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CONTINUOUS BOUNDED COHOMOLOGY AND APPLICATIONS TO RIGIDITY THEORY

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Introduction and Statement of the Results

We present a theory of continuous bounded cohomology of locally compact groups with coefficients in Banach modules. A central rôle is played by amenable actions, as they give rise to relatively injective resolutions.

Further, we propose a substitute for the Mautner property, based on the virtual subgroup viewpoint, and we show (Theorem 6) that all compactly generated locally compact groups, e.g. finitely generated groups, satisfy it. This, together with the cohomological characterization of amenable actions, leads to a refined version of a higher degree Lyndon–Hochschild–Serre exact sequence (Theorem 13), which entails a stronger Künneth type formula for continuous bounded cohomology in degree two.

We apply this theory to general irreducible lattices in products of locally compact groups: we obtain notably super-rigidity results for bounded cocycles (Theorem 16 and Corollary 23), rigidity results for actions by diffeomorphisms on the circle (Corollary 22) and vanishing of the stable commutator length (Corollary 32). More applications will be published elsewhere.

In the spirit of relative homological algebra, we give for a locally compact second countable group G a functorial characterization of the continuous bounded cohomology of G with coefficients.

The resolutions and the notion of relatively injective objects (Definition 1.4.2) are set up in the category of continuous Banach G-modules, while the coefficients are mainly duals of separable continuous Banach G-modules (henceforth called *coefficient modules*), including notably separable continuous unitary representations, L^∞ spaces and trivial coefficients. We emphasize that on all Banach G-modules, the G-action is isometric. If E is a coefficient module and S a regular measure G-space (see Definition 1.3.1), let $L^\infty_{\mathrm{w*}}(S,E)$ be the space of weak-* measurable essentially bounded maps; we consider the resolution

$$0 \longrightarrow E \xrightarrow{d} L^\infty_{\mathrm{w*}}(S,E) \xrightarrow{d} L^\infty_{\mathrm{w*}}(S^2,E) \xrightarrow{d} L^\infty_{\mathrm{w*}}(S^3,E) \xrightarrow{d} \cdots$$

where d is the standard homogeneous coboundary operator. If $S = G$, we call this the *standard resolution* and define the continuous bounded cohomology $\mathrm{H}^\bullet_{\mathrm{cb}}(G, E)$ to be the cohomology of the associated non-augmented complex of invariants, endowed with the quotient semi-norm. In the functorial approach, we show that this standard resolution is indeed relatively injective. However, for an actual computation of the bounded cohomology, it is desirable to size down the G-space S, while keeping the above resolution relatively injective. Our first result is a necessary and sufficient condition on the G-space S for this to happen.

Theorem 1. *Let G be a locally compact second countable group and S a regular G-space. The following assertions are equivalent*

- *(i) The G-action on S is amenable in the sense of Zimmer [Z3].*
- *(ii) The Banach G-module $L^\infty(S)$ is relatively injective.*
- *(iii) The Banach G-module $L^\infty_{\mathrm{w}*}(S^{n+1}, E)$ is relatively injective for all $n \geq 0$ and every coefficient G-module E.*

Recall that examples of amenable G-spaces are Poisson boundaries of *étalées* measures on locally compact groups [Z2, Corollary 5.3] and homogeneous spaces G/P, where $P < G$ is a closed amenable subgroup [Z3, Proposition 4.3.2].

With this cohomological characterization at hand, we establish the following result, which is indeed the starting point of our applications; we insist on the fact that the claimed isomorphisms between cohomology groups are isometries of semi-normed spaces.

Theorem 2. *Let G be a locally compact second countable group, S an amenable regular G-space and E a coefficient G-module. There is a canonical isometric isomorphism between the continuous bounded cohomology $\mathrm{H}^\bullet_{\mathrm{cb}}(G, E)$ and the cohomology of the complex*

$$0 \longrightarrow L^\infty_{\mathrm{w}*}(S, E)^G \longrightarrow L^\infty_{\mathrm{w}*}(S^2, E)^G \longrightarrow L^\infty_{\mathrm{w}*}(S^3, E)^G \longrightarrow \cdots$$

of bounded measurable invariant cochains on S. The same holds for the subcomplex of alternating bounded measurable invariant cochains.

Example 3. If G is an amenable group, we may take S to be a one point space and deduce $\mathrm{H}^n_{\mathrm{cb}}(G, E) = 0$ for all $n \geq 1$ and every coefficient module E. This is but a new approach to an old result of B.E. Johnson [Jo].

Example 4. Let G be a connected semi-simple Lie group, $\Gamma < G$ a lattice and $P < G$ a minimal parabolic subgroup. Using Theorem 2 we obtain for real coefficients a canonical isometric identification

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma) \cong Z L^\infty_{\mathrm{alt}}((G/P)^3)^\Gamma,$$

where the right-hand side is the space of Γ-invariant alternating measurable bounded cocycles on $(G/P)^3$.

In Example 4, the concrete realization of $\mathrm{H}^2_\mathrm{b}(\Gamma)$ in terms of bounded cocycles on a flag manifold turns out to be essential for our applications to rigidity questions (see also [I]). This realization is a consequence of the ergodicity of the diagonal Γ-action on $G/P \times G/P$, which is itself a consequence of the Mautner property. Recall that for a connected semi-simple Lie group without compact factors the Mautner property states that in a continuous unitary representation of G, any vector invariant under a maximal split torus is G-invariant. We now proceed to generalize this Mautner property to all compactly generated locally compact groups, thereby obtaining an extension of Example 4 to a much wider framework. For this, the following ergodicity property will turn out to be a flexible tool:

DEFINITION 5. Let $\mathfrak{X}$ be any class of coefficient Banach modules, G a locally compact group and S a regular G-space (see 1.3.1). We say that the G-action on S is *doubly $\mathfrak{X}$-ergodic* if for every coefficient G-module F in $\mathfrak{X}$, any weak-* measurable function

$$f:\ S \times S \longrightarrow F$$

which is G-equivariant for the diagonal action is essentially constant.

We synonymously say that S is a doubly $\mathfrak{X}$-ergodic G-space and simply write "doubly F-ergodic" if $\mathfrak{X}$ is reduced to a single coefficient module F.

One of the virtues of this strong ergodicity property is its persistence by passing to closed subgroups $H < G$ of finite invariant co-volume for suitable classes $\mathfrak{X}$, notably the class of unitary representations (Proposition 3.2.4).

In this language, the two classical instances of this generalized Mautner property are the following: let G be a semi-simple connected group or the automorphism group $\mathrm{Aut}(\mathcal{T})$ of a regular tree, and let $Q < G$ be a parabolic subgroup in the first case, the stabilizer of a point in the boundary $\partial_\infty \mathcal{T}$ at infinity in the second case. Then the G-space G/Q with its canonical class of quasi-invariant measures is doubly $\mathfrak{X}^{\mathrm{cont}}$-ergodic, where $\mathfrak{X}^{\mathrm{cont}}$ is the class of all continuous coefficient modules.

Restricting Q to be minimal parabolic in the first case, we have moreover that in both cases the G-action on G/Q is amenable in the sense of Zimmer [Z3].

These two classes of examples, together with the solution to Hilbert's fifth problem, are used to establish the following

Theorem 6. *Let G be a compactly generated locally compact group.*

There exists a canonical topologically characteristic finite index open subgroup $G^* \lhd G$ and a regular G^*-space S such that

(i) The G^*-action on S is amenable.

(ii) The G^*-action on S is doubly $\mathfrak{X}^{\mathrm{sep}}$-ergodic, where $\mathfrak{X}^{\mathrm{sep}}$ is the class of all separable coefficient modules.

Moreover, if G is either connected or totally diconnected (e.g. discrete), then $G^* = G$.

As we shall see (Proposition 1.1.4), a separable coefficient module is necessarily continuous.

Remark 7. Theorem 6 implies that the commensurator super-rigidity results [BuMo1, Theorem 0.1] and [Bu, Theorem 2] hold unconditionally for all lattices Γ in any locally compact second countable group, generalizing Margulis' commensurator super-rigidity.

Remark 8. It will follow from the proof of Theorem 6 and from a result of V. Kaimanovich [K] that we can take S to be the Poisson boundary of (the random walk associated to) an étalée measure on G^*; see Remark 3.5.1 below.

As a rather direct consequence of Theorems 6 and 2, we obtain

Corollary 9. *Let G be a compactly generated locally compact second countable group and $\alpha : E \to F$ an injective adjoint morphism of coefficient modules. Assume F is separable. Then*

(i) $\mathrm{H}^1_{\mathrm{cb}}(G, E) = 0$.

(ii) The induced map $\mathrm{H}^2_{\mathrm{cb}}(G, E) \to \mathrm{H}^2_{\mathrm{cb}}(G, F)$ is injective and both spaces are Banach spaces.

In particular, if F is a separable coefficient module, then $\mathrm{H}^1_{\mathrm{cb}}(G, F) = 0$ and $\mathrm{H}^2_{\mathrm{cb}}(G, F)$ is a Banach space.

Remarks 10. (a) The first statement is well known for reflexive coefficients since in this case it follows from the Ryll–Nardzewski fixed point theorem [Bour1, IV, Appendice, N° 3]. The second statement was previously only known if simultaneously G is discrete and $F = \mathbf{R}$.

(b) The first statement has the following consequence. Let Γ be any group acting by isometries on a separable dual Banach space F. If Γ has a bounded orbit in F, then there is a Γ-fixed point in F (a compactness argument reduces the problem to the case of Γ finitely generated). However, this latter statement follows from N. Bourbaki's general version of Ryll–Nardzewski's theorem, see Lemme 3 in [Bour1, IV, Appendice, N° 3] under the assumption *c)* given therein.

(c) We point out that if F is not separable, both conclusions of Corollary 9 may fail, as one can see e.g. with the identity (Corollary 1.6.6)

$$\mathrm{H}^n_{\mathrm{cb}}\big(G, L^\infty(G)/\mathbf{C}\big) \cong \mathrm{H}^{n+1}_{\mathrm{cb}}(G) \qquad (\forall\, n \geq 1)\,,$$

recalling that $\mathrm{H}^2_{\mathrm{cb}}(G)$ is non-zero (in fact infinite dimensional) for any non-elementary Gromov-hyperbolic group [EF] and that for a non-amenable surface group $\mathrm{H}^3_{\mathrm{cb}}(G)$ is not Hausdorff [So1,2]. The assumption that F be a coefficient module is also crucial: indeed, let Γ be any finitely generated group. Consider the separable coefficient module $\ell^1(\Gamma)$ and its codimension one Banach submodule F consisting of the functions of total sum zero; F is not a coefficient module, and indeed the reader may check that $\mathrm{H}^1_{\mathrm{b}}(\Gamma, F)$ vanishes (if and) only if Γ is finite.

A powerful tool in the study of the ordinary cohomology of, say cocompact, lattices $\Gamma < G$ is provided by the Blanc–Eckmann–Shapiro lemma [Bl], which gives an isomorphism between the cohomology of Γ and the continuous cohomology of G with coefficients in the unitary G-module $L^2(\Gamma\backslash G)$. In specific situations, a good knowledge of the decomposition of $L^2(\Gamma\backslash G)$ into irreducible representations gives in return information about the cohomology of Γ. In the context of bounded cohomology, one checks readily that there is an analogous isomorphism between $\mathrm{H}^\bullet_{\mathrm{cb}}(H)$ and $\mathrm{H}^\bullet_{\mathrm{cb}}(G, L^\infty(H\backslash G))$ for any closed subgroup $H < G$; the drawback however is that very little is known about the G-module $L^\infty(H\backslash G)$. Nonetheless, in degree two, Theorem 6 allows us to fight our way back to unitary representations:

Corollary 11. *Let G be a compactly generated locally compact second countable group and $H < G$ a closed subgroup of finite invariant co-volume. Let E be a separable coefficient H-module. Then the L^2 induction*

$$\mathbf{i} : \mathrm{H}^2_{\mathrm{cb}}(H, E) \longrightarrow \mathrm{H}^2_{\mathrm{cb}}(G, L^2\mathrm{Ind}_H^G E)$$

is injective.

Example 12. The fundamental group $\Gamma = \pi_1\Sigma$ of a surface Σ of genus $g \geq 2$ is Gromov-hyperbolic and hence $\mathrm{H}^2_{\mathrm{b}}(\Gamma)$ is an infinite dimensional Banach space (in this case, the result goes back to [BrS] and [Mit]). But by Corollary 11, any hyperbolization $\Gamma \to G = \mathrm{PSL}_2(\mathbf{R})$ of Σ yields an injection

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma) \longrightarrow \mathrm{H}^2_{\mathrm{cb}}\big(G, L^2(\Gamma\backslash G)\big)\,.$$

This suggests the question of how this infinite dimensional space gets distributed over the spectral decomposition of $L^2(\Gamma\backslash G)$ into irreducible representations. We show in [BuM3] that $\dim\mathrm{H}^2_{\mathrm{cb}}(G, \mathfrak{H}) = 1$ for all spherical representations $\mathfrak{H}$ of G, while $\mathrm{H}^2_{\mathrm{cb}}(G,\mathfrak{H})=0$ for all representations of the discrete series.

We consider now the behaviour of continuous bounded cohomology under group extensions. In view of the vanishing result given by Corollary 9, we are going to establish a higher degree Lyndon–Hochschild–Serre exact sequence, special cases of which were established for discrete groups by G.A. Noskov [N2] and A. Bouarich [Bou]. Taking advantage of Theorem 6, we will then obtain the following refinement in which the new feature is the term $\mathrm{H}^2_{\mathrm{cb}}(N,F^{Z_G(N)})^Q$, where $Z_G(N)$ is the centralizer of N in G:

Theorem 13. *Let $1\to N\to G\to Q\to 1$ be an exact sequence of locally compact second countable groups, with N compactly generated. Let (π,F) be a separable coefficient G-module. Then we have an exact sequence*

$$0\longrightarrow \mathrm{H}^2_{\mathrm{cb}}(Q,F^N)\xrightarrow{\ \mathrm{inf}\ }\mathrm{H}^2_{\mathrm{cb}}(G,F)\xrightarrow{\ \mathrm{res}\ }\mathrm{H}^2_{\mathrm{cb}}(N,F^{Z_G(N)})^Q\longrightarrow$$

$$\longrightarrow \mathrm{H}^3_{\mathrm{cb}}(Q,F^N)\xrightarrow{\ \mathrm{inf}\ }\mathrm{H}^3_{\mathrm{cb}}(G,F)\,.$$

Our main application of Theorem 13 is to a Künneth type formula for continuous bounded cohomology in degree two, with separable coefficient modules. More precisely, let $G=G_1\times\cdots G_n$ be a product of compactly generated locally compact second countable groups G_j and F a separable coefficient G-module, e.g. a continuous unitary representation in a separable Hilbert space. Write $G'_j=\prod_{i\neq j}G_i$; then $\sum_{j=0}^n F^{G'_j}$ is closed and even weak-* closed in F (Lemma 4.4.2), hence is a coefficient G-module. In this setting, we have

Theorem 14. *There is a natural isomorphism of topological vector spaces*

$$\mathrm{H}^2_{\mathrm{cb}}(G,F)\cong\mathrm{H}^2_{\mathrm{cb}}\Big(G,\sum_{j=1}^n F^{G'_j}\Big)\cong\bigoplus_{j=1}^n\mathrm{H}^2_{\mathrm{cb}}(G_j,F^{G'_j})\,.$$

Now let (M,μ) be a regular G-space, where μ is a G-invariant probability measure; assume the number n of factors of G is at least two. We say that G acts irreducibly on M if G'_j acts ergodically for every $1\leq j\leq n$. As a consequence of Corollary 9 and Theorem 14, we obtain

COROLLARY 15. *In this setting, the inclusion of constants $\mathbf{C}\to L^\infty(M)$ induces an isomorphism $\mathrm{H}^2_{\mathrm{cb}}(G)\to\mathrm{H}^2_{\mathrm{cb}}(G,L^\infty(M))$.*

Let $\Gamma<G=G_1\times\cdots G_n$ be a lattice. Here and in the sequel we will say that Γ is *irreducible* if the projection $\mathrm{pr}_j(\Gamma)$ is dense in G_j for all $1\leq j\leq n$; this is easily seen to be equivalent to the irreducibility of the G-action on the probability space $\Gamma\backslash G$. Therefore, Corollary 15

applied to the induction module $L^\infty(\Gamma\backslash G)$ alluded to above would already yield an isomorphism of Banach spaces $\mathrm{H}^2_{\mathrm{b}}(\Gamma) \cong \mathrm{H}^2_{\mathrm{cb}}(G)$. The latter space decomposes as $\bigoplus_{j=1}^n \mathrm{H}^2_{\mathrm{cb}}(G_j)$ by Theorem 14.

Using now L^2 induction, we bring in once again the double ergodicity and proceed to generalize this isomorphism to continuous bounded cohomology with coefficients in separable coefficient modules. Let thus F be a separable coefficient Γ-module and let F_j be the maximal Γ-submodule of F such that the restriction $\pi|_{F_j}$ extends continuously to G, factoring through $G \twoheadrightarrow G_j$; this is well defined because $\overline{\mathrm{pr}_j(\Gamma)} = G_j$. Thus we have a G-action on the sum $\sum_{j=1}^n F_j$; we shall see (Lemma 5.1.2) that the latter space is again a coefficient G-module. In this setting we have

Theorem 16. *There are canonical topological isomorphisms*

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma, F) \cong \bigoplus_{j=1}^n \mathrm{H}^2_{\mathrm{cb}}(G_j, F_j) \cong \mathrm{H}^2_{\mathrm{cb}}\Big(G, \sum_{j=1}^n F_j\Big).$$

Remark 17. At first sight, there is a striking analogy between the above statement and Y. Shalom's super-rigidity for irreducible lattices [Sh2]. However, it turns out that both the actual contents and the methods of proof are completely different. For applications to rigidity theory, the interplay of Shalom's results with ours appears to be very fruitful – some instances of this are shown below.

Remark 18. Theorem 16, applied to a cohomology class constructed by Y. Shalom and the second named author, yields a super-rigidity statement for action of irreducible lattices on negatively curved metric spaces. This generalization of a result known [BuMo1] in the arithmetic case will appear elsewhere [MoS].

Remark 19. We shall actually prove the Theorem 16 for any closed subgroup $H < G$ such that G/H has finite invariant measure and with $\overline{\mathrm{pr}_j(H)} = G_j$. We also point out that the isomorphism from the rightmost to the leftmost term is realized by the restriction map.

In Theorem 16, the special case where F is a unitary representation of Γ and all G_i are algebraic groups generalizes the main results that we established in [BuM1,2] for *co-compact* lattices:

Theorem 20. *Let $\Gamma < G = \prod_{\alpha\in\mathrm{A}} \mathbf{G}_\alpha(k_\alpha)$ be an irreducible lattice, where $(k_\alpha)_{\alpha\in\mathrm{A}}$ is a finite family of local fields and the $\mathbf{G}_\alpha$ are connected simply connected k_α-almost simple groups of positive k_α-rank. Then the comparison map*

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathfrak{H}) \longrightarrow \mathrm{H}^2(\Gamma, \mathfrak{H})$$

is injective for any non-degenerate unitary representation $(\pi, \mathfrak{H})$ *of* Γ.

Here, *non-degenerate* refers to the (necessary) condition that the induced representation $\mathrm{Ind}_\Gamma^G \pi$ does not contain a subrepresentation factoring non-trivially through a rank one factor of G (if any such).

In accordance with our definition of irreducibility, the assumption in the above theorem implies that A contains at least two elements. Otherwise, we are in the almost simple case; there also, the result given in [BuM1] for co-compact lattices in a single algebraic group of higher rank can be generalized to non-uniform lattices:

Theorem 21. *Let* Γ *be a lattice in* $\mathbf{G}(k)$, *where* $\mathbf{G}$ *is a connected, simply connected, almost simple* k*-isotropic group and* k *a local field.*

If $\mathbf{G}$ *has* k*-rank at least two, then the natural map*

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathfrak{H}) \longrightarrow \mathrm{H}^2(\Gamma, \mathfrak{H})$$

is injective for any unitary representation $(\pi, \mathfrak{H})$ *of* Γ.

In order to dispose of the co-compactness assumption, we use notably the results of Lubotzky, Mozes and Raghunathan [LMR] on the word metrics of such lattices.

We turn now to applications to actions by homeomorphisms on the circle $\mathbf{S}^1$. Recall that if $\pi : \Gamma \to \mathrm{Homeo}^+(\mathbf{S}^1)$ is an action by orientation-preserving homeomorphisms, then the Euler class $e_\pi \in \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbf{Z})$ is a complete invariant of semi-conjugacy [Gh]. Denoting by $e_{\pi,\mathbf{R}}$ its image in $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbf{R})$, we record the following consequence of É. Ghys' result [Gh]:

The bounded cohomology class $e_{\pi,\mathbf{R}}$ vanishes if and only if π is semi-conjugated to a Γ-action by rotations.

