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Bounded cohomology of lattices in higher rank Lie groups

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Abstract. We prove that the natural map $H^2_{\mathrm{b}}(\Gamma) \to H^2(\Gamma)$ from bounded to usual cohomology is injective if Γ is an irreducible cocompact lattice in a higher rank Lie group. This result holds also for nontrivial unitary coefficients, and implies finiteness results for Γ: the stable commutator length vanishes and any C^1–action on the circle is almost trivial. We introduce the continuous bounded cohomology of a locally compact group and prove our statements by relating $H^\bullet_{\mathrm{b}}(\Gamma)$ to the continuous bounded cohomology of the ambient group with coefficients in some induction module.

1. Introduction

If one considers only bounded cochains in the standard resolution for group cohomology, one obtains a subcomplex defining the so-called *bounded cohomology* $H^\bullet_{\mathrm{b}}(\Gamma; -)$ of a group Γ. This complex inclusion determines a natural map

$$H^\bullet_{\mathrm{b}}(\Gamma; -) \longrightarrow H^\bullet(\Gamma; -)$$

which in general is neither injective nor surjective.

This interesting new invariant has been shown to be relevant to geometry by M. Gromov in his work on minimal volume [21]; moreover, the space $H^2_{\mathrm{b}}(\Gamma; \mathbf{R})$ has remarkable algebraic and dynamical significance as we shall see later.

Bounded cohomology comes equipped with a natural seminorm; this provides the classes which are in the image of the natural map with a numerical invariant. This feature has been used by Gromov to give a proof of Mostow's rigidity theorem. In this context we mention a claim of Gromov [22] recently proved by I. Mineyev [31]: for hyperbolic groups, the natural map $H^n_{\mathrm{b}} \to H^n$ is surjective in every degree $n > 1$.

However, few results are known about the size of the bounded cohomology of groups; the first is a theorem of B.E. Johnson's [26]: the bounded cohomology of any amenable group vanishes. On the other hand, one knows

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now large classes of groups for which H_{b}^2 is infinite dimensional; this includes notably non–elementary Gromov hyperbolic groups (see [12], [15] and [32]).

Our aim is to give finiteness results for certain groups; our main results are the following:

Theorem 1.1. *Let $\Gamma < G$ be an irreducible cocompact lattice in a finite product $G = \prod_{a\in A} \mathbf{G}_a(k_a)$, where $\mathbf{G}_a$ are connected, simply connected, almost simple k_a–isotropic groups and k_a are local fields.*

If $\sum_{a\in A} \mathrm{rank}_{k_a}\mathbf{G}_a \ \geq\ 2$, then the natural map

$$H_{\mathrm{b}}^2(\Gamma;\mathfrak{H}) \ \longrightarrow\ H^2(\Gamma;\mathfrak{H})$$

is injective for any separable Hilbert space $\mathfrak{H}$ with unitary Γ–action.

Theorem 1.2. *Let $\mathcal{T}_1$, $\mathcal{T}_2$ be regular or biregular locally finite trees and Γ be a cocompact lattice in $\mathrm{Aut}\mathcal{T}_1 \times \mathrm{Aut}\mathcal{T}_2$.*

If the closure $\overline{\mathrm{pr}_i(\Gamma)}$ of the canonical projection $\Gamma \to \mathrm{Aut}\mathcal{T}_i$ acts transitively on the boundary $\mathcal{T}_i(\infty)$ for $i = 1, 2$, then the natural map

$$H_{\mathrm{b}}^2(\Gamma;\mathfrak{H}) \ \longrightarrow\ H^2(\Gamma;\mathfrak{H})$$

is injective for any separable Hilbert space $\mathfrak{H}$ with unitary Γ–action.

In particular, the Theorem 1.2 applies to the new family of finitely presented simple groups constructed in [7].

The bounded cohomology carries crucial information in degree two via its connection with *quasimorphisms*; recall that a (real-valued) quasimorphism of a group Γ is a map $q : \Gamma \to \mathbf{R}$ satisfying

$$\sup_{x,y\in\Gamma} \big|q(x) + q(y) - q(xy)\big| \ <\ \infty.$$

The kernel $EH_{\mathrm{b}}^2(\Gamma;\mathbf{R})$ of the natural map identifies canonically with the space of quasimorphisms modulo those that are at finite distance of an actual homomorphism. Therefore, we have the

Corollary 1.3. *Let Γ be as in Theorem 1.1 or 1.2 above. Then any quasimorphism $\Gamma \to \mathbf{R}$ is bounded.*

The Corollary 1.3 has a consequence of algebraic flavor on the *commutator subgroup* $[\Gamma,\Gamma]$, that is, the subgroup of Γ generated by the set S of all commutators of pairs of elements in Γ. Indeed, let $\|\cdot\|$ be the word metric on $[\Gamma,\Gamma]$ associated to S, and let

$$\ell_{\mathrm{s}}(\gamma) \ =\ \lim_{n\to\infty} \frac{\|\gamma^n\|}{n}$$

be the *stable length* of $\gamma \in [\Gamma, \Gamma]$. Ch. Bavard proves in [3] that the stable length ℓ_s is identically zero if and only if every quasimorphism of Γ is at bounded distance from a homomorphism, that is, $EH_b^2(\Gamma; \mathbf{R}) = 0$. Thus we deduce

Corollary 1.4. *Let Γ be as in Theorem 1.1 or 1.2 above. Then the stable length on $[\Gamma, \Gamma]$ vanishes.*

This suggests the following

Question. Let Γ be an irreducible lattice in a group G as in Theorem 1.1 above; does there exist a constant C_Γ such that every element in $[\Gamma, \Gamma]$ is a product of at most C_Γ commutators ?

The answer is affirmative for $\Gamma = \mathrm{SL}_n(\mathbf{Z})$ with $n \geq 3$, and more generally also when $\mathbf{Z}$ is replaced by certain number rings (see [9]). Still more general rings are considered in [36].

The Corollary 1.3 also implies a result of dynamical flavor. Let $\Gamma \to \mathit{Homeo}^+\big(\mathbf{S}^1\big)$ be an action of Γ by orientation preserving homeomorphisms of the circle, and $e \in H^2\left(\mathit{Homeo}^+\big(\mathbf{S}^1\big); \mathbf{Z}\right)$ be the Euler class. In [17], É. Ghys observed that e is a bounded cohomology class and that its restriction $e_\Gamma \in H_b^2(\Gamma; \mathbf{Z})$ is, as bounded cohomology class, a complete invariant of semi–conjugacy.

Corollary 1.5. *Let Γ be as in Theorem 1.1 or 1.2 above, and assume $H^2(\Gamma; \mathbf{R}) = 0$. Then any Γ–action by C^1–diffeomorphisms on the circle factors via a finite group.*

In the case of (not necessarily cocompact) irreducible lattices in higher rank real Lie groups, this finiteness result for C^1–actions on the circle has been obtained independently by Ghys in [18]. If Γ is associated to an algebraic group of higher $\mathbf{Q}$–rank, D. Witte has obtained in [39] this finiteness result even for actions by homeomorphisms.

In the context of real Lie groups, one can deduce the following corollary from the Theorem 1.1:

Corollary 1.6. *Let X be an irreducible symmetric space of non-compact type, and $\Gamma < \mathrm{Is}(X)$ a torsion free cocompact lattice. Assume that the rank of X is at least three.*

(i) If X is not hermitian symmetric, then $H_b^2(\Gamma; \mathbf{R}) = 0$.
(ii) If X is hermitian symmetric, then $H_b^2(\Gamma; \mathbf{R})$ is one-dimensional, generated by the Kähler class.

For a computation of the sup–norm of the Kähler class, see [11].

While it is well-known that the vanishing of ℓ^2–cohomology is a quasi-isometry invariant, we have:

Corollary 1.7. *For finitely generated groups, the vanishing or finite dimensionality of* $H^2_{\mathrm{b}}(\Gamma; \mathbf{R})$ *is not a quasi-isometry invariant.*

Indeed, our Theorem 1.1 applies to irreducible cocompact lattices in $\mathrm{PSL}_2(\mathbf{Q}_p) \times \mathrm{PSL}_2(\mathbf{Q}_q)$. On the other hand, reducible lattices in this product are virtually a product of non-abelian free groups, and therefore have an infinite dimensional H^2_{b}.

As an example of a group isomorphic to an irreducible cocompact lattice in $\mathrm{PSL}_2(\mathbf{Q}_p) \times \mathrm{PSL}_2(\mathbf{Q}_q)$, one can consider the group $\mathrm{SO}_3\Big(\mathbf{Z}[\frac{1}{pq}]\Big)$, where p and q are distinct primes congruent to 1 modulo 4. Notice also that there is a finite index subgroup $\Gamma < \mathrm{SO}_3\Big(\mathbf{Z}[\frac{1}{pq}]\Big)$ which is an amalgam $\Gamma = A *_C B$ of free groups A, B, C; our Theorem 1.1 implies $EH^2_{\mathrm{b}}(\Gamma) = 0$, which is in contrast to a result of K. Fujiwara [14] and R.I. Grigorchuk [20] asserting that $EH^2_{\mathrm{b}}(A *_C B)$ is infinite dimensional provided $|B/C| \geq 2$ and $|C\backslash A/C| \geq 3$. Indeed, for the mentioned group Γ, there are only two double classes of C in A and B.

The proof of our main theorems leads us to introduce *continuous* bounded cohomology with coefficients and to consider a commutative diagram of the type:

$$\begin{array}{ccc} H^2_{\mathrm{b}}(\Gamma; \mathfrak{H}) & \longrightarrow & H^2_{\mathrm{b,cont.}}\big(G; L^2(G; \mathfrak{H})^\Gamma\big) \\ \downarrow & & \downarrow \\ H^2(\Gamma; \mathfrak{H}) & \longrightarrow & H^2_{\mathrm{cont.}}\big(G; L^2(G; \mathfrak{H})^\Gamma\big) \end{array} \qquad (D)$$

Here the lower arrow is the analogue of the Eckmann–Shapiro isomorphism, and the upper arrow is a similar induction map for bounded cohomology. With this picture in mind, we shift the original problem concerning Γ over to the corresponding question about G, thus being left with two different kinds of questions:

(i). The injectivity of the induction map (upper arrow in (D)). The coefficient space $L^2(G; \mathfrak{H})^\Gamma$ suits us best, but is not quite the right analogue to the Eckmann–Shapiro induction module. We can however establish injectivity by realizing the bounded cohomology by measurable bounded cochains on a *Furstenberg boundary* G/P (P amenable), bringing into play the ergodicity of Γ on $(G/P)^2$. The higher rank assumption is not needed here.

(ii). The injectivity of the comparison map $H^2_{\mathrm{b,cont.}}(G; -) \to H^2_{\mathrm{cont.}}(G; -)$. For trivial coefficients, this is a rather simple matter for the groups G under consideration, and doesn't involve the higher rank assumption. However, the induction module $L^2(G; \mathfrak{H})^\Gamma$ has non-trivial G–action even for trivial $\mathfrak{H}$, and this is the central point where higher rank phenomena appear. Indeed, since $H^2_{\mathrm{b}}(\Gamma)$ is infinite dimensional for a lattice in a rank one group G, the

point *(i)* implies that

$$H^2_{\mathrm{b,cont.}}\big(G; L^2(\Gamma\backslash G)\big) \longrightarrow H^2_{\mathrm{cont.}}\big(G; L^2(\Gamma\backslash G)\big)$$

is not injective, while it would be if $L^2(\Gamma\backslash G)$ is replaced by any space with trivial G–action. We settle this point *(ii)* by appealing to properties of the regular representation of G in $L^2(\Gamma\backslash G)$ particular to the higher rank situation.

Notice also that the diagramm (D) shows why we are led to consider continuous bounded cohomology with non-trivial coefficients even to settle the case of $H^2_{\mathrm{b}}(\Gamma; \mathbf{C})$.

The structure of this paper is as follows. After introducing in Sect. 2 the *continuous* bounded cohomology of a locally compact group, we devote Sect. 3 to constructing new resolutions for (continuous or not) bounded cohomology. As a consequence we point out the Corollaries 3.8 and 3.9, allowing us to realize bounded cohomology on Furstenberg boundaries.

In the Sect. 4, we prove an induction result linking the bounded cohomology of a lattice $\Gamma < G$ with the continuous bounded cohomology of G with coefficients in various induction modules.

In Sect. 5, we introduce a technical hypothesis (**A**) under which the comparison map for the ambient group G is injective.

Now, putting everything together along the lines drafted in the diagram (D) above, we complete the proof of Theorem 1.1 in Sect. 6 and the proof of Theorem 1.2 in Sect. 7. In the last section, we prove the corollaries stated in the present introduction.

Remark 1.8. We shall stay to the following notational conventions throughout the paper:

All *locally compact* topological spaces will be Hausdorff (T_2).

