PUBLICATIONS MATHÉMATIQUES DE L'I.H.É.S.

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Publications mathématiques de l'I.H.É.S., tome 92 (2000), p. 151-194.
<http://www.numdam.org/item?id=PMIHES_2000__92__151_0>

LATTICES IN PRODUCT OF TREES

by Marc BURGER *and* Shahar MOZES

Introduction

The aim of this paper is to study the structure of lattices in products $\mathrm{Aut}\,T_1 \times \mathrm{Aut}\,T_2$ of automorphism groups of regular trees. These lattices have a rich structure theory both parallel to the theory of lattices in semisimple Lie groups as well as exhibiting some new phenomena. A basic difference is that cocompact lattices $\Gamma < \mathrm{Aut}\,T_1 \times \mathrm{Aut}\,T_2$ never have dense projections. The class of lattices considered here are those whose projections in each factor satisfy various transitivity conditions, in particular that of being locally quasiprimitive. The structure theory of locally quasiprimitive subgroups of $\mathrm{Aut}\,T$ is developped in [B-M]$_3$ and is used in an essential way in this paper. The main consequence of the theory outlined in this paper is the existence and construction of lattices which are finitely presented, torsion free, simple groups; the corresponding quotients of $T_1 \times T_2$ are finite aspherical complexes with simple fundamental group, thus answering a question of G. Mess ([PLT] Probl. 5.11 (c)). Furthermore, using the action of these lattices on each of the tree factors, we show that they are free amalgams $F *_C F$ of finitely generated free groups; this answers a question of P. M. Neumann [Ne] (see also [K-N] Problem 4.45). M. Bhattacharjee constructed in [Ba] a free amalgam $L *_K H$ of finitely generated free groups with no finite index subgroup; on the geometric side, D. Wise [W] constructed a finite complex with no finite (non trivial) coverings, and covered by a product of two trees.

Torsion free discrete subgroups of semisimple groups are fundamental groups of locally symmetric spaces. In Chapter 1 we show that the object corresponding to a torsion free, discrete subgroup of $\mathrm{Aut}\,T_1 \times \mathrm{Aut}\,T_2$ is a square complex, with additional structure. In fact, torsion free, cocompact lattices correspond to finite square complexes whose link at every vertex is a complete, bipartite graph. Such a complex X inherits, from the product structure of its universal covering, a decomposition of its 1-skeleton $X^{(1)}$ into a "horizontal" $X_h^{(1)}$, and "vertical" $X_v^{(1)}$ 1-skeleton. The link condition enables one to define, for every vertex x, an action of $\pi_1(X_h^{(1)}, x)$ on the set $E_v(x)$ of vertical edges with origin x, defining thus a "vertical" permutation group $P_v(x) < \mathrm{Sym}\,E_v(x)$; one obtains analogously a horizontal permutation group $P_h(x) < \mathrm{Sym}\,E_h(x)$. These are basic invariants associated to the square complex X; they give information of "local" nature on the action of $\pi_1(X)$ on the factors of $\widetilde{X}$; for example, the projections of the lattice $\pi_1(X)$ are both locally (quasi) primitive ([B-M]$_3$) precisely when the above finite permutation groups are (quasi) primitive.

A lattice $\Gamma < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ is reducible, if it is commensurable to a product $\Gamma_1 \times \Gamma_2$ of lattices $\Gamma_i < \mathrm{Aut}\, T_i$; for a square complex, this amounts to the existence of a finite covering which is a product of graphs. In Chapter 1 we give a computable sufficient condition for the irreducibility of a complex, based on the Thompson-Wielandt Theorem.

In Chapter 2 we turn to irreducible cocompact lattices with locally quasi-primitive projections. We prove that if such a lattice meets one of the factors, then it is not a residually finite group. Using a geometric method and the results in $[\text{B-M}]_3$, we construct examples of such non residually finite lattices. The existence of such lattices constitutes a fundamental difference with the case of Lie groups.

In Chapter 3 we obtain, using a method of P. Pansu (see [Pa]), certain cohomological vanishing results for irreducible lattices with locally quasi-primitive projections. We deduce, for example (see Prop. 3.1), that if $N \triangleleft \Gamma$ is a normal subgroup in such a lattice Γ, and if N has non-discrete projections, then Γ/N has property (T). This is an analogue of a theorem of G. A. Margulis in the Lie-group case (see [Ma]). It is used in the proof of the normal subgroup theorem, in Chapter 4.

In Chapter 4 we prove one of the main results of this paper, namely the normal subgroup theorem. This concerns lattices Γ whose projections satisfy stronger transitivity conditions, in particular they are locally ∞-transitive, and asserts that any nontrivial normal subgroup of Γ is of finite index. The strategy of the proof is borrowed from Margulis' normal subgroup theorem (see [Ma]); it rests on the characterization of finite groups as being those which at the same time are amenable and have property (T). While there are many (elementary) methods of showing that certain groups cannot be finite, there seem to be few methods of showing that certain groups cannot be infinite.

Along the way we prove that closed, locally ∞-transitive subgroups of Aut T enjoy the Howe-Moore property.

A natural class of closed subgroups of the automorphism group $\mathrm{Aut}\,\mathcal{T}_d$ of the d-regular tree, introduced in $[\text{B-M}]_3$, are the groups $U(F) < \mathrm{Aut}\,\mathcal{T}_d$, associated to a permutation group $F < S_d$. Any vertex transitive subgroup of $\mathrm{Aut}\,\mathcal{T}_d$, whose local action at every vertex is permutation isomorphic to $F < S_d$, is conjugate to a subgroup of U(F). If $F < S_d$ is a 2-transitive permutation group, then U(F) is ∞-transitive. In particular, given F_1, F_2, 2-transitive permutation groups, the normal subgroup theorem applies to all cocompact lattices $\Gamma < U(F_1) \times U(F_2)$, with dense projections. In Chapter 5, we give effective sufficient conditions (based on $[\text{B-M}]_3$ Chapt. 3) on a finite square complex X, ensuring that its fundamental group $\Gamma = \pi_1(X)$ is of the above type.

In Chapter 6 we construct, for every $n \geqslant 15$, $m \geqslant 19$, a square complex $X_{n,m}$ on one vertex, whose fundamental group $\pi_1(X_{n,m}) < U(A_{2n}) \times U(A_{2m})$ has dense projections. We introduce certain geometric operations, joining and surgery, on one vertex square complexes. Using these operations, we show that any finite collection of one vertex square complexes whose links are complete bipartite graphs, embeds into a square complex Y whose fundamental group $\pi_1(Y)$ is a cocompact lattice in $U(A_k)\times U(A_\ell)$, with dense projections. Starting with a one-vertex square complex with non-residually finite fundamental group, (see Chapt. 2) the fundamental group of the resulting square complex Y is virtually simple, that is, it contains a simple group of finite index. A more elaborate construction leads to an infinite family of square complexes on 4 vertices, with simple fundamental groups; an analogue of the Mostow rigidity theorem can be used to show that the above groups are pairwise non-isomorphic.

We end by stating a few properties that any of the simple groups Γ constructed in Chapter 6 enjoys (see Theorem 5.5):

(1) Γ is finitely presented, torsion-free.
(2) Γ is a CAT(0)-group.
(3) Γ is of cohomological dimension 2.
(4) Γ is biautomatic.
(5) Γ is isomorphic to an amalgam $F*_E F$ of free groups over a subgroup of finite index.

Acknowledgement. — It is a pleasure to express the intellectual debt we owe to G. A. Margulis. We thank J. Thévenaz for helpful discussions on the structure of 2-transitive groups, IHES for its hospitality, and the Israel Academy of Sciences which supported the second named author. Finally, we express our gratitude to the referees for their numerous remarks and suggestions for improvement.

0. Preliminaries

0.1. A permutation group $F < \mathrm{Sym}(\Omega)$ of a set Ω is quasiprimitive if every nontrivial normal subgroup $e \neq N \lhd F$ acts transitively on Ω. Let $F^+ = \langle F_\omega : \omega \in \Omega \rangle$ denote the normal subgroup of F generated by the stabilizers F_ω of points $\omega \in \Omega$. We have the following implications:

$$F \text{ is 2-transitive} \Rightarrow F \text{ is primitive} \Rightarrow F \text{ is quasiprimitive} \Rightarrow \begin{cases} F = F^+ \\ \text{or} \\ F \text{ is simple and regular (that is, simply transitive) on } \Omega. \end{cases}$$

Recall that a permutation group $F < \mathit{Sym}\,\Omega$ is called *primitive* if it is transitive and if every F-invariant partition of Ω is either the partition into points or the trivial partition $\{\Omega\}$. An equivalent condition which is often used in the sequel is that F is transitive and the stabilizer F_ω of a point $\omega \in \Omega$ is a maximal subgroup of F. See [Di-Mo] Chapt. 4 for the structure of primitive and [Pr] §5 for the structure of quasiprimitive groups.

0.2. For notations and notions pertaining to graph theory we adopt the viewpoint of Serre's book ([Se]). Let $\mathfrak{g} = (X, Y)$ be a graph with vertex set X and edge set Y, let $E(x) = \{y \in Y : o(y) = x\}$ denote the set of edges with origin x; for a subgroup $H < \operatorname{Aut} \mathfrak{g}$ let $H(x) = \operatorname{Stab}_H(x)$ and $\underline{H}(x) < \operatorname{Sym}(E(x))$ be the permutation group obtained by restricting to $E(x)$ the action of $H(x)$ on Y. We say that H is locally "P" if for every $x \in X$, the permutation group $\underline{H}(x) < \operatorname{Sym}(E(x))$ satisfies one of the following properties "P" : transitive, quasiprimitive, primitive, 2-transitive. We say that H is locally n-transitive ($n \geqslant 3$) if for every $x \in X$, the group $H(x)$ acts transitively on the set of reduced paths (i.e. without back-tracking) of length n and origin x. Observe that H is locally 2-transitive iff, for every $x \in X$, $H(x)$ acts transitively on the set of reduced paths of length 2 and origin x. We say that $H < \operatorname{Aut} \mathfrak{g}$ is n-transitive if H acts transitively on the set of oriented paths of length n without back-tracking; $H < \operatorname{Aut} \mathfrak{g}$ is locally ∞-transitive if it is locally n-transitive for all $n \geqslant 1$.

For a connected graph $\mathfrak{g}$ and $H < \operatorname{Aut} \mathfrak{g}$, we have $H < \operatorname{Sym}(Y)$, and H^+ denotes the subgroup generated by edge stabilizers; ${}^+H$ denotes the subgroup generated by all vertex-stabilizers. If $\mathfrak{g} = (X, Y)$ is connected and locally finite, the group $\operatorname{Aut} \mathfrak{g} < \operatorname{Sym}(Y)$ is locally compact for the topology of pointwise convergence on Y.

Let d denote the combinatorial distance on X, $n \geqslant 1$ and $x_1, \ldots, x_k \in X$;

$$H_n(x_1, \ldots, x_k) = \left\{ g \in H : \begin{array}{l} g \text{ is the identity on the subgraph} \\ \text{spanned by all vertices } y \in X \\ \text{with } d(y, \{x_1, \ldots, x_k\}) \leqslant n \end{array} \right\}$$

and for $x \in X$, we set

$$\underline{H}_n(x) = H_n(x)/H_{n+1}(x).$$

For $x, y \in X$ adjacent vertices, set $H(x, y) := H(x) \cap H(y)$.

0.3. For a totally disconnected group H, we define $H^{(\infty)} := \bigcap_{L<H} L$, where the intersection is taken over all closed subgroups $L < H$ of finite index and $QZ(H) = \{h \in H : Z_H(h) \text{ is open}\}$. Let T be a locally finite tree and $H < \operatorname{Aut} T$ a closed subgroup; assume that H is locally quasiprimitive: it follows from [B-M]$_3$, Proposition 1.2.1, that $H/H^{(\infty)}$ is compact, that $QZ(H) \lhd H$ is discrete and that for any closed normal subgroup $N \lhd H$, one has either $N \supset H^{(\infty)}$ or $N \subset QZ(H)$. Let $H^{(\infty)} < G < H$ be a closed subgroup; then $G/\overline{[G, G]}$ is compact and for any open normal subgroup $N \lhd G$, one has $N \supset H^{(\infty)}$ (see [B-M]$_3$ Corollary 1.2.2).

0.4. (See 3.1 in [B-M]$_3$.) Let $T = (X, Y)$ be a locally finite tree. For a closed subgroup $H < \operatorname{Aut} T$, the following properties are equivalent: (1) H is locally ∞-transitive, (2) $H(x)$ is transitive on $T(\infty)$ for all $x \in X$, (3) H is non-compact and transitive on $T(\infty)$, (4) H is 2-transitive on $T(\infty)$. Any of the preceeding properties imply, (5) $\underline{H}(x) < \mathrm{Sym}\, E(x)$ is 2-transitive and H is non-discrete.

Finally we mention ([B-M]$_3$ Proposition 3.1.2) that if $H < \mathrm{Aut}\, T$ is closed and locally ∞-transitive then,

(1) $QZ(H) = e$,

(2) $H^{(\infty)}$ is locally ∞-transitive and topologically simple.

1. Square complexes and lattices

A convenient and powerful way of describing and studying a group acting on a tree is via the associated graph of groups. Similarly one can associate with a group acting on a product of trees a complex of groups (see [Ha]). In general one has to restrict to groups satisfying certain technical conditions such as the "no inversions" in the case of an action on a single tree. We shall restrict ourselves here to the case of free actions. Thus we will be able to construct groups acting freely on a product of trees as the fundamental groups of certain two-dimensional cell complexes. To describe these we set the following notations: Circ_4 is the circuit of length 4, that is the graph with vertex set $\{1, 2, 3, 4\}$ and edge set $\{ [i,j] : i-j = \pm 1 \bmod 4,\ 1 \leqslant i,j \leqslant 4\}$. We consider the dihedral group D_4 as subgroup of the symmetric group S_4; the group D_4 acts then by automorphisms on Circ_4 and, for any graph $\mathfrak{g}$, the group D_4 acts on the set $\mathrm{Mor}(\mathrm{Circ}_4, \mathfrak{g})$, of graph morphisms $\mathrm{Circ}_4 \to \mathfrak{g}$. Recall that the set of edges E of a graph $\mathfrak{g} = (V, E)$ comes equipped with a free action of the dihedral group D_2, denoted $y \to \bar{y}, y \in E$.

A square complex X is given by a graph $X^{(1)} = (V, E)$, a set S with a free action of D_4 and a map $\partial : S \to \mathrm{Mor}(\mathrm{Circ}_4, X^{(1)})$ which is equivariant w.r.t. the actions of D_4 on source and target. We sometimes denote V by $X^{(0)}$; the sets $E, S, D_2\backslash E, D_4\backslash S$ are respectively the sets of: edges, squares, geometric edges and geometric squares. For later use we let σ denote the fixed point free involution on S given by the action of the transposition $(2, 4) \in D_4$.

The link $\mathrm{Lk}(x)$ of a vertex $x \in X^{(0)}$ is a graph with vertex set $E(x)$ and edge set $S_x = \{s \in S : \partial s(1) = x\}$; origin and terminus maps are given by $o_x(s) = \partial s([1, 2])$, $t_x(s) = \partial s([1, 4])$ $s \in S_x$ and the fixed point free involution on S_x is given by the restriction $\sigma | S_x$. A morphism $F : X_1 \to X_2$ of square complexes $X_i = (V_i, E_i, S_i)$ is a pair (φ, ϕ), where $\varphi : X_1^{(1)} \to X_2^{(1)}$ is a morphism of graphs and $\phi : S_1 \to S_2$ is a map, such that $\partial \cdot \phi = \varphi^{(4)} \cdot \partial$, where $\varphi^{(4)} : \mathrm{Mor}(\mathrm{Circ}_4, X_1^{(1)}) \to \mathrm{Mor}(\mathrm{Circ}_4, X_2^{(1)})$ denotes the map induced by φ. There are obvious notions of composition of morphisms, of monomorphisms, epimorphisms and isomorphisms. A morphism $F : X_1 \to X_2$ of square complexes induces for every $x \in X_1^{(0)}$ a morphism of graphs $F_x : \mathrm{Lk}(x) \to \mathrm{Lk}(\varphi(x))$, $F = (\varphi, \phi)$. We say that a subgroup $G < \mathrm{Aut}\, X$ acts freely on X if the corresponding actions of G on $V, D_2\backslash E, D_4\backslash S$ are free: in this case there is a quotient square complex $Y = G\backslash X$ and a canonical morphism $P : X \to Y$, $P = (\pi, \prod)$, such that $P_x : \mathrm{Lk}(x) \to \mathrm{Lk}(\pi(x))$ is an isomorphism for every $x \in X^{(0)}$.

Example. — Product of two graphs: given graphs $\mathfrak{G}_i = (V_i, E_i)$, we define the square complex $X = \mathfrak{G}_1 \times \mathfrak{G}_2$, $X = (V, E, S)$ as follows:

$$V = V_1 \times V_2,\ E = (V_1 \times E_2) \sqcup (E_1 \times V_2),\ S = (E_1 \times E_2) \sqcup (E_2 \times E_1);$$

the graph structure on $X^{(1)} = (V, E)$ is given by $o(v_1, e_2) = (v_1, o(e_2))$, $t(v_1, e_2) = (v_1, t(e_2))$, $o(e_1, v_2) = (o(e_1), v_2)$, $t(e_1, v_2) = (t(e_1), v_2)$, $\overline{(v_1, e_2)} = (v_1, \overline{e}_2)$, $\overline{(e_1, v_2)} = (\overline{e}_1, v_2)$, for $v_i \in V_i$, $e_i \in E_i$. The action of D_4 on S is uniquely defined by $\sigma(e_1, e_2) = (e_2, \overline{e}_1)$, where σ corresponds to $(2, 4)$ and $c(e_1, e_2) = (e_2, e_1)$, where c corresponds to $(1, 2, 3, 4) \in D_4$. Finally, $\partial(e_1, e_2)$, $\partial(e_2, e_1)$ are given by the sequence of consecutive edges

$$(e_1, o(e_2)),\ (t(e_1), e_2),\ (\overline{e}_1, t(e_2)),\ (o(e_1), \overline{e}_2),$$

$$\text{resp. } (t(e_1), e_2),\ (\overline{e}_1, t(e_2)),\ (o(e_1), \overline{e}_2),\ (e_1, o(e_2)).$$

Observe that in this case, $\mathrm{Lk}(x)$ is a complete bipartite graph, $\forall x \in X^{(0)}$.

Finally, we say that a square complex X is connected when $X^{(1)}$ is connected; if X is connected and $x \in X^{(0)}$, we have the notion of a combinatorial fundamental group $\pi_1(X, x)$ and X is the quotient of a simply connected square complex $\widetilde{X}$, its universal covering, by a free action of $\pi_1(X, x) < \mathrm{Aut}\,\widetilde{X}$. We record the following basic fact:

Proposition **1.1.** — *The universal covering* $\widetilde{X}$ *of a connected square complex* X *is a product of trees if and only if* $\mathrm{Lk}(x)$ *is a complete bipartite graph for all* $x \in X^{(0)}$.

Square complexes satisfying the link condition of Proposition 1.1 will be called T-complexes. We say that a T-complex $X = \Gamma\backslash(T_h \times T_v)$, where $\widetilde{X} = T_h \times T_v$, is VH, if no $\gamma \in \Gamma$ interchanges the factors of $\widetilde{X}$. Equivalently, there is a partition $E = E_h \sqcup E_v$ such that if $E_h(x) = E_h \cap E(x)$, $E_v(x) = E_v \cap E(x)$, $x \in X^{(0)}$, then $E_h(x)$, $E_v(x)$ gives at every vertex $x \in X^{(0)}$ the bipartite structure on $\mathrm{Lk}(x)$. Notice that every T-complex has a two-fold covering which is VH.