We obtain thus

Corollary 22. *Let* $\Gamma < G = G_1 \times \cdots \times G_n$ *be an irreducible lattice and assume* $\mathrm{H}^2_{\mathrm{cb}}(G_j) = 0$ *for* $1 \leq j \leq n$.

Then any Γ*-action by orientation-preserving homeomorphisms of* $\mathbf{S}^1$ *is semi-conjugated to a* Γ*-action by rotations.*

If in addition the Abelianization Γ_{Ab} is finite, which happens for instance if Γ is co-compact and $\mathrm{Hom}_{\mathrm{cont}}(G_j) = 0$ for all j (see Y. Shalom [Sh2]), then the corollary can be strengthened to

(i) *Any* Γ*-action by orientation preserving homeomorphisms of the circle has a finite orbit.*

(ii) *Any* Γ*-action by orientation preserving* C^1 *diffeomorphisms of the circle factors through a finite group.*

The fact that (i) implies (ii) uses W.P. Thurston's stability theorem [T] and has been observed by several authors independently, see e.g. [W2].

Next we turn to an application to extension properties for quasimorphisms; recall that a *quasimorphism* of a group H is a function $f : H \to \mathbf{C}$ such that the map $\delta f : H \times H \to \mathbf{C}$ defined by $\delta f(x,y) = f(x)+f(y)-f(xy)$ is bounded. Combining now Theorem 16 with a result of Y. Shalom [Sh2, Theorem 0.8], we obtain

Corollary 23. *Assume that the irreducible lattice $\Gamma < G$ is co-compact. Then any quasimorphism $f : \Gamma \to \mathbf{C}$ extends to a continuous quasimorphism $f_{\mathrm{ext}} : G \to \mathbf{C}$.*

In view of the above results, it is clearly desirable to gain an understanding of $\mathrm{H}^2_{\mathrm{cb}}(G)$ for natural classes of locally compact groups. For semi-simple Lie groups over local fields and for certain groups of tree automorphisms, the second continuous bounded cohomology can be explicitly determined (see the proof of Corollaries 24 and 26 in section 5.3). This together with Theorem 16 leads to the following two corollaries.

Corollary 24. *Let $\Gamma < G = \prod_{\alpha \in \mathrm{A}} \mathbf{G}_\alpha(k_\alpha)$ be an irreducible lattice, where (k_α) is a finite set of local fields and the $\mathbf{G}_\alpha$ are connected simply connected k_α-almost simple groups of positive k_α-rank. Assume $|\mathrm{A}| \geq 2$.*

Then the comparison map from bounded to ordinary cohomology induces an isomorphism

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma) \longrightarrow \mathrm{H}^2(\Gamma)^{\mathrm{inv}},$$

where the latter is the image in $\mathrm{H}^2(\Gamma)$ under restriction of the continuous cohomology $\mathrm{H}^2_{\mathrm{c}}(G)$. Both spaces have the dimension of the number of Hermitian factors of G.

Remark 25. Let $\Gamma < G = \mathrm{SL}_2(\mathbf{R}) \times \mathrm{SL}_2(\mathbf{R})$ be a co-compact torsion free irreducible lattice. Then $\mathrm{H}^2_{\mathrm{b}}(\Gamma) \cong \mathrm{H}^2(\Gamma)^{\mathrm{inv}}$ has dimension two while

$$\dim \mathrm{H}^2(\Gamma) = c \,\mathrm{Vol}(\Gamma \backslash G) - 2\,,$$

wherein c is an absolute constant. This is in contrast with the case of Gromov-hyperbolic groups, where the comparison map in degree two (and higher) is known to be surjective [Mi1,2].

The next corollary concerns lattices in the product of automorphisms groups of locally finite regular (or bi-regular) trees. Such lattices are never irreducible in the sense of our definition (see [BuMo4]), therefore it is necessary to consider the closures $\overline{\mathrm{pr}_j(\Gamma)}$ of the canonical projections.

COROLLARY 26. *Let* $\Gamma < \mathrm{Aut}(\mathcal{T}_1) \times \cdots \times \mathrm{Aut}(\mathcal{T}_n)$ *be a lattice such that the closure* $\overline{\mathrm{pr}_j(\Gamma)}$ *acts transitively on* $\partial_\infty \mathcal{T}_j$ *for all* j. *Then we have*

$$\mathrm{H}^2_\mathrm{b}(\Gamma) = 0\,.$$

REMARK 27. In the above corollary, the assumptions on Γ depend only on its commensurability class (see [BuMo3]). In contrast to the vanishing of $\mathrm{H}^2_\mathrm{b}(\Gamma) = 0$, one has

$$\dim \mathrm{H}^2(\Gamma) \geq c\,\mathrm{Vol}(\Gamma\backslash G) - 1$$

for co-compact lattices $\Gamma < G = \mathrm{Aut}(\mathcal{T}_1) \times \mathrm{Aut}(\mathcal{T}_1)$, where $c > 0$ is some absolute constant.

A classical set of examples for Corollary 26 is provided by co-compact lattices $\Gamma < G = \prod_{\alpha\in \mathrm{A}} \mathbf{G}_\alpha(k_\alpha)$, where all $\mathbf{G}_\alpha$ have k_α-rank one and all k_α are non-Archimedean; indeed $\mathbf{G}_\alpha(k_\alpha)$ sits (modulo its centre) in the automorphism group of the associated Bruhat–Tits tree. Those lattices are linear and hence in particular residually finite.

In contrast to this class of linear examples, the following was shown in [BuMo2,4]:

For every $n \geq 109$, $m \geq 150$, there exists a torsion free co-compact lattice $\Gamma < \mathrm{Aut}(\mathcal{T}_1) \times \mathrm{Aut}(\mathcal{T}_2)$, where $\mathcal{T}_1$ and $\mathcal{T}_2$ are regular of degree $2n$ respectively $2m$, such that

(i) The closures $\overline{\mathrm{pr}_j(\Gamma)}$ act transitively on $\partial_\infty \mathcal{T}_j$.
(ii) Γ has a subgroup of finite index which is simple.

In particular, the latter simple groups provide also examples for Corollary 26 as well as for Corollary 22 and its strengthening.

We observe incidentally that adélization techniques provide us with a special class of lattices, which are irreducible in the sense introduced above because of the Strong Approximation Theorem for almost simple groups [M, II.6.8]. The situation differs slightly from the setting of Theorem 16 because we have to exhaust the *infinite* family of factors associated to all places of K:

Theorem 28. *Let* K *be a global field and* $\mathbf{G}$ *a simply connected semi-simple linear algebraic group over* K. *Denote by* $\mathcal{V}_\infty$ *the collection of Archimedean places of* K.

There are canonical topological isomorphisms

$$\mathrm{H}^2_\mathrm{b}(\mathbf{G}(K)) \cong \bigoplus_{v\in\mathcal{V}_\infty} \mathrm{H}^2_\mathrm{cb}(\mathbf{G}(K_v)) \cong \bigoplus_{v\in\mathcal{V}_\infty} \mathrm{H}^2_\mathrm{c}(\mathbf{G}(K_v))\,.$$

REMARK 29. In ordinary cohomology, A. Borel and J. Yang [BoY] prove the analogous statement for any positive degree. In particular, the rightmost term in the above statement is in return isomorphic to $\mathrm{H}^2(\mathbf{G}(K))$ and thus the natural map

$$\mathrm{H}^2_{\mathrm{b}}(\mathbf{G}(K)) \longrightarrow \mathrm{H}^2(\mathbf{G}(K))$$

is injective.

EXAMPLES 30. (i) If $d \in \mathbf{N}$ is not a square, then $\mathrm{H}^2_{\mathrm{b}}(\mathrm{SL}_2(\mathbf{Q}[\sqrt{d}]))$ has dimension two.

(ii) Let $\mathbf{G}$ be a simply connected semi-simple linear group defined over $\mathbf{Q}$. Then the restriction map

$$\mathrm{H}^2_{\mathrm{cb}}(\mathbf{G}(\mathbf{R})) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\mathbf{G}(\mathbf{Q}))$$

is an isomorphism. Thus the dimension of $\mathrm{H}^2_{\mathrm{b}}(\mathbf{G}(\mathbf{Q}))$ is exactly the number of factors of Hermitian type in $\mathbf{G}(\mathbf{R})$. We observe however that the $\mathbf{G}(\mathbf{Q})$-action on the Furstenberg boundary of $\mathbf{G}(\mathbf{R})$ is not amenable [Z4].

Notice further that since $\mathcal{V}_\infty = \varnothing$ when K has positive characteristic, the Theorem 28 implies immediately

COROLLARY 31. *Let K be a global field of positive characteristic and $\mathbf{G}$ a simply connected semi-simple linear algebraic group over K. Then*

$$\mathrm{H}^2_{\mathrm{b}}(\mathbf{G}(K)) = 0\,. \qquad \square$$

Recall that for a group Γ the *stable length* of an element $\gamma \in [\Gamma, \Gamma]$ is $\ell(\gamma) = \lim_{n\to\infty} \|\gamma^n\|/n$, where $\|\gamma\|$ is the word metric associated to the set of commutators. Ch. Bavard has given in [B] the following characterization:

Theorem (Bavard [B]). *For a discrete group Γ, the following assertions are equivalent:*

(i) The natural map $\mathrm{H}^2_{\mathrm{b}}(\Gamma) \to \mathrm{H}^2(\Gamma)$ is injective.

(ii) The stable length function ℓ of the commutator subgroup $[\Gamma, \Gamma]$ vanishes.

Thus we may apply our above results and deduce:

COROLLARY 32. *Let Γ be either*

(i) a lattice as in Theorem 21 or any of the Corollaries 22, 24, 26,

or

(ii) $\Gamma = \mathbf{G}(K)$ as in Theorem 28.

Then the stable length on the commutator subgroup $[\Gamma, \Gamma]$ vanishes. $\square$

Finally, concerning the relation between the complex and integral bounded cohomology, we observe that for any group Γ the following properties are equivalent (see the proof of Corollary 33):

(a) The comparison map $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbf{Z}) \to \mathrm{H}^2(\Gamma, \mathbf{Z})$ is injective.

(b) The comparison map $\mathrm{H}^2_{\mathrm{b}}(\Gamma) \to \mathrm{H}^2(\Gamma)$ is injective and the Abelianization Γ_{Ab} is a torsion group.

With this at hand, we conclude:

Corollary 33. *Let Γ be either*

(i) a lattice as in Theorem 21 or Corollary 24,

or

(ii) a lattice as in Corollary 26 but being moreover co-compact,

or

(iii) $\Gamma = \mathbf{G}(K)$ as in Theorem 28.

Then the comparison map $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbf{Z}) \to \mathrm{H}^2(\Gamma, \mathbf{Z})$ is injective.

Location of the proofs. Theorem 1 is proved in section 2.2, Theorem 2 is completed in section 2.3. Theorem 6 and Corollary 9 are established in section 3.5, while Corollary 11 is deduced in section 3.6. The proof of Theorem 13 is completed in section 4.3, the proofs of Theorem 14 and Corollary 15 in section 4.4. For Theorem 16 see section 5.1, for Theorems 20 and 21 section 5.2. Theorem 28 is handled in section 5.4. The Corollaries 22, 23, 24, 26 and 33 are all proved in section 5.3.

1 On Continuous Bounded Cohomology

(A more detailed and general discussion of this theory can be found in the second named author's thesis, available [Mo] in Springer's Lecture Notes.)

1.1 Banach modules.

Let G be a locally compact group (e.g. discrete). We shall work within the category of Banach G-modules, which are Banach spaces endowed with an isometric G-action. For the sake of simplicity, we leave aside the study of non-isometric uniformly bounded actions.

Definitions 1.1.1. A *Banach G-module* is a pair (π, E) where E is a Banach space over $\mathbf{R}$ or $\mathbf{C}$ and π is a (not necessarily continuous) homomorphism from G to the group of isometric automorphisms of E. Thus modules are always left modules, right modules being understood as left modules over G^{op}, the opposite group.

A map $\alpha : E \to F$ between Banach G-modules is a *G-morphism* provided it is linear, continuous and G-equivariant; $\|\alpha\|$ is its operator norm. Mind that the category we just defined is not Abelian.

The Banach G-module (π, E) is *continuous* if the action map $G \times E \to E$ is continuous; equivalently, if for all $v \in E$ the map $G \to E$, $g \mapsto \pi(g)v$ is continuous. When no confusion can arise, we simply write E for the module and gv for $\pi(g)v$.

A *dual* Banach G-module is the dual Banach space of a continuous Banach G^{op}-module endowed with dual structure; in particular, the action map is weak-* continuous but in general not norm continuous. The *contragredient* Banach G-module $(\pi^\sharp, E^\sharp)$ to a continuous Banach G-module (π, E) is the dual Banach G-module obtained via the topological isomorphism $G \to G^{\mathrm{op}}, g \mapsto g^{-1}$ (thus $E^\sharp = E^*$ as spaces, the notation emphasizing the action).

In order to avoid heavy terminology, we introduce the following concept, which will be basic in this paper.

DEFINITION 1.1.2. A *coefficient G-module* is a Banach G-module (π, E) contragredient to some separable continuous Banach G-module denoted $(\pi^\flat, E^\flat)$. The choice of $E^\flat$ is part of the data. A morphism or G-morphism of Banach modules $\alpha : E \to F$ between coefficient modules is called *adjoint* if it is the adjoint of a morphism $\alpha^\flat : F^\flat \to E^\flat$, or equivalently if it is weak-* continuous. We say synonymously that α is a *morphism (or G-morphism) of coefficient modules.*

REMARK 1.1.3. We insist that a coefficient module includes by definition the choice of a pre-dual; for it may happen that $(\pi^\flat, E^\flat)$ is not uniquely determined by its contragredient. All the same, the above definition entitles us to speak of *the* weak-* topology of a coefficient module.

The *projective product* $E\widehat{\otimes}F$ of two Banach G-modules E, F is the Schatten–Grothendieck projective tensor product endowed with the diagonal tensor action. We refer to [J, III 15] (or [Gro1, I §1.1]) for the virtues and flaws of this product. The projective product of continuous Banach G-modules is again continuous. The canonical linear form on $E^\sharp\widehat{\otimes}E$ is G-invariant; the corresponding pairing will always be denoted $\langle\cdot|\cdot\rangle$. We recall that the Banach space $(E\widehat{\otimes}F)^*$ identifies canonically isometrically with the space $\mathcal{L}(E, F^*)$ of linear continuous operators endowed with the operator norm ([DU, Corollary VIII.2.2]); endowing the latter with the obvious action, this yields an identification $(E\widehat{\otimes}F)^\sharp \cong \mathcal{L}(E, F^\sharp)$.

For any Banach G-module E we define the *maximal continuous submodule* by

$$\mathcal{C}E = \{v \in V : G \to E, g \mapsto gv \text{ is continuous } \}\,.$$

One checks that $\mathcal{C}E$ is closed in E, hence is a continuous Banach G-module. If $\alpha : E \to F$ is a G-morphism, one has $\alpha(\mathcal{C}E) \subset \mathcal{C}F$ because of the equivariance, so that $\mathcal{C}$ is a retract functor on the full subcategory of continuous Banach G-modules.

Whenever a confusion on the group is possible, we write $\mathcal{C}_G E$.

Basic examples include $\mathbf{C}$ or $\mathbf{R}$ with the trivial action, unitary representations and the various Lebesgue spaces $L^p(G)$ (for $1 \le p \le \infty$ and a left Haar measure) with translation action. The latter is in general not continuous for $p = \infty$, but is a coefficient module if G is second countable.

We record the following observation:

PROPOSITION 1.1.4. *Let G be a Baire topological group, e.g. a locally compact group. Then every separable coefficient G-module is continuous.*

Proof. A standard argument using Baire's category theorem shows that a representation of a Baire group by isometries of a separable Banach space with Borel orbital maps is continuous (for the norm topology). On the other hand, the representation in a coefficient module is by definition weak-* continuous. However, the Banach–Alaoğlu theorem implies that the weak-* and normic Borel structures coincide for *separable* Banach spaces: indeed norm-open sets are countable unions of open balls, and the latter are countable unions of closed balls; these are weak-* compact hence weak-* closed. □

1.2 Integration matters.

Bochner's integral. Let (π, E) be a continuous Banach G-module and suppose either E separable or G second countable. Given a left Haar measure m on G, one can turn E into a $L^1(G)$-module by the formula

$$\pi(\psi)v = \int_G \psi(g)\pi(g)v \qquad\qquad (\psi \in L^1(G), v \in E)\,, \tag{1}$$

the above integral being well defined in the sense of Bochner because of Pettis' theorem, see [DU, Theorem II.1.2]. The action map $L^1(G) \times E \to E$ is continuous and compatible with G in the sense that $\pi(g)\pi(\psi) = \pi(\lambda(g)\psi)$ and $\pi(\psi)\pi(g) = \pi(\varrho(g^{-1})\psi)$, where $(\lambda(g)\psi)(h) = \psi(g^{-1}h)$ and $(\varrho(g)\psi)(h) = \Delta(g)\psi(hg)$ are the two isometric translation actions (Δ is the modular function).

An important feature of the Bochner integral is that it commutes with continuous linear operators; in particular, for any G-morphism $\alpha : E \to F$ one has $\alpha\pi_E(\psi) = \pi_F(\psi)\alpha$.

The canonical inversion isomorphism $G \to G^{\mathrm{op}}$ induces an isomorphism $L^1(G) \to L^1(G^{\mathrm{op}}) \cong (L^1(G))^\sim$, where $\psi^\sim(g) = \Delta(g^{-1})\psi(g^{-1})$. Therefore, $E^\sharp$ has a natural $L^1(G)$-module structure defined by $\pi^\sharp(\psi) = (\pi(\psi^\sim))^*$. Thus the action map $L^1(G) \times E^\sharp \to E^\sharp$ is continuous – mind however that in general $G \times E^\sharp \to E^\sharp$ is not continuous, nor measurable, nor even weakly measurable; it is only weak-* continuous.

The Gelfand–Dunford integral. One can also define the contragredient $L^1(G)$-module structure on $E^\sharp$ by a formula analogous to (1) above, but now the integral must be taken in the Gelfand–Dunford sense. Since we shall need Gelfand–Dunford integration, we recall a few facts.

Let (S, μ) be a measure space and $f : S \to E^\sharp$ a weak-* integrable map – that is, $\langle f|v\rangle \in L^1(\mu)$ for all $v \in E$. The formula

$$\left\langle \int_S f(s)d\mu(s) \;\middle|\; x \right\rangle = \int_S \langle f(s) \mid x\rangle d\mu(s)$$

defines an element $\int_S f(s)d\mu(s)$ of the *algebraic* dual of E; the Gelfand–Dunford theorem (see [Bour2, chap. VI §1.4 Théorème 1]) precisely states that $\int_S f(s)d\mu(s)$ belongs to the topological dual $E^\sharp$. Provided this, the following are simple verifications:

Lemma 1.2.1. *(i) If T is a weak-* continuous linear operator, then $\int_S Tf(s)d\mu(s) = T\int_S f(s)d\mu(s)$.*

(ii) If f is bounded and $\psi \in L^1(\mu)$, then

$$\left\| \int_S \psi(s)f(s)d\mu(s) \right\| \leq \|\psi\|_1 \cdot \|f\|_\infty \,. \qquad \square$$

A major drawback of the Gelfand–Dunford integral is that it does usually not commute with continuous linear operators. This is a source of complications for us, since the operators appearing in amenability issues are precisely not weak-* continuous. Another difficulty is that there is no general principle of the kind $\| \int_S \| \leq \int_S \| \cdot \|$ generalizing (ii). The maximal continuous submodule will be of help:

Proposition 1.2.2. *Let (π, E) be a continuous Banach G-module with either E separable or G second countable.*

(i) $\mathcal{C}E^\sharp$ coincides with the image $L^1(G)E^\sharp$ of $E^\sharp$ under $\pi^\sharp$.

(ii) $\mathcal{C}E^\sharp$ is weak- dense in $E^\sharp$.*

As to (ii), recall that $\mathcal{C}E^\sharp$ is norm closed. Point (i) implies that $\mathcal{C}E^\sharp$ is the *essential part* of $E^\sharp$ in the sense of [DoW].

Proof of Proposition 1.2.2. For point (i), fix $w \in E^\sharp$, $\varphi \in L^1(G)$ and a net (x) converging to $e \in G$. Let's check that $\pi^\sharp(x)\pi^\sharp(\varphi)w$ converges to $\pi^\sharp(\varphi)w$ in norm:

$$\begin{aligned}\big\|\pi^\sharp(x)\pi^\sharp(\varphi)w - \pi^\sharp(\varphi)w\big\|_{E^\sharp} &= \sup_{\|u\|_E=1} \big|\langle(\pi^\sharp(\lambda(x)\varphi) - \pi^\sharp(\varphi))w|u\rangle\big| \\ &= \sup_{\|u\|_E=1} \big|\langle w|\pi((\lambda(x)\varphi - \varphi)^\sim)u\rangle\big| \\ &\leq \sup_{\|u\|_E=1} \big(\|w\|_{E^\sharp} \cdot \|\pi((\lambda(x)\varphi - \varphi)^\sim)u\|_E\big) \\ &\leq \|w\|_{E^\sharp} \cdot \big\|(\lambda(x)\varphi - \varphi)^\sim\big\|_1 \\ &= \|w\|_{E^\sharp} \cdot \big\|\lambda(x)\varphi - \varphi\big\|_1 ,\end{aligned}$$

which converges to zero since $L^1(G)$ is a continuous Banach G-module (the second inequality is justified because it concerns a Bochner integral, see [DU, Theorem II.2.4 (ii)]).