We denote by ϱ the *right translation* on function spaces over a group or semigroup G, that is the action defined by the rule $(\varrho(g)\, f)(x) = f(xg)$. For groups, the *left translation* is defined by $(\lambda(g)\, f)(x) = f(g^{-1}x)$. A continuous function f is *right uniformly continuous* if the orbital map $G \ni g \mapsto \varrho(g)\, f$ is continuous (for the Fréchet structure of local boundedness). Mind that some authors use the opposite convention.

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2. Continuous bounded cohomology

In his work [26], Johnson defined the cohomology of Banach algebras by giving an analogue to Hochschild's construction in the more delicate *topological* context, replacing e.g. the algebraic tensor product with

Grothendieck's projective product. He could therefore consider the cohomology of the group algebra $L^1(G)$ of a locally compact group G, and this will turn out to be what we call the *continuous bounded cohomology* $H^\bullet_{\mathrm{b,cont.}}(G; -)$ of G.

When G is a discrete group, this reduces to the bounded cohomology $H^\bullet_{\mathrm{b}}(G; -)$, to which there is a more functorial approach; that is, the bounded cohomology of a group will be defined by the invariants of any resolution with an universal injectivity property in an appropriate category. This functorial definition has been introduced by N.V. Ivanov [25] and later in a slightly different form by G.A. Noskov [33].

We give a definition of continuous bounded cohomology and then recall the functorial setting for plain bounded cohomology. Many of the following definitions are analogous to the standard setup:

First definitions. Let G be a locally compact group. A *Banach G–module* (or module for short) is a Banach space E on which G acts by continuous linear operators. The module E will be said *bounded* (resp. *isometric*) if G acts by operators of uniformly bounded norm (resp. acts by isometric operators). A *morphism* of modules is a continuous linear map; a G–morphism is an equivariant one. Notice that every bounded module E is isomorphic to an isometric module E_{is} by replacing its norm with the equivalent norm

$$\|v\|_{\mathrm{is}} = \sup_{g\in G} \|gv\| \qquad (v \in E).$$

When we consider simply a Banach space E, it is understood that E has the isometric G–module structure defined by the trivial G–action.

Resolutions. A *G–resolution* of a bounded G–module E is a sequence

$$0 \longrightarrow E \xrightarrow{d_0} E_0 \xrightarrow{d_1} E_1 \xrightarrow{d_2} E_2 \longrightarrow \cdots$$

of bounded G–modules $(E_n)_{n\geq 0}$ and G–morphisms $(d_n)_{n\geq 0}$, together with a *contracting homotopy*, that is a sequence of morphisms $h_n : E_n \to E_{n-1}$ (where $E_{-1} = E$) satisfying $h_{n+1}d_{n+1} + d_n h_n = \mathrm{Id}$ and $h_0 d_0 = \mathrm{Id}$. We denote such a resolution by $(E_\bullet, d_\bullet, h_\bullet)$ or simply by $E_\bullet$.

Likewise, an *isometric G–resolution* of an isometric module E is a sequence of isometric G–modules as above but with $\|h_n\| \leq 1$ for all n.

The elements of E are called *coefficients*, the map d_0 is the *augmentation* and the other d_n are *coboundary* maps; subscripts will often be omitted.

The *cohomology* of G with respect to $E_\bullet$, denoted by $H^\bullet(G; E_\bullet)$, is the cohomology of the subcomplex of G–invariants

$$0 \longrightarrow E_0^G \longrightarrow E_1^G \longrightarrow E_2^G \longrightarrow \cdots$$

More precisely, the space of *n–cocycles* is $Z^n(G; E_\bullet) = \mathrm{Ker}(d_{n+1}|_{E_n^G})$, the space of *n–coboundaries* is $B^n(G; E_\bullet) = \mathrm{Im}(d_n|_{E_{n-1}^G})$ and we set $H^n(G;E_\bullet) = Z^n(G;E_\bullet)/B^n(G;E_\bullet)$. We endow the vector space $H^n(G; E_\bullet)$ with the quotient seminorm.

Cochain morphisms and homotopies. A *cochain morphism* from a G–resolution $(E_\bullet, d_\bullet, h_\bullet)$ of a Banach G–module E to another resolution $(E'_\bullet, d'_\bullet, h'_\bullet)$ of a Banach G–module E' is a sequence $\varphi_\bullet = (\varphi_n)_{n\geq -1}$ of G–equivariant continuous linear maps such that the diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & E & \xrightarrow{d_0} & E_0 & \xrightarrow{d_1} & E_1 & \xrightarrow{d_2} & E_2 & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \varphi_{-1}} & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \\ 0 & \longrightarrow & E' & \xrightarrow{d'_0} & E'_0 & \xrightarrow{d'_1} & E'_1 & \xrightarrow{d'_2} & E'_2 & \longrightarrow & \cdots \end{array}$$

commutes. In case the coefficients coincide and φ_{-1} is the identity, one says that $\varphi_\bullet$ is *augmentation preserving*. The following is immediate:

Proposition 2.1. *The morphism $\varphi_\bullet$ induces a sequence of maps at the level of cohomology*

$$H^\bullet\varphi_\bullet : \ H^\bullet(G; E_\bullet) \ \longrightarrow \ H^\bullet(G; E'_\bullet),$$

where each $H^n\varphi_\bullet$ is a continuous linear map of norm at most $\|\varphi_n\|$. □

Let $\varphi_\bullet, \varphi'_\bullet$ be two cochain morphisms from $E_\bullet$ to $E'_\bullet$. A (G–equivariant) *homotopy* from $\varphi_\bullet$ to $\varphi'_\bullet$ is a sequence of (G–equivariant) morphisms $\sigma_n : E_n \to E'_{n-1}$ (where $E'_{-1} = E'$) satisfying

$$\sigma_{n+1}d_{n+1} + d_n\sigma_n \ = \ \varphi'_n - \varphi_n \quad \text{and} \quad \sigma_0 d_0 \ = \ \varphi'_{-1} - \varphi_{-1}.$$

If there is such a G–equivariant homotopy, the morphisms $\varphi_\bullet, \varphi'_\bullet$ induce indeed the same maps at the level of cohomology.

The standard resolution. Now consider a bounded resp. isometric separable Banach G–module E. For each $n \geq 0$, we consider the space $L^\infty(G^{n+1}; E)$ of essentially bounded measurable map classes $f : G^{n+1} \to E$. The sup-norm relative to E turns $L^\infty(G^{n+1}; E)$ into a Banach space, in general non separable. We endow $L^\infty(G^{n+1}; E)$ with a structure of bounded resp. isometric Banach G–module via the diagonal *left regular* action defined by

$$(g\,f)(g_0, \ldots, g_n) = g\Big(f(g^{-1}g_0, \ldots, g^{-1}g_n)\Big)$$

for $f \in L^\infty(G^{n+1}; E)$ and almost all $g, g_0, \ldots, g_n \in G$. Notice in particular that if the action on E is trivial, this coincides with diagonal left translation.

One gets a sequence

$$0 \longrightarrow E \xrightarrow{d_0} L^\infty(G; E) \xrightarrow{d_1} L^\infty(G^2; E) \xrightarrow{d_2} \cdots$$

by the usual formula $d_n = \sum_{i=0}^{n}(-1)^i d_{n,i}$ where $d_{n,i}$ simply omits the i^{th} variable; the map d_0 is defined by $d_0 v(g) = gv$. A contracting homotopy is provided as in [5], Proposition 3.2.1, by the integration of the first variable against a fixed function $\varphi \in C_{00}^+(G)$ of integral one for a left Haar measure m, that is

$$h_n f(g_0, \ldots , g_{n-1}) = \int_G \varphi(g) \cdot f(g, g_0, \ldots , g_{n-1})\, dm(g)$$

for almost all $g_0, \ldots , g_{n-1} \in G$. With this homotopy, the resolution above is an isometric G–resolution of E, and we call it the *standard homogenous resolution*. The cohomology of G with respect to this resolution is the *continuous bounded cohomology* of G, and we denote it by $H^\bullet_{\text{b,cont.}}(G; E)$. As to terminology, see the Proposition 2.4 below.

When we take the complex field $\mathbf{C}$ endowed with the trivial G–module structure as coefficient module, we use the shorter notation $H^\bullet_{\text{b,cont.}}(G)$; likewise, we follow the general convention and write simply $L^\infty(G)$, $C(G)$, $C_{00}(G)$ etc. when the coefficients are complex numbers.

Remark 2.2. The relation with Johnson's Banach algebra cohomology is as follows. Suppose E is the dual of a Banach space with jointly continuous G–action (in particular, the G–action on E becomes weak-* continuous). Since we have taken E to be separable, the spaces L^∞ above coincide with the spaces of weak-* measurable bounded maps. Therefore, a proposition of Johnson's ([26], Proposition 2.3) implies that $H^\bullet_{\text{b,cont.}}(G; E)$ is the Banach algebra cohomology of $L^1(G)$ with coefficients in such modules E.

A coefficient formula. The following statement can be derived *verbatim* as in the classical case:

Proposition 2.3. *Let G be a locally compact group and let E, F be two separable bounded G–modules. Then the canonical isomorphism of topological vector spaces*

$$H^n_{\text{b,cont.}}(G; E \oplus F) \cong H^n_{\text{b,cont.}}(G; E) \oplus H^n_{\text{b,cont.}}(G; F)$$

holds for all $n \geq 0$. □

Continuous cochains. Considering for every $n \geq 0$ the subspace $C_{\text{b}}(G^{n+1}; E)$ of continuous bounded functions, one gets another G–resolution of E:

$$0 \longrightarrow E \xrightarrow{d_0} C_{\text{b}}(G; E) \xrightarrow{d_1} C_{\text{b}}(G^2; E) \xrightarrow{d_2} \cdots \qquad (*)$$

Notice that although (*) is a homotopy preserving subresolution of the former, a simpler contracting homotopy $h'_\bullet$ can be given for this resolution by

$$h'_n f(x_0, \dots, x_{n-1}) \;=\; f(e, x_0, \dots, x_{n-1}).$$

Proposition 2.4. *The complex*

$$0 \longrightarrow C_\mathrm{b}(G; E)^G \longrightarrow C_\mathrm{b}(G^2; E)^G \longrightarrow C_\mathrm{b}(G^3; E)^G \longrightarrow \cdots$$

of G–invariant continuous bounded cochains realizes the continuous bounded cohomology $H^\bullet_{\mathrm{b,cont.}}(G; E)$.

More precisely, the inclusion maps $\iota_n : C_\mathrm{b}(G^n; E) \subset L^\infty(G^n; E)$ induce isometric isomorphisms at the level of cohomology.

Proof. The proof uses a standard regularization technique, almost identical to what is exposed in [5], § 4. However, we describe explicitly a few (tedious) steps which will be of later use.

Fix a compactly supported continuous non-negative function $\psi \in C^+_{00}(G)$ of integral one for a left Haar measure m and define the *regularization* maps

$$R_n : L^\infty(G^{n+1}; E) \longrightarrow C_\mathrm{b}(G^{n+1}; E)$$

by the convolution $R_n\alpha = \alpha * \psi^{\otimes(n+1)}$ for all $\alpha \in L^\infty(G^{n+1}; E)$. This yields a G–morphism of complexes of norm one, another being given by the inclusion maps $\iota_\bullet$. To see that these induce mutually inverse isomorphisms of the cohomology spaces, one shows that $R_\bullet \circ \iota_\bullet$ and $\iota_\bullet \circ R_\bullet$ are equivariantly homotopic to the identity. To this end, define first for all $-1 \le i \le n$ the partial regularizations $R_{n,i}$ by $R_{n,i}\alpha = \alpha * \psi^{\otimes(i+1)} \otimes \delta_e^{\otimes(n-i)}$ (where δ_e is the convolution identity), so that $R_{n,-1} = \mathrm{Id}$ and $R_{n,n} = R_n$. Now define the *stuttering* maps

$$\sigma_{n,i} : L^\infty(G^{n+1}; E) \longrightarrow L^\infty(G^n; E)$$

for each $0 \le i \le n-1$ in the following way: for every $\alpha \in L^\infty(G^{n+1}; E)$ and $x_0, \dots, x_n \in G$, let

$$\sigma_{n,i}\alpha\,(x_0, \dots, x_n) \;= \;= \int_{G^{n+2}} \psi^{\otimes(n+2)}(y) \cdot \alpha(x_0 y_0^{-1}, \dots, x_i y_i^{-1}, x_i, \dots, x_n)\, dm^{\otimes(n+2)}(y),$$

where $y = (y_0, \dots, y_{n+1})$. Every $\sigma_{n,i}$ is a well-defined G–equivariant continuous linear operator. A direct calculation yields the "simplicial" relations

$$\begin{aligned} \sigma_{n,i}\, d_{n,j} &= d_{n-1,j-1}\, \sigma_{n-1,i} && \forall\, i \le j-2,\\ \sigma_{n,i}\, d_{n,j} &= d_{n-1,j}\, \sigma_{n-1,i-1} && \forall\, i \ge j+1,\\ \sigma_{n,i}\, d_{n,i} &= R_{n-1,i-1},\\ \sigma_{n,i}\, d_{n,i+1} &= R_{n-1,i}. \end{aligned}$$

Now define a G–equivariant continuous linear operator

$$\sigma_n : L^\infty(G^{n+1}; E) \longrightarrow L^\infty(G^n; E)$$

by $\sigma_n = \sum_{i=0}^{n-1} (-1)^i \sigma_{n,i}$. Using the simplicial relations above, one checks that

$$d_n \, \sigma_n + \sigma_{n+1} \, d_{n+1} = \mathrm{Id} - R_n,$$

thus establishing the homotopy from $\iota_\bullet \circ R_\bullet$ to Id and, by restriction, from $R_\bullet \circ \iota_\bullet$ to Id. □

The comparison map. The resolution (*) can be viewed as a subcomplex of the sequence of vector spaces

$$0 \longrightarrow E \xrightarrow{d_0} C(G; E) \xrightarrow{d_1} C(G^2; E) \xrightarrow{d_2} \cdots$$

which realizes the *usual* continuous cohomology $H^\bullet_{\mathrm{cont.}}(G; E)$ of G (see [23], § 2). This inclusion of complexes induces for each n a map

$$H^n_{\mathrm{b,cont.}}(G; E) \longrightarrow H^n_{\mathrm{cont.}}(G; E)$$

which, in general, is neither injective nor surjective.

