb{N}$, and let $f \in \mathbb{Z}[X]$ be the minimal polynomial of α over $\mathbb{Q}$. The preceding argument shows that, for all primes p but a finite number, the polynomial f splits completely into linear factors over $\mathbb{F}_p$. By lemma 3.3, this means that f has degree 1, so that $\alpha \in \mathbb{Z}$, and $\tau(e) \in \mathbb{Q}$. ∎

Remark 3.4. Compared with the original proof of Zalesskii [23], the main simplification in the above proof lies in lemma 3.2, which allows us to assume immediately, when the characteristic of F is 0, that F is a number field (a similar argument also based on the Nullstellensatz appears in [2], Corollary 8.3). In this way one bypasses the results in commutative algebra saying that the Jacobson radical of finitely generated, commutative domain is zero, and that the quotient of such a domain by a maximal ideal is a *finite* field. Also, lemma 3.3 makes clear that only a very modest part of the Frobenius density theorem is needed in the lifting argument from characteristic p to characteristic 0 (for cyclotomic extensions, lemma 3.3 was probably known to Dirichlet).

Proof of Corollary 1.2: Let e be an idempotent in $\mathbb{C}\Gamma \otimes M_n(\mathbb{C})$; we have to show that $(\tau_\Gamma \otimes \mathrm{Tr}_n)(e)$ is rational. Let H be a finite group which has an irreducible representation of degree n; we view $\mathbb{C}\Gamma \otimes M_n(\mathbb{C})$ as a subalgebra of $\mathbb{C}(\Gamma \times H)$. Then $(\tau_\Gamma \otimes \mathrm{Tr}_n)(e) = \frac{|H|}{n} . \tau_{\Gamma \times H}(e)$ is a rational number, since $\tau_{\Gamma \times H}(e)$ is. ∎

4. On the conjecture of idempotents

For a group Γ, we define a set N_Γ of positive integers as follows:

$$N_\Gamma = \{n \in \mathbb{N} - \{0,1\} : \text{ there exists } x \in \Gamma - \{1\} \text{ which is conjugate to } x^n\}.$$

The method of proof of the next lemma is due to Formanek [8].

Lemma 4.1. *Let F be a field of positive characteristic p. Assume that Γ has no p-torsion and, for every $k \geq 1$: $p^k \notin N_\Gamma$. Let e be an idempotent in $F\Gamma$. Then $\tau(e) = 0$ or 1.*

Proof. For $x \in \Gamma - \{1\}$, denote by C_x the conjugacy class of x, and define a trace Tr_x on $F\Gamma$ by

$$\mathrm{Tr}_x(a) = \sum_{\gamma \in C_x} a(\gamma) \quad (a \in F\Gamma).$$

Write $e = \sum_{\gamma\in\Gamma} e(\gamma).\gamma$; since augmentation $F\Gamma \to F$ is a character, we have $\sum_{\gamma\in\Gamma} e(\gamma) \in \{0, 1\}$. Now

$$\sum_{\gamma\in\Gamma} e(\gamma) = \tau(e) + \sum_{[x]} \mathrm{Tr}_x(e),$$

where the last sum is over a set of representatives for non-trivial conjugacy classes. So it is enough to show

$$\mathrm{Tr}_x(e) = 0.$$

By formula (2), we have for all $k \geq 1$:

$$\mathrm{Tr}_x(e) = \mathrm{Tr}_x(e^{p^k}) = \sum_{\gamma\in\Gamma} e(\gamma)^{p^k}\, \mathrm{Tr}_x(\gamma^{p^k}) = \sum_{\gamma\in \mathrm{supp}\, e;\gamma^{p^k}\in C_x} e(\gamma)^{p^k}.$$

We notice that, for a fixed $\gamma \in \Gamma$, there is at most one $k \geq 1$ such that $\gamma^{p^k} \in C_x$. Indeed, suppose by contradiction that γ^{p^j} and γ^{p^k} belong to C_x, for $j < k$. Then γ^{p^j} is conjugate to $(\gamma^{p^j})^{p^{k-j}}$, and since $p^{k-j} \notin N_\Gamma$ this implies $\gamma^{p^j} = 1$; since Γ has no p-torsion this means that $\gamma = 1$, which contradicts $x \neq 1$.

This remark shows, by taking k large enough, that $\mathrm{Tr}_x(e) = 0$, which concludes the proof of the lemma. ∎

At this point we re-obtain a result of Formanek ([8], Theorem 9; see also [17], Theorem 3.9 in Chapter 2).

Proposition 4.2. *Suppose that, for infinitely many primes p, one has $p^k \notin N_\Gamma$ for every $k \geq 1$. Then $\mathbb{C}\Gamma$ has no non-trivial idempotent.*

Proof. We first notice that the assumption implies that Γ is torsion-free. Indeed, if Γ admits an element x of order $N \geq 2$, then for every prime p not dividing N and every integer $k \geq 1$ such that $p^k \equiv 1 \pmod{N}$, we have $p^k \in N_\Gamma$ since $x^{p^k} = x$.

Let now e be an idempotent in $\mathbb{C}\Gamma$; in view of Kaplansky's theorem, it is enough to show that $\tau(e) = 0$ or 1. By lemma 3.2, we may assume that $e \in F\Gamma$, where F is a finite algebraic extension of $\mathbb{Q}$. Denote by $\Re$ the ring of integers of F. Let p be a prime as in the assumption, not dividing denominators of coefficients of e, and let $\wp$ be a maximal ideal of $\Re$ lying above p; reducing modulo $\wp$, we obtain an idempotent $\bar{e} \in (\Re/\wp)\Gamma$ to which lemma 4.1 applies. So, for infinitely many $\wp$'s, we have $\tau(e) \equiv 0$ or $1 \pmod{\wp}$; hence the result. ∎

Remark 4.3. Let Γ be a torsion-free group which is hyperbolic in the sense of Gromov; it is then known that N_Γ is empty, so that $\mathbb{C}\Gamma$ has no non-trivial idempotent. Note that more is true in this case; indeed, Ji [11] showed that the Banach algebra $\ell^1(\Gamma)$ has no non-trivial idempotent; and Delzant [5] proved that $\mathbb{C}\Gamma$ has no zero divisor for many torsion-free hyperbolic groups.