Let $X = \Gamma\backslash(T_h \times T_v)$ be a VH-T-complex and $E = E_h \sqcup E_v$; in $X^{(1)}$ we have two subgraphs, the horizontal $X_h^{(1)} = (X^{(0)}, E_h)$ and the vertical $X_v^{(1)} = (X^{(0)}, E_v)$. For $x \in X^{(0)}$, let $x = \pi(x_h, x_v)$ where $\pi : T_h \times T_v \to X$ is the canonical projection, and define

$$\Gamma^{x_v} = \{\gamma = (\gamma_h, \gamma_v) : \gamma_v(x_v) = x_v\},$$

$$\Gamma_{x_h} = \{\gamma = (\gamma_h, \gamma_v) : \gamma_h(x_h) = x_h\}.$$

Then $\Gamma^{x_v}\backslash(T_h\times\{x_v\})$ is the connected component of x in $X_h^{(1)}$, while

$$\Gamma_{x_h}\backslash(\{x_h\}\times T_v)$$

is the connected component of x in $X_v^{(1)}$. Associating to every $\gamma\in\Gamma^{x_v}$ the projection in $X_h^{(1)}$ of the horizontal path connecting (x_h, x_v) to $(\gamma_h(x_h), x_v)$ defines an isomorphism

$$\Gamma^{x_v}\longrightarrow\pi_1\left(X_h^{(1)}, x\right);$$

one obtains in a similar way an isomorphism

$$\Gamma_{x_h}\longrightarrow\pi_1\left(X_v^{(1)}, x\right).$$

The group Γ^{x_v} (resp. Γ_{x_h}) induces for every $n\geqslant 1$ a finite permutation group of the sphere $S(x_v, n)$ (resp. $S(x_h, n)$) in T_v (resp. T_h); we give now a direct geometric description of these permutation groups:

Every $e\in E_h$, respectively $e\in E_v$, gives rise to a bijection

$$\varphi_e : E_v(o(e))\longrightarrow E_v(t(e)),\quad \text{resp.}$$

$$\varphi_e : E_h(o(e))\longrightarrow E_h(t(e)),$$

defined by $\varphi_e(e')=e''$ where $e''=\partial s([2, 3])$, and $s\in S$ in the unique square such that:

$$\partial s([1, 2])=e,\qquad \partial s([1, 4])=e';$$

and $\varphi_e(e')=e''$ where $e''=\partial s([4, 3])$, and $s\in S$ is the unique square such that:

$$\partial s([1, 4])=e,\qquad \partial s([1, 2])=e'.$$

More generally, let $E_v^{(n)}(x)$, respectively $E_h^{(n)}(x)$, denote the set of vertical, resp. horizontal paths, without backtracking, of length $n\geqslant 1$ and origin $x\in X^{(0)}$; in a similar way, every $e\in E_h(x)$, resp. $e\in E_v(x)$, gives rise to a bijection

$$\varphi_e^{(n)} : E_v^{(n)}(x)\longrightarrow E_v^{(n)}(t(e)),\quad \text{resp.}$$

$$\varphi_e^{(n)} : E_h^{(n)}(x)\longrightarrow E_h^{(n)}(t(e)).$$

Composing these bijections, we get a system of homomorphisms

$$m_{h,x}^{(n)} : \pi_1\left(X_h^{(1)}, x\right)\longrightarrow \operatorname{Sym} E_v^{(n)}(x)$$

$$m_{v,x}^{(n)} : \pi_1\left(X_v^{(1)}, x\right)\longrightarrow \operatorname{Sym} E_h^{(n)}(x)$$

which is compatible with respect to the canonical projection maps $E_v^{(n)}(x)\to E_v^{(n-1)}(x)$, resp. $E_h^{(n)}(x)\to E_h^{(n-1)}(x)$.

Returning to the description of X as a quotient $\Gamma\backslash(T_h \times T_v)$, one obtains natural bijections,

$$S(x_v, n) \longrightarrow E_v^{(n)}(x), \quad \text{resp.}$$
$$S(x_h, n) \longrightarrow E_h^{(n)}(x)$$

which are equivariant w.r.t. the isomorphism

$$\Gamma^{x_v} \longrightarrow \pi_1\left(X_h^{(1)}, x\right), \quad \text{resp.}$$

$$\Gamma_{x_h} \longrightarrow \pi_1\left(X_v^{(1)}, x\right).$$

Let $P_v^{(n)}(x)$, resp. $P_h^{(n)}(x)$, denote the image of $m_{h,x}^{(n)}$, resp. $m_{v,x}^{(n)}$; for $n=1$ we will write $P_v(x)$, $P_h(x)$; let $H^{(v)}$, resp. $H^{(h)}$ denote the closure of the projection of Γ into $\operatorname{Aut} T_v$, resp. $\operatorname{Aut} T_h$. The above description implies that the groups $P_v^{(n)}(x) < \operatorname{Sym} E_v^{(n)}(x)$ and $H^{(v)}(x_v)/H_n^{(v)}(x_v) < \operatorname{Sym} S(x_v, n)$ are permutation isomorphic, and that the same holds for

$$P_h^{(n)}(x) < \operatorname{Sym} E_h^{(n)}(x) \text{ and } H^{(h)}(x_h)/H_n^{(h)}(x_h) < \operatorname{Sym} S(x_h, n).$$

In particular, $P_v(x) < \operatorname{Sym} E_v(x)$ is permutation isomorphic to $\underline{H}^{(v)}(x_v) < \operatorname{Sym} E(x_v)$, and $P_h(x) < \operatorname{Sym} E_h(x)$ is permutation isomorphic to $\underline{H}^{(h)}(x) < \operatorname{Sym} E(x_n)$. This simple but fundamental observation enables us to control the "local action" of $H^{(v)}$ and $H^{(h)}$ in terms of the complex X.

We turn now to the notion of reducible lattice. Let T_1, T_2 be locally finite trees.

Definition. — *A lattice* $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ *is reducible, if it is commensurable to a product* $\Gamma_1 \times \Gamma_2$ *of lattices* $\Gamma_i < \operatorname{Aut} T_i$. *The lattice* Γ *is called irreducible, if it is not reducible.*

Proposition **1.2.** — *For a cocompact lattice* $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$, *the following properties are equivalent:*

(a) *There exists* $i \in \{1, 2\}$ *such that* $pr_i(\Gamma) < \operatorname{Aut} T_i$ *is discrete.*

(b) $\Gamma_1 = \{\gamma \in \operatorname{Aut} T_1 : (\gamma, e) \in \Gamma\}$, *resp.* $\Gamma_2 = \{\eta \in \operatorname{Aut} T_2 : (e, \eta) \in \Gamma\}$ *are lattices in* $\operatorname{Aut} T_1$, *resp.* $\operatorname{Aut} T_2$ *and* $\Gamma_1 \times \Gamma_2$ *is of finite index in* Γ.

Proof. — Assume that $pr_1(\Gamma) < \operatorname{Aut} T_1$ is discrete; then $\Gamma \cdot \operatorname{Aut} T_2$ is closed and hence $\Gamma \cap \operatorname{Aut} T_2$ is a lattice in $\operatorname{Aut} T_2$. The group $pr_2(\Gamma) < \operatorname{Aut} \Gamma_2$ normalizes the cocompact lattice $\Gamma \cap \operatorname{Aut} T_2$ and hence ([B-M]$_3$, 1.3.6) is discrete. Thus $pr_2(\Gamma)$ is discrete, hence as above, $\Gamma \cap \operatorname{Aut} T_1$ is a lattice in $\operatorname{Aut} T_1$ which shows that (a) implies (b). The converse (b) $\Rightarrow$ (a) is obvious; notice that both projections are discrete. □

We say that a VH-T-complex $X = \Gamma\backslash(T_h \times T_v)$ is reducible, if $\Gamma < \operatorname{Aut} T_h \times \operatorname{Aut} T_v$ is reducible, otherwise it is called irreducible.

In geometric terms, the complex X is reducible if and only if X admits a finite covering which is a product of two graphs. However, we do not have an algorithm deciding if a given finite VH-T-complex is reducible, nor do we know whether such an algorithm exists. Nevertheless, we can, using Proposition 1.2 and the Thompson-Wielandt theorem, give a sufficient condition ensuring irreducibility. More precisely, let X be a VH-T-complex; for $e \in E_h$, resp. $e \in E_v$, $x = o(e)$, let $L_h(e) = \mathrm{Stab}_{P_h(x)}(e)$, resp. $L_v(e) = \mathrm{Stab}_{P_v(x)}(e)$; let $\mathscr{E}_h(e) \subset E_h^{(2)}(x)$, resp. $\mathscr{E}_v(e) \subset E_v^{(2)}(x)$, denote the set of horizontal, resp. vertical, paths of length two starting at e, and:

$$K_h(e) = \left\{ \tau \in P_h^{(2)}(x) : \tau\Big|_{\mathscr{E}_h(e) \cup E_h(x)} = id \right\},$$

$$K_v(e) = \left\{ \tau \in P_v^{(2)}(x) : \tau\Big|_{\mathscr{E}_v(e) \cup E_v(x)} = id \right\}.$$

When $X_h^{(1)}$ and $X_v^{(1)}$ are connected, then the permutation groups $P_v^{(n)}(x)$, $P_h^{(n)}(x)$, $L_v(x)$, $L_h(x)$, are independent of x, up to permutation isomorphism; in this case we omit the "x"; when P_v and P_h are transitive, $K_v(e)$ and $K_h(e)$ are independent of e, up to permutation isomorphism, and we omit the "e".

Proposition **1.3.** — *Let* X *be a finite* VH-T-*complex; we assume that* $X_v^{(1)}$, $X_h^{(1)}$ *are connected and that* P_v, P_h *are primitive permutation groups. If either* K_v *or* K_h *is not a p-group, then* X *is an irreducible* VH-T-*complex.*

Proof. — Let $X = \Gamma \backslash (T_h \times T_v)$; $H^{(h)}$, $H^{(v)}$ as above and x_h, y_h, resp. x_v, y_v, adjacent vertices in T_h, resp. T_v. The assumptions imply that $H^{(h)}$, $H^{(v)}$ are both vertex transitive and locally primitive. If Γ is reducible, then $H^{(h)}$, $H^{(v)}$ are discrete and hence by Thompson-Wielandt (see [Th],[Wi]$_2$, [B-C-N] or [B-M]$_3$ Chapt. 2), $H_1^{(h)}(x_h, y_h)$ and $H_1^{(v)}(x_v, y_v)$ are p-groups; since K_h, K_v are homomorphic images of the latter, they would also be p-groups, a contradiction. □

Observe that if $X = \Gamma \backslash (T_h \times T_v)$ is reducible, then Γ contains a subgroup of finite index which is a product of two free groups of finite rank; in particular the group Γ is linear over **C**. In this context we mention the following consequence of an arithmeticity result proved in [B-M-Z], which shows that under the hypothesis of Proposition 1.3, the group Γ is not linear over any field, and hence an irreducible lattice.

Theorem **1.4.** (See [B-M-Z].) — *Let* X *be a finite* VH-T-*complex; we assume that* $X_v^{(1)}$, $X_h^{(1)}$ *are connected and that* P_v, P_h *are primitive permutation groups. If either* K_v *or* K_h *is not a p-group then, over any field, any finite dimensional linear representation of* $\pi_1(X)$ *has finite image.*

2. A Criterium for non-residual finiteness

2.1. A basic question concerning the structure of a group is whether it is residually finite. We shall show that certain lattices in $\mathrm{Aut}\,T_1 \times \mathrm{Aut}\,T_2$, with locally quasiprimitive projections, are not residually finite; the results from [B-M]$_3$ on the structure of locally quasiprimitive groups needed here and in the sequel are recalled in 0.3. The basis for constructing these examples is given by the following

Proposition **2.1.** — *Let* $H_i < \mathrm{Aut}\,T_i$ *be closed, non discrete and locally quasiprimitive; let* $\Gamma < H_1 \times H_2$ *be a cocompact lattice with* $H_i^{(\infty)} \subset \overline{pr_i(\Gamma)} \subset H_i$. *Then,*

$$\Gamma^{(\infty)} \supset \left[H_1^{(\infty)}, \Lambda_1\right] \cdot \left[H_2^{(\infty)}, \Lambda_2\right],$$

where $\Lambda_i = \Gamma \cap H_i$. *In particular, if* $\Lambda_1 \cdot \Lambda_2 \neq e$, *then* Γ *is not residually finite.*

Proof. — Let $\Gamma' \lhd \Gamma$ be a normal subgroup of finite index in Γ, then $\Lambda_1' = H_1 \cap \Gamma'$ is of finite index in Λ_1 and both Λ_1, Λ_1' are normal in $\overline{pr_1(\Gamma)} > H_1^{(\infty)}$; since $H_1^{(\infty)}$ has no proper open normal subgroups ([B-M]$_3$ Prop. 1.2.1), the action by conjugation of $H_1^{(\infty)}$ on Λ_1/Λ_1' is trivial and thus $\Lambda_1' \supset \left[H_1^{(\infty)}, \Lambda_1\right]$. Since $H_1^{(\infty)} \backslash T_1$ is finite, $\mathscr{Z}_{\mathrm{Aut}\,T_1}\left(H_1^{(\infty)}\right) = e$ and therefore $\left[H_1^{(\infty)}, \Lambda_1\right] \neq e$ provided $\Lambda_1 \neq e$. □

With the notations of the above proposition, we have the inclusion $\Lambda_i \subset QZ(H_i)$; the next result gives information about the size of $QZ(H_i)/\Lambda_i$.

Proposition **2.2.** — *Let* $H_i < \mathrm{Aut}\,T_i$ *be non-discrete, closed, locally quasiprimitive and* $\Gamma < H_1 \times H_2$ *a cocompact lattice with* $\overline{pr_i(\Gamma)} = H_i$; *let* $\Lambda_i = \Gamma \cap H_i$. *Then, the group* $QZ(H_i)/\Lambda_i$ *is locally finite, meaning that every finitely generated subgroup is finite.*
In particular, $QZ(H_i) \neq e$ *iff* $\Lambda_i \neq e$.

Proof. — Let $S \subset QZ(H_1)$ be a finite subset and $U < H_1$ be an open compact subgroup commuting with S. Then, the intersection $\Gamma \cap (U \times H_2)$ is a cocompact lattice in $U \times H_2$, hence finitely generated. Let $\Delta \subset \Gamma \cap (U \times H_2)$ be a finite generating set; then the centralizer $\mathscr{Z}_\Gamma(\Delta)$ of Δ in Γ is a cocompact lattice in $\mathscr{Z}_{H_1}(\Delta) \times H_2$; since $pr_2(\Delta)$ generates a cocompact lattice in H_2 and $pr_1(\Delta)$ generates a dense subgroup of U, we obtain $\mathscr{Z}_{H_1 \times H_2}(\Delta) = \mathscr{Z}_{H_1}(U) \times (e) \subset QZ(H_1) \times (e)$. Thus, $\mathscr{Z}_{H_1}(U) \times (e)$ is discrete, and $\mathscr{Z}_{H_1}(U) \times (e)/\mathscr{Z}_\Gamma(\Delta)$ is finite. Since $S \subset \mathscr{Z}_{H_1}(U)$ and $\mathscr{Z}_\Gamma(\Delta) \subset \Lambda_1 \times (e)$, it follows that S generates a finite subgroup of $QZ(H_1)/\Lambda_1$. □

Question. — Is $QZ(H_i)/\Lambda_i$ a finite group?

Corollary **2.3.** — *Let* $H_i < \mathrm{Aut}\,T_i$ *be closed, non-discrete, locally primitive groups; assume that* $QZ(H_i) \neq (e)$, *for some* $i = 1, 2$. *Then any cocompact lattice* $\Gamma < H_1 \times H_2$ *with* $\overline{pr_i(\Gamma)} = H_i$ *is not residually finite.*

2.2. We describe now how combining Proposition 2.1 with [B-M]$_3$ Section 1.8 leads to examples of non residually finite lattices.

Let E denote the tree consisting of a single edge, fix morphisms $\varphi_i : T_i \to E$ and let $\eta_i : \mathrm{Aut}\, T_i \to \mathrm{Aut}\, E (\simeq \mathbf{Z}_2)$ be the induced homomorphisms. Let $\mathscr{D}_i = T_i \times_{\varphi_i} T_i$ be the graph of diagonals, that is, the fiber product of T_i with itself relative to $\varphi_i : T_i \to E$ (see [B-M]$_3$ Section 1.6 and 1.8). Recall that $\mathrm{Aut}\, T_i \times_{\eta_i} \mathrm{Aut}\, T_i < \mathrm{Aut}\, \mathscr{D}_i$.

Let $\Gamma < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ be a cocompact, torsion free lattice; let $H_i := \overline{pr_i(\Gamma)}$, define $\psi : \Gamma \times \Gamma \to (H_1 \times H_1) \times (H_2 \times H_2)$ by $\psi((\gamma_1, \gamma_2), (\gamma'_1, \gamma'_2)) = ((\gamma_1, \gamma'_1), (\gamma_2, \gamma'_2))$ and set

$$\Lambda := \psi(\Gamma \times \Gamma) \cap \left[(H_1 \times_{\eta_1} H_1) \times (H_2 \times_{\eta_2} H_2) \right].$$

Then Λ is of index $1, 2$ or 4 in $\psi(\Gamma \times \Gamma)$, it acts as a group of covering transformations on $\mathscr{D}_1 \times \mathscr{D}_2$, and gives rise to a finite VH-T-square complex:

$$Y := \Lambda \backslash (\mathscr{D}_1 \times \mathscr{D}_2).$$

We have the exact sequence

$$1 \longrightarrow \pi_1(\mathscr{D}_1) \times \pi_1(\mathscr{D}_2) \longrightarrow \pi_1(Y) \longrightarrow \Gamma \times \Gamma$$

where the last arrow has image of index $1, 2$ or 4 in $\Gamma \times \Gamma$.

Proposition **2.4.** — *Assume that* H_i *is non-discrete and that* $H_i^{(\infty)}$ *is locally* 2*-transitive,* $i = 1, 2$. *Then we have:*

$$\pi_1(Y)^{(\infty)} \supset \pi_1(\mathscr{D}_1) \times \pi_1(\mathscr{D}_2)$$

and the quotient is isomorphic to $\Gamma^{(\infty)} \times \Gamma^{(\infty)}$.