Bounded cohomology. Considering any group G, denote G_δ the locally compact group consisting of G endowed with the discrete topology. The *bounded cohomology* $H^\bullet_{\mathrm{b}}(G; E)$ of G with coefficients in a separable module E is the continuous bounded cohomology $H^\bullet_{\mathrm{b,cont.}}(G_\delta; E)$ of G_δ. Since G_δ is discrete, there is no difficulty in considering non separable modules E aswell.

The resolution (*) takes the more familiar form

$$0 \longrightarrow E \xrightarrow{d_0} \ell^\infty(G; E) \xrightarrow{d_1} \ell^\infty(G^2; E) \xrightarrow{d_2} \cdots \qquad (**)$$

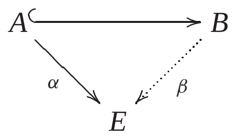
The comparison map now reduces to

$$H^\bullet_{\mathrm{b}}(G; E) \longrightarrow H^\bullet(G; E),$$

that is, connects the bounded cohomology to the usual cohomology $H^\bullet$. We call this map the *natural map* from $H^\bullet_{\mathrm{b}}$ to $H^\bullet$; this terminology will be justified by the Proposition 2.6 below. The kernel $EH^\bullet_{\mathrm{b}}(G; E)$ of the natural map is called the *exact part* of $H^\bullet_{\mathrm{b}}(G; E)$.

Injectivity. We recall now the functorial definition of bounded cohomology for a *discrete* group G. A G–morphism is *admissible* if it has a (left) section which is a morphism (not necessarily G–equivariant). An *admissible submodule* is a submodule (i.e. a closed G–invariant subspace) for which the inclusion map is admissible. A bounded module E is *injective* if any

G–morphism $\alpha : A \to E$ from a bounded admissible submodule $A \subset B$ of a bounded module B can be extended:



Likewise, a G–morphism is *isometrically admissible* if it has a (left) section which is a morphism of norm ≤ 1. An *isometrically admissible submodule* is a submodule for which the inclusion map is isometrically admissible. An isometric module E is *isometrically injective* if any G–morphism $\alpha : A \to E$ from an isometric isometrically admissible submodule $A \subset B$ of an isometric module B can be extended as above but with $\|\beta\| \leq \|\alpha\|$.

The resolution $E_\bullet$ is *injective* if all E_n are; it is *isometrically injective* if all E_n are and $\|h_n\| \leq 1$ for all n. As expected, if $E_\bullet$ is an injective resolution of a G–module E, $H^n(G; E_\bullet)$ doesn't depend, as a topological vector space, on the choice of the resolution; see the Remark 2.5 below.

The point of these considerations is that the standard resolution is injective; more precisely, putting together results of [25] and [33], one has that $\ell^\infty(G^n; E)$ is injective for any bounded module E and isometrically injective if E is isometric.

While $H_\mathrm{b}^\bullet$ comes with a canonical topology, mind that the quotient *seminorm* on $H^n(G; E_\bullet)$ does depend on the resolution. Therefore one defines the *canonical seminorm* on $H_\mathrm{b}(G; E)$ to be the infimum seminorm over all *isometrically* injective resolutions. This seminorm is realized by the standard resolution ([25], Theorem 3.6). Notice also that while $H_\mathrm{b}^n(G; \mathbf{R})$ is Hausdorff for $n \leq 2$ (see [30]), this is not necessarily the case in higher degree, as has been shown by T. Soma in [37].

Remark 2.5. Let us be more precise about functoriality (following [25], Lemma 3.3.2 and below). Consider two isometric resolutions $E_\bullet$ and $E'_\bullet$ of an isometric G–module E and suppose $E_\bullet$ isometrically injective. Then there is an augmentation preserving morphism $\varphi_\bullet : E'_\bullet \to E_\bullet$, and moreover any two such morphisms are (G–equivariantly) homotopic. Endowing $H^\bullet(G; E_\bullet)$ with the canonical seminorm and $H^\bullet(G; E'_\bullet)$ with its quotient seminorm, the induced maps at the level of cohomology are of norm at most one. In particular, if $E'_\bullet$ is also isometrically injective, we get canonical isomorphisms, isometric for the canonical seminorms.

In particular, we insist that the symbol $H_\mathrm{b}^\bullet(G; E)$ stands for the cohomology associated to the resolution (**), while there is a canonical isomorphism from the cohomology of any other isometrically injective resolution of E to $H_\mathrm{b}^\bullet(G; E)$. Likewise, we choose $H^\bullet$ to stand for the cohomology associated to the resolution by the spaces $C(G^n; E)$.

These consideration will be understood whenever we mention the *canonical isomorphisms*. Notice the obvious analogues for non-isometric injectivity.

The natural map. Let us turn back to the *natural map* mentioned above, that is the comparison map $H^\bullet_{\mathrm{b}}(G; E) \to H^\bullet(G; E)$ induced by the inclusions $\ell^\infty(\Gamma^n; E) \subset C(\Gamma^n; E)$. It turns out that this map is completely canonical. This is due to the fact that bounded cohomology is defined in a subcategory of the category used to define usual cohomology; more precisely, a resolution in the sense of bounded cohomology is in particular a resolution in the usual sense, a module as defined in this paper is in particular also a module in the usual cohomological context, and so on for morphisms and homotopies. It is actually this inclusion of categories that determines the natural map:

Proposition 2.6. *Let G be a discrete group and E a bounded G–module.*

Let $\psi_\bullet : E_\bullet \to E'_\bullet$ be an augmentation preserving morphism from any injective resolution $E_\bullet = 0 \to E \to E_0 \to E_1 \to \cdots$ to any $E'_\bullet = 0 \to E \to E'_0 \to E'_1 \to \cdots$ which is an injective resolution in the sense of usual *cohomology.*

Then the canonical isomorphisms $H^\bullet(G; E_\bullet) \cong H^\bullet_{\mathrm{b}}(G; E)$ and $H^\bullet(G; E'_\bullet) \cong H^\bullet(G; E)$ intertwin $H^\bullet\varphi_\bullet$ with the natural map.

Notice that this implies in particular that *any* augmentation preserving morphism from the standard resolution $\ell^\infty(G^\bullet; E)$ to the usual resolution $C(G^\bullet; E)$ induces the natural map.

Proof of Proposition 2.6. The injectivity conditions (in each category) yield the existence of augmentation preserving morphisms $u_\bullet : \ell^\infty(G^\bullet; E) \to E_\bullet$ and $v_\bullet : E'_\bullet \to C(G^\bullet; E)$ inducing the corresponding canonical isomorphisms.

The usual injectivity implies that any augmentation preserving morphism from $\ell^\infty(G^\bullet; E)$ to $C(G^\bullet; E)$ is equivariantly homotopic to the inclusion $\iota_\bullet$. Apply this to $v_\bullet \circ \iota_\bullet \circ u_\bullet$. □

Remark 2.7. At least part of this argumentation can be carried over to the comparison map for continuous bounded cohomology; indeed, there is a functorial setting for (usual) continuous cohomology (see [5] or [6]). The spaces $C(G^n; E)$ and $L^p_{\mathrm{loc}}(G^n; E)$ (for $1 \leq p < \infty$) are injective in the appropriate sense, so that the Proposition 2.4 implies that the inclusions

$$L^\infty(G^n; E) \longrightarrow L^p_{\mathrm{loc}}(G^n; E)$$

also induce the comparison map $H^\bullet_{\mathrm{b,cont.}}(G, E) \to H^\bullet_{\mathrm{cont.}}(G, E)$.

A few injective resolutions. The isometric injectivity of $\ell^\infty(G; E)$ can be immediately generalized to the following setting:

Lemma 2.8. *Let G be a group acting (say on the left) freely on a set X. Then the space $\ell^\infty(X; E)$ endowed with the left regular G–action is an isometrically injective bounded G–module for every isometric G–module E.*

This statement (aswell as some others below) has an obvious analogue for *bounded* G–modules E; the adjustment is left to the reader.

Proof of Lemma 2.8. Since the action on X is free, there is a fundamental domain $F \subset X$ and a G–equivariant map $s : X \to G$ such that $s(x)^{-1}x \in F$ for all $x \in X$. If we endow the Banach space $\ell^\infty(F; E)$ with the G–action on the coefficients E, then we know that the Banach space $\ell^\infty(G; \ell^\infty(F; E))$ endowed with the left regular action is isometrically injective. Now one checks that the map

$$A :\ \ell^\infty(X; E) \longrightarrow \ell^\infty\big(G; \ell^\infty(F; E)\big)$$

defined by $Af(g)(x) = f(gx)$ is well-defined, G–equivariant, linear and isometric. One checks also that the map

$$B :\ \ell^\infty\big(G; \ell^\infty(F; E)\big) \longrightarrow \ell^\infty(X; E)$$

defined by $Bf(x) = f(s(x))(s(x)^{-1}x)$ is an inverse of A. Thus $\ell^\infty(X^{n+1})$ is also isometrically injective. □

There is a measure–theoretic version of the Lemma 2.8 if the group under consideration is countable:

Lemma 2.9. *Let G be a countable group acting on a measure space X. Suppose that G preserves the measure class and that there is a measurable section $s : X \to G$.*

Then the space $L^\infty(X; E)$ endowed with the left regular G–action is an isometrically injective bounded G–module for every isometric G–module E.

By *measurable section*, we understand a measurable G–equivariant map $s : X \to G$ as in the proof of Lemma 2.8; in particular $F = s^{-1}(e)$ is a fundamental domain.

Proof of Lemma 2.9. The proof goes as for the Lemma 2.8 by considering the maps

$$A :\ L^\infty(X; E) \longrightarrow \ell^\infty\big(G; L^\infty(F; E)\big)$$

defined by $Af(g)(x) = f(gx)$ and

$$B :\ \ell^\infty\big(G; L^\infty(F; E)\big) \longrightarrow L^\infty(X; E)$$

defined by $Bf(x) = f(s(x))(s(x)^{-1}x)$; but we have to check measurability. The map $Af(g)$ is clearly measurable for all g, and as to Bf, notice that for every measurable set $U \subset E$ we have

$$(Bf)^{-1}(U) = \bigcup_{\gamma \in G} \Big(s^{-1}(g) \bigcap (f(g))^{-1}(gU) \Big)$$

hence we are done by the countability of G. □

This yields immediatly the following

Corollary 2.10. *Let G be a separable locally compact group, $\Gamma < G$ a countable closed subgroup, E a separable isometric Γ–module. Then*

(i) the bounded cohomology $H_{\mathrm{b}}^{\bullet}(\Gamma; E)$ of Γ is realized by the complex of invariants

$$0 \longrightarrow E \longrightarrow L^{\infty}(G^2; E)^{\Gamma} \longrightarrow L^{\infty}(G^3; E)^{\Gamma} \longrightarrow \cdots$$

(ii) any augmentation preserving Γ–equivariant cochain morphism $\varphi_\bullet$ or $\psi_\bullet$

$$\begin{array}{ccccccccc}
0 \longrightarrow & E & \longrightarrow & L^{\infty}(G; E) & \longrightarrow & L^{\infty}(G^2; E) & \longrightarrow & L^{\infty}(G^3; E) & \longrightarrow \cdots \\
 & \| & & \varphi_0 \downarrow \uparrow \psi_0 & & \varphi_1 \downarrow \uparrow \psi_1 & & \varphi_2 \downarrow \uparrow \psi_2 & \\
0 \longrightarrow & E & \longrightarrow & \ell^{\infty}(\Gamma; E) & \longrightarrow & \ell^{\infty}(\Gamma^2; E) & \longrightarrow & \ell^{\infty}(\Gamma^3; E) & \longrightarrow \cdots
\end{array}$$

induces an isometric isomorphism at the level of cohomology; the isomorphism doesn't depend on the choice of $\varphi_\bullet$ resp. $\psi_\bullet$.