Thus we have already $L^1(G)E^\sharp \subset \mathcal{C}E^\sharp$. Fix a bounded approximate identity (ψ) (by which we mean a two-sided positive continuous approximate identity bounded by one, considered as a (generally uncountable) net. This exists for any locally compact group, see e.g. [DoW, Theorem 13.4]). Since $\mathcal{C}E^\sharp$ is continuous, $\pi^\sharp(\psi)w$ converges to w for all $w \in \mathcal{C}E^\sharp$, hence $L^1(G)\mathcal{C}E^\sharp$ is dense in $\mathcal{C}E^\sharp$. But Cohen's factorization theorem, as stated in [DoW, Theorem 16.1], implies that $L^1(G)\mathcal{C}E^\sharp$ is norm closed in $\mathcal{C}E^\sharp$. Therefore $L^1(G)\mathcal{C}E^\sharp = \mathcal{C}E^\sharp$, which completes the proof of (i).

Point(ii): let (ψ) be a bounded approximate identity for $L^1(G)$. Since $\pi(\psi)u$ converges to u in norm for all $u \in E$, we see that $\pi^\sharp(\psi^\sim)w$, which is in $\mathcal{C}E^\sharp$ by (i), weak-* converges to w for all $w \in E^\sharp$, whence (ii). □

1.3 L^∞ spaces.

Let S be a standard measure space, E a dual Banach space with separable pre-dual. We denote by $L^\infty_{\mathrm{w}*}(S, E)$ the space of classes of weak-* measurable essentially bounded maps $S \to E$ endowed with the essential supremum norm. The separability of the pre-dual implies that $\|s \mapsto f(s)\|_E \in L^\infty(S)$ for $f \in L^\infty_{\mathrm{w}*}(S, E)$. Suppose now G acts on S; in order to yield a well defined translation action on $L^\infty_{\mathrm{w}*}(S, E)$, the action must preserve the measure class on S; hence the Radon–Nikodým derivatives are in $L^1(S)$. If moreover we are given a isometric representation π on E, we define a G-representation λ_π on $L^\infty_{\mathrm{w}*}(S, E)$ by

$$\big(\lambda_\pi(g)f\big)(s) = \pi(g)f(g^{-1}s) \qquad (s\text{-a.e.})$$

(in case π is trivial, we simply write λ). *In view of the nature of this action, we shall sometimes term an invariant element as equivariant.*

Definition 1.3.1. A *regular* G-space is a standard Borel G-space endowed with a G-invariant *class* with the following property:

the class contains a probability measure μ such that the isometric G-action $\lambda^\flat$:

$$(\lambda^\flat(g)\varphi)(s) = \varphi(g^{-1}s)\tfrac{dg^{-1}\mu}{d\mu}(s) \qquad (\varphi \in L^1(\mu)\,,\ s \in S)$$

is continuous. (The notation hints to the fact that λ is contragredient to $\lambda^\flat$.)

Examples: a locally compact second countable group G endowed with the class of a Haar measure is a regular G-space, a finite product of regular G-spaces with the diagonal action is again a regular G-space, Poisson and Furstenberg boundaries are regular G-spaces. A compact polish space with continuous action of a second countable group G, endowed with a radon measure μ with continuous Radon–Nikodým derivatives $dg\mu/d\mu$ is a regular G-space. A consequence of the requirement that (S,μ) be a standard measure space is the separability of $L^1(\mu)$.

The following amounts to well-known functional analysis based on the Dunford–Pettis theorem, see [DuS, VI.8], [Gro1, I §2.2] and [J, III 17.6].

Proposition 1.3.2. *Let G be a locally compact second countable group, let $(S_j)_{j=1}^n$ be regular G-spaces and (π, E) a coefficient G-module. Then*

$$L^\infty_{\mathrm{w}*}(S_1 \times \cdots \times S_n, E) \quad \text{endowed with } \lambda_\pi$$

is a coefficient G-module, canonically contragredient to

$$L^1(\mu_1)\widehat{\otimes}\cdots\widehat{\otimes}L^1(\mu_n)\widehat{\otimes}E^\flat$$

for any $(\mu_j)_{j=1}^n$ as in Definition 1.3.1. In particular, one has the canonical coefficient G-module identification

$$L^\infty_{\mathrm{w}*}(S_1 \times \cdots \times S_n, E) \cong L^\infty_{\mathrm{w}*}\big(S_1, L^\infty_{\mathrm{w}*}(S_2 \times \cdots \times S_n, E)\big)\,. \qquad \square$$

Remark 1.3.3. In the setting of Proposition 1.3.2, one has also a canonical isomorphism between $L^1(\mu_n)\widehat{\otimes}E^\flat$ and the Bochner–Lebesgue space $L^1(G, E^\flat)$, which induces on the latter a G-action to which λ_π is contragredient.

1.4 Relative injectivity.

We turn to the interplay between the categories of Banach spaces and Banach G-modules:

Definition 1.4.1. A G-morphism $\eta : A \to B$ of Banach G-modules is *admissible* if there is a continuous linear map $\sigma : B \to A$ with $\|\sigma\| \leq 1$ and $\eta\sigma\eta = \eta$.

In particular, an injective G-morphism is admissible if and only if it has a left inverse $\{e\}$-morphism of norm at most one.

In the non-topological case, an analogue of the following definition has been considered by Ivanov [Iv].

DEFINITION 1.4.2. A Banach G-module E is *relatively injective* (with respect to G) if for every injective admissible G-morphism $\iota : A \to B$ of continuous Banach G-modules A, B and every G-morphism $\alpha : A \to E$ there is a G-morphism $\beta : B \to E$ satisfying $\beta\iota = \alpha$ and $\|\beta\| \leq \|\alpha\|$.

$$\begin{tikzcd} A \arrow[rr, hook, "\iota"'] \arrow[dr, "\alpha"'] & & B \arrow[ll, bend right, "\sigma"'] \arrow[dl, dotted, "\beta"] \\ & E & \end{tikzcd}$$

REMARK 1.4.3. A purist would restrict the above definition to continuous Banach G-modules E to stay in the same category as A, B; but anyway, with our definition, one checks easily that E is relatively injective if and only if $\mathcal{C}E$ is so (recall $\alpha(A) \subset \mathcal{C}E$).

As an immediate consequence of the definition, we have

LEMMA 1.4.4. *Let $\upsilon : E \to F$ be a norm one G-morphism of Banach G-modules admitting a left inverse G-morphism of norm one.*

If F is relatively injective, then so is E. $\square$

For practical purposes, the fundamental property of relatively injective modules is the following.

LEMMA 1.4.5. *Let $\eta : A \to B$ be an admissible G-morphism of continuous Banach G-modules and let E be a relatively injective Banach G-module. Then for any G-morphism $\alpha : A \to E$ with $\mathrm{Ker}(\alpha) \supset \mathrm{Ker}(\eta)$ there is a G-morphism $\beta : B \to E$ with $\beta\eta = \alpha$ and $\|\beta\| \leq \|\alpha\|$.* $\square$

The next proposition provides us with the first example of relatively injective modules.

PROPOSITION 1.4.6. *Let G be a locally compact second countable group, (π, E) a dual coefficient G-module. Then $L^\infty_{\mathrm{w}*}(G, E)$ is relatively injective.*

Proof. We contend that $\mathcal{C}L^\infty_{\mathrm{w}*}(G, E)$ is contained in the space of classes of weak-* continuous E-valued maps on G.

Indeed, let $f : G \to E$ represent a class in $\mathcal{C}L^\infty_{\mathrm{w}*}(G, E)$ and fix a bounded approximate identity (ψ) on G. For every v in the fixed pre-dual $E^\flat$ of E, $\langle f(\cdot)|v\rangle$ is in $L^\infty(G)$ and hence the net $\psi * \langle f(\cdot)|v\rangle$ is equicontinuous. Therefore, Ascoli's theorem (in the generality of [Bour4, X §2, No. 5]) implies uniform convergence of $\psi' * \langle f(\cdot)|v\rangle$ to a continuous function for some subnet (ψ'). Appealing to Tychonoff's theorem, we may fix another subnet (ψ'')

for which convergence takes place for all $v \in E^\flat$. On the other hand, for all $v \in E^\flat$, the net $\psi'' * \langle f(\cdot)|v\rangle$ converges pointwise almost everywhere to $\langle f(\cdot)|v\rangle$. Restricting this to a *countable* dense subset of elements $v \in E^\flat$, we conclude that f coincides a.e. with a weak-* continuous map, establishing the claim.

Consider now

$$\begin{tikzcd}
A \arrow[rr, hook, "\iota"'] \arrow[dr, bend right, "\alpha"'] & & B \arrow[ll, bend right, "\sigma"'] \arrow[dl, dotted, bend left, "\beta?"] \\
& \mathcal{C}L^\infty_{\mathrm{w}*}(G,E) &
\end{tikzcd}$$

as in Definition 1.4.2. For $b \in B$ and $g \in G$, the continuity claim above allows us to define an element of E by

$$\beta(b)(g) = \pi(g)\big(\alpha\sigma(g^{-1}b)(e)\big)\,.$$

Since $g \mapsto \alpha\sigma(g^{-1}b)(e)$ is norm continuous (B is continuous), $\beta(b)$ is weak-* continuous; moreover, $\|\beta(b)(g)\|_\infty \leq \|\alpha\| \cdot \|b\|_B$, so that we have a map β from B to $L^\infty_{\mathrm{w}*}(G,E)$. It is straightforward to check that β is equivariant (hence ranges in the maximal continuous submodule), $\|\beta\| \leq \|\alpha\|$ and $\beta\iota = \alpha$. This completes the proof. □

We deduce immediately the following corollary, which will notably apply to the case $S = G^n$:

COROLLARY 1.4.7. *Let G be a locally compact second countable group, S a regular G-space and (π, E) a coefficient G-module. Then $L^\infty_{\mathrm{w}*}(G \times S, E)$ is relatively injective.*

Proof. Using Proposition 1.3.2, we may identify $L^\infty_{\mathrm{w}*}(G \times S, E)$ with $L^\infty_{\mathrm{w}*}(G, L^\infty_{\mathrm{w}*}(S,E))$. Now apply Proposition 1.4.6 with $L^\infty_{\mathrm{w}*}(S,E)$ instead of E. □

1.5 Functorial definition of bounded cohomology. In this section we introduce a functorial definition of the continuous bounded cohomology of a locally compact second countable group G, with coefficients. The defining machinery bears certain analogies with Hochschild's relative homological algebra [Ho]. We point out that the functorial characterization of continuous bounded cohomology extends to all topological groups (not necessarily locally compact), but this extension is not necessary for the present paper and is not suited to the study of L^∞ spaces.

REMARK 1.5.1. For *discrete* groups, Johnson already alluded in [Jo] to the possibility of such a theory; the task has been completed by Ivanov

[Iv] (and Noskov [N1]). However, it remained unclear whether anything of this kind was possible for topological groups, even for trivial coefficients (compare with the remark on p. 37 in [Jo]).

A resolution $E_\bullet$ of a Banach G-module E is an acyclic sequence

$$\begin{tikzcd}
E_\bullet : & 0 \arrow[r] & E \arrow[r, "d_0"] & E_0 \arrow[r, "d_1"] & E_1 \arrow[r, "d_2"] & E_2 \arrow[r] & \cdots
\end{tikzcd}$$

of G-morphisms of Banach G-modules. It is said relatively injective, continuous, etc. if all E_n $(n \geq 0)$ are so (disregarding E). We define G-morphisms of resolutions and G-homotopies of such morphisms in the obvious way. One associates as usual to any resolution $E_\bullet$ the cohomology of the corresponding (non-augmented) subcomplex of invariants

$$E_\bullet^G : \ 0 \longrightarrow E_0^G \longrightarrow E_1^G \longrightarrow E_2^G \longrightarrow \cdots$$

and endows these cohomology spaces with the quotient semi-norm. The resolution $E_\bullet$ is *admissible* if there is also a sequence (h_n) of continuous linear maps of norm at most one

$$\begin{tikzcd}
E_\bullet : & 0 \arrow[r] & E \arrow[r, "d_0"] & E_0 \arrow[l, bend left, "h_0"] \arrow[r, "d_1"] & E_1 \arrow[l, bend left, "h_1"] \arrow[r, "d_2"] & E_2 \arrow[l, bend left, "h_2"] \arrow[r] & \cdots \arrow[l, bend left]
\end{tikzcd}$$

satisfying $h_n d_n + d_{n-1} h_{n-1} = Id_{E_{n-1}}$ for all $n \geq 0$ (with the convention $d_{-1}, h_{-1} = 0$). In particular, d_n is an admissible G-morphism. We call the sequence (h_n) a *contracting homotopy.* A resolution is *strong* if the subcomplex $\mathcal{C}E_\bullet : 0 \to \mathcal{C}E \to \mathcal{C}E_0 \to \cdots$ of maximal continuous submodules is an admissible resolution.

The definitions of relative injectivity and strong resolutions are adjusted to each other so that the following proposition becomes a standard verification using Lemma 1.4.5 and the obvious observation that for any resolution $E_\bullet$ one has $E_\bullet^G = (\mathcal{C}E_\bullet)^G$.

Proposition 1.5.2. *Let $E_\bullet$ be a strong resolution of a Banach G-module E and $F_\bullet$ a relatively injective resolution of a Banach G-module F. Then any G-morphism $\alpha : \mathcal{C}E \to F$ extends to a G-morphism of resolutions $\mathcal{C}E_\bullet \to F_\bullet$ which is unique up to G-homotopy; hence α induces functorially a sequence of continuous linear maps on the corresponding cohomology spaces.*

In particular, if $E = F$ and both resolutions are strong and relatively injective, then any G-morphism of resolutions which is the identity on E induces a canonical isomorphism of topological vector spaces between the corresponding cohomology spaces. □

We shall now deduce:

COROLLARY 1.5.3. *Let E be a coefficient G-module, and let $E_\bullet$ be any strong relatively injective resolution of E. Then the cohomology of $E_\bullet^G$ is canonically isomorphic to* $\mathrm{H}^\bullet_{\mathrm{cb}}(G,E)$.

More precisely, there is a G-morphism $\mathcal{C}E_\bullet \to L^\infty_{\mathrm{w}*}(G^{\bullet+1},E)$ extending the inclusion $\mathcal{C}E \subset E$, any two such are G-homotopic and they induce a topological isomorphism in cohomology.

REMARK 1.5.4. In certain cases, one can show that the induced map in cohomology is isometric for the quotient semi-norm, but this does not follow from the above; see section 2.3 below.

Proof of Corollary 1.5.3. The maximal continuous submodules $\mathcal{C}L^\infty_{\mathrm{w}*}(G^{\bullet+1},E)$ constitute a subcomplex of the standard resolution, and obviously

$$\big(\mathcal{C}L^\infty_{\mathrm{w}*}(G^{\bullet+1},E)\big)^G = \big(L^\infty_{\mathrm{w}*}(G^{\bullet+1},E)\big)^G.$$

Thus the cohomology associated to the continuous subcomplex coincides canonically with $\mathrm{H}^\bullet_{\mathrm{cb}}(G,E)$ (notice also that any G-morphism has to range in the continuous subcomplex). Since $\mathcal{C}L^\infty_{\mathrm{w}*}(G^{\bullet+1},E)$ are all injective by the Corollary 1.4.7, it remains only to see that this continuous subcomplex admits a contracting homotopy and is hence an admissible resolution. This is taken care of by Lemma 1.5.6 below, so one can apply Proposition 1.5.2. □

COROLLARY 1.5.5. *Let G be a locally compact second countable group and E a relatively injective coefficient G-module. Then $\mathrm{H}^n_{\mathrm{cb}}(G,E) = 0$ for all $n \geq 1$.*

Proof. Apply Corollary 1.5.3 to the resolution

$$0 \longrightarrow E \xrightarrow{Id} E \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

which is indeed strong. □

The standard resolution (defined in the Introduction) is the simplest example of a large family of resolutions: let S be a regular G-space and (π, E) a coefficient G-module. Endow the spaces $L^\infty_{\mathrm{w}*}(S^{n+1},E)$ $(n \geq 0)$ with the action(s) λ_π. Define coboundary maps $d_n : L^\infty_{\mathrm{w}*}(S^n,E) \to L^\infty_{\mathrm{w}*}(S^{n+1},E)$ by $d_n = \sum_{i=0}^n (-1)^i d_{n,i}$, where $d_{n,i}$ omits the i^{th} variable and $(d_0 v)(g) = v$; it is standard to verify $d_{n+1}d_n = 0$. The map d_0 is also called the *co-augmentation.*

LEMMA 1.5.6. *There exists a contracting homotopy $h_\bullet$ turning*

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}E \arrow[r, shift left, "d_0"] & \mathcal{C}L^\infty_{\mathrm{w}*}(S,E) \arrow[l, shift left, "h_0"] \arrow[r, shift left, "d_1"] & \mathcal{C}L^\infty_{\mathrm{w}*}(S^2,E) \arrow[l, shift left, "h_1"] \arrow[r, shift left, "d_2"] & \mathcal{C}L^\infty_{\mathrm{w}*}(S^3,E) \arrow[l, shift left, "h_2"] \arrow[r, shift left] & \cdots \arrow[l, shift left]
\end{tikzcd}$$

into an admissible resolution of $\mathcal{C}E$ *(so the resolution with* $L^\infty_{\mathrm{w}*}(S^{n+1},E)$ *is strong).*

Proof. Fix a probability measure μ on S as in Definition 1.3.1. For any coefficient G-module (γ,F) define

$$h_F:\mathcal{C}L^\infty_{\mathrm{w}*}(S,F)\longrightarrow F\,,\quad h_F(f)=\int_S f(s)d\mu(s)\,,\quad f\in\mathcal{C}L^\infty_{\mathrm{w}*}(S,F)$$

(Gelfand–Dunford integral). We claim that h_F ranges in $\mathcal{C}F$.

To this end, notice first that since $\gamma(g)$ $(g\in G)$ is an adjoint operator, we may apply Lemma 1.2.1 (i) and commute it with the Gelfand–Dunford integral

$$\gamma(g)h_F(f)=\int_S\gamma(g)(f(s))d\mu(s)=\int_S\frac{dg\mu}{d\mu}(s)(\lambda_\gamma(g)f)(s)d\mu(s)$$

(recalling that the Radon–Nikodým derivative $dg\mu/d\mu$ is in $L^1(\mu)$). Using this, if (g) is a net converging to $e\in G$,

$$\begin{aligned}\big\|\gamma(g)h_F(f)-h_F(f)\big\|_F\le&\left\|\int_S f(s)d\mu(s)-\int_S\big(\lambda_\gamma(g)f\big)(s)d\mu(s)\right\|_F\\&+\left\|\int_S\big(\lambda_\gamma(g)f\big)(s)d\mu(s)-\int_S\frac{dg\mu}{d\mu}(s)\big(\lambda_\gamma(g)f\big)(s)d\mu(s)\right\|_F.\end{aligned}$$

Using Lemma 1.2.1 (ii), we bound the first term by $\|f-\lambda_\gamma(g)f\|_\infty$, which converges to zero because f is in $\mathcal{C}L^\infty_{\mathrm{w}*}(S,F)$. The second term can be bounded by

$$\|f\|_\infty\cdot\big\|\mathbb{1}_S-\tfrac{dg\mu}{d\mu}\big\|_1\,.$$

The fact that the right hand side factor converges to zero is part of Definition 1.3.1. The claim is proved.