Proof. — Set $L_i = \left(H_i \times_{\eta_i} H_i \right) \rtimes \langle \tau_i \rangle$, where $\tau_i \in \mathrm{Aut}\left(T_i \times_{\varphi_i} T_i \right)$ is the automorphism exchanging the factors, let $\widetilde{\mathscr{D}}_i$ be the universal covering tree of $\mathscr{D}_i$ and

$$1 \longrightarrow \pi_1(\mathscr{D}_i) \longrightarrow G_i \xrightarrow{\omega_i} L_i \longrightarrow 1$$

be the associated exact sequence. Let $\omega = \omega_1 \times \omega_2 : G_1 \times G_2 \to L_1 \times L_2$ and

$$\Delta := \pi_1(Y) = \omega^{-1}(\Lambda) > \pi_1(\mathscr{D}_1) \times \pi_1(\mathscr{D}_2).$$

We claim that $\overline{pr_i(\Delta)} \supset G_i^{(\infty)}$; considering Λ as a subgroup of $L_1 \times L_2$, we have $\overline{pr_i(\Delta)} = \omega_i^{-1}\left(\overline{pr_{L_i}(\Lambda)} \right)$ since Δ contains $\mathrm{Ker}\, \omega_1 \times \mathrm{Ker}\, \omega_2$; the group $pr_{L_i}(\Lambda) = pr_{H_i \times H_i}(\Lambda)$ is normal of finite index in $pr_{H_i \times H_i}(\psi(\Gamma \times \Gamma))$ and the latter subgroup being dense in $H_i \times H_i$, we conclude that $\overline{pr_{L_i}(\Lambda)}$ is normal of finite index in $H_i \times_{\mu_i} H_i$. Observe now that τ_i normalizes $pr_{L_i}(\Lambda)$, hence $\overline{pr_{L_i}(\Lambda)}$ is normal of finite index in L_i, which implies the same property for $\overline{pr_i(\Delta)}$ in G_i and proves the claim. Using that G_i is locally primitive non-discrete (Prop. 1.8.1, [B-M]$_3$ and Prop. 2.1), we conclude

$$\Delta^{(\infty)} \supset \left[G_1^{(\infty)}, \pi_1(\mathscr{D}_1)\right] \times \left[G_2^{(\infty)}, \pi_1(\mathscr{D}_2)\right] = \pi_1(\mathscr{D}_1) \times \pi_1(\mathscr{D}_2) = \operatorname{Ker}\omega.$$

Since $\Delta^{(\infty)}$ contains the kernel of ω, we deduce that $\Delta^{(\infty)}/\operatorname{Ker}\omega$ equals $\Lambda^{(\infty)}$, which in turn equals $\psi(\Gamma \times \Gamma)^{(\infty)} \simeq \Gamma^{(\infty)} \times \Gamma^{(\infty)}$ since Λ is of finite index in $\psi(\Gamma \times \Gamma)$. □

2.3. We briefly describe a way of constructing the complex Y of 2.2 in terms of a fiber product of square complexes. Thus, let $X = \Gamma\backslash(T_1 \times T_2)$ be a finite VH-T-complex, $\varphi_i : T_i \to E$, $\eta_i : \operatorname{Aut} T_i \to \operatorname{Aut} E$ the morphisms defined in 2.2, and let $S := E \times E$ be the square complex consisting of 1 geometric square and 4 vertices. Let

$$\varphi_1 \times \varphi_2 : T_1 \times T_2 \longrightarrow S$$

$$\eta_1 \times \eta_2 : \operatorname{Aut} T_1 \times \operatorname{Aut} T_2 \longrightarrow \operatorname{Aut} S$$

be the corresponding product morphisms; let $\Gamma_0 := \Gamma \cap \operatorname{Ker}(\eta_1 \times \eta_2)$. Then,

$$\widehat{X} := \Gamma_0\backslash(T_1 \times T_2)$$

is a finite covering $p : \widehat{X} \to X$ with Galois group $G := \Gamma/\Gamma_0$ and the above product morphisms induce morphisms

$$\varphi : \widehat{X} \longrightarrow S$$

$$\eta : G \longrightarrow \operatorname{Aut} S,$$

where φ is equivariant w.r.t. η. The fiber product $\widehat{Y} := \widehat{X} \times_\eta \widehat{X}$ comes equipped with an action of $G \times_\eta G$, and the complex Y in 2.2 is then isomorphic to the quotient $(G \times_\eta G)\backslash\widehat{Y}$. The projection maps p_1, p_2 of $\widehat{Y}$ on both factors, composed with the covering map $p : \widehat{X} \to X$, give rise to homomorphisms

$$h_i : \pi_1(Y) \longrightarrow \pi_1(X), \quad i = 1, 2.$$

Using the isomorphism $Y \xrightarrow{\sim} (G \times_\eta G)\backslash\widehat{Y}$, whose explicit construction is left to the reader, one gets $\operatorname{Ker} h_i = \pi_1(\mathscr{D}_i)$. We denote by $X \boxtimes X = Y$ the complex constructed in this way and let

$$h : \pi_1(X \boxtimes X) \longrightarrow \pi_1(X) \times \pi_1(X)$$

denote the product homomorphism $h_1 \times h_2$. We now obtain the following

Corollary **2.5.** — *Let* X *be a finite, irreducible VH-T-complex. Assume that* $X_h^{(1)}, X_v^{(1)}$ *are connected and that* P_h, P_v *are* 2*-transitive permutation groups with* 2*-transitive socles. Let*

$$h : \pi_1(X \boxtimes X) \to \pi_1(X) \times \pi_1(X)$$

be the canonical homomorphism. Then, the image of h has index 2 *or* 4 *and,*

$$\pi_1(X \boxtimes X)^{(\infty)} \supset \mathrm{Ker}\, h.$$

Remark. — The socle of a finite group is the product of all its minimal normal subgroups. A celebrated theorem of Burnside (see [D-M], 4.1) states that a 2-transitive group has a unique minimal normal subgroup which is either elementary abelian regular, or primitive and simple. Furthermore, in the case of non-abelian socle, it follows from the classification of 2-transitive groups that the socle is 2-transitive, except in one case (see [Ca] p. 624).

2.4. An explicit example of such lattices may be constructed as follows. Let $p \neq q$ be odd primes, both congruent to 1 mod 4, with $(\frac{p}{q}) = (\frac{q}{p}) = 1$, $\mathrm{H}(\mathbf{Q})$ the usual Hamilton quaternion algebra over $\mathbf{Q}$ with basis $1, i, j, k$, and

$$Q := \{x \in \mathrm{H}(\mathbf{Z}) : \mathrm{N}(x) = p^a q^b,\ a,\ b \in \mathbf{N},\ x \equiv 1 (\mathrm{mod}\, 2)\}.$$

Fix $\varepsilon_p \in \mathbf{Q}_p$, $\varepsilon_q \in \mathbf{Q}_q$ with $\varepsilon_p^2 = -1$, $\varepsilon_q^2 = -1$, and let $\Gamma_{p,q}$ be the image of the map

$$\varphi : Q \longrightarrow \mathrm{PGL}(2, \mathbf{Q}_p) \times \mathrm{PGL}(2, \mathbf{Q}_q)$$
$$x \longrightarrow \left(\begin{pmatrix} x_0 + x_1\varepsilon_p, & x_2 + x_3\varepsilon_p \\ -x_2 + x_3\varepsilon_p, & x_0 - x_1\varepsilon_p \end{pmatrix}, \begin{pmatrix} x_0 + x_1\varepsilon_q, & x_2 + x_3\varepsilon_q \\ -x_2 + x_3\varepsilon_q, & x_0 - x_1\varepsilon_q \end{pmatrix} \right).$$

Then $\Gamma_{p,q} < \mathrm{Aut}\,\mathcal{T}_{p+1} \times \mathrm{Aut}\,\mathcal{T}_{q+1}$ is a torsion free cocompact lattice, acting simply transitively on the set of vertices of $\mathcal{T}_{p+1} \times \mathcal{T}_{q+1}$ and satisfying $\overline{pr_1(\Gamma_{p,q})} = \mathrm{H}_p$, $\overline{pr_2(\Gamma_{p,q})} = \mathrm{H}_q$, where for a prime ℓ we define (see also [B-M]$_3$, 1.8)

$$\mathrm{H}_\ell = \left\{ g \in \mathrm{PSL}(2, \mathbf{Q}_\ell) : \frac{\det g}{|\det g|} \in (\mathbf{Z}_\ell^*)^2 \right\}.$$

The density of projections follows easily from the fact that $\Gamma_{p,q}$ is an irreducible lattice and that H_ℓ contains the index 2 subgroup $\mathrm{H}_\ell^+ = \mathrm{PSL}(2, \mathbf{Q}_\ell)$ which is simple. Via the homomorphism

$$\psi : \Gamma_{p,q} \times \Gamma_{p,q} \longrightarrow (\mathrm{H}_p \times \mathrm{H}_p) \times (\mathrm{H}_q \times \mathrm{H}_q)$$
$$((\gamma_1, \gamma_2), (\gamma_3, \gamma_4)) \longrightarrow (\gamma_1, \gamma_3, \gamma_2, \gamma_4)$$

the group $\Gamma_{p,q} \times \Gamma_{p,q}$ acts simply transitively on the set of vertices of

$$\mathcal{T}_{p+1} \times \mathcal{T}_{p+1} \times \mathcal{T}_{q+1} \times \mathcal{T}_{q+1}$$

and the subgroup $\Lambda_{p,q} < \psi(\Gamma_{p,q}\times\Gamma_{p,q})$ acts simply transitively on the set of vertices of $\mathscr{D}_p\times\mathscr{D}_q$. The quotient

$$\mathscr{D}_{p,q} := \Lambda_{p,q}\backslash(\mathscr{D}_p\times\mathscr{D}_q)$$

is a square complex with one vertex. As $\Gamma_{p,q}$ is residually finite it follows from Corollary 2.5 and Proposition 2.4 that:

$$\pi_1(\mathscr{D}_{p,q})^{(\infty)} = \pi_1(\mathscr{D}_p)\times\pi_1(\mathscr{D}_q).$$

Let us denote the one vertex VH-T-square complex corresponding to the arithmetic lattice $\Gamma_{p,q} < \mathrm{PGL}(2, \mathbf{Q}_p)\times\mathrm{PGL}(2, \mathbf{Q}_q) < \mathrm{Aut}\,\mathscr{T}_{p+1}\times\mathrm{Aut}\,\mathscr{T}_{q+1}$ by $\mathscr{A}_{p,q} = \Gamma_{p,q}\backslash \mathscr{T}_{p+1}\times\mathscr{T}_{q+1}$. Both complexes $\mathscr{A}_{p,q}$ and $\mathscr{D}_{p,q}$ will be used later for constructing various other square complexes and lattices.

3. Cohomological properties of lattices

In this section we turn to certain cohomological properties of fundamental groups of finite T-square complexes. The main results are Proposition 3.1 and 3.2 below; Proposition 3.1 enters in an essential way in the proof of the normal subgroup theorem (see §4) while Proposition 3.2 will imply certain cohomological vanishing results for irreducible lattices with locally quasiprimitive projections.

Proposition **3.1.** — *Let* T_1, T_2 *be locally finite trees,* $\Gamma < \mathrm{Aut}\,T_1\times\mathrm{Aut}\,T_2$ *a discrete subgroup such that* $\Gamma\backslash(T_1\times T_2)$ *is finite and* $N \lhd \Gamma$ *a normal subgroup such that the quotient graphs* $pr_i(N)\backslash T_i$, $i = 1, 2$, *are finite trees. Then* Γ/N *has property* (T).

Proposition **3.2.** — *Let* T_1, T_2, Γ *be as in Proposition 3.1 and* $H_i := \overline{pr_i(\Gamma)} < \mathrm{Aut}\,T_i$.

(a) *The homomorphism* $\mathrm{Hom}_c(H_1\times H_2, \mathbf{C}) \to \mathrm{Hom}(\Gamma, \mathbf{C})$ *mapping* χ *to* $\chi|_\Gamma$ *is an isomorphism.*
(b) *Let* (π, V) *be an irreducible finite dimensional unitary representation of* Γ *with* $H^1(\Gamma, \pi) \neq 0$. *Then* π *extends continuously to* $H_1\times H_2$, *factoring via one of the projections.*

The following result is an application of Proposition 3.2 to irreducible lattices with locally quasiprimitive projections:

Corollary **3.3.** — *Let* T_1, T_2 *be locally finite trees,* $H_i < \mathrm{Aut}\,T_i$ *closed non-discrete locally quasiprimitive subgroups and*

$$\Gamma < \mathrm{Aut}\,T_1\times\mathrm{Aut}\,T_2$$

a cocompact lattice with $H_i^{(\infty)} \subset \overline{pr_i(\Gamma)} \subset H_i$.

(1) $\mathrm{Hom}(\Gamma, \mathbf{C}) = 0$.

(2) $\mathrm{H}^1(\Gamma, \pi) = (0)$, *for every unitary, finite dimensional representation* (π, V) *of* Γ.

Proof. — Set $\mathrm{G}_i = \overline{pr_i(\Gamma)}$.

(1) Follows from Proposition 3.2 (a) and the fact (see [B-M]$_3$ Corollary 1.2.2) that $\mathrm{G}_i/\overline{[\mathrm{G}_i, \mathrm{G}_i]}$ is compact.

(2) We distinguish two cases:

(a) π extends continuously to $\mathrm{G}_1 \times \mathrm{G}_2$, factoring via one of the projections. W.l.o.g., let $\omega : \mathrm{G}_1 \to \mathrm{U(V)}$ be a continuous unitary representation with $\pi(\gamma_1, \gamma_2) = \omega(\gamma_1)$, $\gamma = (\gamma_1, \gamma_2) \in \Gamma$. Since G_1 has small subgroups and $\mathrm{U(V)}$ is a real Lie group, $\mathrm{Ker}\,\omega$ is open in G_1 and since $\mathrm{G}_1 \supset \mathrm{H}_1^{(\infty)}$, it follows (see [B-M]$_3$ Prop. 1.2.1.5)) that $\mathrm{Ker}\,\omega \supset \mathrm{H}_1^{(\infty)}$; thus $\mathrm{G}_1/\mathrm{Ker}\,\omega$ being compact and discrete, is finite. This implies that $\pi(\Gamma)$ is finite. For any subgroup $\Gamma' < \Gamma$ of finite index, there is $\Gamma'' \lhd \Gamma$ of finite index with $\Gamma'' \subset \Gamma'$. Then, $\overline{pr_i(\Gamma'')} \lhd \overline{pr_i(\Gamma)}$ is normal open and hence (Cor. 1.2.2.2. [B-M]$_3$), $\overline{pr_i(\Gamma'')}$ contains $\mathrm{H}_i^{(\infty)}$, in particular $\overline{pr_i(\Gamma')} \supset \mathrm{H}_i^{(\infty)}$. Thus (1) applied to Γ' implies $\mathrm{Hom}(\Gamma', \mathbf{C}) = 0$, for any subgroup Γ' of finite index in Γ, which together with the finiteness of $\pi(\Gamma)$ implies $\mathrm{H}^1(\Gamma, \pi) = (0)$.

(b) π does not satisfy the extension property (a); it follows then from Proposition 3.2 (b) that $\mathrm{H}^1(\Gamma, \pi) = (0)$. □

The proofs of Proposition 3.1 and 3.2 rely on the study of a certain complex of cochains associated to a group acting properly on a square complex. More precisely, let $\mathrm{X} = (\mathrm{V}, \mathrm{E}, \mathrm{S})$ be a VH-complex, $\Gamma < \mathrm{Aut}\,\mathrm{X}$ a discrete subgroup preserving the VH-structure, acting in a clean way on X, and $\pi : \Gamma \to \mathrm{U}(\mathscr{H})$ a unitary representation of Γ into a Hilbert space $\mathscr{H}$. The assumption on the action of Γ means that, whenever an element $\gamma \in \Gamma$ fixes a geometric edge, resp. a geometric square (see §1 for definition), then it fixes all edges, resp. squares, in this equivalence class. Let Γ_x denote the stabilizer of $x \in \mathrm{V} \sqcup \mathrm{E} \sqcup \mathrm{S}$ and $n(x) = |\Gamma_x|$. Define for $\mathrm{Y} \in \{\mathrm{V}, \mathrm{E}, \mathrm{S}\}$:

$$\mathscr{L}_\mathrm{Y} = \{f : \mathrm{Y} \longrightarrow \mathscr{H} : f \text{ is } \Gamma\text{-equivariant and } \|f\|^2 := \sum_{\Gamma\backslash \mathrm{Y}} \frac{1}{n(y)} \|f(y)\|^2 < +\infty\}.$$

We obtain a complex $\mathscr{L}_\mathrm{V} \xrightarrow{d} \mathscr{L}_\mathrm{E} \xrightarrow{\mathrm{D}} \mathscr{L}_\mathrm{S}$ of Hilbert spaces, where the bounded operators d, D are given by:

$$df(e) := f(t(e)) - f(o(e)), \quad f \in \mathscr{L}_\mathrm{V}$$

$$\mathrm{DF}(\sigma) := \mathrm{F}(\partial\sigma[1, 2]) + \mathrm{F}(\partial\sigma[2, 3]) + \mathrm{F}(\partial\sigma[3, 4]) + \mathrm{F}(\partial\sigma[4, 1]), \quad \mathrm{F} \in \mathscr{L}_\mathrm{E}.$$

The space of 1-cocycles,

$$\mathscr{Z}^1(\mathscr{L}) = \{F \in \mathscr{L}_E : DF = 0,\ F(e) + F(\bar{e}) = 0, \quad \forall e \in E\}$$

is a closed subspace of $\mathscr{L}_E$; the orthogonal complement of $\operatorname{Im} d$ in $\mathscr{Z}^1(\mathscr{L})$ is $\operatorname{Ker}(\delta|_{\mathscr{Z}^1})$, where $\delta : \mathscr{L}_E \to \mathscr{L}_V$ is the adjoint of d and our first task is to establish a formula for $\|\delta F\|^2$, $F \in \mathscr{Z}^1(\mathscr{L})$, which takes into account the VH-structure and the link condition. To this end, let $\mathscr{L}_{E_h}$, resp. $\mathscr{L}_{E_v}$, be the subspace of maps $f \in \mathscr{L}_E$ having their support in E_h, resp. E_v. Let P_h, resp. P_v, be the orthogonal projection on $\mathscr{L}_{E_h}$, resp. $\mathscr{L}_{E_v}$; let $\delta_h = \delta P_h$, $\delta_v = \delta P_v$, $D_h = DP_h$, $D_v = DP_v$.

Proposition **3.4.** — *For all* $F \in \mathscr{Z}^1(\mathscr{L})$, *the following equality holds:*

$$\|\delta F\|^2 = \|\delta_h F\|^2 + \|\delta_v F\|^2 + \|D_h F\|^2 + \|D_v F\|^2.$$

Proof. — Let $F \in \mathscr{Z}^1(\mathscr{L})$; since $\delta = \delta_v + \delta_h$, we have to compute the following quantity:

$$\langle \delta_h F, \delta_v F \rangle = \sum_{x \in \Gamma\backslash V} \frac{1}{n(x)} \langle \delta_h F(x), \delta_v F(x) \rangle.$$

Since $F(e) + F(\bar{e}) = 0$, we have

$$\langle \delta_h F(x), \delta_v F(x) \rangle = 4 \left\langle \sum_{e \in E_h(x)} F(e), \sum_{e' \in E_v(x)} F(e') \right\rangle, \quad \text{for all } x \in V.$$

Taking into account the link condition, we obtain:

$$\langle \delta_h F, \delta_v F \rangle = \sum_{x \in \Gamma\backslash V} \frac{1}{n(x)} \sum_{\sigma \in S_x} u(\sigma),$$

where $S_x = \{\sigma \in S : \sigma(0) = x\}$ and $u(\sigma) = \langle F(\partial\sigma[1, 2]), F(\partial\sigma[1, 4]) \rangle$.
Using $\sum_{\sigma \in S_x} u(\sigma) = \sum_{\sigma \in \Gamma_x\backslash S_x} \frac{n(x)}{n(\sigma)} u(\sigma)$, we obtain,

$$\langle \delta_h F, \delta_v F \rangle = 4 \sum_{\sigma \in \Gamma\backslash S} \frac{1}{n(\sigma)} u(\sigma).$$

Let R be a set of representatives of the set of geometric squares in $\Gamma\backslash S$, then

$$4 \sum_{\sigma \in \Gamma\backslash S} \frac{1}{n(\sigma)} u(\sigma) = 4 \sum_{\sigma \in R} \frac{1}{n(\sigma)} \{u(\sigma) + u(\tau_h\sigma) + u(\tau_v\sigma) + u(\tau_h\tau_v\sigma)\}.$$

Using $DF = 0$ and $F(e) + F(\bar{e}) = 0$, $\forall e \in F$, a computation leads to:

$$u(\sigma) + u(\tau_h\sigma) + u(\tau_v\sigma) + u(\tau_h\tau_v\sigma) = \|D_hF(\sigma)\|^2 = \|D_vF(\sigma)\|^2,$$

and hence $\langle \delta_h F, \delta_v F\rangle = \|D_hF\|^2 = \|D_vF\|^2$, which proves the lemma. □

Remark. — P. Pansu obtained in [Pa] an equivalent formula by a different method.

Let now T_1, T_2, Γ, N be as in Proposition 3.1 and $\pi : \Gamma \to U(\mathscr{H})$ be a unitary representation of Γ with $\operatorname{Ker}\pi \supset N$; let $\mathscr{L}$ be the complex associated to the Γ action on $X = T_1 \times T_2$ and to π.

In this situation we have:

Lemma **3.5.** — *There is a constant $c > 0$, such that*

$$\|F\| \leqslant c\|\delta F\|, \quad \forall F \in \mathscr{Z}^1(\mathscr{L}).$$

Proof. — We recall the following elementary fact: let $S = (X, Y)$ be a finite tree, $i : Y \to \mathbf{N}^*$ an edge indexing and $\mathscr{H}$ a Hilbert space. For $w \in \mathscr{H}^Y$ define

$$\delta^S w(x) = \sum_{o(e) = x} i(e) w(e), \quad x \in X;$$

then there exists a constant $c > 0$, such that $\|w\| \leqslant c\|\delta^S w\|$, for all $w \in \mathscr{H}^Y$, satisfying $w(e) + w(\bar{e}) = 0$, $\forall e \in Y$.