In this statement, the coboundary maps are still the maps $(d_n)_{n \in \mathbf{N}}$ (or d for short) defined above, as will always be the case unless otherwise stated.

Proof of Corollary 2.10. Concerning point (i), the existence of the map s needed to apply the Lemma 2.9 with $X = G^n$ is a consequence of G being separable; so it remains only to exhibit a contracting homotopy. This is achieved by setting

$$h_n f(g_0, \ldots, g_{n-1}) = \int_G f(g, g_0, \ldots, g_{n-1})\, \varphi(g)\, dm(g),$$

for any fixed positive compactly supported continuous function φ of integral one for a left Haar measure m on G. The integration is justified by the separability of E.

As to point (ii), this is the functorial property of injectivity mentioned in [25], Sect. 3.7. □

We have proved in Proposition 2.4 that the resolutions $L^\infty(G^\bullet; E)$ and $C_\mathrm{b}(G^\bullet; E)$ of a G–module E are G–equivariantly homotopic; therefore, considering E as a Γ–module, we deduce the following from the Corollary 2.10:

Corollary 2.11. *Let G be a separable locally compact group, $\Gamma < G$ a countable closed subgroup, E a separable isometric Γ–module.*

The bounded cohomology $H^\bullet_\mathrm{b}(\Gamma; E)$ of Γ is realized by the complex of invariants

$$0 \longrightarrow E \longrightarrow C_\mathrm{b}(G^2; E)^\Gamma \longrightarrow C_\mathrm{b}(G^3; E)^\Gamma \longrightarrow \cdots$$

□

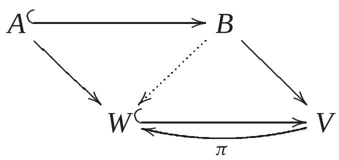
Here is another source of new injective resolutions:

Lemma 2.12. *Let V be an injective G–module and W a closed subspace of V admitting a G–equivariant continuous projection $\pi : V \to W$.*

Then W is also injective.

Moreover, if $\|\pi\| = 1$ and V is isometrically *injective, then so is W.*

Proof. Apply the definition(s) of injectivity to the following diagram:



□

3. Amenability, harmonicity and resolutions

In this section, we will apply amenability methods to semigroups of measures in order to obtain new resolutions for bounded cohomology.

Definitions. *A* semigroup *is a set S endowed with an associative composition law $S \times S \to S$ (written multiplicatively); neither inverses nor a neutral element are required.*

A semitopological semigroup *is a semigroup endowed with a topology for which the multiplication is* separately *continuous, that is for each $s \in S$ the right and left multiplication by s are continuous maps $S \to S$.*

A semitopological semigroup S is right amenable *if the space $C_{\mathrm{b,ru}}(S)$ of right uniformly continuous bounded functions admits a right invariant mean, that is a continuous linear form of norm one*

$$\mathfrak{M} : C_{\mathrm{b,ru}}(S) \longrightarrow \mathbf{C}$$

invariant under right translations and with $\mathfrak{M}(\mathbb{1}_S) = 1$.

For a survey on amenable semigroups, see [27]. This reference considers *left* amenability of S, which is equivalent to right amenability of the opposite semigroup S^o (that is, the semigroup with multiplication $(s, s') \mapsto s's$). Beware however that countrary to the case of groups, a semigroup needs not to be isomorphic to its opposite; and it may indeed happen that S is right amenable while S^o is not. Nonetheless, we will often omit to specify that we consider *right* amenability.

For technical reasons, we introduce a further notion: a semitopological semigroup S is *right* C*–amenable* if the space $C_{\mathrm{b}}(S)$ of continuous bounded functions admits a right invariant mean. It is not known whether C–amenability is really different from amenability (see [27], problem 1). However, one has the results stated in 3.1 below.

Finally, we say that a semigroup S is amenable or C–amenable whenever it is the case for the semitopological semigroup S_δ consisting of S endowed with the discrete topology.

We now summarize a few facts on amenability.

Proposition 3.1.
(i) C–amenable $\Longrightarrow$ *amenable.*
(ii) For discrete semigroups, C–amenable $\Longleftrightarrow$ *amenable.*
(iii) For locally compact groups, C–amenable $\Longleftrightarrow$ *amenable.*
(iv) All commutative semitopological semigroups are amenable.
(v) A finite product of amenable semitopological semigroups is amenable.

Proof. Since $C_{\mathrm{b,ru}}(S)$ is a subspace of $C_{\mathrm{b}}(S)$, we get (i) by restricting the mean. In (ii), these spaces coincide. As to (iii), an amenable locally compact group G has an invariant mean on the even larger space $L^\infty(G)$ (see [19]). The last two statements are well-known, see [10] and [27]. □

Convolution semigroups on a locally compact group. Let G be a locally compact group and denote by $M(G)$ the convex set of positive Radon measures of norm one (probability measures). This is a semigroup for the convolution defined by

$$(\mu * \nu)(f) = \int_G \int_G f(xy)\, d\mu(x) d\nu(y)$$

for $\mu, \nu \in M(G)$ and $f \in C_{\mathrm{b}}(G)$. Recall that the *narrow* topology on $M(G)$ is defined by integration of bounded continuous functions, which means that a net $(\mu_a)_{a\in A}$ converges to μ if and only if $\mu_a(f) \to \mu(f)$ for all $f \in C_{\mathrm{b}}(G)$. This topology turns $M(G)$ into a semitopological semigroup (see [35], Proposition 24.1.3).

For every separable Banach space E, the semigroup $M(G)$ acts on $L^\infty(G;E)$ by *right* convolution, that is

$$(f*\mu)(x) \;=\; \int_G f(xy^{-1})\,d\mu(y) \;=\; \mu\Big(\varrho(x^{-1})\,f^\vee\Big),$$

for $f\in L^\infty(G;E)$ and $\mu\in M(G)$. This is an action by linear operators of norm one which leaves the constants invariant. Moreover, given a G–action on E by continuous linear operators, the associated left action on $L^\infty(G;E)$ defined for $f\in L^\infty(G;E)$ and almost all $g\in G$ by $(g\cdot f)(x) \;=\; g\big(f(g^{-1}x)\big)$ commutes with $M(G)$ (this amounts to Bochner's theorem, see e.g. [41], V.5 Corollary 2). Likewise, viewing the direct product $M(G)^n$ as a subset of $M(G^n)$, one has for every n an $M(G)^n$–action on $L^\infty(G^n;E)$ which commutes with the left diagonal regular G–action.

Notice also that the action of $M(G)$ preserves the subspaces $C_{\mathrm{b}}(G;E)$ and $C_{\mathrm{b,lu}}(G;E)$ of continuous (resp. left uniformly continuous) bounded functions.

We consider for any subsemigroup S of $M(G)$ the space $L^\infty_S(G^n;E)$ of all functions which are S–invariant in every variable, that is the subspace of S^n–invariant functions. We call these functions *S–harmonic* in each variable, or *S–pluriharmonic* (when $n\geq 2$). More generally, any S^n–convolable function will be called S–pluriharmonic if it is S^n–invariant.

Proposition 3.2. *Let G be a locally compact separable group and S a subsemigroup of $M(G)$ endowed with the narrow topology.*

If S is C*–amenable, then there is a G–equivariant continuous projection of norm one*

$$\pi:\; L^\infty(G)\longrightarrow L^\infty_S(G)$$

onto the subspace of S–harmonic functions.

Moreover, π preserves left uniform continuity.

Proof. As is pointed by M.E.B. Bekka in [4], it is enough to define π on the subspace $C_{\mathrm{b,lu}}(G)$ of bounded left uniformly continuous functions, that is

$$\pi:\; C_{\mathrm{b,lu}}(G)\longrightarrow C_{\mathrm{b,lu}}(G)\cap L^\infty_S(G).$$

For more details, see the proof of Theorem 1 in [4].

Fix $f\in C_{\mathrm{b,lu}}(G)$ and $x\in G$. We define a function $f_x:S\to\mathbf{C}$ by setting $f_x(\mu)=(f*\mu)(x)$. If $(\mu_a)_{a\in A}$ is a net converging narrowly to μ, one has

$$f_x(\mu_a) \;=\; \mu_a\Big(\varrho(x^{-1})\,f^\vee\Big) \;\to\; \mu\Big(\varrho(x^{-1})\,f^\vee\Big) \;=\; f_x(\mu),$$

hence f_x is continuous on S. But f_x is bounded by $\|f\|_\infty$ and hence, choosing an invariant mean $\mathfrak{M}$ on $C_{\mathrm{b}}(S)$, we may average and define $\pi f(x) = \mathfrak{M}(f_x)$. We have obtained a composed function

$$\begin{array}{ccccc} G & \longrightarrow & C_{\mathrm{b}}(S) & \xrightarrow{\mathfrak{M}} & \mathbf{C} \\ x & \longmapsto & f_x & \longmapsto & \pi f(x) \end{array}$$

where the secound arrow is norm-continuous, while the bound

$$\begin{aligned} \|f_x - f_{yx}\|_\infty &= \sup_{\mu\in S} \Big|\mu\Big(\varrho(x^{-1}) f^\vee\Big) - \mu\Big(\varrho(x^{-1}y^{-1}) f^\vee\Big)\Big| \leq \\ &\leq \|f^\vee - \varrho(y^{-1}) f^\vee\|_\infty = \|f - \lambda(y^{-1}) f\|_\infty, \end{aligned}$$

together with the left uniform continuity of f, implies that the first arrow is left uniformly continuous. Therefore we have shown that πf is left uniformly continuous. On the other hand, πf is bounded by $\|f\|_\infty$ and π is clearly linear in f, so that we have a continuous map of norm one (π preserves the constant functions). It is straightforward that π is left G–equivariant, and hence we proceed to show that πf is S–harmonic.

To this end, pick $\mu \in S$. Now we have

$$(\pi f * \mu)(x) = \int_G \mathfrak{M}(f_{xy^{-1}})\, d\mu(y),$$

and since the map $x \mapsto f_{xy^{-1}}$ is continuous, it has separable range in the Banach space $C_{\mathrm{b}}(S)$ and hence (see again [41]) we may commute the integral with the continuous form $\mathfrak{M}$ and continue with Bochner integrals:

$$\begin{aligned} (\pi f * \mu)(x) &= \mathfrak{M}\left(\int_G f_{xy^{-1}}\, d\mu(y)\right) = \\ &= \mathfrak{M}\left(\nu \mapsto \int_G f * \nu(xy^{-1})\, d\mu(y)\right) = \\ &= \mathfrak{M}\Big(\nu \mapsto f * \nu * \mu(x)\Big) = \mathfrak{M}(\varrho(\mu) f_x). \end{aligned}$$

By right S–invariance of $\mathfrak{M}$, this last expression is equal to $\mathfrak{M}(f_x)$ which is $\pi f(x)$. This shows the S–harmonicity of πf. Since it follows from the definition of π that it leaves harmonic functions unchanged, the proof is complete. □

The result above can be extended as follows. Take $n \geq 0$ and suppose S^{n+1} is C–amenable; this is e.g. the case when S is amenable and discrete or is an amenable locally compact group. Then the proposition above yields a projection

$$\pi_n : L^\infty(G^{n+1}) \longrightarrow L^\infty_S(G^{n+1})$$

onto the subspace of S–pluriharmonic functions. Furthermore, turning back to a separable Banach space E with G–action by continuous linear operators (i.e. a Banach G–module), consider the associated S^{n+1}–action on the (left diagonal regular) Banach G–module $L^\infty(G^{n+1}; E)$. For every continuous linear form $\Lambda \in E^*$ and every $f \in L^\infty(G^{n+1}; E)$, the function $\Lambda \circ f$ is in $L^\infty(G^{n+1})$ and we define for almost all $x \in G$ an element $\pi f(x)$ of the bidual E^{**} by the formula

$$\pi f(x)(\Lambda) \;=\; \pi(\Lambda \circ f)(x).$$

In particular, for E reflexive, it is straightforward to check that this yields a G–equivariant projection of norm one

$$\pi_n : \; L^\infty(G^{n+1}; E) \longrightarrow L^\infty_S(G^{n+1}; E)$$

onto the subspace of S–harmonic functions. Moreover, one can choose the invariant mean on $C_{\mathrm{b}}(S^{n+1})$ in a way compatible with the canonical projections $S^{n+1} \to S$, so that the maps π_n commute with the coboundary. To sum up, we have obtained the following result:

Corollary 3.3. *Let G be a locally compact separable group, E a separable reflexive Banach G–module and S a subsemigroup of $M(G)$. Suppose S is either amenable as discrete semigroup or is an amenable locally compact group when endowed with the narrow topology.*