Recall that, for an arbitrary group Γ, we defined a set P_Γ of primes by

$$P_\Gamma = \{p : \Gamma \text{ embeds in a finite extension of a pro-}p\text{-group}\}.$$

Lemma 4.4. *Let Γ be a non-trivial torsion-free group. If $p \in P_\Gamma$ and $n \in N_\Gamma$, then p does not divide n.*

Proof. Since $p \in P_\Gamma$, there exists a decreasing sequence $(\Gamma^{(k)})_{k\geq 0}$ of finite index normal subgroups of Γ, with $\Gamma^{(0)} = \Gamma$, $\bigcap_{k=0}^{\infty} \Gamma^{(k)} = \{1\}$ and $\Gamma^{(1)}/\Gamma^{(k)}$ a finite p-group. Set $a_p = [\Gamma : \Gamma^{(1)}]$ and $p^{b_k} = [\Gamma^{(1)} : \Gamma^{(k)}]$. Let $x \in \Gamma - \{1\}$ be conjugate to x^n; denote by $|x|_k$ the order of the image of x in the quotient-group $\Gamma/\Gamma^{(k)}$. Since Γ is torsion-free, one has

$$\lim_{k\to+\infty} |x|_k = +\infty.$$

On the other hand, $|x|_k$ divides $a_p.p^{b_k}$, meaning that, for k large enough, p divides $|x|_k$. Now $|x|_k = |x^n|_k$, so that n and $|x|_k$ are relatively prime; in particular p does not divide n. ∎

Proof of Theorem 1.5: Lemma 4.4 ensures that, if $p \in P_\Gamma$ and $k \geq 1$, then $p^k \notin N_\Gamma$. The desired result then follows from Proposition 4.2. ∎

Proof of Corollary 1.6: If Γ is a finitely generated subgroup of $GL_n(\mathbb{C})$, then all but a finite number of primes belong to P_Γ, by a result of Merzljakov [15]; see also [22], Theorem 4.7; [13], lemma 3. ∎

References

[1] Alperin, R. C., *An elementary account of Selberg's lemma*, L'Enseignement Mathématique **33** (1987), 269–273.

[2] Bass, H., *Euler characteristics and characters of discrete groups*, Invent. Math **35** (1976), 155–196.

[3] Baum, P., and Connes, A., Geometric K-theory for Lie groups and foliations, Unpublished IHES preprint, 1982.

[4] Borevitch, Z. I., and Chafarevitch, I. R., "Théorie des nombres," Gauthier-Villars, 1967.

[5] Delzant, T, *Sur l'anneau d'un groupe hyperbolique*, C.R. Acad. Sci. Paris **324** (1997), 381–384.

[6] Farkas, D. R., *Group rings: an annotated questionnaire*, Comm. in Algebra **8** (1980), 585–602.

[7] Fack, T., and Maréchal, O., *Application de la K-théorie algébrique aux C*-algèbres*, in: Algèbres d'opérateurs, Springer LNM **725** (1979), 144–169.

[8] Formanek, E., *Idempotents in noetherian group rings*, Can. J. Math. **25** (1973), 366–369.

[9] Frobenius, F. G., *Ueber Beziehungen zwischen den Primidealen eines algebraischen Kőrpers und den Substitutionen seiner Gruppe*, in: Gesammelte Abhandlungen II, Springer, 1968.

[10] Higson, N., and Kasparov, G. G., *Operator K-theory for groups which act properly and isometrically on Hilbert space*, Preprint, October 1997.

[11] Ji, R., *Nilpotency of Connes' periodicity operator, and the idempotent conjectures*, K-theory **9**(1995), 59–76.

[12] Kaplansky, I., "Fields and rings," Chicago Lect. in Maths, Univ. of Chicago Press, 1965.

[13] Lubotzky, A., *A group theoretic characterization of linear groups*, J. of Algebra **113** (1988), 207–214.

[14] Mal'cev, A. I., *On the faithful representations of infinite groups by matrices*, Amer. Math. Soc. Transl. **45** (1965), 1–18.

[15] Merzljakov, Ju. I., *Central series and commutator series in matrix groups*, Algebra i Logika **3** (1964), 49–59.

[16] Montgomery, M. S., *Left and right inverses in group algebras*, Bull. Amer. Math. Soc. **75** (1969), 539–540.

[17] Passman, D., "The algebraic structure of group rings," Krieger Publishing Company, 1985.

[18] Samuel, P., "Théorie algébrique des nombres," Hermann, 1971.

[19] Stevenhagen, P., and Lenstra, H. W., Jr, *Chebotarëv and his density theorem*, Math. Intelligencer **18**(1996), 26–27.

[20] Tu, J.-L., *La conjecture de Baum-Connes pour les feuilletages moyennables*, Preprint, 1997.

[21] Valette, A., *The conjecture of idempotents: a survey of the C^*-algebraic approach*, Bull. Soc. Math. Belg. **XLI**(1989), 485–521.

[22] Wehrfritz, B. A. F., "Infinite linear groups," Springer-Verlag, 1973.

[23] Zalesskii, A. E., *On a problem of Kaplansky*, Soviet Math. **13** (1972), 449–452.

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