Now we can define h_n via the identification

$$L^\infty_{\mathrm{w}*}(S^{n+1},E)\cong L^\infty_{\mathrm{w}*}\big(S,L^\infty_{\mathrm{w}*}(S^n,E)\big)$$

by letting $F=L^\infty_{\mathrm{w}*}(S^n,E)$ (and $F=E$ for h_0). We have $\|h_n\|\le 1$ because of Lemma 1.2.1 (ii). Moreover, for all $0\le i\le n$, we have $d_{n,i}h_n=h_{n+1}d_{n+1,i+1}$: indeed, the linear map $d_{n,i}$ is weak-* continuous because it is induced by one of the canonical projections $S^{n+1}\to S^n$; thus we may commute it with h_n, which gives $d_{n+1,i+1}$ via the above identification, whence the relation $d_{n,i}h_n=h_{n+1}d_{n+1,i+1}$. This, together with the analogous $h_{n+1}d_{n+1,0}=Id$, implies immediately that $h_\bullet$ is a contracting homotopy. □

The natural map. Let (π,E) be a dual Banach G-module. The usual continuous cohomology $\mathrm{H}^\bullet_{\mathrm{c}}(G,E)$ is defined with resolutions by modules satisfying an appropriate injectivity condition; call it *c-injectivity.* It is shown in [Bl] that the standard resolution by locally p-summable functions is c-injective for all $1 \leq p < \infty$. Now if E is separable, then we have

$$L^\infty_{\mathrm{w}*}(G^{n+1}, E) \;=\; L^\infty(G^{n+1}, E) \;\subset\; L^p_{\mathrm{loc}}(G^{n+1}, E)\,,$$

determining a cochain complex inclusion, and therefore a map

$$\mathrm{C}^{(\bullet)} : \mathrm{H}^\bullet_{\mathrm{cb}}(G, E) \longrightarrow \mathrm{H}^\bullet_{\mathrm{c}}(G, E)\,.$$

We call the above map the *natural map* for the following reason: if $E_\bullet$ is a strong relatively injective resolution of E and $F_\bullet$ is a c-injective resolution of E, then there is a G-complex morphism $E_\bullet \to F_\bullet$ extending the identity Id_E and every such extension induces the above map at the cohomological level. This follows indeed immediately from Proposition 1.5.2 and its analogue in continuous cohomology. The kernel of the natural map is written $\mathrm{EH}^\bullet_{\mathrm{cb}}(G, E)$.

Contravariance. Let $\psi : G \to H$ be a morphism of locally compact second countable groups, that is a continuous group homomorphism. Any Banach H-module F becomes a G-module by pull-back, and we observe that in this way both $\mathcal{C}_G F$ and $\mathcal{C}_H F$ are Banach G-modules, the latter being contained in the former.

Now let (π, E) be a coefficient H-module. If $E_\bullet$ is a strong relatively injective resolution for the H-module (π, E), then $\mathcal{C}_H E_\bullet$ is in particular a strong resolution for the G-module $\mathcal{C}_H E$ by the above observation. Applying Proposition 1.5.2, one gets a natural map

$$\mathrm{H}^\bullet_{\mathrm{cb}}(\psi, E) : \mathrm{H}^\bullet_{\mathrm{cb}}(H, E) \longrightarrow \mathrm{H}^\bullet_{\mathrm{cb}}(G, E)\,.$$

The particular case of the restriction is considered again in section 2.4.

1.6 Coefficient sequence. Continuous bounded cohomology admits also long exact coefficient sequences:

PROPOSITION 1.6.1. *Let G be a locally compact second countable group and let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an adjoint exact sequence of coefficient G-modules. Then there is a family of continuous maps (τ^n) so that the infinite sequence*

$$\cdots \xrightarrow{\tau^n} \mathrm{H}^n_{\mathrm{cb}}(G, A) \longrightarrow \mathrm{H}^n_{\mathrm{cb}}(G, B) \longrightarrow \mathrm{H}^n_{\mathrm{cb}}(G, C) \xrightarrow{\tau^{n+1}} \mathrm{H}^{n+1}_{\mathrm{cb}}(G, A) \longrightarrow \cdots$$

is exact. Moreover, if α (or equivalently β) has a left (respectively right) inverse G-morphism, then $\tau^n = 0$ for all $n \geq 0$.

REMARKS 1.6.2. (i) In the second statement, the left (or right) inverse is *not* supposed adjoint.

(ii) The long exact sequence depends naturally on the short exact sequence and on G.

(iii) It is possible to use E. Michael's selection theorem in order to establish a long exact sequence for more general Banach modules, see [Mo, 8.2].

Proof of Proposition 1.6.1. The proof is a straightforward adaptation of the classical argument based on the "snake lemma" (here, the latter is a consequence of the open mapping theorem), with one *caveat*: in order to apply the snake lemma, one needs Lemma 1.6.3 below. $\square$

LEMMA 1.6.3. *Let G be a locally compact second countable group and let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an adjoint short exact sequence of G-morphisms of coefficient G-modules. Then the induced sequence*

$$0 \longrightarrow L^\infty_{\mathrm{w}*}(G^{n+1}, A)^G \xrightarrow{\alpha_*} L^\infty_{\mathrm{w}*}(G^{n+1}, B)^G \xrightarrow{\beta_*} L^\infty_{\mathrm{w}*}(G^{n+1}, C)^G \longrightarrow 0 \tag{2}$$

is also exact for all $n \geq 0$.

REMARK 1.6.4. We point out that the closed range theorem implies that an adjoint sequence of Banach spaces is exact if and only if its pre-dual is exact.

Proof of Lemma 1.6.3. For any coefficient G-module (ϱ, D) the Fubini–Lebesgue theorem implies that the map

$$U^n : \; L^\infty_{\mathrm{w}*}(G^n, D) \longrightarrow L^\infty_{\mathrm{w}*}(G^{n+1}, D)^G$$

defined almost everywhere by

$$(U^n f)(g_0, \dots, g_n) = \varrho(g_0) f(g_0^{-1} g_1, \dots, g_{n-1}^{-1} g_n)$$

is an isomorphism. Since U^n is natural in D with respect to G-morphisms, it intertwines (2) with

$$0 \longrightarrow L^\infty_{\mathrm{w}*}(G^n, A) \xrightarrow{\alpha_*} L^\infty_{\mathrm{w}*}(G^n, B) \xrightarrow{\beta_*} L^\infty_{\mathrm{w}*}(G^n, C) \longrightarrow 0\,, \tag{3}$$

in particular the case $n = 0$ is clear. For $n > 0$, the exactness in the middle follows from the open mapping theorem and hence the only non-trivial point is the surjectivity of β_* in (3). Denoting $\beta^\flat : C^\flat \to B^\flat$ the map of pre-duals to which β is adjoint, this amounts to the injectivity of the map of Bochner L^1 spaces

$$\beta^\flat_* : L^1(G^n, C^\flat) \longrightarrow L^1(G^n, B^\flat)\,,$$

where we recall that it is the Dunford–Pettis theorem [DuS, VI.8] that yields the duality between $L^1(G^n, C^\flat)$ and $L^\infty_{\mathrm{w}*}(G^n, C)$. $\square$

REMARK 1.6.5. The property of the predual L^1 spaces that we used in the proof of Lemma 1.6.3 actually characterizes such spaces [Gro2].

Applying Proposition 1.6.1 and Corollary 1.5.5 to the sequence

$$0 \longrightarrow F \longrightarrow L^\infty_{\mathrm{w}*}(G,F) \longrightarrow L^\infty_{\mathrm{w}*}(G,F)/F \longrightarrow 0\,,$$

we deduce the *dimension shifting* statement

COROLLARY 1.6.6. *There is for all $n \geq 1$ an isomorphism* $\mathrm{H}^{n+1}_{\mathrm{cb}}(G,F) \cong \mathrm{H}^{n}_{\mathrm{cb}}\big(G, L^\infty_{\mathrm{w}*}(G,F)/F\big)$. □

1.7 Alternating and continuous cochains.

Let S be a regular G-space, E a coefficient G-module, and consider the complex

$$0 \longrightarrow E \longrightarrow L^\infty_{\mathrm{w}*,\mathrm{alt}}(S,E) \longrightarrow L^\infty_{\mathrm{w}*,\mathrm{alt}}(S^2,E) \longrightarrow L^\infty_{\mathrm{w}*,\mathrm{alt}}(S^3,E) \longrightarrow \cdots$$

of alternating bounded measurable cochains; the contracting homotopy of Lemma 1.5.6 preserves this subcomplex. The inclusions

$$\iota_n : L^\infty_{\mathrm{w}*,\mathrm{alt}}(S^{n+1},E) \;\subset\; L^\infty_{\mathrm{w}*}(S^{n+1},E)$$

determine isometric isomorphisms at the level of cohomology because the usual alternation operators

$$\mathrm{Alt}_n = \frac{1}{(n+1)!}\sum_{\pi\in\mathcal{S}_{n+1}} \mathrm{sign}(\pi)\pi^* \quad \big(\pi^*(\cdot) = \cdot\circ\pi^{-1}\big)\,,$$

where the symmetric group $\mathcal{S}_{n+1}$ acts by permutation of the coordinates, are norm one G-homotopy inverses for the inclusions.

When the module E is a separable Banach space, the usual regularization procedure establishes a G-homotopy equivalence between the standard resolution and the subcomplex of (norm-)continuous cochains. That is, the complex

$$0 \longrightarrow C_{\mathrm{b}}(G,E)^G \longrightarrow C_{\mathrm{b}}(G^2,E)^G \longrightarrow C_{\mathrm{b}}(G^3,E)^G \longrightarrow \cdots$$

of G-invariant continuous bounded cochains realizes the continuous bounded cohomology $\mathrm{H}^\bullet_{\mathrm{cb}}(G,E)$ in the sense that the inclusions

$$C_{\mathrm{b}}(G^{n+1},E) \;\subset\; L^\infty(G^{n+1},E)$$

induce isometric isomorphisms at the level of cohomology. The proof can be taken verbatim from our Proposition 2.4 in [BuM1].

Since Alt_n preserves continuity, one can also use the the subcomplex of alternating G-invariant continuous bounded cochains.

1.8 Cup product. A *pairing* of Banach G-modules is a triple (A, B, C) of Banach G-modules together with a G-morphism

$$A \widehat{\otimes} B \longrightarrow C$$

of norm one. Echoing the usual Alexander–Whitney construction, we get a graded bilinear map

$$\wedge : \mathrm{H}^\bullet_{\mathrm{cb}}(G, A) \times \mathrm{H}^\bullet_{\mathrm{cb}}(G, B) \longrightarrow \mathrm{H}^\bullet_{\mathrm{cb}}(G, C) \tag{4}$$

for all coefficient Banach G-modules A, B, C. Indeed, the coefficient pairing induces a pairing (symmetric cochain cup product)

$$\times : L^\infty_{\mathrm{w}*}(G^{n+1}, A) \,\widehat{\otimes}\, L^\infty_{\mathrm{w}*}(G^{m+1}, B) \longrightarrow L^\infty_{\mathrm{w}*}(G^{n+m+1}, C)$$

defined almost everywhere by

$$\alpha \times \beta(x_0, \dots, x_{n+m}) = \big\langle \alpha(x_0, \dots, x_n) | \beta(x_n, \dots, x_{n+m}) \big\rangle$$

and which restricts to the respective maximal continuous submodules. The symbol $\wedge$ denotes both the antisymmetrized version of $\times$ on the graded group of alternating cochains and the quotient structure (4). The same construction is retained for non-topological groups.

According to these definitions, the natural map intertwines the bounded cup product with the usual one. In particular, for trivial coefficients, the natural map $\mathrm{H}^\bullet_{\mathrm{b}} \to \mathrm{H}^\bullet$ (or $\mathrm{H}^\bullet_{\mathrm{cb}} \to \mathrm{H}^\bullet_{\mathrm{c}}$) determines a natural transformation of contravariant functors from the category of groups (respectively locally compact second countable groups) to the category of graded algebras.

As an illustration, we present the following remark.

PROPOSITION 1.8.1. *Let $\omega \in \mathrm{H}^2_{\mathrm{b}}(T)$ be the Euler class for Thompson's simple group T. Then the n-fold cup product $\omega \wedge \ldots \wedge \omega$ is non-trivial in $\mathrm{H}^{2n}_{\mathrm{b}}(T)$ for all n.*

Recall that

$$\begin{aligned} T = \big\langle a, b, c \;\big|\; &[ab^{-1}, a^{-1}ba],\ [ab^{-1}, a^{-2}ba^2],\ c^{-1}ba^{-1}cb\,, \\ &(a^{-1}cba^{-1}ba)^{-1}ba^{-2}cb^2,\ a^{-1}c^{-1}(a^{-1}cb)^2,\ c^3 \big\rangle \end{aligned}$$

can be viewed as the group of all orientation preserving piecewise affine transformations of $\mathbf{R}/\mathbf{Z}$ which have dyadic breaking points and whose slopes are integral powers of two. We refer to [CFP] for a careful introduction to this group.

Proof of Proposition 1.8.1. Since the cup product preserves boundedness and the natural map is an algebra morphism, the statement reduces to the corresponding assertion for the image of ω in $\mathrm{H}^{2n}(T)$. For rational coefficients, this is a result of É. Ghys and V. Sergiescu ([GhS, Théorème D]). One concludes with the dual universal coefficients theorem. □

1.9 Remarks on Banach algebra cohomology. Let G be a locally compact second countable group and E a separable continuous Banach G-module. Then the continuous bounded cohomology $\mathrm{H}^\bullet_{\mathrm{cb}}(G, E^\sharp)$ coincides with Johnson's Banach algebra cohomology $\mathcal{H}^\bullet(L^1(G), E^\sharp)$ (see Proposition 2.3 in [Jo]), for which Johnson's memoir does not give a functorial characterization.

After the completion of the present paper, we became aware of Helemskiĭ's monographs [H1] and [H2], where Johnson's cohomology is characterized by an analogue of the classical derived functors $\mathrm{Ext}^\bullet$.

We have seen in section 1.2 how $E^\sharp$ can be given the structure of a Banach $L^1(G)$-module; $E^\sharp$ is not neo-unital in general, but $\mathcal{C}E^\sharp$ is so. Now if $E^\sharp$ is relatively injective in the sense of Definition 1.4.2, then one can check that it is an injective $L^1(G)_+$-module in the sense of Definition III 1.13 in [H1]. Here $L^1(G)_+$ is the unitized algebra $L^1(G) \oplus \mathbf{C}$ (endowed with sum norm) and our claim relies on the fact that the canonical morphism

$$(L^1(G)_+)^* \longrightarrow (L^1(G))^*$$

is a retraction over $L^1(G)$ since $L^1(G)$ admits a bounded approximate identity.

The above gives a connection between continuous bounded cohomology and Banach cohomology, although the latter does not carry with it any isometric information of the kind of our Theorem 2.

WARNING. The interplay between Banach G-modules and $L^1(G)$-modules is not as straightforward as is sometimes assumed in the literature: of basic importance in Banach algebra cohomology are $L^1(G)$-morphisms defined on the $L^1(G)$-module $L^\infty(G)$, in particular morphisms which are not weak-* continuous. The (generally non-continuous) corresponding G-module $L^\infty(G)$ admits G-morphisms that are *not* $L^1(G)$-morphisms. An example of this situation is given by W. Rudin in [R, Theorem 4.1]. Our Theorem 2.2.4 gives an instance where such phenomena are ruled out.

2 Amenable Actions

2.1 Amenability. To begin with, we remark that some amenability issues already came in through the back door while we were discussing relative injectivity.

To make this more precise, we consider for a while a *discrete* group Γ, and recall that Γ is said to be *amenable* if one of the following two equivalent conditions holds:

(A1) Every non-empty convex compact Γ-invariant subset of a Fréchet space on which Γ acts by continuous linear operators contains a fixed point.

(A2) There is an *invariant mean* on $\ell^\infty(\Gamma)$, i.e. there is a Γ-invariant left inverse of norm one to the natural inclusion $\mathbf{R} \to \ell^\infty(\Gamma)$.

Now let E be a Banach Γ-module; the natural inclusion $E \to \ell^\infty(\Gamma, E)$ is an admissible embedding since the evaluation at any fixed element of Γ yields a (non-equivariant) left inverse of norm one. Therefore, considering the diagram

$$\begin{tikzcd}
E \arrow[rr, hook] \arrow[dr, equal] & & \ell^\infty(\Gamma, E) \arrow[dl, dotted] \\
& E &
\end{tikzcd}$$

we see that if E happens to be relatively injective, then we have indeed an *equivariant mean* on $\ell^\infty(\Gamma, E)$, that is, a Γ-equivariant left inverse of norm one to the natural inclusion $E \to \ell^\infty(\Gamma, E)$.

Conversely, since for *discrete* groups $\ell^\infty(\Gamma, E)$ is relatively injective regardless of the nature of the Banach Γ-module E (see [Iv, Lemma 3.2.2]), the presence of such an equivariant mean forces E to be relatively injective by Lemma 1.4.4. Thus we have shown

PROPOSITION 2.1.1. *Let Γ be a discrete group and E a Banach Γ-module. The following assertions are equivalent:*

(i) E is relatively injective.

(ii) There is an equivariant mean on $\ell^\infty(\Gamma, E)$.

In particular, the trivial module $\mathbf{R}$ (or $\mathbf{C}$) is relatively injective if and only if Γ is amenable. □

2.2 A characterization of amenable actions. The purpose of this section is twofold: generalize the above proposition to locally compact groups (which draws us into the issues of section 1.2), and give a connection with amenable actions. With this in mind, we recall now how R.J. Zimmer defined amenable group actions, generalizing the idea of (A1) above:

DEFINITION 2.2.1 (Zimmer). Let G be a locally compact group and S a standard Borel space with measure class preserving Borelian G-action. The G-action on S is said to be *amenable* if for every separable Banach space E and every Borelian (right) cocycle $\alpha : S \times G \to \mathrm{Isom}(E)$ the following holds for the dual α^*-twisted action on E^*:

Any α^*-invariant Borelian field $\{A_s\}_{s\in S}$ of non-empty convex weak-* compact subsets A_s of the unit ball in E^* admits an α^*-invariant Borelian section.

For more details, see [Z3]; it is important to us to have at our disposal a criterion more in the spirit of (A2). Despite Zimmer's early partial result in [Z1], the task has been completed only quite recently:

Theorem 2.2.2 (Zimmer, Adams–Elliott–Giordano). *Let G be a locally compact separable group and S a regular G-space. The following assertions are equivalent:*

(i) G acts amenably on S.

(ii) The canonical inclusion $L^\infty(S) \to L^\infty(G \times S)$ admits a left inverse G-morphism of norm one.

Proof. (i)⇒(ii) is Theorem 3.4 in [AEG], while for (ii)⇒(i), according to [AEG], the proof in [Z1] with G discrete holds without change in the continuous case. □

REMARKS 2.2.3. (i) In the references given, the second condition above is expressed in terms of conditional expectations; both formulations are easily seen to be equivalent.

(ii) The above theorem is already contained in S. Adam's unpublished notes [A].

An important step in the proof of Theorem 1 is the following Theorem 2.2.4, which can be considered as a generalization of both Proposition 2.1.1 and of a classical result of Greenleaf to our Banach setting. However, difficulties arise from the lack of continuity of the coefficient space; we will tackle them with Proposition 1.2.2.

Analogously to the classical scalar case, we say that a *mean* on a function space is a continuous linear left inverse of norm one to the coefficient inclusion.

Theorem 2.2.4. *Let G be a locally compact second countable group and (π, E) a coefficient G-module.*

The following assertions are equivalent:

(i) (π, E) is a relatively injective Banach G-module.

(ii) There is a G-equivariant mean $\mathfrak{m} : L^\infty_{\mathrm{w}*}(G, E) \to E$.

(iii) There is a G-equivariant mean $\mathfrak{m} : \mathcal{C}L^\infty_{\mathrm{w}*}(G, E) \to \mathcal{C}E$.

(iv) There is an $L^1(G)$-equivariant mean $\mathfrak{m} : L^\infty_{\mathrm{w}*}(G, E) \to E$.

(v) There is an $L^1(G)$-equivariant mean $\mathfrak{m} : \mathcal{C}L^\infty_{\mathrm{w}*}(G, E) \to \mathcal{C}E$.

Proof. Recall the notation $(\pi^\flat, E^\flat)$ of Definition 1.1.2, so that $\lambda_\pi = \lambda^\sharp_{\pi^\flat}$. Among the equivalences of the conditions (ii) to (v), the crux is the implication

(v)⇒(iv): fix an $L^1(G)$-equivariant mean $\mathfrak{m} : \mathcal{C}L^\infty_{\mathrm{w}*}(G,E) \to \mathcal{C}E$ and some bounded approximate identity (φ) on G. The Proposition 1.2.2 applied to $L^\infty_{\mathrm{w}*}(G,E)$ allows us to consider the composition

$$\mathfrak{m}\lambda_\pi(\varphi) : L^\infty_{\mathrm{w}*}(G,E) \longrightarrow \mathcal{C}E \subset E$$

(see Remark 1.3.3). Using the identification

$$\mathcal{L}\big(L^\infty_{\mathrm{w}*}(G,E),E\big) \cong \big(L^\infty_{\mathrm{w}*}(G,E)\widehat{\otimes}E^\flat\big)^*,$$

we apply the theorem of Bourbaki–Alaoğlu and conclude to the existence of an (other) approximate identity (ψ) such that for all $f \in L^\infty_{\mathrm{w}*}(G,E)$ the net $\mathfrak{m}\lambda_\pi(\psi)f$ weak-* converges in E to some element that we denote by $\overline{\mathfrak{m}}f$. It is straightforward that this yields a linear operator $\overline{\mathfrak{m}} : L^\infty_{\mathrm{w}*}(G,E) \to E$ with $\|\overline{\mathfrak{m}}\| \leq 1$. If f has constant essential value $w \in E$, then $\lambda_\pi(\psi)f$ is essentially constant of value $\pi(\psi)w$; applying now Proposition 1.2.2 to E, $\pi(\psi)w$ is in $\mathcal{C}E$ and hence $\mathfrak{m}\lambda_\pi(\psi)f$ equals $\pi(\psi)w$, which weak-* converges to w.