Then there is a morphism of complexes

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & E & \longrightarrow & L^\infty(G; E) & \longrightarrow & L^\infty(G^2; E) & \longrightarrow & L^\infty(G^3; E) & \longrightarrow & \cdots \\
 & & \| & & \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} & & \\
0 & \longrightarrow & E & \longrightarrow & L^\infty_S(G; E) & \longrightarrow & L^\infty_S(G^2; E) & \longrightarrow & L^\infty_S(G^3; E) & \longrightarrow & \cdots
\end{array}$$

consisting of G–equivariant continuous projections of norm one. □

The contracting homotopy defined in Sect. 2 preserves S–pluriharmonicity; therefore the spaces $L^\infty_S(G^{n+1}; E)$ determine a subresolution of E. In the case that G is a discrete group, the Corollary 3.3 implies in particular (via Lemma 2.12) that every $L^\infty_S(G^{n+1}; E)$ ($= \ell^\infty_S(G^{n+1}; E)$) is isometrically injective (for E isometric), and therefore we have the

Corollary 3.4. *Let G be a countable group, E a separable reflexive isometric G–module and S a subsemigroup of $M(G)$. Suppose S is either amenable as discrete semigroup or is an amenable locally compact group when endowed with the narrow topology.*

Then each $\ell^\infty_S(G^{n+1}; E)$ is isometrically injective and thus the complex

$$0 \longrightarrow \ell^\infty_S(G; E)^G \longrightarrow \ell^\infty_S(G^2; E)^G \longrightarrow \ell^\infty_S(G^3; E)^G \longrightarrow \cdots$$

of G–invariant S–pluriharmonic bounded cochains realizes the bounded cohomology $H^\bullet_{\mathrm{b}}(G; E)$*.*

More precisely, the inclusions $\ell^\infty_S(G^n; E) \subset \ell^\infty(G^n; E)$ *induce a canonical isometric isomorphism at the level of cohomology.* □

More generally, the Corollary 2.10 now implies

Corollary 3.5. *Let G be a locally compact separable group, E a separable reflexive isometric G–module, S a subsemigroup of* $M(G)$ *and* $\Gamma < G$ *a countable closed subgroup. Suppose S is either amenable as discrete semigroup or is an amenable locally compact group when endowed with the narrow topology.*

Then each $L^\infty_S(G^n; E)$ *is isometricallly injective and thus the complex*

$$0 \longrightarrow L^\infty_S(G; E)^\Gamma \longrightarrow L^\infty_S(G^2; E)^\Gamma \longrightarrow L^\infty_S(G^3; E)^\Gamma \longrightarrow \cdots$$

of measurable Γ*–invariant S–pluriharmonic bounded cochains realizes the bounded cohomology* $H^\bullet_{\mathrm{b}}(\Gamma; E)$*.*

More precisely, the inclusions $L^\infty_S(G^n; E) \subset L^\infty(G^n; E)$ *induce a canonical isometric isomorphism at the level of cohomology.* □

We do not have a satisfactory notion of injective module for *continuous* bounded cohomology that would also be compatible with the morphism $\pi_\bullet$; but we obtain nevertheless a corresponding statement for $H^\bullet_{\mathrm{b,cont.}}$ by constructing a homotopy. First, observe that if we let S^{n+1} act on $L^\infty(G^{n+1}; E)$ via the canonical projections

$$S^{n+1} \longrightarrow S^{n-i} \cong \{\delta_e\}^{i+1} \times S^{n-i} \qquad (-1 \le i \le n),$$

we obtain corresponding projections $\pi_{n,i}$ onto the subspaces of those functions harmonic in the last $n - i$ variables. In particular, $\pi_{n,-1} = \pi_n$ and $\pi_{n,n} = \mathrm{Id}$. Having choosen the invariant means compatible with projections, it is a matter of computation to check the following lemma:

Lemma 3.6. *For all* $n \ge 1$, $-1 \le i \le n$ *and* $0 \le j \le n$ *the relations*

$$\begin{aligned} \pi_{n,i}\, d_{n,j} &= d_{n,j}\, \pi_{n-1,i-1} && (i \ge j) \\ \pi_{n,i}\, d_{n,j} &= d_{n,j}\, \pi_{n-1,i} && (i \le j-1) \end{aligned}$$

hold. □

We are now ready to prove

Proposition 3.7. *Let G be a locally compact separable group, E a separable reflexive Banach G–module and S a subsemigroup of* $M(G)$*. Suppose S is either amenable as discrete semigroup or is an amenable locally compact group when endowed with the narrow topology.*

Then the complex

$$0 \longrightarrow L^\infty_S(G;E)^G \longrightarrow L^\infty_S(G^2;E)^G \longrightarrow L^\infty_S(G^3;E)^G \longrightarrow \cdots$$

of measurable G–invariant S–pluriharmonic bounded cochains realizes the continuous bounded cohomology $H^\bullet_{\mathrm{b,cont.}}(G;E)$.

More precisely, the inclusions $L^\infty_S(G^n;E) \subset L^\infty(G^n;E)$ *induce an isometric isomorphism at the level of cohomology.*

Proof. Fix a function $\psi \in C^+_{00}(G)$ of integral one for a left Haar measure m and recall from the proof of Proposition 2.4 the definitions of the operators $R_{n,i}$ and $\sigma_{n,i}$.

Now we define a G–equivariant continuous linear operator

$$\tau_n : L^\infty(G^{n+1};E) \longrightarrow L^\infty(G^n;E)$$

by

$$\tau_n = \sum_{i=0}^{n-1}(-1)^i \sigma_{n,i}\,\pi_{n,i}.$$

The simplicial relations of the proof of Proposition 2.4 together with Lemma 3.6 yield after a calculation the relation

$$d_n\,\tau_n + \tau_{n+1}\,d_{n+1} = R_{n,-1}\,\pi_{n,-1} - R_{n,n}\,\pi_{n,n} = \pi_n - R_n.$$

Therefore, denoting by ι_n the inclusion $L^\infty_S(G^{n+1};E) \subset L^\infty(G^{n+1};E)$, we conclude that $\tau_\bullet$ is a G–equivariant homotopy from $\iota_\bullet \circ R_\bullet$ to $\iota_\bullet \circ \pi_\bullet$, and by restriction also from $R_\bullet \circ \iota_\bullet$ to $\pi_\bullet \circ \iota_\bullet$. Combining this with the homotopy $\sigma_\bullet$ from $\iota_\bullet \circ R_\bullet$ (resp. $R_\bullet \circ \iota_\bullet$) to the identity constructed in the proof of Proposition 2.4 above, we have shown that the cochain maps $i_\bullet$ and $\pi_\bullet$ induce the identity at the level of cohomology. □

We draw now a few consequences of the preceding proposition, by taking specific semigroups for $S \subset M(G)$.

Amenable subgroups. We begin with the generalization of a result which is well-known for discrete groups.

Corollary 3.8. *Let G be a locally compact separable group, P a closed amenable subgroup of G and E a separable reflexive Banach G–module.*

The complex

$$0 \longrightarrow L^\infty(G/P;E)^G \longrightarrow L^\infty((G/P)^2;E)^G \longrightarrow L^\infty((G/P)^3;E)^G \longrightarrow \cdots$$

of measurable G–invariant componentwise right P–invariant bounded cochains realizes the continuous bounded cohomology $H^\bullet_{\mathrm{b,cont.}}(G;E)$.

More precisely, the inclusions $L^\infty((G/P)^n;E) \subset L^\infty(G^n;E)$ *induce an isometric isomorphism at the level of cohomology.*

Proof. We can view P as locally compact subgroup of $M(G)$ via the map $p \mapsto \delta_p$ wich assigns to $p \in P$ its point measure. This is a topological isomorphism of groups on its image endowed with the narrow topology, hence we may apply the Proposition 3.7. □

Using the Corollary 3.5, we also deduce:

Corollary 3.9. *Let G, P be as in the Corollary 3.8 above and E be a separable reflexive isometric G–module.*

If $\Gamma < G$ is a countable closed subgroup, then each $L^\infty((G/P)^n; E)$ is isometrically injective and thus the complex

$$0 \longrightarrow L^\infty(G/P; E)^\Gamma \longrightarrow L^\infty((G/P)^2; E)^\Gamma \longrightarrow L^\infty((G/P)^3; E)^\Gamma \longrightarrow \cdots$$

of measurable Γ–invariant P–invariant bounded cochains realizes the bounded cohomology $H_{\mathrm{b}}^\bullet(\Gamma; E)$.

More precisely, the inclusions $L^\infty((G/P)^n; E) \subset L^\infty(G^n; E)$ induce a canonical isometric isomorphism at the level of cohomology. □

As an important application of this result, suppose that the Γ–action on $(G/P)^2$ is ergodic. Let us then point out an explicit reformulation of the above for complex coefficients:

Corollary 3.10. *Let Γ, G, P and E be as in the Corollary 3.9 above. If the Γ–action on $(G/P)^2$ is ergodic, then the space $H_{\mathrm{b}}^2(\Gamma)$ is realized as the quotient of the space*

$$ZL^\infty((G/P)^3)^\Gamma = \left\{c \in L^\infty((G/P)^3, \nu^{\otimes 3})^\Gamma : dc = 0\right\}$$

of Γ–invariant measurable bounded cocycles by the subspace of constant functions. □

An illustration. Let $G = \mathrm{Sp}(2n, \mathbf{R})$ be the group of symplectic automorphisms on $\mathbf{R}^{2n}$, and denote by Λ_n the *Lagrangian Grassmannian* manifold consisting of all Lagrangian subspaces of $\mathbf{R}^{2n}$ (see [2]). Denote by

$$\omega : \Lambda_n \times \Lambda_n \times \Lambda_n \longrightarrow \mathbf{Z}$$

the Maslov index; then ω is a bounded G–invariant cocycle with set of values $[-n, n] \cap \mathbf{Z}$. Let F_n be the space of maximal isotropic flags in $\mathbf{R}^{2n}$ and $\pi : F_n \to \Lambda_n$ the canonical projection. Then F_n is the Furstenberg boundary of G, and $\kappa = \omega \circ (\pi \times \pi \times \pi)$ is in $ZL^\infty(F_n^3)^G$. Since κ is not essentially constant, Corollary 3.10 implies that κ defines a nonzero element in $H_{\mathrm{b}}^2(\Gamma)$ for every lattice $\Gamma < G$. Moreover, if $n \geq 3$ and if Γ is cocompact, then it follows from Corollary 1.6 that κ is a generator of $H_{\mathrm{b}}^2(\Gamma)$.

Discrete Poisson transform. Suppose Γ is a countable discrete group. If we take S to be the semigroup generated by a single measure μ, then S

is commutative and hence amenable as discrete semigroup. The associated spaces ℓ^∞_μ of *μ–pluriharmonic* functions realize therefore the bounded cohomology of G.

Now, one may associate (see [16] or [1]) to the pair (Γ, μ) its *Poisson boundary* which is a standard measure space (B, ν) acted upon by Γ in such a way that various Poisson transform isomorphisms hold:

Proposition 3.11. *Let Γ be a countable discrete group, $\mu \in M^1(\Gamma)$ a probability measure and (B, ν) the corresponding boundary.*
For every separable reflexive Γ–module E and all $n \geq 0$, there is a Γ–equivariant isometric isomorphism

$$\mathcal{P}^{(n)} : L^\infty(B^{n+1}, \nu^{\otimes(n+1)}; E) \longrightarrow \ell^\infty_\mu(\Gamma^{n+1}; E)$$

defined by

$$\mathcal{P}^{(n)} F(p) = \int_{B^{n+1}} F(p\xi) d\nu^{\otimes(n+1)}(\xi).$$

via the product action of Γ^{n+1} on B^{n+1}.

Remark. The map $\mathcal{P}^{(n)}$ is obviously Γ^{n+1}–equivariant, but this is not the action we want to emphasize. Remark also that the sequence $\mathcal{P}^{(\bullet)}$ is a (augmentation preserving) cochain map for the usual coboundary maps, that is $\mathcal{P}^{(n+1)} d_{n+1} = \nu(B) \cdot d_{n+1} \mathcal{P}^{(n)} = d_{n+1} \mathcal{P}^{(n)}$.

Proof of Proposition 3.11. Suppose first $E = \mathbf{C}$. The case $n = 0$ is well-known, so we argue by induction. The injectivity is done componentwise, so the point here is surjectivity. Pick $f \in \ell^\infty_\mu(\Gamma^{n+1})$; by the induction hypothesis, there is for each $\gamma \in \Gamma$ a map $F_\gamma \in L^\infty(B^n, \nu^{\otimes n})$ with $f(\gamma, \cdot) = \mathcal{P}^{(n-1)} F_\gamma$ and $\|F_\gamma\|_\infty \leq \|f\|_\infty$. Now for all $p \in \Gamma^n$ the μ–pluriharmonicity reads

$$\sum_{\eta \in \Gamma} \mu(\eta) f(\gamma\eta^{-1}, p) = f(\gamma, p),$$

and therefore

$$\int_{B^n} \sum_{\eta \in \Gamma} \mu(\eta) F_{\gamma\eta^{-1}}(p\xi) \, d\nu^{\otimes n}(\xi) = \int_{B^n} F_\gamma(p\xi) \, d\nu^{\otimes n}(\xi).$$

The injectivity of $\mathcal{P}^{(n-1)}$ now implies

$$\sum_{\eta \in \Gamma} \mu(\eta) F_{\gamma\eta^{-1}}(p\xi) = F_\gamma(p\xi) \qquad \nu\text{–a.-e. } (\xi).$$

In other words, the map $\gamma \mapsto F_\gamma(\xi)$ is μ–harmonic for almost every $\xi \in B^n$, and hence admits a Poisson representation $F_\gamma(\xi) = \mathcal{P}^{(1)} F_\xi(\gamma)$ with $F_\xi \in L^\infty(B, \nu)$. So we have a class $F \in L^\infty(B^{n+1}, \nu^{\otimes(n+1)})$ defined by $F(\xi_0, \ldots, \xi_n) = F_{(\xi_1,\ldots,\xi_n)}(\xi_0)$; observe that $\|F\|_\infty = \|f\|_\infty$ and that F is measurable by Fubini's theorem.