Let now $T_i = (X_i, Y_i)$ and let $S_1 = (V_1, E_1)$ be a finite subtree of T_1 which is a strict fundamental domain for the action of $pr_1(N)$ on T_1. The set $E_{S_1}(v)$, $(v \in V_1)$ of edges in E_1 with origine v is a fundamental domain for the $pr_1(N)(v)$-action on $E(v)$ and we define an edge indexing $i_1 : E_1 \to \mathbf{N}^*$ by $i_1(e) = |pr_1(N)(v)e|$; δ^{S_1} denotes then the corresponding operator on $\mathscr{H}^{E_1}$. Let $W_2 \subset Y_2$ be a finite set of vertices such that $E_1 \times W_2$ surjects modulo Γ onto the set of horizontal edges of $\Gamma\backslash(T_1 \times T_2)$. According to the above general fact, there exists $c_1 > 0$ such that

$$\sum_{e \in E_1} \|F(e, w)\|^2 \leqslant c_1 \left\|\delta^{S_1} F(\cdot, w)\right\|^2, \quad \forall w \in Y_2, \forall F \in \mathscr{Z}^1(\mathscr{L}).$$

For every $e \in E_{S_1}(v_1)$, $v_1 \in V_1$, choose a finite set $F_e \subset N$, $|F_e| = i_1(e)$, such that $pr_1(F_e)e = pr_1(N)(v_1)(e)$. Then, we have for every $v_1 \in V_1$, $w \in W_2$:

$$\begin{aligned}\delta_h F(v_1, w) &= \sum_{e' \in E(v_1)} F(e', w) \\ &= \sum_{e \in E_{S_1}(v_1)} \sum_{(n_1, n_2) \in F_e} F(n_1 e, w)\end{aligned}$$

thus

$$\delta^{S_1}F(v_1, w) - \delta_h F(v_1, w) = \sum_{e \in E_{S_1}(v_1)} \sum_{(n_1, n_2) \in F_e} (F(e, w) - F(n_1 e, w))$$

which, by N-invariance of F, equals

$$= \sum_{e \in E_{S_1}(v_1)} \sum_{(n_1, n_2) \in F_e} (F(e, w) - F(e, n_2^{-1} w)).$$

Then we observe that

$$\|F(e, w) - F(e, n_2^{-1} w)\|^2 \leqslant C_2 \|D_h F\|^2, \tag{2}$$

where C_2 only depends on the distance $d(w, n_2 w)$.

From (1) and (2) we deduce

$$\sum_{e \in E_1, w \in W_2} \|F(e, w)\|^2 \leqslant C_3(\|\delta_h F\|^2 + \|D_h F\|^2)$$

where C_3 depends on N, Γ and the choice of W_2. The same argument applied to the $pr_2(N)$-action on T_2, together with Lemma 3.4 implies Lemma 3.5. □

Proof of Proposition 3.1. — We have to show that $H^1(\Gamma, \pi) = 0$ for any unitary representation π of Γ such that $\operatorname{Ker} \pi \supset N$. In the notation preceeding Lemma 3.5, we have $H^1(\Gamma, \pi) \simeq \mathscr{Z}^1(\mathscr{L})/\operatorname{Im} d$.

We may assume that π has no invariant vectors, thus $\operatorname{Ker} d = 0$. Then Lemma 3.5 implies that $\delta : \mathscr{Z}^1 \to \mathscr{L}_V$ is a continuous isomorphism and thus $d : \mathscr{L}_V \to \mathscr{Z}^1$ is an isomorphism which implies $H^1(\Gamma, \pi) = 0$. □

Proof of Proposition 3.2. — Let $(\pi, \mathscr{H})$ be a finite dimensional unitary representation of Γ and $\mathscr{L}$ the complex associated to the Γ-action on $T_1 \times T_2$ and π. Since $\dim \pi < +\infty$, $H^1(\Gamma, \pi)$ is isomorphic to $\operatorname{Ker} \delta|_{\mathscr{Z}^1(\mathscr{L})}$, which by Lemma 3.4 coincides with $\{F \in \mathscr{Z}^1(\mathscr{L}) : \delta_h F = \delta_v F = D_h F = D_v F = 0\}$.

a) Suppose $\pi = \mathrm{Id}$, $\mathscr{H} = \mathbf{C}$; let $\chi \in H^1(\Gamma, \pi)$ and $F \in \operatorname{Ker} \delta|_{\mathscr{Z}^1(\mathscr{L})}$ correspond to χ under the above isomorphism; let $f \in \mathbf{C}^{X_1 \times X_2}$, with $df = F$; then $\chi(\gamma) = f \circ \gamma - f$. Since $D_h F = D_v F = 0$, there are functions $F_i : Y_i \to \mathbf{C}$, with $F(e_1, v_2) = F_1(e_1)$, $F(v_1, e_2) = F_2(e_2)$, $\forall v_i \in X_i$, $e_i \in Y_i$. Since F is Γ-invariant, F_i is $pr_i(\Gamma)$-invariant, and hence $H_i = \overline{pr_i(\Gamma)}$-invariant. Fix a base point $b = (b_1, b_2) \in X_1 \times X_2$, and choose $f_i \in \mathbf{C}^{X_i}$ with $df_i = F_i$ and $f(b_1, b_2) = f_1(b_1) + f_2(b_2)$. Let $\chi_i : H_i \to \mathbf{C}$ be the continuous homomorphism defined by:

$$\chi_i(h_i) = f_i \circ h_i - f_i.$$

It follows then that $f(v_1, v_2) = f_1(v_1) + f_2(v_2)$, $\forall (v_1, v_2) \in X_1 \times X_2$, and hence $\chi((\gamma_1, \gamma_2)) = \chi_1(\gamma_1) + \chi_2(\gamma_2)$, $\forall (\gamma_1, \gamma_2) \in \Gamma$. This shows that the restriction map $\mathrm{Hom}_c(H_1 \times H_2, \mathbf{C}) \to \mathrm{Hom}(\Gamma, \mathbf{C})$ is surjective; on the other hand, since Γ is cocompact, this map is clearly injective.

b) Let π be an irreducible representation, with $H^1(\Gamma, \pi) \neq 0$ and take $F \in \mathrm{Ker}(\delta|_{\mathscr{Z}^1})$, $F \neq 0$. As above, there are maps $F_i : Y_i \to \mathscr{H}$, such that $F(e_1, v_2) = F_1(e_1)$, $F(v_1, e_2) = F_2(e_2)$; without loss of generality, we may assume that $F_1 \neq 0$. Using the Γ-equivariance of F and the fact that π is irreducible, finite dimensional, we may find $e_1, \ldots, e_n$ in Y, such that $F_1(e_1), \ldots, F_1(e_n)$ is a basis of $\mathscr{H}$. Observe now that $F_1(\gamma_1 e) = \pi(\gamma_1, \gamma_2)(F_1(e))$, $\forall (\gamma_1, \gamma_2) \in \Gamma$, $\forall e \in Y$. This implies that there is a representation $\omega : pr_1(\Gamma) \to U(\mathscr{H})$, such that $\pi(\gamma_1, \gamma_2) = w(\gamma_1)$, $\forall (\gamma_1, \gamma_2) \in \Gamma$. Let N be the normal open subgroup of H_1 generated by $H_1(e_1) \cap \ldots \cap H_1(e_n)$; then $pr_1(\Gamma) \cdot N = H_1$, and $\omega|_{pr_1(\Gamma) \cap N} = \mathrm{Id}_{\mathscr{H}}$; therefore ω extends continuously to H_1. □

4. The normal subgroup theorem

4.1. The main result of this section is the following analogue of Margulis' normal subgroup theorem.

Theorem **4.1.** — *Let* $\Gamma < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ *be a cocompact lattice such that* $H_i := \overline{pr_i(\Gamma)}$ *is locally* ∞*-transitive and* $H_i^{(\infty)}$ *is of finite index in* H_i. *Then, any non-trivial normal subgroup of* Γ *has finite index.*

Remark. — The results from $[\text{B-M}]_3$ on the structure of locally ∞-transitive groups needed in this chapter are recalled in 0.4.

4.1. *Various decompositions*

Let $T = (X, Y)$ be a locally finite tree and $H < \mathrm{Aut}\, T$ a closed, locally ∞-transitive subgroup. For $\xi \in T(\infty)$, let $P_\xi := \mathrm{Stab}_H(\xi)$; since $H(x)$ acts transitively on $T(\infty)$, we have

$$H = H(x) \cdot P_\xi \qquad (\mathbf{4.1})$$

For $\xi \in T(\infty)$, let $\beta_\xi : X \times X \to \mathbf{Z}$ denote the Busemann cocycle, that is $\beta_\xi(x, y) := \lim_{p \to \xi}(d(x, p) - d(y, p))$. The map

$$\begin{aligned} \chi_\xi : P_\xi &\longrightarrow \mathbf{Z} \\ g &\longmapsto \beta_\xi(x, gx) \end{aligned}$$

is independent of $x \in X$, and defines a continuous homomorphism. For $g \in \operatorname{Aut} T$, let $\ell(g) := \min_{x \in X} d(x, gx)$; an element $a \in P_\xi$ is hyperbolic iff $\chi_\xi(a) \neq 0$, in which case $\chi_\xi(a) = \ell(a)$ if ξ is the attracting fixed point of a and $\chi_\xi(a) = -\ell(a)$ if ξ is the repelling fixed point of a. Let $\chi_\xi(P_\xi) = \ell_\xi \mathbf{Z}$, with $\ell_\xi \geqslant 1$, $P_\xi^0 = \operatorname{Ker} \chi_\xi$ and $A = \langle a \rangle$, where $|\chi_\xi(a)| = \ell_\xi$, then we have the semidirect product decomposition:

$$P_\xi = A \cdot P_\xi^0. \tag{4.2}$$

Since H is 2-transitive on $T(\infty)$ ([B-M]₃ Lemma 3.1.1), the group P_ξ^0 acts transitively on $T(\infty) \backslash \{\xi\}$ and thus

$$H = P_\xi \sqcup P_\xi^0 \sigma P_\xi, \quad \text{for any } \sigma \notin P_\xi. \tag{4.3}$$

Let $a \in P_\xi$ be a hyperbolic element with $\ell := \chi_\xi(a) > 0$ and $r : \mathbf{Z} \to X$ a parametrization of its axis, with $r(+\infty) = \xi$. Then:

$$\left.\begin{array}{ll} P_\xi^0(r(k)) \subset P_\xi^0(r(k+1)), & \forall k \in \mathbf{Z} \\ a P_\xi^0(r(k)) a^{-1} = P_\xi^0(r(k+\ell)), & \forall k \in \mathbf{Z} \\ \bigcup_{k \geqslant j} P_\xi^0(r(k)) = P_\xi^0, & \forall j \in \mathbf{Z} \end{array}\right\}. \tag{4.4}$$

Since H is locally ∞-transitive, ℓ_ξ does not depend on $\xi \in T(\infty)$; for the common value ℓ_H we have $\ell_H = 1$ iff H is vertex transitive and $\ell_H = 2$, otherwise. For $a \in H$ a hyperbolic element with $\ell(a) = \ell_H$ and $x, y \in X$ adjacent vertices on the axis of a we have

$$\begin{array}{ll} H = H(x) A^+ H(x) & \text{if } \ell_H = 1 \\ H = H(x) A^+ [H(x) \cup H(y)] & \text{if } \ell_H = 2 \end{array} \tag{4.5}$$

where $A^+ = \{a^n : n \geqslant 0\}$.

4.2. *The Howe-Moore property*

Proposition **4.2.** — *Let* $H < \operatorname{Aut} T$ *be a closed, locally ∞-transitive subgroup and* $(\pi, \mathcal{H})$ *a continuous unitary representation of* H *with no nonzero* $H^{(\infty)}$*-invariant vectors. Then all coefficients of* π *vanish at infinity.*

The proof of Proposition 4.2 depends on Proposition 4.3 below which treats the following general situation: Let $B = A \cdot N$ be a locally compact group which is the semidirect product of the closed subgroups $A = \langle a \rangle \simeq \mathbf{Z}$ and a totally discontinuous group $N \lhd B$; assume that there is an open compact subgroup $C_0 \subset N$ such that $C_0 \subset a C_0 a^{-1}$ and $N = \bigcup_{k \geqslant 0} a^k C_0 a^{-k}$, in this setting we have:

Proposition **4.3.** — *Let* $(\pi, \mathscr{H})$ *be a continuous unitary representation of* B *and* $v \in \mathscr{H}$ *such that*

$$\mathbf{N} \longrightarrow \mathbf{C}$$
$$n \longmapsto \langle \pi(a^n)v, v\rangle$$

does not vanish at infinity. Then there exists $w \in \mathscr{H}$, $w \neq 0$, *such that* $\mathrm{Stab}_{\mathrm{N}}(w)$ *is of finite index in* N.

Proof. — The subspace $\mathscr{H}^{(\infty)} = \{w \in \mathscr{H}, \mathrm{Stab}_{\mathrm{N}}(w) \text{ is open}\}$ is dense in $\mathscr{H}$: fix a decreasing sequence $(\mathrm{K}_n)_{n\in\mathbf{N}}$ of compact open subgroups of N, such that $\bigcap_{n\geqslant 1} \mathrm{K}_n = (e)$. The sequence $(f_n)_{n\geqslant 1}$, where $f_n = \frac{1}{m(\mathrm{K}_n)}\cdot\chi_{\mathrm{K}_n}$ is an L^1-approximation of the identity, and hence $\lim_{n\to\infty} \pi(f_n)v = v$ for every $v \in \mathscr{H}$. Observing that $\pi(f_n)v \in \mathscr{H}^{(\infty)}$, we conclude that $\mathscr{H}^{(\infty)}$ is dense in $\mathscr{H}$. Thus there exists $u \in \mathscr{H}^{(\infty)}$, such that the function $n \to \langle \pi(a^n)u, u\rangle$ does not vanish at infinity; there exists therefore a subsequence $(n_i)_{i\in\mathbf{N}}$, with $\lim_i n_i = +\infty$ and $w \in \mathscr{H}$, $w \neq 0$ such that $(\pi(a^{n_i})u)_{i\geqslant 1}$ converges weakly to w. Let $\mathrm{K} \lhd \mathrm{C}_0$ be an open subgroup (of finite index) such that $\pi(k)u = u$, $\forall k \in \mathrm{K}$. Then $\pi(a^{n_i})u$ is fixed by $a^{n_i}\mathrm{K}a^{-n_i}$ and, passing to a subsequence, one may assume that the sequence of subgroups $(a^{n_i}\mathrm{K}a^{-n_i})_{i\in\mathbf{N}}$ converges in Chabeauty topology (that is uniformly on compacts) to a closed subgroup $\mathrm{L} < \mathrm{N}$.

Claim **1.** — w is L-invariant.
For $\ell \in \mathrm{L}$ and $m_i \in a^{n_i}\mathrm{K}a^{-n_i}$ with $\lim_{i\to\infty} m_i = \ell$, we have,

$$\begin{aligned}\langle \pi(\ell)w, w\rangle - \langle w, w\rangle = (\langle \pi(\ell)w, w\rangle - \langle \pi(\ell)w, \pi(a^{n_i})u\rangle)+\\ + (\langle \pi(\ell)w, \pi(a^{n_i})u\rangle - \langle \pi(m_i)w, \pi(a^{n_i})u\rangle)+\\ + (\langle w, \pi(a^{n_i})u\rangle - \langle w, w\rangle).\end{aligned}$$

The first and last summand tend to zero since $(a^{n_i})u$ converges weakly to w, and the second summand tends to zero since it is bounded by $\|\pi(\ell)w - \pi(m_i)w\| \cdot \|w\|$ and $\lim_{i\to\infty} m_i = \ell$. This shows Claim 1.

Claim **2.** — The subgroup L is of finite index in N. Let d be the index of K in C_0 and pick $x_1, ..., x_r \in \mathrm{N}$ with $r > d$; since the increasing union $\bigcup_{i\geqslant 1} a^{n_i}\mathrm{C}_0a^{-n_i}$ equals N there exists i_0 such that $\{x_1, ..., x_r\} \subset a^{n_i}\mathrm{C}_0a^{-n_i}$, for all $i \geqslant i_0$; since $r > d$, there exists for every $i \geqslant i_0, j_i \neq k_i$ such that $x_{j_i}^{-1}x_{k_i} \in a^{n_i}\mathrm{K}a^{-n_i}$ and since $r < +\infty$, there exists $j \neq k$ such that $x_j^{-1}x_k \in a^{n_i}\mathrm{K}a^{-n_i}$ for infinitely many $i \geqslant i_o$; this implies $x_j^{-1}x_k \in \mathrm{L}$ and shows that the index of L in N is at most d. □

Now fix $x_0 \in X$ and consider the H-action on $T(\infty)\times\mathbf{Z}$ given by:

$$h(\eta, m) = (h\eta, \beta_\eta(h^{-1}x_0, x_0) + m), \ \eta \in T(\infty), \ m \in \mathbf{Z}, \ h \in H.$$

For this action we have $\mathrm{Stab}_H(\xi, 0) = P^0_\xi$.

Lemma **4.4.** — *The* H*-orbits in* $T(\infty)\times\mathbf{Z}$ *are closed.*

Proof. — Since H acts transitively on $T(\infty)$ it suffices to show that if $\lim h_n(\xi, m) = (\xi, s)$ then $(h_n)_{n\in\mathbf{N}}$ is bounded in H/P^0_ξ. Write $h_n = k_n a^{e(n)} u_n$ with $k_n \in H(x_0)$, $a \in P_\xi$ hyperbolic and $u_n \in P^0_\xi$. For n large we have: $\beta_\xi(h_n^{-1}x_0, x_0) + m = s$, which amounts to $\beta_\xi(a^{-e(n)}x_0, x_0) + m = s$, hence $\ell e(n) + m = s$, which shows that $(a^{e(n)})_{n\geq 1}$ is stationary and thus $(h_n)_{n\in\mathbf{N}}$ is bounded in H/P^0_ξ. □

Proof of Proposition 4.2. — Let $\xi \in T(\infty)$, $a \in P_\xi$ with $\ell = \chi_\xi(a) = \ell_\xi > 0$, $r : \mathbf{Z} \to X$ a parametrization of the axis of a with $r(\infty) = \xi$, and $x_k := r(k)$, $k \in \mathbf{Z}$. Then $P_\xi = A \cdot P^0_\xi$, $A = \langle a \rangle$, and (see (4.4)), Proposition 4.3 applies to P_ξ. Let $(\pi, \mathcal{H})$ be a continuous unitary representation of H and assume that some coefficient of π does not vanish at infinity. Since $H = H(x_0)A^+(H(x_0) \cup H(x_1))$, (see (4.5)), there exists $v \in \mathcal{H}$, $v \neq 0$, such that $n \to \langle \pi(a^n)v, v\rangle$, $n \in \mathbf{N}$, does not vanish at infinity: indeed let $u \neq 0$ and $(g_n)_{n\geq 1}$ be a sequence tending to infinity such that

$$|\langle \pi(g_n)u, u\rangle| \geqslant \varepsilon > 0, \quad \text{for all } n \geqslant 1.$$

Write $g_n = k_n a^{\ell_n} k'_n$, with $k_n \in H(x_0)$, $l_n \geqslant 1$, $k'_n \in H(x_0) \cup H(x_1)$; we may assume that $(k_n)_{n\geq 1}$, $(k'_n)_{n\geq 1}$ converge, say to resp. k, k'. Setting $w' = \pi(k')u$, $w = \pi(k)^{-1}u$ and applying the triangle inequality, we get:

$$\lim_{n\to\infty} \left|\langle \pi(g_n)u, u\rangle - \langle \pi(a^{l_n})w', w\rangle\right| = 0$$

and hence $n \to \langle \pi(a^n)w', w\rangle$ does not vanish an infinity; since any matrix coefficients of a unitary representation is linear combination of diagonal coefficients, we conclude that there exists $v \neq 0$ such that $n \to \langle \pi(a^n)v, v\rangle$ does not vanish at infinity. Hence (Proposition 4.3) there exists $w \in \mathcal{H}$, $w \neq 0$, such that $L := \mathrm{Stab}_{P^0_\xi}(w)$ is of finite index in P^0_ξ.