Thus it remains only to show that $\overline{\mathfrak{m}}$ is $L^1(G)$-equivariant. For $\varphi \in L^1(G)$ and $f \in L^\infty_{\mathrm{w}*}(G,E)$, we have that $\overline{\mathfrak{m}}\lambda_\pi(\varphi)f$ is the weak-* limit of $\mathfrak{m}\lambda_\pi(\psi * \varphi)f$, whilst for all $u \in E^\flat$

$$\begin{aligned}\big\langle \mathfrak{m}\lambda_\pi(\varphi * \psi)f \big| u\big\rangle &= \big\langle \pi(\varphi)\mathfrak{m}\lambda_\pi(\psi)f \big| u\big\rangle\\ &= \big\langle \mathfrak{m}\lambda_\pi(\psi)f \big| \pi^\flat(\varphi^{\sim})u\big\rangle\\ &\to \big\langle \overline{\mathfrak{m}}f \big| \pi^\flat(\varphi^{\sim})u\big\rangle = \big\langle \pi(\varphi)\overline{\mathfrak{m}}f \big| u\big\rangle,\end{aligned}$$

where in the very first equality we used the Lemma 1.2.1 applied to $L^\infty_{\mathrm{w}*}(G,E)$ in order to commute $\mathfrak{m}$ with $\lambda_\pi(\varphi)$ according to the hypothesis (v).

This shows that $\overline{\mathfrak{m}}\lambda_\pi(\varphi)f - \pi(\varphi)\overline{\mathfrak{m}}f$ is the weak-* limit of $\mathfrak{m}\lambda_\pi(\psi * \varphi - \varphi * \psi)f$. On the other hand, since our approximate identity is two-sided, $\psi * \varphi - \varphi * \psi$ norm converges to zero in $L^1(G)$. The continuity of the contragredient *algebra*-representation thus implies that $\lambda_\pi(\psi * \varphi - \varphi * \psi)f$, hence also $\mathfrak{m}\lambda_\pi(\psi * \varphi - \varphi * \psi)f$, converge to zero in norm. Putting everything together, we conclude that $\overline{\mathfrak{m}}\lambda_\pi(\varphi)f = \pi(\varphi)\overline{\mathfrak{m}}f$, completing the proof of (v)⇒(iv).

(iv)⇒(ii): let $\mathfrak{m}$ be an $L^1(G)$-equivariant mean $L^\infty_{\mathrm{w}*}(G,E) \to E$; we claim that $\mathfrak{m}$ is actually G-equivariant. Indeed, let $f \in L^\infty_{\mathrm{w}*}(G,E)$ and $g \in G$ and fix a bounded approximate identity (ψ). Now $\mathfrak{m}\lambda_\pi(g)f$ is the weak-* limit of

$$\begin{aligned}\pi(\psi^{\sim})\mathfrak{m}\lambda_\pi(g)f &= \mathfrak{m}\lambda_\pi(\psi^{\sim})\lambda_\pi(g)f = \mathfrak{m}\lambda_\pi((\lambda(g^{-1})\psi)^{\sim})f\\ &= \pi\big((\lambda(g^{-1})\psi)^{\sim}\big)\mathfrak{m}f = \pi(\psi^{\sim})(\pi(g)\mathfrak{m}f)\,,\end{aligned}$$

which converges weak-* to $\pi(g)\mathfrak{m}f$.

(ii)⇒(iii) is obvious.

(iii)⇒(v): let $\mathfrak{m}$ be as in (iii). Since $\mathcal{C}L^\infty_{\mathrm{w}*}(G,E)$ is continuous and G separable, Pettis' theorem implies that the Gelfand–Dunford integral is a Bochner integral, hence commutes with $\mathfrak{m}$. Thus conditions (ii) to (v) are equivalent.

(i)⇒(iii): considering the diagram

$$\begin{tikzcd}\mathcal{C}E \arrow[rr, hook, "\iota"] \arrow[dr, equal] & & \mathcal{C}L^\infty_{\mathrm{w}*}(G,E) \arrow[dl, dotted, "\mathfrak{m}?"] \\ & \mathcal{C}E & \end{tikzcd}$$

we see that it is enough to show that ι is admissible. But this is exactly the content of the initial claim in the proof of Lemma 1.5.6 (with $S=G$ and $F=E$).

(ii)⇒(i) (or (iii)⇒(i), see Remark 1.4.3): combine Proposition 1.4.6 with Lemma 1.4.4. This completes the proof of the Theorem 2.2.4. □

Proof of Theorem 1. (i)⇒(iii): by the Proposition 4.3.4 of [Z3], G acts amenably on S^{n+1}, so that we may as well suppose $n=0$. Using Theorem 2.2.2, we get a left inverse G-morphism $\mathfrak{m}_0$ of norm one to the inclusion of $L^\infty(S)$ in $L^\infty(G\times S)$. For every $f\in L^\infty_{\mathrm{w}*}(G\times S,E)$ we define a bilinear form $\mathfrak{m}f$ on $L^1(S)\times E^\flat$ by

$$\mathfrak{m}f(\psi,v)=\big\langle \mathfrak{m}_0\langle f(\cdot)|v\rangle\big|\psi\big\rangle \quad (\psi\in L^1(S)\,,\ v\in E^\flat)\,.$$

The estimate $|\mathfrak{m}f(\psi,v)|\le \|f\|_\infty\cdot\|v\|_{E^\flat}\cdot\|\psi\|_1$ shows at once that the bilinear form $\mathfrak{m}f$ is continuous and that the corresponding linear map

$$\mathfrak{m}: L^\infty_{\mathrm{w}*}(G\times S,E)\longrightarrow \big(L^1(S)\widehat{\otimes}E^\flat\big)^\sharp \cong L^\infty_{\mathrm{w}*}(S,E)$$

is continuous of norm at most one. Using the relation

$$\big\langle\lambda_\pi(g)f(\cdot)|v\big\rangle=\lambda(g)\big\langle f(\cdot)|\pi^\flat(g^{-1})v\big\rangle\,,$$

one checks readily that $\mathfrak{m}$ is G-equivariant. Recalling that the pairing on $L^\infty_{\mathrm{w}*}(S,E)\times L^1(S)\widehat{\otimes}E^\flat$ is obtained by Gelfand–Dunford integration over S of the pairing on $E\times E^\flat$, one verifies that $\mathfrak{m}$ is a left inverse G-morphism to the inclusion of $L^\infty_{\mathrm{w}*}(S,E)$ in $L^\infty_{\mathrm{w}*}(G\times S,E)$. Now by Corollary 1.4.7, $L^\infty_{\mathrm{w}*}(G\times S,E)$ is relatively injective; finally apply Lemma 1.4.4.

(iii)⇒(ii) is obvious.

(ii)⇒(i): set $E = L^\infty(S)$ in Theorem 2.2.4 to deduce the existence of an equivariant mean on $L^\infty_{\mathrm{w}*}(G, E)$. Using the canonical identification $L^\infty_{\mathrm{w}*}(G, L^\infty(S)) \cong L^\infty(G\times S)$, this is the same as a left inverse G-morphism of norm one to the canonical inclusion $L^\infty(S) \to L^\infty(G\times S)$. Thus we may apply the Theorem 2.2.2. □

Remark 2.2.5. The statement of Theorem 1 does not hold for arbitrary Banach G-modules in condition (iii). Indeed, G.A. Noskov considers in [N1] the Banach $\mathbf{Z}$-module $\mathcal{A}^\mu_\rho$ of 2π-periodic functions that are analytic in the strip $|\mathrm{im}(z)| < \rho$ ($\rho > 0$) and continuous in the closure of the strip, endowed with the translation by multiples of $2\pi\mu$ ($\mu \in \mathbf{R}$) and sup-norm. He shows that results of Arnold imply $\dim \mathrm{H}^1_{\mathrm{b}}(\mathbf{Z}, \mathcal{A}^\mu_\rho) = \infty$ for $2^{\aleph_0}$ many $\mu \in \mathbf{R}$ (we read Arnold's relevant results in the translation [Ar1, chap. 3 §12]; there is an English version [Ar2]).

Now $\mathbf{Z}$ acts amenably on S = one point, so that if $L^\infty(S^{n+1}, \mathcal{A}^\mu_\rho)$ were injective, the general principles of section 1.5 would imply $\mathrm{H}^1_{\mathrm{b}}(\mathbf{Z}, \mathcal{A}^\mu_\rho) = 0$.

Remark 2.2.6. Suppose S is an amenable G-space and $N \lhd G$ a normal closed subgroup. Let T be the point realization of $L^\infty(S)^N$. By Theorem 1, $L^\infty(S)$ is G-relatively injective. This implies immediately that $L^\infty(S)^N$ is G/N-relatively injective, and so applying again Theorem 1 we conclude that T is an amenable G/N-space.

2.3 Relatively injective resolutions and the semi-norm.

Let G be a locally compact second countable group and E a coefficient G-module. The outcome of the functorial constructions of section 1.5 is that for any strong relatively injective resolution $E_\bullet$ there is a natural isomorphism of topological vector spaces between the associated cohomology of invariants

$$E^G_\bullet : 0 \longrightarrow E^G_0 \longrightarrow E^G_1 \longrightarrow E^G_2 \longrightarrow \cdots$$

and $\mathrm{H}^\bullet_{\mathrm{cb}}(G, E)$; however, this isomorphism is in general not isometric – we recall that the semi-normed spaces $\mathrm{H}^n_{\mathrm{cb}}(G, E)$ are *defined* via the standard resolution. The point is that the G-morphisms of complexes granted by Corollary 1.5.3 need not preserve the norm since the coboundary maps are in general not of norm one (the standard d_n is of norm $n + 1$).

Since the semi-norm is an important cohomological invariant, we shall show that the natural isomorphisms are isometric in the case of resolutions on amenable regular G-spaces (Corollary 2.3.2 below). This is due to the tensorial nature of the standard coboundary; the technical ingredient is the following proposition.

Proposition 2.3.1. *Let G be a locally compact second countable group and S,T regular G-spaces. If there is a norm one G-morphism $\mathfrak{m}_0 : L^\infty(S) \to L^\infty(T)$ such that $\mathfrak{m}_0(\mathbb{1}_S) = \mathbb{1}_T$, then for every coefficient G-module E there is a G-morphism of complexes*

$$\begin{tikzcd}
0 \arrow[r] & E \arrow[r] \arrow[d, equal] & L^\infty_{\mathrm{w}*}(S,E) \arrow[r] \arrow[d, "\mathfrak{m}_{0,E}"'] & L^\infty_{\mathrm{w}*}(S^2,E) \arrow[r] \arrow[d, "\mathfrak{m}_{1,E}"'] & L^\infty_{\mathrm{w}*}(S^3,E) \arrow[r] \arrow[d, "\mathfrak{m}_{2,E}"'] & \cdots \\
0 \arrow[r] & E \arrow[r] & L^\infty_{\mathrm{w}*}(T,E) \arrow[r] & L^\infty_{\mathrm{w}*}(T^2,E) \arrow[r] & L^\infty_{\mathrm{w}*}(T^3,E) \arrow[r] & \cdots
\end{tikzcd}$$

with all $\mathfrak{m}_{n,E}$ of norm at most one.

Proof. Choose measures μ, ν on S, T as in Definition 1.3.1 and consider the corresponding canonical isometric G-equivariant isomorphisms

$$\mathcal{L}\big(L^\infty(S), L^\infty(T)\big) \cong \big(L^\infty(S)\widehat{\otimes}L^1(T)\big)^\sharp \cong \mathcal{L}\big(L^1(T), L^\infty(S)^\sharp\big)\,.$$

Denote by $\mathfrak{m}_0'$ the invariant element of the closed unit ball in the right-hand side which corresponds to $\mathfrak{m}_0$. One can fix a directed set A such that for each $\varphi \in L^1(T)$ there is a net $(M_0^\alpha(\varphi))_{\alpha\in A}$ in $L^1(S)$ converging weak-* in its bi-dual $L^\infty(S)^\sharp$ to $\mathfrak{m}_0'(\varphi)$. Moreover, we may suppose $\mu(M_0^\alpha(\varphi)) = \nu(\varphi)$ since $\mathfrak{m}_0(\mathbb{1}_S) = \mathbb{1}_T$. Let $n \geq 0$ and write $C_{n,E}$ for the closed unit ball of $L^\infty_{\mathrm{w}*}(S^{n+1}, E)$ endowed with the weak-* topology which is part of the data of the coefficient module E. The product space

$$C = \prod_{n=0}^{\infty} \prod_{\substack{\varphi_j \in L^1(T)\\ 0\leq j\leq n}} \prod_{v\in E^\flat} C_{n,E}$$

is compact by the theorems of Bourbaki–Alaoğlu and Tychonoff. We define a net $(M_{n,E}^\alpha)_{\alpha\in A}$ in C by assigning to $M_{n,E}^\alpha(\varphi_0, \ldots, \varphi_n, v)$ the image of

$$M_0^\alpha(\varphi_0) \otimes \cdots \otimes M_0^\alpha(\varphi_n) \otimes v \ \in L^1(S^{n+1}, E^\flat)$$

under the canonical embedding into the bi-dual. By compactness of C, there is an accumulation point $(\mathfrak{m}_{n,E}')_{n=0}^\infty$, which must be linear in v and the φ_j. Therefore, we view it as simultaneous weak-* accumulation points $\mathfrak{m}_{n,E}'$ of nets $(M_{n,E}^\alpha)_{\alpha\in A}$ in

$$\mathcal{L}\big(L^1(T)\widehat{\otimes}\cdots\widehat{\otimes}L^1(T)\widehat{\otimes}E^\flat, L^\infty_{\mathrm{w}*}(S^{n+1}, E)^\sharp\big)\,.$$

We claim that the maps $\mathfrak{m}_{n,E}$ corresponding to $\mathfrak{m}_{n,E}'$ under the identification of the latter space with

$$\mathcal{L}\big(L^\infty_{\mathrm{w}*}(S^{n+1}, E), L^\infty_{\mathrm{w}*}(T^{n+1}, E)\big)$$

have all required properties. The only point that is not an immediate consequence of the weak-* continuity of the G-module structures is that the coboundaries intertwine $\mathfrak{m}_{n,E}$ with $\mathfrak{m}_{n-1,E}$. We shall actually show that each summand $d_{n,j}$ of the coboundary d_n (see section 1.5) intertwines them. Under the above identification, this reduces to show that for every $\varphi \in L^1(T)$, $\psi \in L^1(T^n, E^\flat)$ and $\chi \in L^\infty_{\mathrm{w}*}(S^n, E)$ the relation

$$\mathfrak{m}'_{n,E}(\varphi \otimes \psi)(\mathbb{1}_S \otimes \chi) = \langle \mathbb{1}_T | \varphi \rangle \mathfrak{m}'_{n-1,E}(\psi)(\chi) \tag{5}$$

holds. Indeed, the standard coboundary map is but an alternating sum of various tensorisations against $\mathbb{1}$, and our definition of $\mathfrak{m}_{n,E}$ is compatible with permutation of the factors. We conclude the proof with the remark that (5) follows from

$$M_0^\alpha(\varphi)(\mathbb{1}_S) = \mu(M_0^\alpha(\varphi)) = \nu(\varphi) = \langle \mathbb{1}_T | \varphi \rangle\,. \qquad \square$$

Corollary 2.3.2. *Let G be a locally compact second countable group, S an amenable regular G-space and E a coefficient G-module. Then the canonical isomorphism between $\mathrm{H}^\bullet_{\mathrm{cb}}(G, E)$ and the cohomology of the complex*

$$0 \longrightarrow L^\infty_{\mathrm{w}*}(S, E)^G \longrightarrow L^\infty_{\mathrm{w}*}(S^2, E)^G \longrightarrow L^\infty_{\mathrm{w}*}(S^3, E)^G \longrightarrow \cdots$$

of bounded measurable invariant cochains is isometric. The same holds for the subcomplex of alternating bounded measurable invariant cochains.

Proof. Since the inclusions ι_n and alternation operators Alt_n of section 1.7 are of norm one, it is sufficient to consider the non-alternating complexes. In this case, an application of the Proposition 2.3.1 (with $T = G$) provides us with a G-morphism of complexes of norm at most one. By Corollary 1.5.3, the corresponding cohomology map is the canonical isomorphism, which is thus of norm at most one. Interchanging the rôle of S and T, we conclude that the canonical isomorphism has an inverse of norm at most one and thus is isometric. $\square$

This completes the proof of Theorem 2 stated in the Introduction.

2.4 Restriction and inflation. Let H be a closed subgroup of a locally compact second countable group G, and let E be a coefficient G-module; the inclusion $H \to G$ induces a dual Banach H-module structure on E. The corresponding natural cohomology map in the sense of section 1.5 is called the *restriction*

$$\mathrm{res} : \mathrm{H}^\bullet_{\mathrm{cb}}(G, E) \longrightarrow \mathrm{H}^\bullet_{\mathrm{cb}}(H, E)\,.$$

By Theorem 1 and Lemma 1.5.6, a strong relatively injective resolution for the H-module E is given by the spaces $L^\infty_{\mathrm{w}*}(G^{n+1}, E)$ viewed as H-modules because G is an amenable regular H-space. Therefore the restriction map is induced by the inclusions

$$L^\infty_{\mathrm{w}*}(G^{n+1}, E)^G \longrightarrow L^\infty_{\mathrm{w}*}(G^{n+1}, E)^H \quad (n \geq 0)\,.$$

Applying the Corollary 2.3.2, it is apparent on this realization that the restriction map does not increase the semi-norm.

For usual cohomology (resp. continuous cohomology) of groups, it is well known that the restriction is injective if H is of finite index (resp. co-compact with invariant measure on the quotient). In bounded cohomology, we have a stronger statement:

Proposition 2.4.1. *Let H be a closed subgroup of a locally compact second countable group G. If there is a (right) invariant mean on $L^\infty(H\backslash G)$, then the restriction*

$$\mathrm{res}:\mathrm{H}^\bullet_{\mathrm{cb}}(G,E)\longrightarrow\mathrm{H}^\bullet_{\mathrm{cb}}(H,E)$$

is isometrically injective for every coefficient G-module (π,E).

Proof. Recall that an invariant mean $\mathfrak{m}$ is an invariant norm one linear form on $L^\infty(H\backslash G)$ satisfying $\mathfrak{m}(\mathbb{1})=1$. We shall show that there is a *transfer* map

$$\mathrm{trans}_{\mathfrak{m}}:\mathrm{H}^\bullet_{\mathrm{cb}}(H,E)\longrightarrow\mathrm{H}^\bullet_{\mathrm{cb}}(G,E)$$

such that $\mathrm{trans}_{\mathfrak{m}}\circ\mathrm{res}=Id$.

First we claim that for every coefficient G-module F there is an *adjointly natural* G-equivariant mean

$$\mathfrak{m}_F:\ L^\infty_{\mathrm{w}*}(G/H,F)\longrightarrow F\,.$$

By adjointly natural, we mean that any adjoint G-morphism $\alpha:F\to F'$ of coefficient G-modules induces a commutative diagram

$$\begin{tikzcd}
L^\infty_{\mathrm{w}*}(G/H,F) \arrow[r, "\mathfrak{m}_F"] \arrow[d, "\alpha_*"] & F \arrow[d, "\alpha"] \\
L^\infty_{\mathrm{w}*}(G/H,F') \arrow[r, "\mathfrak{m}_{F'}"] & F'
\end{tikzcd}$$

Mind that $\mathfrak{m}_F$ itself is not adjoint in general.

Indeed, if for $f\in L^\infty_{\mathrm{w}*}(G/H,F)$ and u in the chosen predual $F^\flat$ of F we define $f_u\in L^\infty(G/H)$ almost everywhere by $f_u(\cdot)=\langle f(\cdot)|u\rangle$, we obtain the desired $\mathfrak{m}_F$ by $\langle\mathfrak{m}_F(f)|u\rangle=\mathfrak{m}(f_u)$; as $F^\flat$ is separable, it is enough to consider countably many elements u, settling the "almost everywhere" problem. If now $\alpha:F\to F'$ is as above, with predual $\alpha^\flat:F'^\flat\to F^\flat$, we check for $v\in F'^\flat$ the relation

$$\big\langle\alpha\mathfrak{m}_F(f)|v\big\rangle=\big\langle\mathfrak{m}_F(f)|\alpha^\flat v\big\rangle=\mathfrak{m}(f_{\alpha^\flat v})=\mathfrak{m}(\alpha_*f_v)=\big\langle\mathfrak{m}_{F'}(\alpha_*f)|v\big\rangle\,,$$

where the third equality follows from $\langle f(\cdot)|\alpha^\flat v\rangle=\langle(\alpha_*f)(\cdot)|v\rangle$. This proves the claim.