For more general modules E, we proceed as in the discussion preceding the Corollary 3.3, that is, for $F \in L^\infty(B^{n+1}, \nu^{\otimes(n+1)}; E)$ and $p \in \Gamma^{n+1}$ we define an element $\mathcal{P}^{(n)}F(p)$ of $E^{**} = E$ by

$$\mathcal{P}^{(n)}F(p)\,(\Lambda) \;=\; \mathcal{P}^{(n)}(\Lambda \circ F)\,(p)$$

for all $\Lambda \in E^*$. □

The Proposition 3.11 together with Corollary 3.4 now imply

Corollary 3.12. *The complex of invariants*

$$0 \longrightarrow L^\infty(B, \nu; E)^\Gamma \longrightarrow L^\infty(B^2, \nu^{\otimes 2}; E)^\Gamma \longrightarrow L^\infty(B^3, \nu^{\otimes 3}; E)^\Gamma \longrightarrow \cdots$$

realizes the bounded cohomology $H^\bullet_{\mathrm{b}}(\Gamma; E)$ for every separable reflexive isometric Γ–module E.

Moreover, the Poisson transform $\mathcal{P}^{(\bullet)}$ induces an isometric isomorphism at the level of cohomology. □

Another corollary concerns the quasimorphisms $q \in \mathrm{QM}(\Gamma)$ of Γ. If the group Γ is finitely generated, there is a natural bi-Lipschitz equivalence class of left invariant metrics canonically attached to Γ: the various *word lengths* associated to finite generating sets. A quasimorphism has at most linear growth with respect to this class. The *first moment* of a measure μ on Γ for a metric d is $\int_\Gamma d(e, \gamma)\, d\mu(\gamma)$, and its finiteness doesn't depend on the choice of d. So if μ has finite first moment, the convolution $q * \mu$ makes sense for any quasimorphism q.

But we need a lemma. Notice that the spaces $\ell^\infty_{\mathrm{alt.}}$ (resp. $\ell^\infty_{\mu,\mathrm{alt.}}$) of *alternating* (pluriharmonic) cochains determine a subcomplex of the standard resolution. We have:

Lemma 3.13. *The complexes of alternating invariant cochains*

$$0 \longrightarrow \ell^\infty_{\mathrm{alt.}}(\Gamma; E)^\Gamma \longrightarrow \ell^\infty_{\mathrm{alt.}}(\Gamma^2; E)^\Gamma \longrightarrow \ell^\infty_{\mathrm{alt.}}(\Gamma^3; E)^\Gamma \longrightarrow \cdots$$

and of μ–pluriharmonic alternating invariant cochains

$$0 \longrightarrow \ell^\infty_{\mu,\mathrm{alt.}}(\Gamma; E)^\Gamma \longrightarrow \ell^\infty_{\mu,\mathrm{alt.}}(\Gamma^2; E)^\Gamma \longrightarrow \ell^\infty_{\mu,\mathrm{alt.}}(\Gamma^3; E)^\Gamma \longrightarrow \cdots$$

realize the bounded cohomology of Γ.

Moreover, the inclusions

$$\ell^\infty_{\mathrm{alt.}}(\Gamma^n; E) \subset \ell^\infty(\Gamma^n; E) \quad \textit{and} \quad \ell^\infty_{\mu,\mathrm{alt.}}(\Gamma^n; E) \subset \ell^\infty_\mu(\Gamma^n; E)$$

determine isometric isomorphisms at the level of cohomology.

Proof. Use the classical projectors

$$A: \ \ell^\infty(\Gamma^n; E) \longrightarrow \ell^\infty_{\text{alt.}}(\Gamma^n; E)$$

defined by the formula

$$A\ f(x_1, \ldots, x_n) \ = \ \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \text{sign}(\sigma)\, f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

wherein $\mathcal{S}_n$ is the permutation group. Apply the Lemma 2.12. □

Corollary 3.14. *Let Γ be a finitely generated group and μ a symmetric probability measure on Γ with finite first moment. Then any quasimorphism is at bounded distance of a μ–harmonic one.*

Proof. We begin with the following observation. Let $\alpha : \Gamma \times \Gamma \to \mathbf{C}$ be an alternating Γ–invariant map such that the coboundary $d\alpha$ is in $\ell^\infty_\mu(\Gamma^3)$; then we claim that α is μ–biharmonic. Indeed, denote by $*_1$ the convolution at the first variable; α being μ-summable in each variable, we may compute

$$\begin{aligned}
(\alpha *_1 \mu - \alpha)(x, y) \ &= \ \sum_{z \in \Gamma} \mu(z)\Big(\alpha(xz^{-1}, y) - \alpha(x, y)\Big) \\
&= \ \sum_{z \in \Gamma} \mu(z)\Big(d\alpha(x, xz^{-1}, y) - \alpha(x, xz^{-1})\Big) \\
&= \ \underbrace{d\alpha(x, x, y)}_{=0} - \sum_{z \in \Gamma} \mu(z)\alpha(e, z^{-1}).
\end{aligned}$$

Letting $\Gamma_0 = \{\gamma \in \Gamma \ : \ \gamma = \gamma^{-1}\}$, one can decompose Γ as $\Gamma = \Gamma_0 \sqcup \Gamma_1 \sqcup \Gamma_1^{-1}$. But α being Γ–invariant alternating and μ being symmetric, we have

$$\begin{aligned}
\sum_{z \in \Gamma_1} \mu(z)\alpha(e, z^{-1}) \ &= \ \sum_{z \in \Gamma_1^{-1}} \mu(z)\alpha(e, z) \ = \\
&= \ \sum_{z \in \Gamma_1^{-1}} \mu(z)\alpha(z^{-1}, e) \ = \ - \sum_{z \in \Gamma_1^{-1}} \mu(z)\alpha(e, z^{-1}),
\end{aligned}$$

while the summand $\mu(z)\alpha(e, z^{-1})$ is zero for $z \in \Gamma_0$. Therefore, we have $\alpha *_1 \mu - \alpha = 0$, so α is harmonic in the first variable. Likewise, α is harmonic in the second.

Now we pick a quasimorphism q of Γ. Observing that $q(\gamma)$ differs from $\big(q(\gamma) - q(\gamma^{-1})\big)/2$ by at most $\big(|q(e)| + \|dq\|_\infty\big)/2$, we may assume q antisymmetric. Define a bounded alternating cocycle $\omega = d\psi$ by letting $\psi(x, y) = q(x^{-1}y)$. Since the inclusion $\ell^\infty_{\mu,\text{alt.}}(\Gamma^\bullet) \subset \ell^\infty(\Gamma^\bullet)$ induces the identity at the level of cohomology, there is a bounded cochain β such that $\omega + d\beta$ is μ–pluriharmonic. Now the reasoning above applied to $\alpha = \psi + \beta$ yields that $\psi + \beta$ is biharmonic. Therefore we have a harmonic quasimorphism q' at bounded distance of q by letting $q'(x) = (\psi + \beta)(e, x)$. □

We turn back to the case of a locally compact group G and give another class of resolutions arising from the Proposition 3.7.

Gelfand pairs. Recall that a *Gelfand pair* consists of a locally compact group G with a compact subgroup K such that the convolution algebra $C_{00}(K\backslash G/K)$ of bi–K–invariant compactly supported continuous functions is commutative. For an introduction to Gelfand pairs, see [13] and [35] § 24.8.

An example of Gelfand pair is $G(k)$ where G is a k–almost simple simply connected group over a field k of characteristic zero together with a good maximal compact subgroup K (see [34] or [29]).

Geometrically, if G is a group acting doubly transitively on a proper metric space and K is the stabilizer of a point, then (G, K) is another example of a Gelfand pair (see [35], Proposition 24.8.3).

Recall also that if (G, K) is a Gelfand pair, then G is unimodular ([35], Proposition 24.8.1).

By the *trivial character* of $C_{00}(K\backslash G/K)$, we mean the character χ_0 defined by $\chi_0(f) = \int_G f$. Let E be a separable G–module; the algebra $C_{00}(K\backslash G/K)$ acts on $L^1_{\mathrm{loc}}(G; E)$ by right convolution. This action preserves $C(G/K)$ and is G–equivariant. Recall the following

Definition. *A function $\varphi \in C(G/K; E)$ is* K–harmonic *if it is a joint eigenfunction of $C_{00}(K\backslash G/K)$ with joint eigenvalue χ_0, that is $\varphi * f = \chi_0(f) \cdot \varphi$ for all $f \in C_{00}(K\backslash G/K)$.*

A function $\varphi \in C((G/K)^n; E)$ is K–pluriharmonic *if K–harmonic in each variable. We denote by $\mathcal{H}^\infty_K(G^n; E)$ the space of* bounded K–*pluriharmonic functions.*

Notice that a K–harmonic function is continuous, right uniformly continuous and right K–invariant.

Now we state another corollary of the Proposition 3.7.

Corollary 3.15. *Let (G, K) be a Gelfand pair with G separable and E a separable reflexive Banach G–module.*

The complex

$$0 \longrightarrow \mathcal{H}^\infty_K(G; E)^G \longrightarrow \mathcal{H}^\infty_K(G^2; E)^G \longrightarrow \mathcal{H}^\infty_K(G^3; E)^G \longrightarrow \cdots$$

of G–invariant K–pluriharmonic bounded cochains realizes the continuous bounded cohomology $H^\bullet_{\mathrm{b,cont.}}(G; E)$.

More precisely, the inclusions $\mathcal{H}^\infty_K(G^n; E) \subset L^\infty(G^n; E)$ *induce an isometric isomorphism at the level of cohomology.*

Proof. Fix a right Haar measure on G (we won't use the fact that this measure has to be left invariant aswell). Denote by S the semigroup of non-negative functions in $C_{00}(K\backslash G/K)$ having integral one; the Proposition 3.7 applies to this S, so that it remains only to show that S–invariance implies harmonicity. Since scalar multiplication commutes with $C_{00}(K\backslash G/K)$ and with χ_0, we have $C^+_{00}(K\backslash G/K)$–invariance. But the standard decomposition $\varphi = \varphi^+ - \varphi^-$ of a function into two non–negative parts preserves bi–K–invariance, so that we get harmonicity. □

Again, we use the Corollary 3.5 to deduce:

Corollary 3.16. *Let* (G, K) *be as in the Corollary 3.15 above and* E *be a separable reflexive isometric* G*–module.*

If $\Gamma < G$ *is a countable closed subgroup, then each* $\mathcal{H}^\infty_K(G^n; E)$ *is isometrically injective and thus the complex*

$$0 \longrightarrow \mathcal{H}^\infty_K(G; E)^\Gamma \longrightarrow \mathcal{H}^\infty_K(G^2; E)^\Gamma \longrightarrow \mathcal{H}^\infty_K(G^3; E)^\Gamma \longrightarrow \cdots$$

of Γ*–invariant* K*–pluriharmonic bounded cochains realizes the bounded cohomology* $H^\bullet_b(\Gamma; E)$.

More precisely, the inclusions $\mathcal{H}^\infty_K(G^n; E) \subset L^\infty(G^n; E)$ *induce a canonical isometric isomorphism at the level of cohomology. This isomorphism is also induced by the restriction maps* $\mathcal{H}^\infty_K(G^n; E) \to \ell^\infty(\Gamma; E)$. □

Notice that the above restriction maps are well–defined because of the continuity of K–harmonic functions; the unicity of the induced cohomological isomorphism is again the consequence of the functorial property of injective resolutions.

4. Cochain induction

We begin by recalling the topological analogue of the "Frobenius reciprocity" as stated in [5], Proposition 8.6. With our terminology, this yields

Proposition 4.1. (Ph. Blanc) *Let* Γ *be a discrete subgroup of a second countable locally compact group* G*. Suppose* G *acts continuously on a second countable locally compact space* X *and preserves a Radon measure* μ.

Then the map

$$i : L^p_{\mathrm{loc}}(X, \mu; E)^\Gamma \longrightarrow L^p_{\mathrm{loc}}\big(X, \mu; L^p_{\mathrm{loc}}(G; E)^\Gamma\big)^G$$

defined for $f \in L^p_{\mathrm{loc}}(X, \mu; E)^\Gamma$ *almost everywhere by* $i f(x)(g) = f(gx)$ *is a topological isomorphism of Fréchet spaces for every* Γ*–module* E *and every* $1 \le p < \infty$. □

In this statement, the groups G and Γ act as usual by left regular action on the spaces $L^p_{\mathrm{loc}}(X;-)$, but we take G to act by *right translation* on $L^p_{\mathrm{loc}}(G;E)^\Gamma$. We call the map i the *induction map*, and call the G–module $L^p_{\mathrm{loc}}(G;E)^\Gamma$ the *induction module*.