Now we claim that there exist sequences $(\ell_i)_{i\in\mathbf{N}}$, $(\ell'_i)_{i\in\mathbf{N}}$ in L, $(h_i)_{i\in\mathbf{N}}$ in $H(x_0)$ and an element $a' \in P_\xi$ with $\lim_{i\to\infty} \ell_i h_i \ell'_i = a'$ and $\chi_\xi(a') = 2\ell$. For every $k \geqslant 1$, choose $\eta_k \in T(\infty)$ with $(\eta_k \cdot \xi)_{x_0} = k$, where $(\alpha \cdot \beta)_y$ denotes Gromov's scalar product, that is, $(\alpha \cdot \beta)_y$ is the distance from y to the geodesic joining α, β; in particular $\lim_{k\to\infty} \eta_k = \xi$. Choose $n_k \in P^0_\xi(x_{k+\ell})$ such that $(n_k\eta_k \cdot \xi)_{x_0} = k + \ell$; with these choices we have $n_k(\eta_k, 0) = (n_k\eta_k, 2\ell)$ and hence $\lim_{k\to\infty} n_k(\eta_k, 0) = (\xi, 2\ell)$. Choose a sequence $(h_k)_{k\in\mathbf{N}}$ in $H(x_0)$ such that $\lim_{k\to\infty} h_k = e$

and $h_k(\xi, 0) = (\eta_k, 0)$; with this we have $\lim_{k\to\infty} n_k h_k(\xi, 0) = (\xi, 2\ell) = a^2(\xi, 0)$. Since the H-orbits in $T(\infty)\times\mathbf{Z}$ are closed (Lemma 4.4), there exists $n'_k \in P^0_\xi$ such that $\lim_{k\to\infty} n_k h_k n'_k = a^2$. Since L is of finite index in P^0_ξ, we may assume, passing to a subsequence, that $n_k = \sigma\ell_k$, $n'_k = \ell'_k\sigma'$ where $\sigma, \sigma' \in P^0_\xi$ and $\ell_k, \ell'_k \in L$, for all $k \geqslant 1$. Setting $a' = \sigma^{-1}a^2\sigma'^{-1}$ we have $\lim_{k\to\infty} \ell_k h_k \ell'_k = a'$ and $\chi_\xi(a') = 2\ell$.

This claim implies that

$$\langle\pi(a')w, w\rangle = \lim_{k\to\infty} \langle\pi(\ell_k h_k \ell'_k)w, w\rangle = \lim_{k\to\infty} \langle\pi(h_k)w, w\rangle = \langle w, w\rangle$$

and hence w is $A' := \langle a'\rangle$-invariant. The closed subgroup P' generated by A' and L is of finite index in P_ξ and w is P'-invariant; by (4.3) we have $|P_\xi\backslash H/P_\xi| = 2$ and hence $|P'\backslash H/P'| < +\infty$. The continuous function $h \to \langle\pi(h)w, w\rangle$ is left and right P'-invariant and takes therefore only finitely many values. In particular

$$\mathrm{Stab}_H(w) = \{h \in H : \langle\pi(h)w, w\rangle = \langle w, w\rangle\}$$

is an open subgroup of H; this subgroup is also cocompact in H, since $\mathrm{Stab}_H(w) \supset P'$ and hence $\mathrm{Stab}_H(w) \supset H^{(\infty)}$. □

4.3. In this section we prove the normal subgroup theorem (Theorem 4.1); for this we follow the strategy of Margulis. First we notice the following corollary to Proposition 3.1:

Corollary **4.5.** — *Let* $\Gamma < \mathrm{Aut}\,T_1 \times \mathrm{Aut}\,T_2$ *be a cocompact lattice such that* $H_i := \overline{pr_i(\Gamma)}$ *is locally* ∞*-transitive and* $H_i = H_i^{(\infty)}$; *let* $N \lhd \Gamma$, $N \neq \{e\}$ *be a nontrivial normal subgroup of* Γ. *Then* Γ/N *has property* (T).

Remark 4.3.1. — Assume $H_i := \overline{pr_i(\Gamma)}$ is locally ∞-transitive; for $N \lhd \Gamma$, $N \neq e$, we have $\overline{pr_i(N)} \supset H_i^{(\infty)}$.

Proof. — As $\overline{pr_i(N)} \lhd H_i$, it follows from [B-M]$_3$ Prop. 3.1.2, that either $\overline{pr_i(N)} \supset H_i^{(\infty)}$ or $\overline{pr_i(N)}$ is trivial. The latter is impossible or else we would have a nontrivial discrete normal subgroup of H_{3-i} which is incompatible with H_{3-i} being locally ∞-transitive. □

Proof of Corollary 4.5. — By Remark 4.3.1, we conclude that each $pr_i(N)$ does not act freely on T_i and hence by [B-M]$_3$ Lemma 1.4.2, $pr_i(N)\backslash T_i$ is a finite tree and Proposition 3.1 shows that Γ/N has a property (T). □

Given $H_i < \mathrm{Aut}\,T_i$, closed, locally ∞-transitive subgroups, we will always endow $T_i(\infty)$ with the H_i-invariant measure class and denote by $\mathcal{M}(T_1(\infty))$, $\mathcal{M}(T_2(\infty))$,

$\mathscr{M}(T_1(\infty)\times T_2(\infty))$ the corresponding measure algebras, that is the algebra of classes of Lebesgue measurable sets, where two sets are identified if they differ by a null set (see e.g. [Ma] IV.2).

The next fundamental ingredient in the proof of the normal subgroup theorem is

Theorem **4.6.** — *Let* $\Gamma < \mathrm{Aut}\,T_1\times\mathrm{Aut}\,T_2$ *be a cocompact lattice such that* $H_i:=\overline{pr_i(\Gamma)}$ *is locally* ∞*-transitive and* $H_i^{(\infty)}$ *is of finite index in* H_i. *Any* Γ*-invariant subalgebra of* $\mathscr{M}(T_1(\infty)\times T_2(\infty))$ *is one of the following algebras:*

$$\{\phi,\, T_1(\infty)\times T_2(\infty)\},\ \mathscr{M}(T_1(\infty))\times\{T_2(\infty)\},$$
$$\{T_1(\infty)\}\times\mathscr{M}(T_2(\infty)),\ \mathscr{M}(T_1(\infty)\times T_2(\infty)).$$

First we show how to deduce the normal subgroup theorem from Corollary 4.5 and Theorem 4.6.

Proof of Theorem 4.1. — Let $\{e\}\neq N \lhd \Gamma$, and $\Gamma_0=\Gamma\cap(H_1^{(\infty)}\times H_2^{(\infty)})$, $N_0=N\cap(H_1^{(\infty)}\times H_2^{(\infty)})$; set $G_i=H_i^{(\infty)}$, then G_i is locally ∞-transitive, $G_i^{(\infty)}=G_i$, Γ_0 is of finite index in Γ and (Remark 4.3.1) $\overline{pr_i(\Gamma_0)}=G_i$. Moreover, N_0 is of finite index in N, and N being infinite (Remark 4.3.1), we have $N_0\neq\{e\}$. We claim that Γ_0/N_0 is amenable. Assume the contrary; then ([Ma] IV, Theorem 4.5, Remark 2 and Lemma 4.7) there exists an infinite, Γ_0-invariant subalgebra $\mathscr{B}\subset\mathscr{M}(T_1(\infty)\times T_2(\infty))$ such that $N_0B=B$, for all $B\in\mathscr{B}$. According to Theorem 4.6, we have the following possibilities for $\mathscr{B}$:

$$\mathscr{M}(T_1(\infty)\times T_2(\infty)),\ \mathscr{M}(T_1(\infty))\times\{T_2(\infty)\},\ \{T_1(\infty)\}\times\mathscr{M}(T_2(\infty)).$$

Since $N_0B=B$, for all $B\in\mathscr{B}$, we obtain, respectively, $N_0=\{e\}$, $N_0\subset\{e\}\times G_2$, $N_0\subset G_1\times\{e\}$, neither of which is possible by Remark 4.3.1. Thus Γ_0/N_0 is amenable and (by Corollary 4.5) has property (T). Thus Γ_0/N_0 is finite and so is Γ/N. □

The remainder of this section is devoted to the proof of Theorem 4.6.

Let T be a locally finite tree, $G<\mathrm{Aut}\,T$ a closed, locally ∞-transitive transitive subgroup, $\mu\in M^1(T(\infty))$ the Patterson density for G; in particular $d(g_*\mu)(w)/d\mu(w)=e^{-\delta\beta_w(gb,\,b)}$, where $b\in X$ is a fixed vertex and δ the critical exponent of G; for the properties of the Patterson density used in this section we refer to $[\text{B-M}]_1$ §1, §6, and §7. Let $s\in G$ be a hyperbolic element with attracting, resp. repelling fixed point α, respectively ξ. For $C\in\mathscr{M}(T(\infty))$, set $\psi(C)=T(\infty)$ if $\xi\in C$ and $\psi(C)=\emptyset$ if $\xi\notin C$.

Lemma **4.7.** — *For every* $C\in\mathscr{M}(T(\infty))$ *and almost every* $g\in G$, *the sequence* $s^n(gC)$ *converges to* $\psi(gC)$ *in* μ*-measure.*

Proof. — Let $r : \mathbf{Z} \to X$ be the parametrization of the axis of s, such that $r(\infty) = \alpha$, $r(0) = b$; for $w_1, w_2 \in T(\infty)\setminus\{\alpha\}$, define $d(w_1, w_2) := e^{-(w_1, w_2)}$ where $(w_1, w_2) := \lim_{k\to\infty} [(w_1 \cdot w_2)_{r(k)} - k]$. Then d is a complete metric on $T(\infty)\setminus\{\alpha\}$, and the latter has finite Hausdorff dimension with respect to d. Moreover, $d(sw_1, sw_2) = e^{\ell} d(w_1, w_2)$ for all $w_i \in T(\infty)\setminus\{\alpha\}$, where ℓ is the translation length of s. For $A \in \mathcal{M}(T(\infty))$, set $\nu(A) = \int_A e^{2\delta(w\cdot\alpha)_b} d\mu(w)$; then ν is a Radon-measure on $T(\infty)\setminus\{\alpha\}$, equivalent to μ, and $\nu(sA) = e^{\delta\ell}\nu(A)$. Let $C \in \mathcal{M}(T(\infty))$ and assume that ξ is a density point of C, that is $\lim_{\varepsilon\to 0} \frac{\nu(C \cap B(\xi, \varepsilon))}{\nu(B(\xi, \varepsilon))} = 1$, where $B(\xi, \varepsilon)$ is the ball centered at ξ, of radius ε, for the distance d. For any fixed $R > 0$, taking into account that $s^{-n}B(\xi, R) = B(\xi, e^{-n\ell}R)$, we have

$$\lim_{n\to\infty} \frac{\nu(C \cap s^{-n}B(\xi, R))}{\nu(s^{-n}B(\xi, R))} = 1$$

and hence $\lim \frac{\nu(s^n C \cap B(\xi, R))}{\nu(B(\xi, R))} = 1$. Thus $s^n C$ converges to $T(\infty)$ in ν-measure and hence in μ-measure. We conclude by observing that for almost every $g \in G, g\xi$ is either a density point of C or a density point of $T(\infty)\setminus C$. □

For $i = 1, 2$, let $H_i < \operatorname{Aut} T_i$ be a closed, locally ∞-transitive subgroup, $\mu_i \in M^1(T_i(\infty))$ the Patterson density as above, $s_i \in H_i$ a hyperbolic element, $\xi_i, \alpha_i \in T_i(\infty)$ the repelling, respectively attracting, fixed point of s_i, $s = (s_1, e)$, $t = (e, s_2)$ and $\mu = \mu_1 \times \mu_2$. For $B \in \mathcal{M}(T_1(\infty) \times T_2(\infty))$ set

$$B_w = \{\eta \in T_1(\infty) : (\eta, w) \in B\},\ w \in T_2(\infty)$$
$$B^\eta = \{w \in T_2(\infty) : (\eta, w) \in B\},\ \eta \in T_1(\infty)$$

and $\psi_1(B) = T_1(\infty) \times B^{\xi_1}$, $\psi_2(B) = B_{\xi_2} \times T_2(\infty)$.

Lemma **4.8.** — *For almost every* $g \in H_1 \times H_2$,

$$s^n gB \longrightarrow \psi_1(gB), \qquad t^n gB \longrightarrow \psi_2(gB),$$

and convergence holds in μ*-measure.*

Proof. — $\lim_{n\to\infty} s^n gB = \psi_1(gB)$ in μ-measure is equivalent to

$$(*) \qquad \lim_{n\to\infty} \int_{T_2(\infty)} d\mu_2(w) \int_{T_1(\infty)} d\mu_1(\xi) |\chi_{s^n gB}(\xi, w) - \chi_{\psi_1(gB)}(\xi, w)| = 0.$$

For a.e. $w \in T_2(\infty)$ and $g_2 \in H_2$, the set $B_{g_2^{-1}w}$ is measurable and $(s^n gB)_w = s_1^n(gB)_w$ converges to $\psi((gB)_w)$ (Lemma 4.7). Observe that $\xi_1 \in (gB)_w$ iff $w \in (gB)^{\xi_1}$ which, using the above, implies

$$\lim_{n\to\infty} \int_{T_1(\infty)} |\chi_{s^n gB}(\xi, w) - \chi_{\psi_1(gB)}(\xi, w)| d\mu(\xi) = 0$$

for almost every $w \in T_2(\infty), g_2 \in H_2, g_1 \in H_1$. Fubini's Theorem and the dominated convergence theorem imply then $(*)$. □

Assume now that $H_i = H_i^{(\infty)}$ and let $\Gamma < H_1 \times H_2$ be a cocompact lattice such that $\overline{pr_i(\Gamma)} = H_i$.

Lemma **4.9.** — *The subgroups* $\langle s \rangle$ *and* $\langle t \rangle$ *act ergodically on* $\Gamma\backslash(H_1 \times H_2)$.

Proof. — Consider the unitary representation π given by the regular action of H_1 on $L^2(\Gamma\backslash(H_1 \times H_2))$ and let $A \subset \Gamma\backslash(H_1 \times H_2)$ be an $\langle s \rangle$-invariant measurable subset; thus χ_A is a $\pi(s)$-invariant vector. Let P be the orthogonal projection on the subspace $\mathscr{H}_1$ of $\pi(H_1)$-invariant vectors and $v := \chi_A - P(\chi_A) \in \mathscr{H}_1^{\perp}$. Since $\mathscr{H}_1^{\perp}$ has no nonzero $\pi(H_1)$-invariant vector, Prop. 4.2 implies that $h_1 \to \langle \pi(h_1)v, v\rangle$, $H_1 \to \mathbf{C}$, vanishes at infinity. On the other hand, $h_1 \to \langle \pi(h_1)v, v\rangle$ has constant value $\|v\|^2$ along the unbounded subgroup $\langle s \rangle < H_1$ and hence $\chi_A - P(\chi_A) = v = 0$. From $\overline{\Gamma H_1} = H_1 \times H_2$ we deduce that any H_1-invariant measurable function on $\Gamma\backslash(H_1 \times H_2)$ is $H_1 \times H_2$-invariant and hence essentially constant; in particular $\chi_A = P(\chi_A)$ is essentially constant and therefore A is either of measure zero or of full measure. □

The above lemma together with Birkhoff's ergodic theorem imply,

Lemma **4.10.** — *For almost every* $h \in H_1 \times H_2$, *the sets* $\bigcup_{n\geq 1} \Gamma h s^{-n}$ *and* $\bigcup_{n\geq 1} \Gamma h t^{-n}$ *are dense in* $H_1 \times H_2$.

Lemma **4.11.** — *Let* $\mathscr{B} \subset \mathscr{M}(T_1(\infty) \times T_2(\infty))$ *be a* Γ*-invariant closed subalgebra,* $B \in \mathscr{B}$ *and* $g \in H$. *Then, for almost every* $g' \in H$, *we have*

$$g\psi_1(g'B) \in \mathscr{B}$$

and

$$g\psi_2(g'B) \in \mathscr{B}.$$

Proof. — The set

$$\mathrm{U} = \left\{ g' \in \mathrm{H}_1 \times \mathrm{H}_2 : \begin{array}{l} \overline{\bigcup_{n \geqslant 1} \Gamma g'^{-1} s^{-n}} = \mathrm{H}_1 \times \mathrm{H}_2 \ \text{ and} \\ s^n(g'\mathrm{B}) \longrightarrow \psi_1(g'\mathrm{B}) \end{array} \right\}$$

is of full measure in $\mathrm{H}_1 \times \mathrm{H}_2$ (Lemma 4.10, Lemma 4.8). Given $g \in \mathrm{H}_1 \times \mathrm{H}_2$ and $g' \in \mathrm{U}$, take $\gamma_i \in \Gamma$ and $(\eta_i)_{i \in \mathbf{N}}$, such that $\lim_{i \to \infty} g_i = g$, where $g_i := \gamma_i g'^{-1} s^{-n_i}$. We have $\gamma_i = g_i s^{n_i} g'$, $\gamma_i \mathrm{B} \in \mathscr{B}$, and $\gamma_i \mathrm{B} = g_i(s^{n_i} g'\mathrm{B}) \to g\psi_1(g'\mathrm{B})$. □

Proof of Theorem 4.6. — Replacing Γ by $\Gamma \cap (\mathrm{H}_1^{(\infty)} \times \mathrm{H}_2^{(\infty)})$ and using Remark 4.3.1, we may assume that $\mathrm{H}_i = \mathrm{H}_i^{(\infty)}$.

(a) Let $\mathscr{B}' \subset \mathscr{M}(\mathrm{T}_i(\infty))$ be an H_i-invariant subalgebra. Then $\mathscr{B}' = \{\phi, \mathrm{T}_i(\infty)\}$ or $\mathscr{B}' = \mathscr{M}(\mathrm{T}_i(\infty))$. Let $\mathrm{P}_i = \mathrm{P}_{\xi_i}$; identifying $\mathrm{T}_i(\infty)$ with $\mathrm{H}_i/\mathrm{P}_i$, we deduce ([Ma] IV Proposition 2.4) that there exists a closed subgroup $\mathrm{L}_i < \mathrm{H}_i$ with $\mathrm{L}_i \supset \mathrm{P}_i$, such that $\mathscr{B}' = \mathscr{M}(\mathrm{H}_i/\mathrm{P}_i, \mathrm{L}_i)$. Since H_i acts 2-transitively on $\mathrm{H}_i/\mathrm{P}_i$, P_i is a maximal subgroup of H_i and thus $\mathrm{L}_i = \mathrm{P}_i$ or H_i, which establishes (a).

Let $\mathscr{B} \subset \mathrm{M}(\mathrm{T}_1(\infty) \times \mathrm{T}_2(\infty))$ be a Γ-invariant subalgebra.

(b) Let $\mathrm{B} \in \mathscr{B}$, and assume that the set of $w \in \mathrm{T}_2(\infty)$ s.t. $\mathrm{B}_w \notin \{\phi, \mathrm{T}_1(\infty)\}$, respectively, the set of $\xi \in \mathrm{T}_1(\infty)$ s.t. $\mathrm{B}^\xi \notin \{\phi, \mathrm{T}_2(\infty)\}$ is of positive measure. Then Lemma 4.11 implies that $\mathscr{B} \supset \mathscr{B}_1 \times \{\mathrm{T}_2(\infty)\}$, respectively, $\mathscr{B} \supset \{\mathrm{T}_1(\infty)\} \times \mathscr{B}_2$ where $\mathscr{B}_i \subset \mathscr{M}(\mathrm{T}_i(\infty))$ is an H_i-invariant, non-trivial, subalgebra and thus, by (a), $\mathscr{B}_i = \mathscr{M}(\mathrm{T}_i(\infty))$.