Now the functoriality implies that the restriction $\mathrm{H}^n_{\mathrm{cb}}(G,E) \to \mathrm{H}^n_{\mathrm{cb}}(H,E)$ is realized, together with its operator semi-norm, by the inclusion $\iota^\bullet$ of complexes

$$\begin{tikzcd}
0 \arrow[r] & L^\infty_{\mathrm{w}*}(G,E)^G \arrow[r] \arrow[d, hook, "\iota^0"] & L^\infty_{\mathrm{w}*}(G^2,E)^G \arrow[r] \arrow[d, hook, "\iota^1"] & L^\infty_{\mathrm{w}*}(G^3,E)^G \arrow[r] \arrow[d, hook, "\iota^2"] & \cdots \\
0 \arrow[r] & L^\infty_{\mathrm{w}*}(G,E)^H \arrow[r] & L^\infty_{\mathrm{w}*}(G^2,E)^H \arrow[r] & L^\infty_{\mathrm{w}*}(G^3,E)^H \arrow[r] & \cdots
\end{tikzcd}$$

We set $F^n = L^\infty_{\mathrm{w}*}(G^{n+1},E)$ and consider the corresponding maps $\mathfrak{m}_{F^n}$. We define for every $f \in (F^n)^H$ the element $\tau^n f$ of $L^\infty_{\mathrm{w}*}(G/H, F^n)$ by $\tau^n f(gH) = \lambda_\pi(g)f$. One checks that the norm one map

$$\tau^n : (F^n)^H \longrightarrow L^\infty_{\mathrm{w}*}(G/H, F^n)$$

ranges actually in $L^\infty_{\mathrm{w}*}(G/H, F^n)^G$; moreover one has $\tau^n \iota^n = \epsilon$. Composing τ^n with $\mathfrak{m}_{F^n}$, we see that we have obtained a norm one left inverse $\mathrm{trans}^n_{\mathfrak{m}} = \mathfrak{m}_{F^n}\tau^n$ to the inclusion ι^n realizing the restriction since

$$\mathrm{trans}^n_{\mathfrak{m}} \iota^n = \mathfrak{m}_{F^n}\tau^n\iota^n = \mathfrak{m}_{F^n}\varepsilon = Id\,.$$

On the other hand, we have $\tau^n d^n = (d^n)_* \tau^{n-1}$ so that the naturality claim above ensures that $\mathfrak{m}_{F^\bullet}\tau^\bullet$ is a morphism of complexes because the differentials $d^\bullet$ are adjoint maps. Therefore it induces a left inverse of semi-norm at most one

$$\mathrm{trans}_{\mathfrak{m}} :\ \mathrm{H}^\bullet_{\mathrm{cb}}(H,E) \longrightarrow \mathrm{H}^\bullet_{\mathrm{cb}}(G,E)$$

to the restriction, finishing the proof. □

As an example, we remark that if $\Gamma < G$ is a non-uniform lattice, then $\mathrm{H}^\bullet_{\mathrm{cb}}(G) \to \mathrm{H}^\bullet_{\mathrm{b}}(\Gamma)$ is injective, while $\mathrm{H}^\bullet_{\mathrm{c}}(G) \to \mathrm{H}^\bullet(\Gamma)$ needs not be so.

For any closed *normal* subgroup $N \lhd G$ and coefficient G-module E, the G/N-action on $\mathrm{H}^\bullet_{\mathrm{cb}}(N,E)$ is defined as follows. Let S be any regular G-space on which the N-action is amenable; for instance, one can take for S any amenable G-space. Then the coefficient G-modules $L^\infty_{\mathrm{w}*}(S^{n+1},E)$ are N-relatively injective. The complex

$$0 \longrightarrow L^\infty_{\mathrm{w}*}(S,E) \longrightarrow L^\infty_{\mathrm{w}*}(S^2,E) \longrightarrow L^\infty_{\mathrm{w}*}(S^3,E) \longrightarrow \cdots$$

computing $\mathrm{H}^\bullet_{\mathrm{cb}}(N,E)$ according to Theorem 2 inherits a G/N-action. It follows from the functoriality that the corresponding isometric action on cohomology does not depend upon the choice of S. Using a classical argument, one moreover shows that the same action is induced by the G-action R on $L^\infty_{\mathrm{w}*}(N^{n+1},E)$ defined by

$$R(g)f(x_0,\dots,x_k) = \pi(g)f(g^{-1}x_0g,\dots,g^{-1}x_kg)\,. \tag{6}$$

Therefore the functoriality implies that the restriction ranges always in the space of G/N-invariant classes. However, even for co-compact subgroups, it is not clear whether the range of the restriction is actually the whole of the G/N-invariants: the difficulty here comes from the fact that the continuous bounded cohomology might not be Hausdorff, so that one cannot integrate over G/N unless this quotient is discrete. This latter case will nevertheless be of use later.

PROPOSITION 2.4.2. *Let $N \lhd G$ be a finite index closed normal subgroup of the locally compact second countable group G, and let E be a coefficient G-module. Then the restriction*

$$\mathrm{res} : \mathrm{H}^n_{\mathrm{cb}}(G, E) \longrightarrow \mathrm{H}^n_{\mathrm{cb}}(N, E)^{G/N}$$

is an isometric isomorphism onto $\mathrm{H}^n_{\mathrm{cb}}(N, E)^{G/N}$ for all $n \geq 0$.

Proof. The transfer is now just the averaging over G/N, exactly as in usual cohomology (see e.g. Proposition III.10.4 in [Bro]), so the classical proof goes through without changes. □

3 Double Ergodicity

3.1 We fix notation for the following important classes:

DEFINITION 3.1.1. We write $\mathfrak{X}^{\mathrm{Hilb}}$ for the class of all unitary coefficient modules (i.e. continuous unitary representations in separable Hilbert spaces). Likewise, $\mathfrak{X}^{\mathrm{refl}}$ is the class of all reflexive coefficient modules and $\mathfrak{X}^{\mathrm{sep}}$ the class of all separable coefficient modules. Finally, $\mathfrak{X}^{\mathrm{cont}}$ denotes the class of all continuous coefficient modules.

We observe

$$\mathfrak{X}^{\mathrm{Hilb}} \subset \mathfrak{X}^{\mathrm{refl}} \subset \mathfrak{X}^{\mathrm{sep}} \subset \mathfrak{X}^{\mathrm{cont}}\,. \tag{7}$$

The only non-trivial inclusion is the last one, which follows from Proposition 1.1.4.

3.2 Basics on double ergodicity. We observe first that if $F \neq 0$ is a coefficient G-module with *trivial* G-action and S a regular G-space, then the G-action is ergodic on $S \times S$ if and only if S is doubly F-ergodic: indeed it is enough to evaluate functions $S \times S \to F$ on a *countable* dense subset of the pre-dual of F.

Moreover, one checks readily the

LEMMA 3.2.1. *(i) Let $\mathfrak{X}$ be a class of coefficient modules, G_1, G_2 locally compact groups and S_1, S_2 doubly $\mathfrak{X}$-ergodic G_1- respectively G_2-spaces. Then $S = S_1 \times S_2$ is a doubly $\mathfrak{X}$-ergodic G-space for $G = G_1 \times G_2$.*

(ii) Suppose $\mathfrak{X}$ is closed under taking weak- closed submodules (e.g. any of $\mathfrak{X}^{\mathrm{cont}}$, $\mathfrak{X}^{\mathrm{sep}}$, $\mathfrak{X}^{\mathrm{refl}}$ or $\mathfrak{X}^{\mathrm{Hilb}}$). Let G be a locally compact group, $H \lhd G$ a closed normal subgroup and S a regular G/H-space. Then the G/H action on S is doubly $\mathfrak{X}$-ergodic if and only if the G-action on S defined via $G \to G/H$ is also doubly $\mathfrak{X}$-ergodic.* □

In connection with (i), we recall that if the G_i-action on S_i is amenable for $i = 1, 2$ then the G-action on S is amenable. Concerning (ii), recall that if S is an amenable G/H-space, then the corresponding G-action on S is amenable if and only if H is amenable.

The basic instances motivating our definition of double ergodicity are consequences of the Mautner property:

PROPOSITION 3.2.2. *Let G be a connected semi-simple real Lie group and $P < G$ a parabolic subgroup. Then the G action on G/P is doubly $\mathfrak{X}^{\mathrm{cont}}$-ergodic.*

PROPOSITION 3.2.3. *Let $\mathcal{T}$ be a locally finite regular or bi-regular tree, G its group of automorphisms and P the stabilizer of a point at infinity. Then the G-action on G/P is doubly $\mathfrak{X}^{\mathrm{cont}}$-ergodic.*

Proof of Propositions 3.2.2 and 3.2.3. In both cases, the *ordinary* ergodicity on $G/P \times G/P$ is just a consequence of a Bruhat decomposition. It is then the classical Mautner lemma that comes in to imply the double $\mathfrak{X}^{\mathrm{Hilb}}$-ergodicity: see II.3 in [M] for the Lie group case and [LM] for the tree case. The proof extends without changes to $\mathfrak{X}^{\mathrm{cont}}$. □

An important closure property is the following.

PROPOSITION 3.2.4. *Suppose $\mathfrak{X}$ is either $\mathfrak{X}^{\mathrm{Hilb}}$, $\mathfrak{X}^{\mathrm{refl}}$ or $\mathfrak{X}^{\mathrm{sep}}$. Let G be a locally compact second countable group, S a doubly $\mathfrak{X}$-ergodic G-space and $H < G$ a closed subgroup. If $H\backslash G$ admits a finite invariant measure, then the H-action on S is also doubly $\mathfrak{X}$-ergodic.*

The proof of the above proposition involves *induction*:

Let (π, F) be any coefficient H-module and G, H, S as in Proposition 3.2.4. Since π is isometric, any H-equivariant map $f : G \to F$ yields a well defined function $\|f\|_F : H\backslash G \to \mathbf{R}$ which is measurable since F has separable pre-dual. Now define the L^2 *induction module* $L^2\mathrm{Ind}_H^G F$ to be the space of those H-equivariant elements f for which $\|f\|_F$ is in $L^2(H\backslash G)$, endowed with the right translation G-action.

LEMMA 3.2.5. *Suppose $\mathfrak{X}$ is either $\mathfrak{X}^{\mathrm{Hilb}}$, $\mathfrak{X}^{\mathrm{refl}}$ or $\mathfrak{X}^{\mathrm{sep}}$. Then the norm $\|(\|f\|_F)\|_2$ turns $L^2\mathrm{Ind}_H^G F$ into a coefficient G-module which belongs to the class $\mathfrak{X}$.*

Proof of the lemma. Consider the separable pre-dual $F^\flat$ of F and recall that by the inclusions (7), the module F is continuous. At the level of Banach spaces, we have an isometric isomorphism $L^2\mathrm{Ind}_H^G F \cong L^2(H\backslash G, F)$. In the cases considered for $\mathfrak{X}$, F has the Radon–Nikodým property (see [DU, VII 7]). Therefore, there is a canonical isometric isomorphism $L^2(H\backslash G, F) = L^2(H\backslash G, F^\flat)^*$ (Theorem 1 in [DU, IV 1]). It remains only to verify that the G-action on $L^2\mathrm{Ind}_H^G F$ is continuous, because the above identification shows then at once that $L^2\mathrm{Ind}_H^G F$ is a coefficient module and is in $\mathfrak{X}$. But the separability of F entails that elements of $L^2\mathrm{Ind}_H^G F$ are Bochner measurable, hence normic limits of uniformly continuous maps $G \to F$, whence the continuity. □

We pick now a weak-* measurable H-equivariant map $f : S \times S \to F$. The idea is to associate to f an *induced* map $\mathbf{i}f$ on $S \times S$ ranging in the space of H-equivariant maps $G \to F$, defined by the formula

$$\mathbf{i}f(s,t)(g) = f(gs, gt) \quad (s,t \in S\,,\ g \in G)\,. \tag{8}$$

Then $\mathbf{i}f$ is obviously G-equivariant with respect to right translations in the image. However we have still to show that $\mathbf{i}f$ ranges in $L^2\mathrm{Ind}_H^G F$. To this end, it is sufficient to show that H is ergodic on $S \times S$, since then $\|f\|_F$ is constant and thus $\mathbf{i}f$ is bounded, hence in $L^2\mathrm{Ind}_H^G F$.

Lemma 3.2.6. *The H-action on $S \times S$ is ergodic.*

Proof of the lemma. To test ergodicity, it is enough to consider a *bounded* H-invariant measurable function $b : S \times S \to \mathbf{C}$. This time $\mathbf{i}b$ ranges in $L^2\mathrm{Ind}_H^G \mathbf{C} \cong L^2(H\backslash G)$ and the assumption on S implies that $\mathbf{i}b$, hence also b, is essentially constant. □

Now we can present the

End of proof of Proposition 3.2.4. For an H-equivariant weak-* measurable map $f : S \times S \to F$, the induced map $\mathbf{i}f : S \times S \to L^2\mathrm{Ind}_H^G F$ is weak-* measurable by Fubini–Lebesgue, so by Lemma 3.2.5 we may apply the assumption on S and conclude that $\mathbf{i}f$ is essentially constant. In consequence, its essential value in $L^2\mathrm{Ind}_H^G F$ is of the form $v\mathbb{1}_G$ for some $v \in F$ and hence f is essentially constant too. □

3.3 The group G^*.

First some notation.

If G is a group and $H < G$ a subgroup, $Z_G(H)$ denotes the centralizer of H in G while $Z(H)$ is the centre of H. If $H \lhd G$ is normal, we denote by $K_G(H)$ the kernel of the representation $G \to \mathrm{Out}(H)$ of G in the group of outer automorphisms of H. Thus $K_G(H) = H.Z_G(H) = Z_G(H).H$ and there are canonical quotient maps $K_G(H) \to Z_G(H)/Z(H)$ and $K_G(H) \to H/Z(H)$.

Let G be a locally compact group. The closure properties of the class of amenable locally compact groups imply that there is a unique maximal amenable closed normal subgroup $A(G) \lhd G$ containing all amenable closed normal subgroups of G. For a topological group G, the identity component is denoted by G^0.

DEFINITION 3.3.1. Let G be a locally compact group. We define $G^* = \pi^{-1}\big(K_L(L^0)\big)$, where $L = G/A(G)$ and $\pi : G \to L$ is the quotient map.

In other words, $G^* \lhd G$ is the kernel of the representation $G \to \mathrm{Out}(L^0)$ defined through $\pi : G \to L$. Therefore, if G is connected, we have $G^* = G$ because the map $L \to \mathrm{Out}(L^0)$ is trivial in view of $L^0 = L$. On the other extreme, if G is totally disconnected (for instance if G is discrete), we have again $G^* = G$: indeed, L is also totally disconnected (by Corollaire 3 in [Bour3, III §4 N° 6]); therefore L^0 is trivial and $G^* = G$.

LEMMA 3.3.2. *Let M be a closed normal subgroup of $L = G/A(G)$. Then $A(M)$ is trivial.*

Proof. The map $\pi : G \to L$ yields a topological group extension

$$1 \longrightarrow A(G) \cap \pi^{-1}\big(A(M)\big) \longrightarrow \pi^{-1}\big(A(M)\big) \longrightarrow A(M) \longrightarrow 1\,.$$

The two extreme terms are amenable, hence $\pi^{-1}\big(A(M)\big)$ is amenable. Being further normal in G, it is contained in $A(G)$. Therefore $A(M) = 1$. □

Using the solution to Hilbert's fifth problem [MonZ] and the finiteness of the group of outer automorphisms of connected semi-simple adjoint Lie groups without compact factors, we deduce:

Theorem 3.3.3. *Let G be a locally compact group and define L, G^* as above.*

- *(i) G^* is a topologically characteristic finite index open subgroup of G.*
- *(ii) The group $G^*/A(G) = K_L(L^0)$ is the topological direct product $L^0.Z_L(L^0)$, and L^0 is a connected semi-simple adjoint real Lie group without compact factors.*

Proof. Since L^0 is a connected locally compact group, there is by [MonZ, Theorem 4.6] a compact normal subgroup $K \lhd L^0$ such that L^0/K is a connected real Lie group. Now $A(L^0) = 1$ (Lemma 3.3.2) implies $K = 1$, hence L^0 is a connected real Lie group. The triviality of $A(L^0)$ implies further that L^0 is semi-simple, adjoint and without compact factors. In this situation, the group $\mathrm{Out}(L^0)$ is finite, so G^* is open of finite index in G. Since L^0 has trivial centre, the product $L^0.Z_L(L^0)$ is direct. It is easy to see that G^* is topologically characteristic. □

3.4 The totally disconnected case. Throughout this section, we let G be a compactly generated totally disconnected locally compact group.

Let $U < G$ be a compact open subgroup (which exists by Corollaire 1 in [Bour3, III §4 N° 6]). Fix a compact generating set C of G such that generality $C = UCU$. Define the graph $\mathfrak{g} = (V, E)$ as follows (we use J.-P. Serre's conventions [S] for graphs). The set of vertices is $V = G/U$ and the set of edges is $E = E_+ \sqcup \overline{E_+}$, where

$$E_+ = \big\{(gU, gcU) : g \in G\,,\ c \in C\big\}$$

with obvious boundary maps. The graph $\mathfrak{g}$ is connected and regular of finite degree $r = |C/U|$. Let $\mathcal{T}_r$ be a r-regular tree and $\mathcal{T}_r \to \mathfrak{g}$ a simplicial universal covering projection. The kernel of the G-action on $\mathfrak{g}$ is $K = \bigcap_{g \in G} gUg^{-1}$, which is compact and normal. For the group $G_1 = G/K$, we have an exact sequence

$$1 \longrightarrow \pi_1(\mathfrak{g}) \longrightarrow \widetilde{G} \longrightarrow G_1 \longrightarrow 1\,,$$

where $\widetilde{G}$ is co-compact in $\mathrm{Aut}(\mathcal{T}_r)$. Let $\partial_\infty \mathcal{T}_r$ be the boundary at infinity of the tree $\mathcal{T}_r$ with its $\mathrm{Aut}(\mathcal{T}_r)$-action and let ν_r be the unique $\mathrm{Stab}(x_0)$-invariant probability measure on $\mathcal{T}_r$, where $\mathrm{Stab}(x_0)$ is the stabilizer in $\mathrm{Aut}(\mathcal{T}_r)$ of some vertex x_0 in $\mathcal{T}_r$. We define now the probability G_1-space (B, ν) as the point realization of $L^\infty(\partial_\infty \mathcal{T}_r)^{\pi_1(\mathfrak{g})}$. Recall that B is a regular G_1-space given with a canonical C *-algebra isomorphism between $L^\infty(B)$ and the weak-* closed sub-C *-algebra $L^\infty(\partial_\infty \mathcal{T}_r)^{\pi_1(\mathfrak{g})}$ of $L^\infty(\partial_\infty \mathcal{T}_r)$, the isomorphism being induced by a measurable equivariant map $\partial_\infty \mathcal{T}_r \to B$. We consider B as a regular G-space via the canonical map $G \to G_1$.

PROPOSITION 3.4.1. *(i) The G-action on B is amenable.*

(ii) The G-action on B is doubly $\mathfrak{X}^{\mathrm{sep}}$-ergodic.

(iii) The G-space B is the Poisson boundary of an étalée measure on G.

Proof. The $\mathrm{Aut}(\mathcal{T}_r)$-action on $\partial_\infty \mathcal{T}_r$ is amenable, because $\partial_\infty \mathcal{T}_r$ is a homogeneous space with amenable stabilizers. Thus the $\widetilde{G}$-action is also amenable, and this implies that the $G_1 = \widetilde{G}/\pi_1(\mathfrak{g})$-action on (B, ν) is amenable (we have pointed out in Remark 2.2.6 how this basic fact can be re-interpreted). Therefore, the G-action is amenable since the kernel K of $G \to G_1$ is compact hence amenable.

As for point (ii), it is enough (Lemma 3.2.1) to show that the G_1-action on B is doubly $\mathfrak{X}^{\mathrm{sep}}$-ergodic. If $f : B \times B \to F$ is a G_1-equivariant weak-* measurable map to a separable coefficient G_1-module F, we pull back through $\partial_\infty\mathcal{T}_r \to B$ and obtain a weak-* measurable $\widetilde{G}$-equivariant $f' : \partial_\infty\mathcal{T}_r \times \partial_\infty\mathcal{T}_r \to F$. Applying successively Proposition 3.2.4 and Proposition 3.2.3, we conclude that f' is essentially constant. Hence f is essentially constant.