The Proposition 4.1 applies in particular to the case where X is G^n with a left Haar measure; since i commutes with the corresponding coboundary maps, we can seek an analogue of the Eckmann–Shapiro induction isomorphism. However, the natural induction module would be the (non separable) space $L^\infty(G;E)^\Gamma$, which is of little use to us because of the lack of information about its G–module structure. Therefore, we need a stronger result:

Proposition 4.2. *Let Γ be a lattice in a separable locally compact group G and E a separable bounded Γ–module. Suppose there is an amenable closed subgroup $P < G$ such that the diagonal Γ–action on $(G/P)^k$ is ergodic for some integer $k \geq 1$.*

Then the induction map i induces an injection

$$H^k i_p :\ H^k_{\mathrm{b}}(\Gamma;E) \longrightarrow H^k_{\mathrm{b,cont.}}(G;L^p(G;E)^\Gamma)$$

for all $1 \leq p < \infty$.

We will apply this result to the case $k = 2$ and $p = 2$, so that we can use properties of the unitary representation of G in $L^2(\Gamma\backslash G)$. The point of the statement above is that H^k_{b} is realized by invariant $(k+1)$–cocycles, while Γ is ergodic only on $(G/P)^k$.

Proof of Proposition 4.2. The statement makes no claim for the precise norm of $H^k i_p$, so that we may assume E isometric by replacing it with E_{is} (see Sect. 2). Since $\Gamma\backslash G$ has finite invariant measure, one has $L^\infty(\Gamma\backslash G) \subset L^p(\Gamma\backslash G)$; now E being separable, we can deduce $L^\infty(G;E)^\Gamma \subset L^p(G;E)^\Gamma$ by passing to the norm.

It follows from its definition that i is G^n–equivariant with respect to the right diagonal translation for every $n \geq 1$. Therefore, using the Corollaries 3.8 and 3.9, we realize $H^n i_p$ by

$$i_p :\ L^\infty((G/P)^{n+1};E)^\Gamma \longrightarrow L^\infty\big((G/P)^{n+1};L^p(G;E)^\Gamma\big)^G,$$

where i_p is the composition of i with the range inclusion $L^\infty(G;E)^\Gamma \subset L^p(G;E)^\Gamma$. Thus every element of the kernel of $H^k i_p$ is represented by a cocycle α in $L^\infty((G/P)^{k+1};E)^\Gamma$ such that $i\alpha = d\beta$ for some β in $L^\infty\big((G/P)^k;L^p(G;E)^\Gamma\big)^G$. Now we can view β as a right P^k–invariant element of $L^\infty\big(G^k;L^p_{\mathrm{loc}}(G;E)^\Gamma\big)^G$ and hence of $L^p_{\mathrm{loc}}\big(G^k;L^p_{\mathrm{loc}}(G;E)^\Gamma\big)^G$, so that $\beta = i\beta'$ for some right P^k–invariant $\beta' \in L^p_{\mathrm{loc}}(G^k;E)^\Gamma$. Furthermore, one checks that β' can actually be viewed as element of $L^p_{\rm loc}((G/P)^k; E)^\Gamma$. Since i commutes with the coboundary maps, the Proposition 4.1 implies $\alpha = d\beta'$. Now the ergodicity assumption implies that the norm $\|\beta'\|$ is an essentially constant function, so that β' is essentially bounded. Hence α is trivial in $H^k_{\rm b}(\Gamma; E)$. □

Actually, the above proof yields a stronger statement. Indeed, one can use the standard resolution to define continuous bounded cohomology with coefficients in Fréchet spaces, say for instance $L^p_{\rm loc}(G; E)^\Gamma$. Now the arguments above show in fact that the (furhter) induction

$$H^k_{\rm b}(\Gamma; E) \longrightarrow H^k_{\rm b,cont.}(G; L^p_{\rm loc}(G; E)^\Gamma)$$

is still injective.

5. The comparison map

Lemma 5.1. *Let $(\pi, \mathfrak{H})$ be a continuous unitary representation of a locally compact group G and let $H < G$ be a compactly generated closed subgroup. Let $\alpha : G \times G \to \mathfrak{H}$ be a locally bounded G–invariant map with bounded coboundary $\omega = d\alpha$.*

If $\pi\big|_H$ has no almost invariant vectors, then $\alpha\big|_{Z_G(H)\times Z_G(H)}$ is bounded.

In other words, the lemma states that the expression

$$\sup_{g_1,g_2\in Z_G(H)} \|\alpha(g_1, g_2)\|_{\mathfrak{H}}$$

is finite.

Proof of Lemma 5.1. For shorter notation, we work with the inhomogenous representatives $\bar\alpha$ and $\bar\omega$ of α and ω, that is $\bar\alpha(x) = \alpha(e, x)$ and $\bar\omega(x, y) = \omega(e, x, xy)$. Now $\omega = d\alpha$ reads

$$\bar\omega(x, y) = \pi(x)\bar\alpha(y) - \bar\alpha(xy) + \bar\alpha(x).$$

Therefore, for all $h \in Z_G(H)$ and $t \in H$, the vectors

$$\pi(t)\bar\alpha(ht^{-1}) - \bar\alpha(tht^{-1}) + \bar\alpha(t)$$
$$\text{and } \pi(h)\bar\alpha(t^{-1}) - \bar\alpha(ht^{-1}) + \bar\alpha(h)$$

are norm–bounded by $\|\omega\|_\infty$ independently of h and t. Applying $\pi(t)$ to the second vector before adding it to the first yields that

$$\pi(t)\bar\alpha(h) - \bar\alpha(h) + \pi(t)\pi(h)\bar\alpha(t^{-1}) + \bar\alpha(t)$$

is bounded independently of h and t, hence

$$\left\| \Big(\mathrm{Id} - \pi(t)\Big)\bar{\alpha}(h) \right\|_{\mathfrak{H}} \leq \|\bar{\alpha}(t^{-1})\|_{\mathfrak{H}} + \|\bar{\alpha}(t)\|_{\mathfrak{H}} + C$$

for some $C < \infty$. Since $\pi\big|_H$ has no almost invariant vectors, there is (by [28], IV 3.2, p. 155) a non negative function $\psi = \psi^\vee$ of integral one in $C_{00}(H)$ such that $\|\pi(\psi)\| < 1$. Now we have

$$\left\| \Big(\mathrm{Id} - \pi(\psi)\Big)\bar{\alpha}(h) \right\|_{\mathfrak{H}} \leq \sup_{t \in \mathrm{Supp}(\psi)} \Big(\|\bar{\alpha}(t^{-1})\|_{\mathfrak{H}} + \|\bar{\alpha}(t)\|_{\mathfrak{H}} + C\Big)$$

for all h in $Z_G(H)$. Since $\mathrm{Id} - \pi(\psi)$ is invertible, we conclude that $\bar{\alpha}\big|_{Z_G(H)}$ is bounded, whence the statement. □

We introduce now a technical definition which allows us to state the next developments of this section with a certain generality.

Definition 5.2. *Let P be a closed subgroup of a locally compact group G. We say that the pair (G, P) has the property* (**A**) *if for any continuous unitary representation of G in a Hilbert space $\mathfrak{H}$ and any $g_1, g_2 \in G$ the following holds:*

if $v \in \mathfrak{H}$ is fixed by $g_1 P g_1^{-1} \cap g_2 P g_2^{-1}$, then the diagonal action of the stabilizer $\mathrm{Stab}_G(v)$ *on $(G/P)^2$ is ergodic.*

In particular, considering the trivial representation, one sees that the G–action on $(G/P)^2$ is ergodic for any pair (G, P) with the property (**A**).

Proposition 5.3. *Let G_1, G_2 be separable compactly generated locally compact groups and let $P_i < G_i$ be closed amenable subgroups such that the pairs (G_i, P_i) have property* (**A**)*.*

Let $(\pi, \mathfrak{H})$ be a continuous unitary representation of $G = G_1 \times G_2$ in a separable Hilbert space $\mathfrak{H}$. If $\mathfrak{H}^{G_1} = \mathfrak{H}^{G_2} = 0$, then the comparison map

$$H^2_{\mathrm{b,cont.}}(G; \mathfrak{H}) \longrightarrow H^2_{\mathrm{cont.}}(G; \mathfrak{H})$$

is injective.

Proof. Setting $P = P_1 \times P_2$, one checks that (G, P) has property (**A**). One can find a sequence $(\mathfrak{H}_n)_{n \geq 1}$ of closed G–invariant subspaces of $\mathfrak{H}$ such that

(i) for all n, the restrictions $\pi\big|_{G_1}$ and $\pi\big|_{G_2}$ have no almost invariant vectors in $\mathfrak{H}_n$;
(ii) for all $v \in \mathfrak{H}$, on has $\lim_{n \to \infty} \|v - \mathrm{pr}_{\mathfrak{H}_n}(v)\| = 0$.

Pick $\omega : G^3 \to \mathfrak{H}$ a continuous bounded 3–cocycle and a continuous cochain $\alpha : G^2 \to \mathfrak{H}$ with $d\alpha = \omega$. By the Corollary 3.8, we find $\alpha' \in L^\infty(G^2; \mathfrak{H})^G$ such that $\omega = \pi_2\omega + d\alpha'$, where

$$\pi_m :\ L^\infty(G^{m+1}; \mathfrak{H}) \longrightarrow\ L^\infty\big((G/P)^{m+1}; \mathfrak{H}\big)$$

are the continuous equivariant projections of the Corollary 3.3; thus $\pi_2\omega = d(\alpha - \alpha')$. Set

$$\omega_n = \mathrm{pr}_{\mathfrak{H}_n} \circ \pi_2\omega \quad \text{and} \quad \beta_n = \mathrm{pr}_{\mathfrak{H}_n} \circ (\alpha - \alpha'),$$

and observe that $\alpha - \alpha'$ and hence β_n are locally bounded. Therefore, since $G_1 \subset Z_G(G_2)$ and vice-versa, we may apply the Lemma 5.1 consecutively to G_1 and G_2 to conclude that

$$\sup_{g_1, g_2 \in G} \|\beta_n(g_1, g_2)\| \ <\ \infty$$

for all $n \geq 1$. Now, β_n being bounded, the identity $\omega_n = \pi_2\omega_n = \pi_2 d\beta_n$ implies $\omega_n = d\pi_1\beta_n$ for all $n \geq 1$. So $\pi_1\beta_n$ is in $L^\infty\big((G/P)^2; \mathfrak{H}\big)^G$ and for almost all couples (g_1P, g_2P) in $(G/P)^2$, the vector $\pi_1\beta_n(g_1P, g_2P)$ of $\mathfrak{H}$ is fixed by $g_1Pg_1^{-1} \cap g_2Pg_2^{-1}$. By the property (**A**), there is for all $n \geq 1$ a vector $v_n \in \mathfrak{H}^G$ such that

$$\pi_1\beta_n(g_1P, g_2P) = v_n$$

for almost all (g_1P, g_2P). Therefore we have also

$$\omega_n(g_1P, g_2P, g_3P) = v_n$$

for almost all triples (g_1P, g_2P, g_3P); but the condition (ii) above implies

$$\lim_{n\to\infty} \big\|\pi_2\omega(g_1P, g_2P, g_3P) - \omega_n(g_1P, g_2P, g_3P)\big\| = 0,$$

so $\pi_2\omega$ is essentially constant of value $v = \lim_{n\to\infty} v_n$. Denote by c_v the G–equivariant map $G \times G \to \mathfrak{H}$ of constant value $v \in \mathfrak{H}^G$; recalling that $\omega = \pi_2\omega + d\alpha'$, we may write $\omega = d(c_v + \alpha')$ as coboundary of a bounded map, and therefore we conclude that ω is trivial in $H^2_{\mathrm{b,cont.}}(G; \mathfrak{H})$. ◻

6. Proof of the Theorem 1.1

Setting. Throughout this section, we consider a finite family $(k_a)_{a\in A}$ of local fields, and for each $a \in A$ we consider a connected, simply connected, almost simple k_a–isotropic group $\mathbf{G}_a$. We set

$$G\ =\ \prod_{a\in A} \mathbf{G}_a(k_a), \qquad\qquad \mathrm{rank}\, G\ =\ \sum_{a\in A} \mathrm{rank}_{k_a}\mathbf{G}.$$

Define for any Banach space E the space $\mathrm{QM}_{\mathrm{cont.}}(G;E)$ of continuous quasimorphisms to be the collection of continuous maps $q : G \to E$ satisfying

$$\sup_{x,y\in\Gamma} \|\delta q(x,y)\|_E \ < \ \infty,$$

where $\delta q(x,y) \stackrel{\mathrm{def}}{=} q(x)+q(y)-q(xy)$ (that is, δ is the usual inhomogenous coboundary map). Then δ induces a natural identification

$$\frac{\mathrm{QM}_{\mathrm{cont.}}(G;E)}{C_{\mathrm{b}}(G;E)+\mathrm{Hom}_{\mathrm{cont.}}(G;E)} \ \cong \ \mathrm{Ker}\Big(H^2_{\mathrm{b,cont.}}(G;E) \to H^2_{\mathrm{cont.}}(G;E)\Big).$$

Lemma 6.1. *All continuous quasimorphisms of G are bounded.*

Proof. First notice that *any* quasimorphism q with values in a Banach space E is bounded on a given conjugacy class:

$$\begin{aligned}\big\|q(x^{-1}yx)\big\|_E &\le \big\|q(x^{-1})+q(y)+q(x)\big\|_E + 2\|\delta q\|_\infty \\ &\le \|q(y)\|_E + \|q(e)\|_E + 3\|\delta q\|_\infty\end{aligned}$$

On a product of groups, one can bound quasimorphisms componentwise, thus let $G = \mathbf{G}_a(k_a)$. Now one may write G as a product $G = N_1 \cdots N_\ell$ of unipotent subgroups such that for each N_i there is a semisimple element $s_i \in G$ whose action as inner automorphism contracts N_i to the identity e of G, that is

$$\lim_{k\to\infty} s_i^{-k} u s_i^k = e \qquad\qquad \forall u \in N_i.$$

Therefore, if q is continuous, it is bounded by $2\|q(e)\|_E + 3\|\delta q\|_\infty$ on each N_i and hence is bounded on G. □

In other words, we now know that the comparison map

$$H^2_{\mathrm{b,cont.}}(G;E) \ \longrightarrow \ H^2_{\mathrm{cont.}}(G;E)$$

is injective for every *trivial* module E.