(c) Assume that $\mathscr{B} \neq \{\phi, \mathrm{T}_1(\infty) \times \mathrm{T}_2(\infty)\}$; then (b) implies that either $\mathscr{B} \supset \mathscr{M}(\mathrm{T}_1(\infty)) \times \{\mathrm{T}_2(\infty)\}$ or $\mathscr{B} \supset \{\mathrm{T}_1(\infty)\} \times \mathscr{M}(\mathrm{T}_2(\infty))$. In the first (respectively, the second) case, if moreover $\mathscr{B} \neq \mathscr{M}(\mathrm{T}_1(\infty)) \times \{\mathrm{T}_2(\infty)\}$ (resp., $\mathscr{B} \neq \{\mathrm{T}_1(\infty)\} \times \mathscr{M}(\mathrm{T}_2(\infty))$ we obtain using (b) that

$$\mathscr{B} \supset \{\mathrm{T}_1(\infty)\} \times \mathscr{M}(\mathrm{T}_2(\infty))$$

$$(\text{respectively}, \mathscr{B} \supset \mathscr{M}(\mathrm{T}_1(\infty)) \times \{\mathrm{T}_2(\infty)\})$$

and hence in both cases $\mathscr{B} = \mathscr{M}(\mathrm{T}_1(\infty) \times \mathrm{T}_2(\infty))$. □

5. Applications of the Normal Subgroup Theorem

The normal subgroup theorem applies to lattices in $\mathrm{Aut}\,\mathrm{T}_1 \times \mathrm{Aut}\,\mathrm{T}_2$ whose projections are locally ∞-transitive. Beside rational points of algebraic groups of rank 1, the main source of locally ∞-transitive groups are the universal groups U(F) whose definition we briefly recall (see [B-M]$_3$ 3.2). Let $d \geqslant 3$, $\mathscr{T}_d = (\mathrm{X}, \mathrm{Y})$ be the d-regular tree, $\mathrm{F} < \mathrm{S}_d$ a permutation group and $i : \mathrm{Y} \to \{1, 2, ..., d\}$ a legal colouring.

Then:

$$U(F) = \{g \in \mathrm{Aut}\, T_d : i_{E(gx)} \cdot g \cdot i|_{E(x)}^{-1} \in F, \quad \forall x \in X\}$$

is a closed subgroup of $\mathrm{Aut}\,\mathcal{T}_d$, acting transitively on X and such that, at every vertex $x \in X$, $\underline{U(F)}(x) < \mathrm{Sym}E(x)$ is permutation isomorphic to $F < S_d$. When $F < S_d$ is 2-transitive we have (see [B-M]$_3$ 3.1, 3.2)

(1) $U(F)^+ = U(F)^{(\infty)}$ is of index 2 in U(F) and simple.

(2) $U(F)^+$ is locally ∞-transitive.

Theorem 4.1 implies then

Corollary **5.1.** — *Let* $d_i \geqslant 3$, $F_i < S_{d_i}$ *be* 2*-transitive permutation groups and* $\Gamma < U(F_1) \times U(F_2)$ *be a cocompact lattice with* $\overline{pr_i(\Gamma)} \supset U(F_i)^+$. *Then, any nontrivial normal subgroup* $e \neq N \lhd \Gamma$ *is of finite index in* Γ.

In general, it is difficult to determine whether a lattice has locally ∞-transitive projections; in [B-M]$_3$ Chapter 3, we have shown that under certain additional assumptions of local nature, 2-transitive groups of tree automorphisms are ∞-transitive and, in some cases, their closure is a group U(F) of the above type. We now apply these results to finite VH-T-complexes.

Let X be a finite VH-T-complex; we assume that the horizontal 1-skeleton $X_h^{(1)}$ and the vertical 1-skeleton $X_v^{(1)}$ are connected. Let $d_h = |E_h(x)|$, $d_v = |E_v(x)|$, $\forall x \in X$; then X is a quotient $\Gamma \backslash (\mathcal{T}_{d_h} \times \mathcal{T}_{d_v})$ of a product of regular trees of degrees d_h and d_v. Let $P_h < S_{d_h}$ and $P_v < S_{d_v}$ be the "horizontal", resp. "vertical", permutation groups; it follows from [B-M]$_3$ Proposition 3.2.2 that, up to conjugation, Γ is contained in $U(P_h) \times U(P_v)$. We assume both P_h, P_v to be transitive and let L_h, L_v, K_h, K_v be the finite permutation groups defined in §1. These groups are effectively computable in terms of the finite complex X.

Proposition **5.2.** — *Assume that the permutation groups* P_h, P_v *are* 2*-transitive and that* L_h, L_v *are simple non-abelian. Then,* $K_h \simeq L_h^{a_h}$, $K_v \simeq L_v^{a_v}$ *with* $(a_h, a_v) \in \{(0, 0), (d_h - 1, d_v - 1)\}$.

(1) *If* $(a_h, a_v) = (0, 0)$, *the lattice* Γ *is reducible.*

(2) *If* $(a_h, a_v) = (d_h - 1, d_v - 1)$, *the lattice* Γ *is irreducible and* $\Gamma < U(P_h) \times U(P_v)$ *has dense projections.*

Proof. — Let $H^{(h)} = \overline{pr_1(\Gamma)}$, $H^{(v)} = \overline{pr_2(\Gamma)}$.

Since $X_h^{(1)}$, $X_v^{(1)}$ are connected, the groups $H^{(h)} < \mathrm{Aut}\,\mathcal{T}_{d_h}$, $H^{(v)} < \mathrm{Aut}\,\mathcal{T}_{d_v}$ are both vertex transitive. Fix adjacent vertices x, y in $\mathcal{T}_{d_h}$ and x', y' in $\mathcal{T}_{d_v}$. By hypothesis, the following isomorphisms of permutation groups hold:

$$\underline{\mathrm{H}}^{(h)}(x) \simeq \mathrm{P}_h, \quad \underline{\mathrm{H}}^{(v)}(x') \simeq \mathrm{P}_v,$$

$$\underline{\mathrm{H}}_1^{(h)}(x,y)/\underline{\mathrm{H}}_2^{(h)}(x) \simeq \mathrm{K}_h, \quad \underline{\mathrm{H}}_1^{(v)}(x',y')/\underline{\mathrm{H}}_2^{(v)}(x') \simeq \mathrm{K}_v.$$

Now we apply Prop. 3.3.1 (and the remark following it) to deduce that $\mathrm{K}_h \simeq \mathrm{L}_h^{a_h}$, $\mathrm{K}_v \simeq \mathrm{L}_v^{a_v}$ with $a_h \in \{0, d_h - 1\}$, $a_v \in \{0, d_v - 1\}$; if $a_h = 0$, then $\mathrm{H}^{(h)}$ is discrete, thus (Prop. 1.2) Γ is reducible and $\mathrm{H}^{(v)}$ is discrete; the latter implies (Prop. 3.3.1 [B-M]$_3$) that $a_v = 0$.

If $a_h = d_h - 1$, then (Prop. 2.2.2) $\mathrm{H}^{(h)} = \mathrm{U}(\mathrm{P}_h)$, and (Prop. 1.2) Γ is irreducible which implies that $\mathrm{H}^{(v)}$ is non-discrete and hence $a_v = d_v - 1$ by Prop. 3.3.1. □

Combining Corollary 5.1 and Proposition 5.2, we obtain

Corollary **5.3.** — *Let* X *be a finite VH-T-complex as in Proposition 5.2 and assume moreover that* $(a_h, a_v) = (d_h - 1, d_v - 1)$.
Then, any nontrivial normal subgroup $\mathrm{N} \lhd \pi_1(\mathrm{X})$ *is of finite index in* $\pi_1(\mathrm{X})$.

Let now $\Gamma < \mathrm{H}_1 \times \mathrm{H}_2$ be a cocompact lattice, where $\mathrm{H}_i < \mathrm{Aut}\, \mathrm{T}_i$ are locally ∞-transitive, $\mathrm{H}_i^{(\infty)}$ is assumed of finite index in H_i and $\overline{pr_i(\Gamma)} \supset \mathrm{H}_i^{(\infty)}$, $i = 1, 2$; let, as usual, $\Gamma^{(\infty)}$ be the intersection of all finite index subgroups of Γ: the subgroup $\Gamma^{(\infty)}$ is normal in Γ. Assume that Γ is not residually finite; then, (normal subgroup theorem) $\Gamma^{(\infty)}$ is of finite index in Γ and we claim that $\Gamma^{(\infty)}$ is simple: indeed, since $\Gamma^{(\infty)}$ is of finite index in Γ, we have $\overline{pr_i(\Gamma^{(\infty)})} \supset \mathrm{H}_i^{(\infty)}$ and the normal subgroup theorem applies to $\Gamma^{(\infty)}$; given $e \neq \mathrm{N} \lhd \Gamma^{(\infty)}$, the group N is of finite index in $\Gamma^{(\infty)}$, hence in Γ and thus $\mathrm{N} \supset \Gamma^{(\infty)}$. This line of reasoning leads to

Corollary **5.4.** — *Let* X *be a finite VH-T-complex satisfying the following assumptions:*

(1) *The horizontal and vertical* 1*-skeleton,* $\mathrm{X}_h^{(1)}$ *and* $\mathrm{X}_v^{(1)}$*, are connected.*

(2) *The horizontal and vertical permutation groups* P_h *and* P_v *are alternating groups of degrees* d_h *and* d_v *at least* 6.

(3) $\pi_1(\mathrm{X})$ *is not residually finite.*

Let Y *be the Galois covering of* X *associated to the intersection of all finite index subgroups of* $\pi_1(\mathrm{X})$. *Then*

(1) Y *is a finite, VH-T-complex.*

(2) $\pi_1(\mathrm{Y})$ *is a simple group.*

Proof. — If $\pi_1(X) < \mathrm{Aut}\,\mathcal{T}_{d_h} \times \mathrm{Aut}\,\mathcal{T}_{d_v}$ is not residually finite, it has to be an irreducible lattice, and hence, by Proposition 5.2, Corollary 5.3 applies. □

Theorem **5.5.** — *Let* $\Gamma = \pi_1(Y)$ *be as in Corollary 5.4; the simple group* Γ *enjoys the following properties:*

(1) Γ *is finitely presented, torsion free.*

(2) Γ *is the fundamental group of a finite, locally CAT(0)-complex.*

(3) Γ *is of cohomological dimension* 2.

(4) Γ *is biautomatic.*

(5) Γ *is isomorphic to an amalgam* $F *_E F$ *of free groups over a subgroup of finite index.*

Proof. — (1), (2), and (3) are clear; (4) follows from [Ge-Sh]. To show (5), observe that $\Gamma < U(A_n)^+ \times U(A_m)^+$ projects densely onto each component, in particular both actions of Γ on $\mathcal{T}_n$ and $\mathcal{T}_m$ have an edge as fundamental domain. □

In Section 6 we will show how to construct square complexes X satisfying the assumptions of Corollary 5.4.

6. Embeddings, constructions of complexes and virtually simple groups

6.1. An interesting class of lattices $\Gamma \subset \mathrm{Aut}\,T_1 \times \mathrm{Aut}\,T_2$ are those which act freely (see chapter 1.) and vertex transitively on $T_1 \times T_2$. Recall that the action of a lattice Γ on the square complex $T_1 \times T_2$ is free if and only if Γ torsion free. In the sequel we construct and modify various examples of vertex transitive torsion free lattices. Thus it is convenient to have several ways of presenting such lattices.

Any torsion free cocompact lattice $\Gamma < \mathrm{Aut}\,T_1 \times \mathrm{Aut}\,T_2$ corresponds to a finite VH-T-square complex (see Chapter 1). Vertex transitive torsion free lattices are precisely those corresponding to VH-T-square complexes which have a single vertex. Thus vertex transitive torsion free lattices may be defined geometrically by constructing 1-vertex VH-T-square complexes.

A useful tool in constructing a 1-vertex VH-T-square complex as well as describing its fundamental group, i.e. the corresponding lattice, is provided by a VH-datum.

Definition. — A VH-datum $(A, B, \varphi_A, \varphi_B, R)$ consists of two finite sets A, B, fixed point free involutions $\varphi_A : A \to A$, $\varphi_B : B \to B$ and a subset $R \subset A \times B \times A \times B$ satisfying conditions 1 and 2 below. Let us denote $a^{-1} = \varphi_A(a)$, $a \in A$, $b^{-1} = \varphi_B(b)$, $b \in B$. The group generated by the maps $\sigma, \rho : A \times B \times A \times B \to A \times B \times A \times B$

$$\sigma(a, b, a', b') = (a'^{-1}, b^{-1}, a^{-1}, b'^{-1})$$

$$\rho(a, b, a', b') = (a', b', a, b)$$

is isomorphic to $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$.

1. Each of the 4 projections of R onto the subproducts of the form A×B or B×A are bijective.

2. The subset R is invariant under the action of the group $\langle \sigma, \rho \rangle$ and this action on R is free.

To a given VH-datum $(A, B, \varphi_A, \varphi_B, R)$ one associates a one vertex VH-T-square complex $X = (\{x_0\}, E, S)$ by defining:

$$E_h = A, \quad E_v = B, \quad S = \{s_{(a,b,a',b')}, s_{(b,a',b',a)} \,|\, (a, b, a', b') \in R\}.$$

The origin and terminus maps $E \to V = \{x_0\}$ are both the constant map $u \mapsto x_0$. The map $E \to E$ given by $u \to u^{-1}$ is the orientation reversing map of graphs. For $s = s_{(u_0,u_1,u_2,u_3)} \in S$, let $\partial s([i, i+1]) = u_i$. The maps $\sigma, \rho' : S \to S$ given by $\sigma(s_{(u_0,u_1,u_2,u_3)}) = s_{(u_2^{-1},u_1^{-1},u_0^{-1},u_3^{-1})}$ and $\rho'(s_{(u_0,u_1,u_2,u_3)}) = s_{(u_1,u_2,u_3,u_0)}$ generate a $D_4 = \langle \rho', \sigma \rangle$ action on S. Observe that X is a one vertex VH-T-square complex.

The fundamental group of a finite VH-T-square complex is finitely presented. Starting with a VH-datum $(A, B, \varphi_A, \varphi_B, R)$, a presentation for the fundamental group Γ of the corresponding square complex is given by:

$$\Gamma = \langle A \cup B \,|\, xx^{-1} = e, \ \forall x \in A \cup B, \ aba'b' = e, \ \forall (a, b, a', b') \in R \rangle.$$

Recall the definition in chapter 1 of the local permutation groups $P_h = P_h(x_0)$, $P_v = P_v(x_0)$. The edges of the horizontal 1-skeleton of X are loops corresponding to the elements of A, and similarly those of the vertical 1-skeleton correspond to B. Thus each element $a \in A$, resp. $b \in B$, defines a generator $\sigma_a \in P_v \subset \mathrm{Sym}(B)$, resp. $\sigma_b \in P_h \subset \mathrm{Sym}(A)$. These permutations may be determined from the VH-datum by letting $\sigma_a(b'^{-1}) = b \quad \sigma_b(a^{-1}) = a'$ whenever $(a, b, a', b') \in R$. The fact that this indeed gives a well defined collection of permutations, i.e. maps $A \to \mathrm{Sym}(B)$, $a \mapsto \sigma_a$, $B \to \mathrm{Sym}(A)$, $b \mapsto \sigma_b$, follows directly from conditions 1, 2 above.

Note that these maps $A \to \mathrm{Sym}(B)$, $B \to \mathrm{Sym}(A)$ contain all the combinatorial information needed to construct the 1-vertex square complex X or equivalently the VH-datum; we will refer to these two maps as the structure maps. One may formulate certain compatibility conditions on a pair of maps $A \to \mathrm{Sym}(B)$, $B \to \mathrm{Sym}(A)$ which ensure that they yield a VH-datum and hence a one-vertex VH-T-square complex.

In Section 2.2, 2.3 we described a construction of a fiber product $X \boxtimes X$ of VH-T-square complex with itself. When X is given by a VH-datum $(A, B, \varphi_A, \varphi_B, R)$ then $X \boxtimes X$ is the one-vertex square complex corresponding to $(A{\times}A, B{\times}B, \varphi_A{\times}\varphi_A, \varphi_B{\times}\varphi_B, \widetilde{R})$ where

$$\widetilde{R} = \left\{ \big((a_1, a_2), (b_1, b_2), (a'_1, a'_2), (b'_1, b'_2)\big) \,\middle|\, \begin{matrix} (a_1, b_1, a'_1, b'_1) \in R \\ (a_2, b_2, a'_2, b'_2) \in R \end{matrix} \right\}.$$

6.2. *Mating of complexes*

We define two operations on VH-T-square complexes. The first "joins" several one vertex VH-T-square complexes to produce a new one vertex VH-T-square complex. The second modifies a given VH-T-square complex by performing "surgery like" operations on it. We shall refer to the combination of these operations as "mating of complexes".

6.2.1. *Joining.* — Let ${}^{(i)}X$, $1 \leqslant i \leqslant r$, be one-vertex VH-T-square complexes. Each of the ${}^{(i)}X$ is given by a VH-datum $\left({}^{(i)}A, {}^{(i)}B, \varphi_{{}^{(i)}A}, \varphi_{{}^{(i)}B}, {}^{(i)}R\right)$. Let $X = \bigvee_{i=1}^{M} {}^{(i)}X$ be the one-vertex VH-T-square complex determined by the VH-datum $(A, B, \varphi_A, \varphi_B, R)$ where

$$A = \bigcup_{i=1}^{n} {}^{(i)}A \qquad \varphi_A = \bigcup_{i=1}^{n} \varphi_{{}^{(i)}A}$$

$$B = \bigcup_{i=1}^{n} {}^{(i)}B \qquad \varphi_B = \bigcup_{i=1}^{n} \varphi_{{}^{(i)}B}$$

$$R = \bigcup_{i=1}^{n} {}^{(i)}R \cup \left\{ \left({}^{(i)}a, {}^{(j)}b, {}^{(i)}a^{-1}, {}^{(j)}b^{-1}\right) \,\middle|\, i \neq j, \quad {}^{(i)}a \in {}^{(i)}A, \quad {}^{(j)}b \in {}^{(j)}B \right\}.$$

Geometrically, X is obtained by gluing the square complexes ${}^{(i)}X$ at the single vertex of each and then gluing in a torus for each horizontal loop of ${}^{(j)}X$ and vertical loop of ${}^{(i)}X$ $i \neq j$.

Let ${}^{(i)}\Psi_A : {}^{(i)}A \to \mathrm{Sym}({}^{(i)}B)$, ${}^{(i)}\Psi_B : B \to \mathrm{Sym}({}^{(i)}A)$ be the structure maps of ${}^{(i)}X$, see 6.1. Then the structure maps corresponding to

$$X = \bigvee_{i=1}^{n} {}^{(i)}X$$

are

$$\Psi_A : A \longrightarrow \mathrm{Sym}(B), \quad \Psi_B : B \longrightarrow \mathrm{Sym}(A)$$

where

$$A = \bigcup_{i=1}^{n} {}^{(i)}A, \quad B = \bigcup_{i=1}^{n} {}^{(i)}B$$

and:

$$\text{for } a \in {}^{(i)}A, \quad \begin{aligned} \Psi_A(a)\Big|_{B \setminus {}^{(i)}B} &= id, \\ \Psi_A(a)\Big|_{{}^{(i)}B} &= {}^{(i)}\Psi_A(a), \end{aligned}$$

$$\text{for } b \in {}^{(i)}\mathrm{B}, \quad \begin{array}{l} \Psi_{\mathrm{B}}(b)\big|_{\mathrm{A}\setminus{}^{(i)}\mathrm{A}} = id, \\ \Psi_{\mathrm{B}}(b)\big|_{{}^{(i)}\mathrm{A}} = {}^{(i)}\Psi_{\mathrm{B}}(b). \end{array}$$

Remark. — The complex X comes equipped with monomorphisms of VH-T-square complexes, $X_i \to X$ inducing injections $\pi_1(X_i) \to \pi_1(X)$ at the level of fundamental groups.