For (iii), it is enough to show that B is the Poisson boundary of an étalée measure on G_1. The space $\partial_\infty\mathcal{T}_r$ is the Poisson boundary of an étalée probability measure μ on $\mathrm{Aut}(\mathcal{T}_r)$, and thus also of an an étalée probability measure $\tilde{\mu}$ on the co-compact subgroup $\widetilde{G}$. The projection $p : \widetilde{G} \to G_1$ induces an étalée measure $p_*\tilde{\mu}$ on G_1, and it is straightforward to check that $f \in L^\infty(G_1)$ is $p_*\tilde{\mu}$-harmonic if and only if p^*f is a $\tilde{\mu}$-harmonic function in $L^\infty(\widetilde{G})$. Thus, p^* induces an isomorphism between the $\pi_1(\mathfrak{g})$-invariant $\tilde{\mu}$-harmonic function in $L^\infty(\widetilde{G})$ and the $p_*\tilde{\mu}$-harmonic functions in $L^\infty(G_1)$. By the Poisson transform isomorphism of the latter with $L^\infty(B)$, this realizes B as the Poisson boundary of $p_*\tilde{\mu}$. □

3.5 The general case.

End of proof of Theorem 6. Let G be a locally compact compactly generated group and adopt the notation of Theorem 3.3.3. Since G^* is closed of finite index in G, it is also compactly generated. Hence $K_L(L^0) = G^*/A(G)$ is compactly generated. By the second point of Theorem 3.3.3, $K_L(L^0) = L^0.Z_L(L^0)$ is a direct product, which implies that $Z_L(L^0) = K_L(L^0)/L^0$ is a totally disconnected compactly generated locally compact group. Therefore there is an amenable doubly $\mathfrak{X}^{\mathrm{sep}}$-ergodic regular $Z_L(L^0)$-space B by Proposition 3.4.1.

On the other hand, we know that L^0 is a connected semi-simple adjoint real Lie group without compact factors. Thus Proposition 3.2.2 provides us with an amenable regular L^0-space which is doubly $\mathfrak{X}^{\mathrm{cont}}$-ergodic (this space is of course nothing but the Furstenberg boundary of L^0).

Applying Lemma 3.2.1, we conclude that the direct product $K_L(L^0) = L^0.Z_L(L^0)$ admits an amenable regular $K_L(L^0)$-space S which is doubly $\mathfrak{X}^{\mathrm{sep}}$-ergodic.

We view now S as a G^*-space via the canonical map $G^* \to G^*/A(G) = K_L(L^0)$ and conclude by Lemma 3.2.1 that S is a doubly $\mathfrak{X}^{\mathrm{sep}}$-ergodic G^*-space. Moreover, the G^*-action is amenable because $A(G)$ is amenable. □

REMARK 3.5.1. In the above proof, the L^0-space provided by Proposition 3.2.2 is the Poisson boundary of an étalée measure since it is just the classical Furstenberg boundary of a semi-simple Lie group. On the other hand, the corresponding statement for the $Z_L(L^0)$-space B is point (iii) in

Proposition 3.4.1. Passing to the product, the $K_L(L^0)$-space S is the Poisson boundary of an étalée measure on $K_L(L^0)$. It is a result of Kaimanovich [K, Thm. 2] that this statement passes to amenable extensions; since G^* is by definition an amenable extension of $K_L(L^0)$, we deduce that S is indeed the Poisson boundary of an étalée measure on G^*.

As a first application of Theorem 6, we give the

Proof of Corollary 9. Retain the notation of Corollary 9 and let $G^* \lhd G$ and S be as in Theorem 6. By Theorem 2, the spaces $\mathrm{H}^\bullet_{\mathrm{cb}}(G^*, E)$ and $\mathrm{H}^\bullet_{\mathrm{cb}}(G^*, F)$ together with the map induced by $\alpha : E \to F$ are realized on the complexes

$$\begin{tikzcd}
0 \arrow[r] & L^\infty_{\mathrm{w*,alt}}(S, E)^{G^*} \arrow[r] \arrow[d, "\alpha_*"] & L^\infty_{\mathrm{w*,alt}}(S^2, E)^{G^*} \arrow[r] \arrow[d, "\alpha_*"] & L^\infty_{\mathrm{w*,alt}}(S^3, E)^{G^*} \arrow[r] \arrow[d, "\alpha_*"] & \cdots \\
0 \arrow[r] & L^\infty_{\mathrm{w*,alt}}(S, F)^{G^*} \arrow[r] & L^\infty_{\mathrm{w*,alt}}(S^2, F)^{G^*} \arrow[r] & L^\infty_{\mathrm{w*,alt}}(S^3, F)^{G^*} \arrow[r] & \cdots
\end{tikzcd}$$

where α_* is post-composition by α, and thus is injective in all degrees. The double $\mathfrak{X}^{\mathrm{sep}}$-ergodicity implies $L^\infty_{\mathrm{w*,alt}}(S^2,F)^{G^*}=0$ and hence $L^\infty_{\mathrm{w*,alt}}(S^2,E)^{G^*}$ is zero, too. As a first consequence, we have the vanishing of $\mathrm{H}^1_{\mathrm{cb}}(G^*, F)$ and of $\mathrm{H}^1_{\mathrm{cb}}(G^*, E)$. A second consequence is that $\mathrm{H}^2_{\mathrm{cb}}(G^*, F)$ is identified as a closed subspace of $L^\infty_{\mathrm{w*,alt}}(S^3, F)^{G^*}$, and likewise $\mathrm{H}^2_{\mathrm{cb}}(G^*, E)$ as closed subspace of $L^\infty_{\mathrm{w*,alt}}(S^3, E)^{G^*}$. This, together with the injectivity of α_*, proves the corollary for G^*. The continuity and injectivity of the restriction from G to G^* (Proposition 2.4.1 or 2.4.2) implies that the corollary holds also for G. □

3.6 Induction. We proceed now to establish Corollary 11, which is an analogue of the Eckmann–Shapiro induction lemma (compare [Bl, Théorème 8.7]). The straightforward L^∞ induction isomorphism in (continuous) bounded cohomology would take us away from continuous coefficient modules, therefore we have to use L^2 induction. This is defined as follows. Let $H < G$ be a closed subgroup of the locally compact second countable group G such that $H\backslash G$ admits a finite invariant measure, F a separable coefficient H-module and S an amenable G-space. Then we define a cochain map

$$\mathbf{i} : L^\infty_{\mathrm{w*}}(S^{n+1}, F)^H \longrightarrow L^\infty_{\mathrm{w*}}(S^{n+1}, L^2\mathrm{Ind}_H^G F)^G$$

by the formula (8) but for all $n \geq 0$. In general, one cannot expect any isomorphism in this setting; however, the double ergodicity implies:

Proposition 3.6.1. *Let G be a compactly generated locally compact second countable group and $H < G$ a closed subgroup such that G/H has* *finite invariant measure. Let F be a separable coefficient H-module. Then the L^2 induction*

$$\mathbf{i} : \mathrm{H}^2_{\mathrm{cb}}(H, F) \longrightarrow \mathrm{H}^2_{\mathrm{cb}}(G, L^2\mathrm{Ind}_H^G F)$$

is injective.

Proof. Let $G^* < G$ and S be as in Theorem 6 and set $H' = H \cap G^*$. Since G^* is open in G, there is a restriction morphism

$$r : L^2\mathrm{Ind}_H^G F \longrightarrow L^2\mathrm{Ind}_{H'}^{G^*} F$$

making the following diagram commutative

$$\begin{tikzcd}
\mathrm{H}^2_{\mathrm{cb}}(H, F) \arrow[r, "\mathbf{i}"] \arrow[d, "\mathrm{res}"] & \mathrm{H}^2_{\mathrm{cb}}(G, L^2\mathrm{Ind}_H^G F) \arrow[d, "r_*\mathrm{res}"] \\
\mathrm{H}^2_{\mathrm{cb}}(H', F) \arrow[r, "\mathbf{i}"] & \mathrm{H}^2_{\mathrm{cb}}(G^*, L^2\mathrm{Ind}_{H'}^{G^*} F)
\end{tikzcd}$$

Since H' is of finite index in H, the left restriction arrow is injective (Proposition 2.4.2). Therefore, it is enough to show the injectivity of the lower induction map. An element of its kernel is represented by a map f in $L^\infty_{\mathrm{w}*}(S^3, F)^{H'}$ such that $\mathbf{i}f = db$ for some b in $L^\infty_{\mathrm{w}*}(S^2, L^2\mathrm{Ind}_{H'}^{G^*} F)^{G^*}$. By Fubini–Lebesgue, there is an H'-equivariant weak-* measurable map $b' : S^2 \to F$ such that $\mathbf{i}b' = b$ holds almost everywhere, and hence $f = db'$. It remains only to show that b' is essentially bounded. But H' has finite invariant co-volume in G^*, so by Lemma 3.2.6 the diagonal H'-action on $S \times S$ is ergodic. Since the map $\|b'\|_F : S^2 \to \mathbf{R}$ is measurable and H'-invariant, we conclude that the norm of b' is essentially constant, hence bounded. □

4 A Lyndon–Hochschild–Serre Sequence

4.1 Setup. Since we deal with second countable and hence σ-compact groups, it is a well-known consequence of Baire's category theorem that the sequence $1 \to N \xrightarrow{\iota} G \to Q \to 1$ is topologically isomorphic to $1 \to \iota(N) \to G \to G/\iota(N) \to 1$ (see e.g. the Corollary 3.11 in [DoF, III]). Thus we suppose from now on that N is a normal subgroup of G and that Q is the quotient.

Lemma 4.1.2 below will serve as a pattern for the proof of the following:

Theorem 4.1.1. *Let G be a locally compact second countable group, $N \lhd G$ a closed subgroup and $Q = G/N$ the quotient. Let (π, F) be a coefficient G-module.*

If $\mathrm{H}^1_{\mathrm{cb}}(N, F) = 0$, *then inflation and restriction fit into an exact sequence*

$$0 \longrightarrow \mathrm{H}^2_{\mathrm{cb}}(Q, F^N) \xrightarrow{\mathrm{inf}} \mathrm{H}^2_{\mathrm{cb}}(G, F) \xrightarrow{\mathrm{res}} \mathrm{H}^2_{\mathrm{cb}}(N, F)^Q \longrightarrow$$

$$\longrightarrow \mathrm{H}^3_{\mathrm{cb}}(Q, F^N) \xrightarrow{\mathrm{inf}} \mathrm{H}^3_{\mathrm{cb}}(G, F)\,.$$

Observe that we make no assumption as to whether the spaces $\mathrm{H}^2_{\mathrm{cb}}$ are Hausdorff.

We use standard notation for spectral sequences, see [GM, section III.7].

Let S be an amenable regular G-space and T an amenable regular Q-space. We consider also F as a coefficient N-module, F^N as a coefficient Q-module, S as an amenable N-space and T as a regular G-space. We define a first quadrant double complex $(L^{\bullet,\bullet}, {}^{I}\!d, {}^{II}\!d)$ as follows. For all $p, q \geq 0$ set

$$L^{p,q} = L^\infty_{\mathrm{w}*}(S^{p+1} \times T^{q+1}, F)^G\,.$$

Define ${}^{I}\!d : L^{p,q} \to L^{p+1,q}$ by ${}^{I}\!d = \sum_{j=0}^{p+1}(-1)^j d_j$, where d_j simply omits the j^{th} variable, and similarly define ${}^{II}\!d : L^{p,q} \to L^{p,q+1}$ by ${}^{II}\!d = \sum_{j=p+1}^{p+q+2}(-1)^j d_j$. The total differential ${}^{I}\!d + {}^{II}\!d$ turns the graded total space

$$TL^n = \bigoplus_{p+q=n} L^{p,q}$$

into a cochain complex. The horizontal and vertical filtrations are respectively

$${}^{I}\!F^m TL^n = \bigoplus_{\substack{p+q=n\\ p\geq m}} L^{p,q}\,, \quad {}^{II}\!F^m TL^n = \bigoplus_{\substack{p+q=n\\ q\geq m}} L^{p,q}\,.$$

We get thus two first quadrant spectral sequences ${}^{I}\mathrm{E}^{\bullet,\bullet}_{\bullet}$ and ${}^{II}\mathrm{E}^{\bullet,\bullet}_{\bullet}$ starting respectively with

$${}^{I}\mathrm{E}^{p,q}_1 = \mathrm{H}^{p,q}(L^{p,\bullet}, {}^{II}\!d)\,, \quad {}^{II}\mathrm{E}^{p,q}_1 = \mathrm{H}^{q,p}(L^{\bullet,p}, {}^{I}\!d)\,,$$

and converging both (in the category of linear spaces) to the cohomology of the total complex. Recall that for both spectral sequences the differentials are of the form

$$d : \mathrm{E}^{p,q}_r \to \mathrm{E}^{p+r,q-r+1}_r\,,$$

so that in particular on ${}^{I}\mathrm{E}^{\bullet,\bullet}_1$ the differential is induced by ${}^{I}\!d$ and on ${}^{II}\mathrm{E}^{\bullet,\bullet}_1$ by ${}^{II}\!d$. We recall that any first quadrant spectral sequence $\mathrm{E}^{\bullet,\bullet}_{\bullet}$ converges as follows: for any $r \geq p+1, q+2$ one has $\mathrm{E}^{p,q}_\infty = \mathrm{E}^{p,q}_r$ and hence in particular for all $s \geq 1$ the differential $\mathrm{E}^{0,s-1}_s \to \mathrm{E}^{s,0}_s$ fits into the exact sequence

$$0 \longrightarrow \mathrm{E}^{0,s-1}_\infty \longrightarrow \mathrm{E}^{0,s-1}_s \longrightarrow \mathrm{E}^{s,0}_s \longrightarrow \mathrm{E}^{s,0}_\infty \longrightarrow 0\,. \qquad (9)$$

With the standard notation $\mathrm{E}^n_\infty = \bigoplus_{p+q=n} \mathrm{E}^{p,q}_\infty$, this implies immediately the

LEMMA 4.1.2. *Let* $\mathrm{E}_\bullet^{\bullet,\bullet}$ *be a first quadrant spectral sequence with* $\mathrm{E}_1^{\bullet,1} = 0$. *Then there is a canonical exact sequence*

$$0 \longrightarrow \mathrm{E}_\infty^{2,0} \longrightarrow \mathrm{E}_\infty^{2} \longrightarrow \mathrm{E}_3^{0,2} \longrightarrow \mathrm{E}_3^{3,0} \longrightarrow \mathrm{E}_\infty^{3}\,.$$

Proof. The assumption implies $\mathrm{E}_\infty^{1,1} = 0$, so that the canonical injection $\mathrm{E}_\infty^{2,0} \to \mathrm{E}_\infty^{2}$ fits into the exact sequence

$$0 \longrightarrow \mathrm{E}_\infty^{2,0} \longrightarrow \mathrm{E}_\infty^{2} \longrightarrow \mathrm{E}_\infty^{0,2} \longrightarrow 0\,.$$

Setting $s = 3$ in (9) we have

$$0 \longrightarrow \mathrm{E}_\infty^{0,2} \longrightarrow \mathrm{E}_3^{0,2} \longrightarrow \mathrm{E}_3^{3,0} \longrightarrow \mathrm{E}_\infty^{3,0} \longrightarrow 0\,.$$

Finally, we have the canonical inclusion $0 \to \mathrm{E}_\infty^{3,0} \to \mathrm{E}_\infty^{3}$. Connecting the three exact sequences yields the statement. □

4.2 The first tableaux.

LEMMA 4.2.1. *The first spectral sequence* ${}^{I}\mathrm{E}_\bullet^{\bullet,\bullet}$ *collapses in the first tableau and converges to the continuous bounded cohomology of* G *with coefficients in* F.

Proof. Since N acts trivially on T, we have the identification

$$L^{p,q} \cong L^\infty_{\mathrm{w}*}\big(T^{q+1}, L^\infty_{\mathrm{w}*}(S^{p+1}, F)^N\big)^Q.$$

Since the Q-action on T is amenable, this yields with ${}^{II}d$ a complex as in Theorem 2. Hence there is a canonical isomorphism

$${}^{I}\mathrm{E}_1^{p,q} \cong \mathrm{H}^q_{\mathrm{cb}}\big(Q, L^\infty_{\mathrm{w}*}(S^{p+1}, F)^N\big)\,.$$

By Theorem 1, $L^\infty_{\mathrm{w}*}(S^{p+1}, F)$ is relatively injective for G and hence $L^\infty_{\mathrm{w}*}(S^{p+1}, F)^N$ is relatively injective for Q. This implies by Corollary 1.5.5 that ${}^{I}\mathrm{E}_1^{p,q} = 0$ unless $q = 0$, proving that ${}^{I}\mathrm{E}_1^{\bullet,\bullet}$ collapses, hence this spectral sequence is stationary from the second tableau on. Thus it remains to identify ${}^{I}\mathrm{E}_\infty^{n} = {}^{I}\mathrm{E}_\infty^{n,0} = {}^{I}\mathrm{E}_2^{n,0}$. To this end, observe that

$${}^{I}\mathrm{E}_1^{n,0} \cong \mathrm{H}^0_{\mathrm{cb}}\big(Q, L^\infty_{\mathrm{w}*}(S^{n+1}, F)^N\big) = \big(L^\infty_{\mathrm{w}*}(S^{n+1}, F)^N\big)^Q = L^\infty_{\mathrm{w}*}(S^{n+1}, F)^G$$

and that the differential ${}^{I}\mathrm{E}_1^{n,0} \to {}^{I}\mathrm{E}_1^{n+1,0}$ is induced by ${}^{I}d$, yielding again a complex as in Theorem 2. We conclude ${}^{I}\mathrm{E}_2^{n,0} \cong \mathrm{H}^\bullet_{\mathrm{cb}}(G, F)$. □

PROPOSITION 4.2.2. *Let* $T = Q$.

(i) *There are canonical isomorphisms*

$${}^{II}\mathrm{E}_2^{p,0} \cong \mathrm{H}^p_{\mathrm{cb}}(Q, F^N) \quad \textit{and} \quad {}^{II}\mathrm{E}_2^{0,q} \cong \mathrm{H}^q_{\mathrm{cb}}(N, F)^Q \quad (p, q \geq 0)\,.$$

(ii) *If for some* q *the space* $\mathrm{H}^q_{\mathrm{cb}}(N, F)$ *is Hausdorff, then there is a canonical isomorphism*

$${}^{II}\mathrm{E}_2^{p,q} \cong \mathrm{H}^p_{\mathrm{cb}}\big(Q, \mathrm{H}^q_{\mathrm{cb}}(N, F)\big) \quad (p \geq 0)\,.$$

(iii) If $\mathrm{H}^1_{\mathrm{cb}}(N,F)=0$, *then* ${}^{II}\mathrm{E}_1^{\bullet,1}=0$ *and there is a canonical isomorphism*

$$ {}^{II}\mathrm{E}_3^{p,0}\cong \mathrm{H}^p_{\mathrm{cb}}(Q,F^N)\quad (p\geq 0)\,. $$

LEMMA 4.2.3. *Let* $A\xrightarrow{\alpha}B\xrightarrow{\beta}C$ *be an adjoint sequence of* Q*-morphisms of coefficient* Q*-modules with* $\beta\alpha=0$. *If* $\alpha(A)$ *is closed, then the homology of*

$$ L^\infty_{\mathrm{w}*}(Q^{n+1},A)^Q\xrightarrow{\alpha_*}L^\infty_{\mathrm{w}*}(Q^{n+1},B)^Q\xrightarrow{\beta_*}L^\infty_{\mathrm{w}*}(Q^{n+1},C)^Q $$

is canonically isomorphic to $L^\infty_{\mathrm{w}*}(Q^{n+1},\operatorname{Ker}\beta/\alpha(A))^Q$ *for all* $n\geq 0$.

Proof of the lemma. By the closed range theorem, $\alpha(A)\to\operatorname{Ker}\beta\to\operatorname{Ker}\beta/\alpha(A)$ is adjoint; now use Lemma 1.6.3. □

Proof of Proposition 4.2.2. Point (i): the case of ${}^{II}\mathrm{E}_1^{p,0}$ is contained in (ii) because $\mathrm{H}^0_{\mathrm{cb}}(N,F)=F^N$ is Hausdorff. The term ${}^{II}\mathrm{E}_1^{0,q}$ is defined by

$$ \cdots\xrightarrow{{}^{I}d_*}L^\infty_{\mathrm{w}*}\big(Q,L^\infty_{\mathrm{w}*}(S^{q+1},F)^N\big)^Q\xrightarrow{{}^{I}d_*}\cdots $$

which is intertwined with

$$ L^\infty_{\mathrm{w}*}(S^q,F)^N\xrightarrow{{}^{I}d}L^\infty_{\mathrm{w}*}(S^{q+1},F)^N\xrightarrow{{}^{I}d}L^\infty_{\mathrm{w}*}(S^{q+2},F)^N, $$

by the isomorphism U^0 defined in the proof of Lemma 1.6.3. Hence (Theorem 2) we have ${}^{II}\mathrm{E}_1^{0,q}\cong\mathrm{H}^q_{\mathrm{cb}}(N,F)$. The term ${}^{II}\mathrm{E}_2^{0,q}$ is by definition the kernel of the differential $d:{}^{II}\mathrm{E}_1^{0,q}\to{}^{II}\mathrm{E}_1^{1,q}$. Under the isomorphism U^0, for a cochain $f\in L^\infty_{\mathrm{w}*}(S^{q+1},F)^N$ the class of df in ${}^{II}\mathrm{E}_1^{1,q}$ is represented by the map $q\mapsto qf-f$, so that indeed ${}^{II}\mathrm{E}_2^{0,q}=\mathrm{H}^q_{\mathrm{cb}}(N,F)^Q$.