Corollary 6.2. *Let Γ be an irreducible cocompact latice in G and let $\mathfrak{H}$ be a separable Hilbert space with unitary Γ–action.*

If rank $G \ \geq 2$*, then the comparison map*

$$H^2_{\mathrm{b,cont.}}\big(G; L^2(G;\mathfrak{H})^\Gamma\big) \ \longrightarrow \ H^2_{\mathrm{cont.}}\big(G; L^2(G;\mathfrak{H})^\Gamma\big)$$

is injective.

Recall from Sect. 4 that in the statement above the coefficients are thought of as the continuous unitary representation $\big(L^2(G;\mathfrak{H})^\Gamma,\ \varrho\big)$ defined by right translation.

Proof of Corollary 6.2. The subspace of G–invariants in $L^2(G;\mathfrak{H})^\Gamma$ can be canonicaly identified with the trivial G–module $\mathfrak{H}^\Gamma$. Therefore, if we denote its orthogonal complement by $\widetilde{\mathfrak{H}}$, we have a G–invariant decomposition

$$L^2(G;\mathfrak{H})^\Gamma \;=\; \mathfrak{H}^\Gamma \oplus \widetilde{\mathfrak{H}},$$

and by the Proposition 2.3 we can handle these components separatedly.

Since the G–action is trivial on $\mathfrak{H}^\Gamma$, the Lemma 6.1 implies that the comparison map

$$H^2_{\mathrm{b,cont.}}\big(G;\mathfrak{H}^\Gamma\big) \;\longrightarrow\; H^2_{\mathrm{cont.}}\big(G;\mathfrak{H}^\Gamma\big)$$

is injective, so that we are left with the $\widetilde{\mathfrak{H}}$ component, which has no invariant vectors.

Now there are two cases.

Suppose first $|A| = 1$ and write $G = \mathbf{G}(k)$. Let $G = KSK$ be a Cartan decomposition of G; here $S = \mathbf{S}(k)$ and $\mathbf{S}$ is a maximal k–split torus. Take $\omega : G^3 \to \widetilde{\mathfrak{H}}$ a continuous bounded 3–cocycle and $\alpha : G^2 \to \widetilde{\mathfrak{H}}$ continuous with $d\alpha = \omega$. Since $\mathrm{rank}_k G \geq 2$, we may choose singular tori $\mathbf{S}_1, \ldots, \mathbf{S}_r \subset \mathbf{S}$ with $\mathbf{S}_1 \cdots \mathbf{S}_r = \mathbf{S}$ and $Z_G\big(\mathbf{S}_i(k)\big)$ non amenable. Thus, using the same arguments needed to establish the property (T) for G, one gets that $\varrho\big|_{Z_G(\mathbf{S}_i(k))}$ has no almost invariant vectors. Now the Lemma 5.1 applied to the representation $(\widetilde{\mathfrak{H}}, \varrho)$ yields in particular that $\alpha\big|_{\mathbf{S}_i(k)\times\mathbf{S}_i(k)}$ is bounded. Therefore α is bounded.

In the second case, when there are at least two indices in A, fix $a_0 \in A$ and define

$$G_1 \;=\; \mathbf{G}_{a_0}(k_{a_0}), \qquad\qquad G_2 \;=\; \prod_{a\neq a_0} \mathbf{G}_a(k_a).$$

Now choose for all $a \in A$ a minimal k_a–parabolic subgroup $\mathbf{P}_a$ of $\mathbf{G}_a$ and set

$$P_1 \;=\; \mathbf{P}_{a_0}(k_{a_0}), \qquad\qquad P_2 \;=\; \prod_{a\neq a_0} \mathbf{P}_a(k_a).$$

By Howe–Moore (see [24]), the two pairs (G_i, P_i) have property **(A)**. Moreover, the irreducibility of Γ implies that the canonical projections $\mathrm{pr}_i(\Gamma)$ are dense in G_i, so that $\widetilde{\mathfrak{H}}^{G_i} = 0$. Now we can apply the Proposition 5.3 to $(\widetilde{\mathfrak{H}}, \varrho)$, thus completing the proof. □

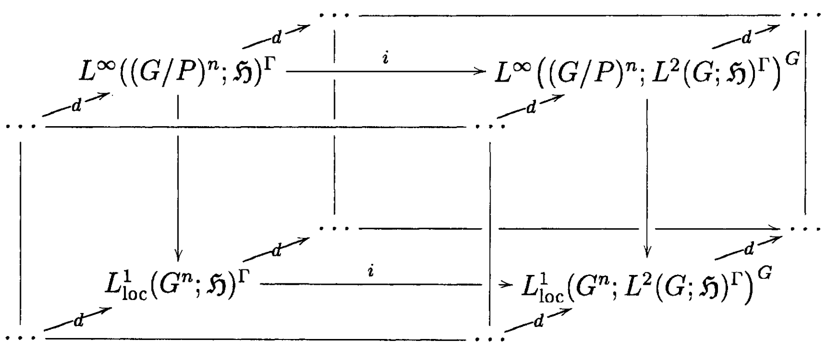
The Theorem 1.1 is a direct consequence of the juxtaposition of this Corollary 6.2 with the Proposition 4.2:

End of the proof of Theorem 1.1. We have a diagramm

$$\begin{array}{ccc} H^2_{\mathrm{b}}(\Gamma;\mathfrak{H}) & \longrightarrow & H^2_{\mathrm{b,cont.}}\big(G;L^2(G;\mathfrak{H})^\Gamma\big) \\ \downarrow & & \downarrow \\ H^2(\Gamma;\mathfrak{H}) & \longrightarrow & H^2_{\mathrm{cont.}}\big(G;L^2(G;\mathfrak{H})^\Gamma\big) \end{array} \qquad (D)$$

in which the upper and rightmost maps are injective; therefore, in order to conclude that the leftmost map is an injection, it is enough to show that the diagram commutes. Notice by the way that the lower map is an isomorphism; this is the content of the more classical Eckmann–Shapiro type lemma in continuous cohomology (see [5] or [6]).

According to Proposition 2.6 and Remark 2.7, the arrows of the diagram (D) are induced by the map sequence



where the vertical arrows are inclusions; this latter diagram commutes. □

7. Proof of the Theorem 1.2

In this section, we consider regular or biregular locally finite trees $\mathcal{T}_1$, $\mathcal{T}_2$ and fix a cocompact lattice Γ in $\mathrm{Aut}\mathcal{T}_1 \times \mathrm{Aut}\mathcal{T}_2$. We define

$$G_i = \overline{\mathrm{pr}_i(\Gamma)} \qquad (i = 1, 2)$$

as the closure of the canonical projection and suppose that each G_i acts transitively on the corresponding boundary at infinity $\mathcal{T}_i(\infty)$. Define $G = G_1 \times G_2$.

Lemma 7.1. *All continuous quasimorphisms of G are bounded.*

Proof. This follows from a Cartan–like decomposition of each of the G_i as given in [8]. More precisely, fix a hyperbolic element $a \in G_1$ and set

$A^+ = \{a^n : n \geq 0\}$. Pick adjacent vertices x, y on the axis of a; the stabilizers $K = \mathrm{Stab}_{G_1}(x)$ and $K' = \mathrm{Stab}_{G_1}(y)$ are compact. One has then

$$\begin{aligned} G_1 &= K \cdot A^+ \cdot K && \text{if } G_1 \text{ is vertex transitive,} \\ G_1 &= K \cdot A^+ \cdot (K \cup K') && \text{otherwise.} \end{aligned}$$

A continuous quasimorphism q with values in a Banach space E is bounded on $K \cup K'$; on the other hand, for every $n \geq 0$ one can write $a^{-n} = ka^m k'$ for $k, k' \in K \cup K'$ and $m \geq 0$. Now the inequalities

$$\left\| q(a^{-n}) - q(a^m) \right\|_E \leq 2\|\delta q\|_\infty + 2\left\| q\big|_{K \cup K'} \right\|_\infty$$

and

$$\left\| q(a^{-n}) + q(a^n) \right\|_E \leq \|\delta q\|_\infty + \|q(e)\|_E$$

yield

$$\left\| q(a^{n+m}) \right\|_E \leq 3\|\delta q\|_\infty + 2\left\| q\big|_{K \cup K'} \right\|_\infty + \|q(e)\|_E,$$

so that q is uniformly bounded on arbitrarily high positive powers of a. Therefore, the quasimorphism $q\big|_{<a>}$ is bounded and hence q is bounded on G_1 and likewise on G_2. □

Corollary 7.2. *The comparison map*

$$H^2_{\mathrm{b,cont.}}\big(G; L^2(G; \mathfrak{H})^\Gamma\big) \longrightarrow H^2_{\mathrm{cont.}}\big(G; L^2(G; \mathfrak{H})^\Gamma\big)$$

is injective.

Proof. As in the proof of the analogous Corollary 6.2, we decompose the coefficients as orthogonal sum

$$L^2(G; \mathfrak{H})^\Gamma = \mathfrak{H}^\Gamma \oplus \widetilde{\mathfrak{H}},$$

and as before, the Lemma 7.1 implies that the comparison map

$$H^2_{\mathrm{b,cont.}}\big(G; \mathfrak{H}^\Gamma\big) \longrightarrow H^2_{\mathrm{cont.}}\big(G; \mathfrak{H}^\Gamma\big)$$

is injective, so that we are left with the $\widetilde{\mathfrak{H}}$ component, which has no invariant vectors.

Now we pick points at infinity $\xi_1 \in \mathcal{T}_1(\infty)$, $\xi_2 \in \mathcal{T}_2(\infty)$ and consider the stabilizers $P_1 = \mathrm{Stab}_{G_1}(\xi_1)$ and $P_2 = \mathrm{Stab}_{G_2}(\xi_2)$. It follows from the Proposition 5 in [7] that the pairs (G_i, P_i) satisfy property (**A**). Now the definition of G_i implies $\widetilde{\mathfrak{H}}^{G_i} = 0$, so that we may again conclude by applying the Proposition 5.3 to $(\widetilde{\mathfrak{H}}, \varrho)$. □

The Corollary 7.2 above together with the Proposition 4.2 completes the proof of the Theorem 1.2. The situation is summarized in the diagram (D) of Sect. 6.

8. Proof of corollaries

The Corollary 1.3 follows from the well-known description of $EH^2_{\mathrm{b}}(\Gamma;\mathbf{R})$ in terms of quasimorphisms (see also the beginning of Sect. 6).

The Corollary 1.4 is *verbatim* the mentioned result of Bavard [3] once we have the Theorems 1.1 and 1.2.

D. Witte shows in [40] how W.P. Thurston's stability theorem in [38] implies the Corollary 1.5 given the injectivity of the natural map. Along the way, one needs the vanishing of $H^1(\Gamma';\mathbf{R})$ for finite index subgroups $\Gamma' < \Gamma$, which is well-known in the setting of Theorem 1.1 and a result of [8] in the setting of Theorem 1.2.

Proof of Corollary 1.6. It follows from [6] that in the case (i) we have $H^2(\Gamma) = 0$, while in case (ii) the space $H^2(\Gamma)$ is one dimensional and generated by the Kähler class. This class being bounded (by [21]), the corollary follows from Theorem 1.1. □

Finally, the Corollary 1.7 is proven in the introduction.

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