6.2.2. *Surgery.* — Let X be a one vertex VH-T-square complex corresponding to a VH-datum $(A, B, \varphi_A, \varphi_B, R)$. Given $a_1, a_2 \in A$, $(a_1, b_i, a_2, b_i') \in R$, $1 \leqslant i \leqslant d$, representing d distinct geometric squares and a permutation $\tau \in S_d$, we define a new VH-datum $(A, B, \varphi_A, \varphi_B, \widetilde{R})$ where $\widetilde{R}$ is obtained from R by replacing for each $1 \leqslant i \leqslant d$,

$$(a_1, b_i, a_2, b_i'), (a_2, b_i', a_1, b_i), (a_2^{-1}, b_i^{-1}, a_1^{-1}, b_i'^{-1}), (a_1^{-1}, b_1'^{-1}, a_2^{-1}, b_i^{-1})$$

by

$$(a_1, b_{\tau(i)}, a_2, b_i'), (a_2, b_i', a_1, b_{\tau(i)}), (a_2^{-1}, b_{\tau(i)}^{-1}, a_1^{-1}, b_i'^{-1}), (a_1^{-1}, b_i'^{-1}, a_2^{-1}, b_{\tau(i)}^{-1}).$$

Observe that this operation indeed produces a VH-datum. Let us denote this operation by $F^{\tau}_{a_1, a_2; b_1 \ldots b_d}$ and in the special case where $a_1 = a_2$, by $F^{\tau}_{a_1; b_1 \ldots b_d}$. One has analogous operations $F^{\tau}_{b_1, b_2; a_1 \ldots a_d}$ exchanging the roles of A and B. Geometrically, such an operation corresponds to taking d squares in the geometric realization of X all having the same horizontal boundaries a_1, a_2 cutting each of them along a vertical interval connecting the midpoints of a_1 and a_2 and pasting the cuts according to the permutation $\tau \in S_d$.

In terms of the structure maps $\Psi_A : A \to \mathrm{Sym}(B)$, $\Psi_B : B \to \mathrm{Sym}(A)$ the operation $F^{\tau}_{a_1, a_2; b_1 \ldots b_d}$ amounts in modifying only the permutations $\Psi_A(a_1), \Psi_A(a_2) \in \mathrm{Sym}\, B$; the new structure maps are given by

$$\widetilde{\Psi}_A(a_1)(b) = \begin{cases} b_{\tau(i)} & \text{if } b = b_i'^{-1} \text{ and } 1 \leqslant i \leqslant d \\ \Psi_A(a_1)(b) & \text{otherwise.} \end{cases}$$

$$\widetilde{\Psi}_A(a_2)(b) = \begin{cases} b_i' & \text{if } b = b_{\tau(i)}^{-1} \text{ and } 1 \leqslant i \leqslant d \\ \Psi_A(a_2)(b) & \text{otherwise.} \end{cases}$$

$$\widetilde{\Psi}_A(a_1^{-1}) = \widetilde{\Psi}_A(a_1)^{-1}, \quad \widetilde{\Psi}_A(a_2^{-1}) = \widetilde{\Psi}_A(a_2)^{-1},$$

$$\widetilde{\Psi}_A(a) = \Psi_A(a), \text{ for all } a \notin \{a_1, a_2\}.$$

$$\widetilde{\Psi}_B = \Psi_B.$$

6.2.3. Useful "building blocks" in the following construction are the 1-vertex VH-T-square complexes $\mathscr{C}_{k,\ell}$, $k, \ell \geqslant 4$ even. They are given by the following VH-datum

$$\left(\mathrm{A}_{k,\ell}, \mathrm{B}_{k,\ell}, \varphi_{\mathrm{A}_{k,\ell}}, \varphi_{\mathrm{B}_{k,\ell}}, \mathrm{R}_{k,\ell}\right) \text{ where } \mathrm{A} = \left\{ a_i^{\pm 1} \,\middle|\, 1 \leqslant i \leqslant \frac{k}{2} \right\},$$

$$\mathrm{B} = \left\{ b_j^{\pm 1} \,\middle|\, 1 \leqslant j \leqslant \frac{\ell}{2} \right\} \text{ and } \mathrm{R} = \left\{ \left(a_i^{\varepsilon}, b_{j+\varepsilon}^{\delta}, a_{i+\delta}^{-\varepsilon}, b_j^{-\delta}\right) \,\middle|\, \begin{array}{l} 1 \leqslant i \leqslant \frac{k}{2} \\ 1 \leqslant j \leqslant \frac{\ell}{2} \\ \varepsilon, \delta \in \{\pm 1\} \end{array} \right\}$$

6.3. The "mating of complexes" construction described in 6.2 leads to the following embedding result:

Proposition **6.1.** — *Let* ${}^{(0)}\mathrm{X}, \ldots, {}^{(n)}\mathrm{X}$ *be 1-vertex* VH-T-*complexes; assume that the vertical permutation group* ${}^{(0)}\mathrm{P}_v < \mathrm{S}_{{}^{(0)}d_v}$ *and horizontal permutation group* ${}^{(0)}\mathrm{P}_h < \mathrm{S}_{{}^{(0)}d_h}$ *are 2-transitive. Then for any even* $k, \ell \geqslant 4$*, there exists a 1-vertex VH-T-complex* X *with the following properties:*

(1) *There exist embeddings* ${}^{(i)}\mathrm{X} \to \mathrm{X}$, $0 \leqslant i \leqslant n$, *and embeddings* $\mathrm{C}_{k,\ell}, \mathrm{C}_{4,4} \to \mathrm{X}$, *whose images intersect pairwise at the single vertex of* X.

(2) *The horizontal and vertical permutation groups* $\mathrm{P}_h < \mathrm{S}_{d_h}$, $\mathrm{P}_v < \mathrm{S}_{d_v}$ *of* X, *are both 2-transitive. We have*

$$d_h = \sum_{i=0}^{n} {}^{(i)}d_h + k + 4,$$

$$d_v = \sum_{i=0}^{n} {}^{(i)}d_v + \ell + 4.$$

(3) a) *Assume* $d_h > 2\,{}^{(0)}d_h$. *If all groups* ${}^{(i)}\mathrm{P}_h$ *consist of even permutations, then* P_h *coincides with the alternating group* A_{d_h}, *and otherwise with the symmetric group* S_{d_h}.

b) *Assume* $d_v > 2\,{}^{(0)}d_v$. *Then the analogous assertion holds for* ${}^{(i)}\mathrm{P}_v$, P_v.

Proof. — Let the VH-datum corresponding to ${}^{(i)}\mathrm{X}$, $0 \leqslant i \leqslant n$, be

$$\left({}^{(i)}\mathrm{A}, {}^{(i)}\mathrm{B}, \varphi_{{}^{(i)}\mathrm{A}}, \varphi_{{}^{(i)}\mathrm{B}}, {}^{(i)}\mathrm{R}\right),$$

and the VH-datum of $\mathrm{C}_{m,n}$ be

$$(\mathrm{A}_{m,n}, \mathrm{B}_{m,n}, \varphi_{\mathrm{A}_{m,n}}, \varphi_{\mathrm{B}_{m,n}}, \mathrm{R}_{m,n})$$

(see 6.1 and 6.23). Let Y be the 1-vertex VH-T-square complex obtained by joining ${}^{(0)}X, {}^{(1)}X, \dots, {}^{(n)}X, C_{k,\ell}$ and $C_{4,4}$. Denote by $({}^{Y}A, {}^{Y}B, \varphi_{{}^{Y}A}, \varphi_{{}^{Y}B}, {}^{Y}R)$ the VH-datum of Y. The complexes ${}^{(i)}X, C_{k,\ell}, C_{4,4}$ are indeed embedded injectively in Y, however, the horizontal and vertical permutation groups of Y are not even transitive. Performing appropriate surgery operations on Y will produce a 1-vertex VH-T-square complex X with the asserted properties. Let us denote the elements of each of the sets ${}^{(i)}A, {}^{(i)}B$, $1 \leqslant i \leqslant n$ by:

$$ {}^{(i)}A = \left\{ {}^{(i)}a_j, {}^{(i)}a_j^{-1} \,\middle|\, 1 \leqslant j \leqslant {}^{(i)}d_h/2 \right\}, $$

$$ {}^{(i)}B = \left\{ {}^{(i)}b_j, {}^{(i)}b_j^{-1} \,\middle|\, 1 \leqslant j \leqslant {}^{(i)}d_v/2 \right\}. $$

Choose some elements ${}^{(0)}a \in {}^{(0)}A$, ${}^{(0)}b \in {}^{(0)}B$ and elements $\widehat{a}_1, \widehat{a}_2 \in A_{k,\ell}$, so that $\widehat{a}_1 \neq \widehat{a}_2, \widehat{a}_2^{-1}$, elements $\widehat{b}_1, \widehat{b}_2 \in B_{k,\ell}$ so that $\widehat{b}_1 \neq \widehat{b}_2, \widehat{b}_2^{-1}$ and $\widetilde{a}_1, \widetilde{a}_2 \in A_{4,4}$ so that $\widetilde{a}_1 \neq \widetilde{a}_2, \widetilde{a}_2^{-1}$, elements $\widetilde{b}_1, \widetilde{b}_2 \in B_{4,4}$ so that $\widetilde{b}_1 \neq \widetilde{b}_2, \widetilde{b}_2^{-1}$. Denote the elements of the set

$$ \left(\bigcup_{i=1}^{r} \left\{ {}^{(i)}a_j \,\middle|\, 1 \leqslant j \leqslant {}^{(i)}d_h/2 \right\} \right) \cup \left\{ {}^{(0)}a, \widetilde{a}_2 \right\} $$

by

$$ \left\{ a_j \,\middle|\, 1 \leqslant j \leqslant t_h = 2 + \sum_{i=1}^{r} {}^{(i)}d_h/2 \right\} $$

and the elements of

$$ \left(\bigcup_{i=1}^{r} \left\{ {}^{(i)}b_j \,\middle|\, 1 \leqslant j \leqslant {}^{(i)}d_v/2 \right\} \right) \cup \left\{ {}^{(0)}b, \widetilde{b}_2 \right\} $$

by $\left\{ b_j \,\middle|\, 1 \leqslant j \leqslant t_v = 2 + \sum_{i=1}^{n} {}^{(i)}d_v/2 \right\}$. Let $\tau_h \in S_{t_h}$ be the cycle $\tau_h = (1, 2, \dots, t_h)$ and $\tau_v \in S_{t_v}$ be the cycle $\tau_v = (1, 2, \dots, t_v)$. Since for each $1 \leqslant i \leqslant t_v$, $(\widehat{a}_1, b_i, \widehat{a}_1^{-1}, b_i^{-1}) \in {}^{Y}R$ and correspond to distinct geometric square, we may perform the surgery $F^{\tau_v}_{\widehat{a}_1; b_1, \dots, b_{t_v}}$ on Y. Observe that we may perform the surgery $F^{\tau_h}_{\widehat{b}_1; a_1, \dots, a_{t_h}}$ on the resulting square complex. Denote the resulting square complex by $\widehat{Y}$. On this square complex we perform the surgery $F^{(1,2)}_{\widetilde{a}_1; {}^{(0)}b, \widehat{b}_2}$ followed by $F^{(1,2)}_{\widetilde{b}_1; {}^{(0)}a, \widehat{a}_2}$. Let X be the 1-vertex VH-T-square complex obtained via these surgery operations and $(A, B, \varphi_A, \varphi_B, R)$ be its VH-datum. Observe that as the surgeries did not involve any of the squares of the VH-T-square complex

${}^{(i)}X$, $0 \leqslant i \leqslant n$, $C_{k,\ell}$ and $C_{4,4}$ we have embeddings ${}^{(i)}X \to X$, $0 \leqslant i \leqslant n$, $C_{k,\ell}$, $C_{4,4} \to X$ as asserted in (1).

To verify that the permutation group P_h is 2-transitive, observe first that the element $\sigma_{\widehat{a}_1} = \Psi_A(\widehat{a}_1) \in \mathrm{Sym}(B)$ contains the cycle $(b_{t_v}, b_{t_v-1}, \dots, t_2, t_1)$. Thus any element of $\bigcup_{i=1}^{n} {}^{(i)}B$ may be moved (using an appropriate power of $\sigma_{\widehat{a}_1}$) into ${}^{(0)}B$. Also using the transitivity properties of $C_{4,4}$ together with $\sigma_{\widehat{a}_1}$, every element of ${}^{(1)}B_{4,4}$ may be brought into ${}^{(0)}B$. We also have $\sigma_{\widetilde{a}_1} = \Psi_A(\widetilde{a}_1) \in \mathrm{Sym}\, B$ which contains the cycle $(b_0, \widehat{b}_2)$. This together with the transitivity properties of $C_{k,\ell}$ allows us also to bring any element of $B_{k,\ell}$ into ${}^{(0)}B$. Moreover, given any pair of elements in B, we may first bring one of them into ${}^{(0)}B$. If now the second element has not already been brought into ${}^{(0)}B$, we may move the image of the first one within ${}^{(0)}B$ to allow bringing the other into ${}^{(0)}B$ without moving out the first element. Thus given any $b, b' \in B$, there exists some $\pi \in P_v$ such that $\pi(b), \pi(b') \in {}^{(0)}B$. Since, by assumption, ${}^{(0)}P_v$ is acting 2-transitively on ${}^{(0)}B$, we conclude that P_v is 2-transitive on B. The argument for P_h is analogous. Thus (2) holds.

Assertion (3) follows from a theorem of Marggraf (1892) (see [Wi]$_1$ Theorem 13.5) using the observation that ${}^{(0)}P_v$ (resp. ${}^{(0)}P_h$) is embedded in P_v (resp. P_h) so that its action on ${}^{(0)}B$ (resp. ${}^{(0)}A$) is transitive and it fixes pointwise the complement $B \setminus {}^{(0)}B$ (resp. $A \setminus {}^{(0)}A$). □

Proposition **6.2.** — *Let* Z *be a 1-vertex VH-T-complex; then there is a 1-vertex VH-T-complex* Y *such that,*

(1) Z *embeds into* Y,

(2) *the groups* P_h^Y *and* P_v^Y *consist of even permutations.*

Proof. — Let $(A, B, \varphi_A, \varphi_B, R)$ be the VH-datum corresponding to Z. Define Y to be the VH-T-square complex corresponding to the VH-datum $(\widetilde{A}, \widetilde{B}, \varphi_{\widetilde{A}}, \varphi_{\widetilde{B}}, \widetilde{R})$ where

$$\widetilde{A} = A \times \{1, 2\}, \widetilde{B} = B \times \{1, 2\}, \varphi_{\widetilde{A}} = \varphi_A \times id, \ \varphi_{\widetilde{B}} = \varphi_B \times id$$

and

$$\widetilde{R} = \{((a, i), (b, j), (a', i), (b', j)) \mid (a, b, a', b') \in R \quad i, j \in \{1, 2\}\} \quad \square$$

6.4. In this section we state and prove the main results of this paper.

Theorem **6.3.** — *For every* $n \geqslant 15$ *and* $m \geqslant 19$*, there exists a torsion free, cocompact lattice* $\Gamma < U(A_{2n}) \times U(A_{2m})$ *with dense projections. Any non trivial, normal subgroup* $N \lhd \Gamma$ *is of finite index in* Γ.

Proof. — We apply Proposition 6.1 to the case where $n = 0$ and ${}^{(0)}X$ is the arithmetic quotient $\mathscr{A}_{13,17}$ (see 2.4); in this case ${}^{(0)}d_h = 14$, ${}^{(0)}P_h < S_{{}^{(0)}d_v}$ is permutation isomorphic to the $PSL(2, \mathbf{F}_{13})$ action on $\mathbf{P}^1(\mathbf{F}_{13})$, ${}^{(0)}d_v = 18$, ${}^{(0)}P_v < S_{{}^{(0)}d_v}$ is permutation isomorphic to the $PSL(2, \mathbf{F}_{17})$ action on $\mathbf{P}^1(\mathbf{F}_{17})$, in particular both groups are 2-transitive and consist of even permutations. Let k, ℓ be even integers with $k \geqslant 12$, $\ell \geqslant 16$ and let X be the complex given by Proposition 6.1. Then $P_h = A_{2n}$, $2n = 18 + k$ and $P_v = A_{2m}$, $2m = 22 + \ell$; moreover, the monomorphism $A_{13,17} \hookrightarrow X$ implies that $X = \Gamma \backslash (\mathscr{T}_{2n} \times \mathscr{T}_{2m})$ is irreducible, and hence (Proposition 5.2) $\Gamma < U(A_{2n}) \times U(A_{2m})$ has dense projections; the last assertion follows then from Corollary 5.1. □

Definition. — *A group* Γ *is virtually simple, if the intersection* $\Gamma^{(\infty)}$ *of all finite index subgroups of* Γ *is of finite index in* Γ *and simple.*

Observe that this amounts to say that Γ *admits a subgroup of finite index which is simple.*

Theorem **6.4.** — *For every* $n \geqslant 109$ *and* $m \geqslant 150$*, there exists a torsion free, cocompact lattice* $\Gamma < U(A_{2n}) \times U(A_{2m})$ *which is virtually simple.*

Proof. — We apply Proposition 6.1 to the case where $n = 1$, ${}^{(0)}X = A_{13,17}$ and ${}^{(1)}X = A_{13,17} \boxtimes A_{13,17}$ (see 2.3), and k, ℓ (even) $\geqslant 4$. As in Theorem 6.3, the resulting complex $X = \Gamma \backslash (\mathscr{T}_{2n} \times \mathscr{T}_{2m})$ is irreducible with $P_h = A_{2n}$, $P_v = A_{2m}$, $2n = 214 + k$, $2m = 346 + \ell$; moreover (Proposition 2.4 and Corollary 2.5), ${}^{(1)}X = A_{13,17} \boxtimes A_{13,17}$ has a non-residually finite fundamental group, injecting into $\pi_1(X)$ which shows that all conditions of Corollary 5.4 are satisfied and thus $\Gamma = \pi_1(X)$ is virtually simple. □

Theorem **6.5.** — *Let* Z *be a* 1*-vertex, VH-T-complex. Then there exists a* 1*-vertex VH-T-complex* X *such that:*

(1) Z *embeds into* X,

(2) $\pi_1(X)$ *is virtually simple.*

In particular, $\pi_1(Z)$ *is isomorphic to a subgroup of* $\pi_1(X)$.

Proof. — Applying Proposition 6.2, we may assume that P_h^Z and P_v^Z consist of even permutations. We apply then Proposition 6.1 to the case $n = 2$, ${}^{(0)}X = A_{13,17}$, ${}^{(1)}X = A_{13,17} \boxtimes A_{13,17}$, ${}^{(2)}X = Z$, $k = \ell = 4$ and argue as in Theorem 6.4. □

6.5. In Section 6.4 we proved the existence of a finite VH-T-square complex whose fundamental group is simple. Note, however, that the construction in 6.4 gives an explicit 1-vertex VH-T-square complex X whose fundamental group is virtually simple; to obtain the square complex having a simple fundamental group one has to consider the maximal finite covering $X^{(\infty)}$ of X. We do not know whether a general procedure for finding the maximal finite covering of a given VH-T-square complex with virtually simple group exists. We note, however, that for the square complexes constructed in the proof of Theorem 6.3, there exists a Turing machine which, given X produces $X^{(\infty)}$; we do not know how to bound the size of $X^{(\infty)}$ in function of X. In this section we give an explicit construction of a finite VH-T-square complex whose fundamental group is simple.

Let Y be a one vertex VH-T-square complex corresponding to a VH-datum $(A, B, \varphi_A, \varphi_B, R)$ and satisfying:

Q1. $\pi_1(Y) < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ is an irreducible lattice. (Where $T_1 \times T_2$ is the universal covering space of Y.)

Q2. The finite permutation groups $P_v < \mathrm{Sym}\, B$ and $P_h < \mathrm{Sym}\, A$, as well as their respective socles are 2-transitive on the corresponding set.

Q3. Let $|A| = 2k$ and $|B| = 2\ell$ with $\ell \geqslant k + 5 + [k(k-1) + k - 1]4 + 4$.

Let D be the one vertex VH-T-square complex obtained from Y via $D = Y \boxtimes Y$, denote its VH-datum by $(A_D, B_D, \varphi_{A_D}, \varphi_{B_D}, R_D)$. Let $\widetilde{Y}$ be a copy of Y associated with VH-datum $(\widetilde{A}, \widetilde{B}, \varphi_{\widetilde{A}}, \varphi_{\widetilde{B}}, \widetilde{R})$ where for each $a \in A$ we let $\widetilde{a} \in \widetilde{A}$ denote the corresponding element, analogously $\widetilde{b} \in \widetilde{B}$ corresponds to $b \in B$, and $\widetilde{R} = \{(\widetilde{a}, \widetilde{b}, \widetilde{a}', \widetilde{b}');\ (a, b, a', b') \in R\}$. We describe next how to mate these three VH-T-square complexes, Y, D and $\widetilde{Y}$, to obtain a new one vertex VH-T-square complex Z so that Z has a covering VH-T-square complex $Z^{(\infty)}$ having 4 vertices and described explicitly, such that its fundamental group $\pi_1(Z)^{(\infty)}$ is a simple group. The idea of the mating is similar to the one used in the previous construction together with the observation that we have certain explicit closed loops in the one skeleton of D representing elements of $\pi_1(D)$ which actually belong to $\pi_1(D)^{(\infty)}$. Using appropriate mating (joining and surgery operations) allows us to use these to construct a square complex Z for which every (necessarily closed) path of length 2 in the horizontal or vertical 1-skeleton represents an element of $\pi_1(Z)^{(\infty)}$.

Let us denote the elements of the sets A, B by:

$$A = \{a_i, a_i^{-1} : 1 \leqslant i \leqslant k\}$$

$$B = \{b_j, b_j^{-1} : 1 \leqslant j \leqslant \ell\}$$

where x and x^{-1} denote the two orientations of the same geometric edge. Correspondingly we have:

$$\widetilde{A} = \{\widetilde{a}_i, \widetilde{a}_i^{-1} : 1 \leqslant i \leqslant k\}$$

$$\widetilde{B} = \{\widetilde{b}_j, \widetilde{b}_j^{-1} : 1 \leqslant j \leqslant \ell\}$$

$$A_D = \left\{ \left(a_i^{\varepsilon}, a_j^{\eta}\right) : 1 \leqslant i, j \leqslant k \quad \varepsilon, \eta \in \{\pm 1\} \right\}$$

$$B_D = \left\{ \left(b_i^{\varepsilon}, b_j^{\eta}\right) : 1 \leqslant i, j \leqslant \ell \quad \varepsilon, \eta \in \{\pm 1\} \right\}.$$

Let ${}^{(0)}Z = Y \vee D \vee \widetilde{Y}$ be the join of the 3 one vertex VH-T-square complexes $Y, D, \widetilde{Y}$. To obtain Z we shall perform a sequence of surgery operations on ${}^{(0)}Z$. All the surgery operations we shall use will be of the following special type: $F^{\tau}_{f;\, c_1, c_2}$ where τ is the transposition (1, 2); for each $i = 1, 2$, the boundary of c_i is the path ∂c_i such that $\partial c_i([0, 1]) = \partial c_i([3, 2]) = f$, $\partial c_i([1, 2]) = \partial c_i([0, 3])) = g_i$. I.e., the geometric realization of the subcomplex consisting of c_1, c_2 and their boundary is a pair of closed tori having the loop corresponding to f in common. Recall that a one vertex VH-T-square complex provides us with a natural presentation of its fundamental group. Observe that prior to the surgery operation $F^{\tau}_{f,\, c_1, c_2}$, we had the following relations: $f^{-1}g_1 f = g_1$, $f^{-1}g_2 f = g_2$ between the generators corresponding to the boundary edges of the squares c_1, c_2. In terms of the presentation of the fundamental group of the square complex, the result of applying the surgery operation $F^{\tau}_{f_i;\, c_1, c_2}$ is refered to by the phrase: "introduce the relations $f^{-1}g_1 f = g_2$, $f^{-1}g_2 f = g_1$".

As we frequently will be introducing relations:

$$\begin{array}{lcll} f^{-1}g_1 f & = & h_1, & f^{-1}h_1 f = g_1 \\ f^{-1}g_2 f & = & h_2, & f^{-1}h_2 f = g_2 \\ \vdots & & & \\ f^{-1}g_r f & = & h_r, & f^{-1}h_r f = g_r. \end{array}$$

It will be useful to denote introducing all these relations by the notation

$$g_1\, g_2 \ldots g_r \overset{f}{\longleftrightarrow} h_1\, h_2 \ldots h_r.$$

Before listing the surgery operations which transform ${}^{(0)}Z$ to Z recall that by Corollary 2.5 for any $u, u', a, a' \in A$ the closed path of length 4 in the 1-skeleton of D consisting of the edges e_1, e_2, e_3, e_4 where $e_1 = (a, u)$, $e_2 = (a^{-1}, u')$, $e_3 = (a'^{-1}, u'^{-1})$ and $e_4 = (a', u^{-1})$ represent an element of $\pi_1(D)$ which belongs to $\pi_1(D)^{(\infty)} = \bigcap N$ where the intersection is over all finite index normal subgroups of $\pi_1(D)$.

6.5.1. *List of surgery operations*

1. For $1 \leqslant i \leqslant k-3$:

$$a_k a_{k-1} a_{k-2} a_i \overset{\widetilde{b}_i}{\longleftrightarrow} (a_k, a_k)(a_k^{-1}, a_i)(a_i^{-1}, a_i^{-1})(a_i, a_k^{-1})$$

2. For $k-2 \leqslant i \leqslant k$:

$$a_4 a_3 a_2 a_i \overset{\widetilde{b}_i}{\longleftrightarrow} (a_1, a_1)(a_1^{-1}, a_i)(a_i^{-1}, a_i^{-1})(a_i, a_1^{-1})$$

3. $$a_k a_{k-1} a_{k-2} a_1^{-1} \overset{\widetilde{b}_{k+1}}{\longleftrightarrow} (a_k, a_k)(a_k^{-1}, a_1^{-1})(a_1, a_1)(a_1^{-1}, a_k^{-1})$$

4. $$a_4 a_3 a_2 a_1^{-1} \overset{\widetilde{b}_{k+2}}{\longleftrightarrow} (a_k, a_k)(a_k^{-1}, a_1^{-1})(a_1, a_1)(a_1^{-1}, a_k^{-1})$$

5. $$a_1 \cdots a_k \overset{(b_1, b_1)}{\longleftrightarrow} \widetilde{a}_1 \widetilde{a}_2 \cdots \widetilde{a}_k$$

6. $$a_1 \widetilde{a}_2 \overset{(b_2, b_2)}{\longleftrightarrow} a_1 a_2$$

7. $$a_1(a_2, a_2) \overset{\widetilde{b}_{k+3}}{\longleftrightarrow} a_1 a_2$$

8. $$\widetilde{a}_1(a_2, a_2) \overset{\widetilde{b}_{k+4}}{\longleftrightarrow} \widetilde{a}_1 \widetilde{a}_2$$

9. for $1 \leqslant i, j \leqslant k$, let

$$(a_i, a_j) \overset{\widetilde{b}_{k+5+[k(i-1)+j-1]4}}{\longleftrightarrow} a_1$$

$$(a_i, a_j) \overset{\widetilde{b}_{k+5+[k(i-1)+j-1]4+1}}{\longleftrightarrow} a_2$$

$$(a_i, a_j^{-1}) \overset{\widetilde{b}_{k+5+[k(i-1)+j-1]4+2}}{\longleftrightarrow} a_1$$

$$(a_i, a_j^{-1}) \overset{\widetilde{b}_{k+5+[k(i-1)+j-1]4+3}}{\longleftrightarrow} a_2$$

10. for $1 \leqslant i \leqslant k$, let

$$\widetilde{a}_i \overset{(b_{2i+1}, b_{2i+1})}{\longleftrightarrow} a_1$$

$$\widetilde{a}_i \overset{(b_{2i+2}, b_{2i+2})}{\longleftrightarrow} a_2$$

11. $$\widetilde{b}_{k+5+[k(k-1)+k-1]4+4} \overset{(a_2, a_2)}{\longleftrightarrow} b_4$$

12. $$\widetilde{b}_{k+5+[k(k-1)+k-1]4+4} \overset{(a_3, a_3)}{\longleftrightarrow} b_5$$

13. $$(b_{2k+3}, b_{2k+3}) \overset{\widetilde{a}_2}{\longleftrightarrow} b_6$$

14. $$(b_{2k+3}, b_{2k+3}) \overset{\widetilde{a}_3}{\longleftrightarrow} b_7$$

These surgery operations performed on $Z^{(0)}$ produce a new one vertex square complex Z with VH-datum $(A_Z, B_Z, \varphi_{A_Z}, \varphi_{B_Z}, R_Z)$.

Let the universal cover of Z be $\mathcal{T}_n \times \mathcal{T}_m$ (as usual $\mathcal{T}_r$ denotes the r-regular tree). Let $\Gamma = \pi_1(Z) < \mathrm{Aut}\mathcal{T}_n \times \mathrm{Aut}\mathcal{T}_m$, $H_i = \overline{pr_i(\Gamma)}$, $i = 1, 2$.

Claim **6.6.** — *Each* H_i *is locally 2-transitive,* $i = 1, 2$.

Proof. — This is proved using property Q2 (2-transitivity for Y) together with surgery operations 9, 10 for the first tree and operations 11-14 for the second tree. □

Recall the following result of A. Bochert:

Theorem **6.7.** ([Bo], cf. $[\mathrm{Wi}]_1$). — *Let* G *be a 2-transitive group of permutations of a set of* d *elements. If the size of the fixed point set of some non trivial* $g \in G$ *exceeds* $\frac{2}{3}d + \frac{2}{3}\sqrt{d}$ *then* G *is either* A_d *or* S_d.

Observe that the element $pr_1(b_1) \in H_1$ fixes a vertex as well as more than $\frac{2}{3}n + \frac{2}{3}\sqrt{n}$ edges out of the n edges at that vertex. Thus we conclude using the theorem of Bochert that for any vertex x the finite group $\underline{H}_1(x)$ is the alternating group A_n (note that all the elements of $\underline{H}_1(x)$ are even permutation). (A similar argument using $pr_2(a_1) \in H_2$ may be used to study H_2 instead of H_1.) Applying now $[\text{B-M}]_3$ Proposition 3.3.1 together with the fact that each H_i is non discrete we conclude:

Claim **6.8.** — *The subgroups* H_i *satisfy*

$$H_1 = U(A_n) \quad H_2 = U(A_m). \quad \square$$

Thus we already know that for the fundamental group $\Gamma = \pi_1(Z)$ we have that $\Gamma^{(\infty)} = \bigcap_{\substack{N \triangleleft \Gamma \\ f.i}} N$ is a finite index simple normal subgroup of Γ. Let $Z^{(\infty)}$ denote the corresponding (maximal) finite cover of Z.

Proposition **6.9.** — *The subgroup* $\Gamma^{(\infty)}$ *is the subgroup of index 4 in* Γ *generated by the elements of* Γ *corresponding to horizontal or vertical paths of length 2.*

This proposition follows using the following claims:

Claim **6.10.** — *Every horizontal path of length 2 in* Z *represents an element belonging to* $\Gamma^{(\infty)}$.

Proof. — Recall that paths of length 2 corresponds to words of length 2 in A_Z. Observe that for $a, a', u, u' \in A$ the path of length 4 determined by $(a, u)(a^{-1}, u')(a'^{-1}, u'^{-1})(a', u^{-1})$ represents a word belonging to $\pi_1(Y_D)^{(\infty)}$ and hence also to $\Gamma^{(\infty)}$. The surgery operations $1, 2, 3, 4$ imply that each of the words $a_k a_{k-1} a_{k-2} a_i$, $1 \leqslant i \leqslant k-3$, $a_4 a_3 a_2 a_j$, $k-2 \leqslant j \leqslant k$, $a_k a_{k-1} a_{k-2} a_1^{-1}$ and $a_4 a_3 a_2 a_1^{-1}$ is conjugate to an element of $\Gamma^{(\infty)}$ and thus is itself in $\Gamma^{(\infty)}$. This however implies that any word of the form aa' for $a, a' \in A$ is in $\Gamma^{(\infty)}$. Indeed, we have

$$a_i^{-1} a_j = (a_k a_{k-1} a_{k-2} a_i)^{-1} (a_k a_{k-1} a_{k-2} a_j) \in \Gamma^{(\infty)} \quad 1 \leqslant i, j \leqslant k-3$$
$$a_1 a_i = (a_k a_{k-1} a_{k-2} a_1^{-1})^{-1} (a_k a_{k-1} a_{k-2} a_i) \in \Gamma^{(\infty)} \quad 1 \leqslant i \leqslant k-3.$$

Thus we see that in $\Gamma/\Gamma^{(\infty)}$ the images of all the elements a_i, a_i^{-1} where $1 \leqslant i \leqslant k-3$ are the same element, say t, and moreover t is its own inverse, $t = t^{-1}$. Now since for each j with $k-2 \leqslant j \leqslant k$ we have $a_4 a_3 a_2 a_j \in \Gamma^{(\infty)}$ we conclude using the fact that $a_4 a_3 \in \Gamma^{(\infty)}$ that $a_2 a_j \in \Gamma^{(\infty)}$ and now it is clear that for any two elements $a, a' \in A = \{a_i, a_i^{-1} : 1 \leqslant i \leqslant k\}$ we have $aa' \in \Gamma^{(\infty)}$. Applying next the conjugacy given by surgery operation 5 we conclude that also paths of the form $\tilde{a}\tilde{a}'$ are in $\Gamma^{(\infty)}$ for any $\tilde{a}, \tilde{a}' \in \tilde{A}$. We turn to paths of length 2 coming from A_D^2 (where A_D is defined in 6.5 and A_D^2 means words of length 2 of elements in A_D). Here we consider again the conjugacy provided by operations 1-4. Observe that from these now follow that the following elements are in $\Gamma^{(\infty)}$:

$$(*) \qquad \begin{cases} (a_k, a_k)\,(a_k^{-1}, a_i) & 1 \leqslant i \leqslant k-3 \qquad (a_k^{-1}, a_i)\,(a_i^{-1}, a_i^{-1}) \\ (a_1, a_1)\,(a_1^{-1}, a_i) & k-2 \leqslant i \leqslant k \qquad (a_1^{-1}, a_i)\,(a_i^{-1}, a_i^{-1}) \\ (a_k, a_k)\,(a_k^{-1}, a_1^{-1}) & \qquad\qquad\qquad (a_k^{-1}, a_1^{-1})\,(a_1, a_1) \end{cases}$$

Recall that for $\Gamma_D = \pi_1(Y_D)$ we have $\Gamma_D/\Gamma_D^{(\infty)} \hookrightarrow \Lambda^2$ where $\Lambda = \pi_1(Y)$. Where the map is the one naturally induced by viewing each edge (x, y) in Y_D as an element of Λ^2. Note also that the image is the index 4 subgroup of Λ^2 consisting of the elements of the form (γ_1, γ_2) such that both "horizontal" and "vertical" length of $\gamma_1\gamma_2$ are even. Thus as $\Gamma^{(\infty)} \cap \Gamma_D$ contains $\Gamma_D^{(\infty)}$ as well as the elements listed in $(*)$ we deduce that $\Gamma^{(\infty)} \cap \Gamma_D$ contains the elements:

$$(**) \qquad \begin{cases} (e, a_k a_i) & 1 \leqslant i \leqslant k-3 \qquad (a_k^{-1} a_i^{-1}, e) \\ (e, a_1 a_i) & k-2 \leqslant i \leqslant k \qquad (a_1^{-1} a_i^{-1}, e) \\ (e, a_k a_1^{-1}) & \qquad\qquad\qquad (a_k^{-1} a_1, e) \end{cases}$$

Thus we may conclude that also horizontal paths of length 2 coming from A_D^2 are in $\Gamma^{(\infty)}$. The conjugacies provided by surgery operations 6, 7, 8 allow us to conclude that any horizontal path of length 2 corresponding to an element of A_Z^2 represent an element belonging to $\Gamma^{(\infty)}$ (and hence corresponds to a closed path in $Z^{(\infty)}$).

Establishing the analogous assertion for vertical paths is provided by the following general observation applied to our VH-T-square complex Z.

Lemma **6.11.** — *Let* W *be a one vertex* VH-T-*square complex. Let* $\Lambda = \pi_1(W) < \mathrm{AutT}_1 \times \mathrm{AutT}_2$, $H_i = \overline{pr_i(\Gamma)}$. *Assume that each* H_i *is locally 2-transitive. Let* $N \lhd \Lambda$ *be a normal subgroup such that* $A_W^2 \subset N$, *i.e. any horizontal path of length 2 namely a path represented by elements of* $A_W^2 = \{aa' : a, a' \in A_w\}$ *correspond to an element of* N. *Then also vertical paths of length 2 correspond to elements of* N, *i.e.* $B_W^2 \subset N$.

Proof. — There exist some $b \in B_W$ and $\alpha \in A_W^2$ such that $b\alpha b^{-1} \notin A_W^2$ in Λ. To see this, observe that for any $a \in A$ and $b \in B$ there exist unique $a' \in A$ and $b' \in B$ such that $ba = a'b'$ thus it follows that for any $b \in B$, any $n \geqslant 1$ and any $\alpha \in A^n$ there is a unique $b' \in B$ and an $\alpha' \in A^n$ (unique when α is "reduced") so that $b\alpha = a'b'$. The assertion is that there is some $b \in B$ and $\alpha \in A^2$ so that the corresponding $b' \in B$ and $a' \in A^2$ such that $b\alpha = a'b'$ satisfy $b' \neq b$. If no such b and α exist then it would follow that given any $b \in B$ the collection of $b' \in B$ for which there exist $\alpha, \alpha \in A^n$ (for some $n \geqslant 1$) so that $b\alpha = a'b'$ consists of at most two elements. This is however impossible since the transitivity of the action of H_2 on the edges at a vertex implies in particular that given any $b' \in B$ there exist some $n \geqslant 1$ and $\alpha, \alpha' \in A^n$ such that $b\alpha = a'b'$. We have then $b\alpha b^{-1} = \alpha'\beta'$ for some $\alpha' \in A_W^2$ $e \neq \beta' \in B_W^2$. This implies that $\beta' \in N$. Using the local 2-transitivity assumption for H_2 we conclude that for any $e \neq \beta \in B_W^2$ there exist some $\alpha_1, \alpha_2 \in A_W^r$ for some $r \in W$ so that we have: $\alpha_1\beta = \beta'\alpha_2 \Rightarrow \alpha_1^{-1}\beta'\alpha_2 = \beta$. Observe that as both α_1, α_2 are of the same length we have $\alpha_1 = \alpha_2$ in Λ/N; hence β is conjugate to $\beta' = e$ in Λ/N. Hence also $\beta \in N$. □

Combining the above we obtain

Corollary **6.12.** — *The subgroup* $\Gamma^{(\infty)}$ *is of index 4 in* Γ. *The corresponding VH-T-square complex* $Z^{(\infty)}$ *is the complex having 4 vertices* v_0, v_1, v_2, v_3. *For each* $a \in A_Z$ *we have an edge* $a^{(b)}$ *with* $t(a^{(b)}) = v_1$, $o(a^{(b)}) = v_0$ *and an edge* $a^{(t)}$ *with* $t(a^{(t)}) = v_3$, $o(a^{(t)}) = v_2$. *For each* $b \in B_Z$ *we have an edge* $b^{(l)}$ *with* $t(b^{(l)}) = v_0$, $o(b^{(l)}) = v_3$ *and an edge* $b^{(r)}$ *with* $t(b^{(r)}) = v_2$, $o(b^{(r)}) = v_1$. *For each* $(a, b, a', b') \in R_Z$ *we have a square with vertices* v_0, v_1, v_2, v_3 *and edges* $a^{(b)}, b^{(r)}, a'^{(t)}, b'^{(l)}$.

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Manuscrit reçu le 17 juin 1999.