Point (ii): the term ${}^{II}\mathrm{E}_1^{p,q}$ is defined by

$$ \cdots\xrightarrow{{}^{I}d_*}L^\infty_{\mathrm{w}*}\big(Q^{p+1},L^\infty_{\mathrm{w}*}(S^{q+1},F)^N\big)^Q\xrightarrow{{}^{I}d_*}\cdots $$

so by Lemma 4.2.3 and Theorem 2 we have a canonical isomorphism ${}^{II}\mathrm{E}_1^{p,q}\cong L^\infty_{\mathrm{w}*}\big(Q^{p+1},\mathrm{H}^q_{\mathrm{cb}}(N,F)\big)^Q$. This isomorphism intertwines

$$ \cdots\xrightarrow{{}^{II}d}L^\infty_{\mathrm{w}*}\big(Q^{p+1},\mathrm{H}^q_{\mathrm{cb}}(N,F)\big)^Q\xrightarrow{{}^{II}d}\cdots $$

with ${}^{II}\mathrm{E}_1^{p-1,q}\to{}^{II}\mathrm{E}_1^{p,q}\to{}^{II}\mathrm{E}_1^{p+1,q}$. Applying Theorem 2 once again we conclude ${}^{II}\mathrm{E}_2^{p,q}\cong\mathrm{H}^p_{\mathrm{cb}}(Q,\mathrm{H}^q_{\mathrm{cb}}(N,F))$.

Point (iii): assume now $\mathrm{H}^1_{\mathrm{cb}}(N,F)=0$. The above consideration gives ${}^{II}\mathrm{E}_1^{p,1}=0$ whence ${}^{II}\mathrm{E}_\bullet^{p,1}=0$. Since by definition ${}^{II}\mathrm{E}_3^{p,0}$ is the (algebraic) cokernel of ${}^{II}\mathrm{E}_2^{p-2,1}\to{}^{II}\mathrm{E}_2^{p,0}$ we have ${}^{II}\mathrm{E}_3^{p,0}={}^{II}\mathrm{E}_2^{p,0}$. □

We can now complete the

Proof of Theorem 4.1.1. Take $T = Q$ and apply Proposition 4.2.2. Consider the exact sequence of Lemma 4.1.2 for ${}^{II}\mathrm{E}_{\bullet}^{\bullet,\bullet}$. We have ${}^{II}\mathrm{E}_{\infty}^{n} \cong {}^{I}\mathrm{E}_{\infty}^{n}$ which is isomorphic to $\mathrm{H}_{\mathrm{cb}}^{n}(G, F)$ by Lemma 4.2.1, so the terms ${}^{II}\mathrm{E}_{\infty}^{2}$ and ${}^{II}\mathrm{E}_{\infty}^{3}$ are identified. Since ${}^{II}\mathrm{E}_{\infty}^{2,0} = {}^{II}\mathrm{E}_{3}^{2,0}$, this term is given by Proposition 4.2.2 point (iii). The same point identifies ${}^{II}\mathrm{E}_{3}^{3,0}$. As for the term ${}^{II}\mathrm{E}_{3}^{0,2}$, it is given as the cohomology of

$$ {}^{II}\mathrm{E}_{2}^{-2,3} \longrightarrow {}^{II}\mathrm{E}_{2}^{0,2} \longrightarrow {}^{II}\mathrm{E}_{2}^{2,1}. \tag{10} $$

The first term here vanishes. On the other hand, $\mathrm{H}_{\mathrm{cb}}^{1}(N, F) = 0$ is indeed Hausdorff, so Proposition 4.2.2 point (ii) identifies ${}^{II}\mathrm{E}_{2}^{2,1}$ as $\mathrm{H}_{\mathrm{cb}}^{2}(Q, 0) = 0$. Thus (10) degenerates and ${}^{II}\mathrm{E}_{3}^{0,2} = {}^{II}\mathrm{E}_{2}^{0,2}$, which is now identified by the first point of Proposition 4.2.2. Thus we have an exact sequence of the required type; unravelling the identifications, we see that except for ${}^{II}\mathrm{E}_{3}^{0,2} \to {}^{II}\mathrm{E}_{3}^{3,0}$, the maps come from inflation and restriction. □

We point out a particular case:

COROLLARY 4.2.4. *Suppose $G = N \rtimes Q$ is a (topological) semi-direct product of the locally compact second countable groups N, Q. Let (π, F) be a coefficient G-module.*

If $\mathrm{H}_{\mathrm{cb}}^{1}(N, F) = 0$, then we have the exact sequence

$$ 0 \longrightarrow \mathrm{H}_{\mathrm{cb}}^{2}(Q, F^{N}) \xrightarrow{\ \mathrm{inf}\ } \mathrm{H}_{\mathrm{cb}}^{2}(G, F) \xrightarrow{\ \mathrm{res}\ } \mathrm{H}_{\mathrm{cb}}^{2}(N, F)^{Q} \longrightarrow 0\,. $$

Proof. There is a topological group homomorphism $\sigma : Q \to G$ with $p\sigma = Id$, where p is the canonical map $G \to Q$. The inflation is precisely the map induced by p. Therefore, by contravariance, the map induced by σ is a left inverse for the inflation, so that the inflation is injective. By exactness at $\mathrm{H}_{\mathrm{cb}}^{3}(Q, F^{N})$ in Theorem 4.1.1, we deduce that the map $\mathrm{H}_{\mathrm{cb}}^{2}(N, F)^{Q} \to \mathrm{H}_{\mathrm{cb}}^{3}(Q, F^{N})$ vanishes, whence the statement. □

4.3 More on $\mathbf{H}_{\mathbf{cb}}^{2}$. An important consequence of double ergodicity is the following.

Theorem 4.3.1. *Let G be a locally compact second countable group, $N \lhd G$ a compactly generated closed normal subgroup and (π, F) a separable coefficient G-module.*

Then the inclusion $F^{Z_G(N)} \to F$ induces a canonical isometric identification $\mathrm{H}_{\mathrm{cb}}^{2}(N, F^{Z_G(N)})^{G} \cong \mathrm{H}_{\mathrm{cb}}^{2}(N, F)^{G}$.

The main step is

PROPOSITION 4.3.2. *Let G be a locally compact second countable group, $N \lhd G$ a closed normal subgroup and (π, F) a coefficient G-module.*

If N admits an amenable doubly F-ergodic regular space S, then the inclusion $F^{Z_G(N)} \to F$ induces a canonical isometric identification $\mathrm{H}^2_{\mathrm{cb}}(N, F^{Z_G(N)}) \cong \mathrm{H}^2_{\mathrm{cb}}(N, F)^{Z_G(N)}$.

Proof of Proposition 4.3.2. We realize $\mathrm{H}^2_{\mathrm{cb}}(N, F)$ on the complex $L^\infty_{\mathrm{w}*}(N^\bullet, F)^N$. The $Z_G(N)$-action π on F is by N-morphisms and hence induces a "coefficient" action on $\mathrm{H}^\bullet_{\mathrm{cb}}(N, F)$. On the other hand, the natural G-action on $\mathrm{H}^\bullet_{\mathrm{cb}}(N, F)$ is given by the operator R of given in section 2.4, equation (6). Yet this operator coincides with π on $Z_G(N)$, so that it induces also the coefficient action when restricted to $Z_G(N)$.

It remains thus to show that any class $\omega \in \mathrm{H}^2_{\mathrm{cb}}(N, F)$ invariant under the coefficient $Z_G(N)$-action is represented by a cocycle ranging in $F^{Z_G(N)}$. We realize $\mathrm{H}^\bullet_{\mathrm{cb}}(N, F)$ on the complex $L^\infty_{\mathrm{w}*}(S^\bullet, F)^N$. Thus ω can be represented by a cocycle $f \in L^\infty_{\mathrm{w}*}(S^3, F)^N$, and for each $z \in Z_G(N)$ there is $b_z \in L^\infty_{\mathrm{w}*}(S^2, F)^N$ with $\pi(z) \circ f = f + db_z$. One can take f and b_z alternating, so that $b_z = 0$ by double F-ergodicity and hence f ranges in $F^{Z_G(N)}$. □

Proof of Theorem 4.3.1. We let $N^* < N$ be as in Theorem 6. We have then the following natural diagram, where α, β, η are the maps induced by the corresponding inclusions of coefficients (observe that $Z_G(N^*) \supset Z_G(N)$ implies $F^{Z_G(N^*)} \subset F^{Z_G(N)}$). The theorem is about α.

$$\begin{tikzcd}
\mathrm{H}^2_{\mathrm{cb}}(N, F^{Z_G(N)})^G \arrow[rr, "\alpha"] \arrow[d, "\mathrm{res}", "\cong"'] & & \mathrm{H}^2_{\mathrm{cb}}(N, F)^G \arrow[d, "\mathrm{res}"', "\cong"] \\
\mathrm{H}^2_{\mathrm{cb}}(N^*, F^{Z_G(N)})^G & \mathrm{H}^2_{\mathrm{cb}}(N^*, F^{Z_G(N^*)})^G \arrow[l, "\beta"'] \arrow[r, "\eta", "\cong"'] & \mathrm{H}^2_{\mathrm{cb}}(N^*, F)^G
\end{tikzcd}$$

The map η is an isomorphism by Proposition 4.3.2, and the restrictions are isomorphisms by Proposition 2.4.2. Since all maps above are obtained either by covariance or contravariance, all possible commutation relations hold. Thus it is enough to show that β is bijective. But $\mathrm{res} = \beta\eta^{-1}\,\mathrm{res}\,\alpha$ implies surjectivity, while $\eta = \mathrm{res}\,\alpha\,\mathrm{res}^{-1}\,\beta$ entails injectivity. □

We can now give the

Proof of Theorem 13. By Corollary 9, we have $\mathrm{H}^1_{\mathrm{cb}}(N, F) = 0$, so Theorem 4.1.1 applies. The Theorem 4.3.1 yields $\mathrm{H}^2_{\mathrm{cb}}(N,F)^G = \mathrm{H}^2_{\mathrm{cb}}(N, F^{Z_G(N)})^G$, finishing the proof. □

4.4 Product formulae. A first immediate application of the above results is the following:

Corollary 4.4.1. *Let $G_1, \dots, G_n$ be compactly generated locally compact second countable groups and let $G = \prod_{j=1}^n G_j$. Let (π, F) be a separable coefficient G-module. Then the inflation and restriction maps yield a canonical topological isomorphism*

$$\mathrm{H}^2_{\mathrm{cb}}(G, F) \cong \bigoplus_{j=1}^{n} \mathrm{H}^2_{\mathrm{cb}}(G_j, F^{G'_j}),$$

where $G'_j = \prod_{i \neq j} G_i$.

Proof. The case $n = 1$ is void. For $n = 2$, combine Corollary 4.2.4 with Theorem 4.3.1 to obtain

$$\mathrm{H}^2_{\mathrm{cb}}(G_1 \times G_2, F) \cong \mathrm{H}^2_{\mathrm{cb}}(G_1, F^{G_2}) \oplus \mathrm{H}^2_{\mathrm{cb}}(G_2, F^{G_1}). \tag{11}$$

If $n \geq 2$, an induction over n reduces the statement to successive applications of the formula (11). □

This statement implies a strong restriction on the range of cohomology classes for a product. In order to formulate this (Corollary 4.4.3 below), we need the

Lemma 4.4.2. *Keep the notation of Corollary 4.4.1. Then $\sum_{j=0}^n F^{G'_j}$ is weak-* closed in F, so that it is again a coefficient G-module.*

Proof. Pick v in the weak-* closure of $\sum_{j=0}^n F^{G'_j}$ and take a sequence $(v_j^k)_{k \in \mathbf{N}}$ of $F^{G'_j}$ such that $v^k = \sum_{j=0}^n v_j^k$ converges weak-* to v. For any $g \in G_1$, we have $\pi(g)v^k - v^k = \pi(g)v_1^k - v_1^k$, which is in $F^{G'_1}$ and yet converges to $\pi(g)v - v$. Since $F^{G'_1}$ is weak-* closed, we conclude that for every $g \in G_1$ the difference $\pi(g)v - v$ is in $F^{G'_1}$. This yields a 1-cocycle for $\mathrm{H}^1_{\mathrm{cb}}(G_1, F^{G'_1})$. This cohomology group vanishes by Corollary 9, so that there is $u_1 \in F^{G'_1}$ with $\pi(g)v - v = \pi(g)u_1 - u_1$ for all $g \in G_1$, and therefore $v - u_1 \in F^{G_1}$.

We may now repeat the argument with $G_2 \times \cdots \times G_n$ instead of G, F^{G_1} instead of F and $v - u_1$ replacing v. This way we obtain by induction that there are $u_j \in F^{G'_j}$ for $j = 1, \dots, n-1$ such that

$$v - u_1 - u_2 \dots - u_{n-1} \ \in \ F^{G_1} \cap F^{G_2} \cap \dots \cap F^{G_{n-1}} = F^{G'_n},$$

and hence v is in $\sum_{j=0}^n F^{G'_j}$. □

Corollary 4.4.3. *Keep the notation of Corollary 4.4.1. There is a canonical topological isomorphism*

$$\mathrm{H}^2_{\mathrm{cb}}(G, F) \cong \mathrm{H}^2_{\mathrm{cb}}\big(G, \textstyle\sum_{j=1}^n F^{G'_j}\big).$$

Proof. Apply Corollary 4.4.1 successively to the coefficient G-modules F and $\sum_{j=1}^n F^{G'_j}$. □

Proof of Theorem 14. The Corollaries 4.4.1 and 4.4.3 yield topological isomorphisms between the terms of Theorem 14. □

Proof of Corollary 15. The irreducibility of the G-action on M implies that for all j one has $L^2(M)^{G'_j} = \mathbf{C}\mathbb{1}_M$. Therefore, considering the diagram induced by $\mathbf{C}\mathbb{1}_M \subset L^\infty(M) \subset L^2(M)$, the Theorem 14 implies that the upper arrow

$$\begin{tikzcd}
\mathrm{H}^2_{\mathrm{cb}}(G) \arrow[rr] \arrow[dr] & & \mathrm{H}^2_{\mathrm{cb}}\big(G, L^2(M)\big) \\
& \mathrm{H}^2_{\mathrm{cb}}\big(G, L^\infty(M)\big) \arrow[ur] &
\end{tikzcd}$$

is an isomorphism. On the other hand, the inclusion $L^\infty(M) \subset L^2(M)$ is an adjoint map and $L^2(M)$ is separable, so that by Corollary 9 the right arrow is injective. Hence all arrows are isomorphisms. □

REMARK 4.4.4. In the statement of Theorems 14, the formula

$$\mathrm{H}^2_{\mathrm{cb}}(G, F) \;\cong\; \bigoplus_{j=1}^{n} \mathrm{H}^2_{\mathrm{cb}}(G_j, F^{G'_j}) \tag{12}$$

actually holds also if there is *one* non-compactly generated factor in the product

$$G_1 \times \cdots \times G_n \,.$$

Indeed, the Lyndon–Hochschild–Serre sequence of Theorem 13 requires only the kernel of the extension to be compactly generated. Therefore, the induction used to prove (12) can be carried out by taking successively all compactly generated groups as kernels, the only non-compactly generated one remaining as last quotient.

5 Remaining Proofs

5.1 Proof of Theorem 16. We turn now to the proof of Theorem 16 in the generality of Remark 19. In other words, G_j are compactly generated locally compact second countable groups ($j = 1, \ldots, n$) and $H < G = G_1 \times \cdots G_n$ is a closed subgroup such that G/H has finite invariant measure and with $\overline{\mathrm{pr}_j(H)} = G_j$ for all j. Let (π, F) be a separable coefficient H-module. The condition on $\mathrm{pr}_j(H)$ implies that there is a unique maximal H-submodule of F such that the restriction $\pi|_{F_j}$ extends continuously to a G-representation π_j factoring through $G \twoheadrightarrow G_j$. Recall the notation $G'_j = \prod_{i \neq j} G_i$.

Lemma 5.1.1. *There is a natural isometric isomorphism of G-modules*

$$F_j \cong \big(L^2\mathrm{Ind}_H^G F\big)^{G'_j}.$$

Proof. Define a map $F_j \to L^2\mathrm{Ind}_H^G F$ by $v \mapsto f_v$, where $f_v(g) = \pi_j(g)v$ is indeed H-equivariant. Since π_j factors through G_j, the map f_v is G'_j-invariant under right translation. Moreover, the map $v \mapsto f_v$ is G-equivariant and it preserves the norm since π_j is isometric and the invariant measure on $H\backslash G$ is normalized. It remains thus to show surjectivity onto the G'_j-invariants. If $f : G \to F$ is in $(L^2\mathrm{Ind}_H^G F)^{G'_j}$, then by Fubini–Lebesgue it is represented by a $\mathrm{pr}_j(H)$-equivariant map $G_j \to F$, which has to be of the form f_v for some $v \in F_j$ by the density of $\mathrm{pr}_j(H)$, since F is continuous by the inclusions (7). □

The Lemma 4.4.2 implies the following

Lemma 5.1.2. *The sum $\sum_{j=0}^n F_j$ is weak-* closed in F, so that it is again a coefficient G-module extending the H-action.*

Proof. The weak-* continuous G-action on the F_j extends to $\sum_{j=0}^n F_j$ and hence to its weak-* closure that we shall denote by F_∞. Applying Lemma 4.4.2 to F_∞ yields the statement since $(F_\infty)^{G'_j} = F_j$. □

End of proof of Theorem 16. We consider the following diagram:

$$\begin{tikzcd}
\mathrm{H}^2_{\mathrm{cb}}(H,F) \arrow[r, "\mathbf{i}"'] & \mathrm{H}^2_{\mathrm{cb}}(G, L^2\mathrm{Ind}_H^G F) \arrow[dr, leftrightarrow] & \\
 & & \bigoplus_{j=1}^n \mathrm{H}^2_{\mathrm{cb}}(G_j, F_j) \\
\mathrm{H}^2_{\mathrm{cb}}\big(H, \sum_{j=1}^n F_j\big) \arrow[uu] & \mathrm{H}^2_{\mathrm{cb}}\big(G, \sum_{j=1}^n F_j\big) \arrow[l, "\mathrm{res}"'] \arrow[uu, "\alpha"] \arrow[ur, leftrightarrow] &
\end{tikzcd}$$

On the right we have a commutative triangle of isomorphisms by Corollaries 4.4.1 and 4.4.3, with the identification provided by Lemma 5.1.1 inducing the map α. The left square is commutative because the formula for α coincides with the composition of restriction and induction. Thus $\mathbf{i}$ is surjective. On the other hand, it is injective by Proposition 3.6.1. Being continuous, it is thus a topological isomorphism because Corollary 9 allows us to apply the open mapping theorem. □

5.2 Higher rank lattices.

In this section, we present the proof of Theorems 20 and 21. The main additional ingredient is the following proposition, based on results of Margulis and Lubotzky–Mozes–Raghunathan in a way similar to Shalom's [Sh2].

PROPOSITION 5.2.1. *Let* Γ, G *be as in Theorem 20 or Theorem 21 and let* $(\pi, \mathfrak{H})$ *be any unitary* Γ*-representation. Then the induction* $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathfrak{H}) \to \mathrm{H}^2_{\mathrm{cb}}(G, L^2\mathrm{Ind}_\Gamma^G\mathfrak{H})$ *maps* $\mathrm{EH}^2_{\mathrm{b}}(\Gamma, \mathfrak{H})$ *to* $\mathrm{EH}^2_{\mathrm{cb}}(G, L^2\mathrm{Ind}_\Gamma^G\mathfrak{H})$.

Proof. A class $[\omega]$ in the kernel $\mathrm{EH}^2_{\mathrm{b}}(\Gamma, \mathfrak{H})$ is given by a Γ-equivariant map $\alpha : \Gamma^2 \to \mathfrak{H}$ such that $\omega = d\alpha$ is bounded. We realize induction as follows. Fix a Borelian fundamental domain $\mathcal{F} \subset G$ for the left Γ-action and denote by $\sigma : G \to \Gamma$ the associated Γ-equivariant retraction. Now define

$$\mathbf{i}\alpha : G^2 \longrightarrow L^2_{\mathrm{loc}}\mathrm{Ind}_\Gamma^G\mathfrak{H}\ , \quad \mathbf{i}\alpha(g_0, g_1)(g) = \alpha\big(\sigma(gg_0), \sigma(gg_1)\big)\,.$$

We know that $d\mathbf{i}\alpha$ ranges in $L^2\mathrm{Ind}_\Gamma^G\mathfrak{H}$ since it coincides with $\mathbf{i}\omega$ (defined by the analogous formula), and since ω is bounded. Therefore, what we have to show is that $\mathbf{i}\alpha$ actually ranges also in $L^2\mathrm{Ind}_\Gamma^G\mathfrak{H}$, that is,

$$\int_{\mathcal{F}} \big\|\alpha(\sigma(gg_0), \sigma(gg_1))\big\|^2 dm(g) < \infty$$

for all g_0, g_1 (m a left Haar measure). Equivalently, setting $\psi(\gamma) = \alpha(\gamma, e)$ and $\kappa(g, g') = \sigma(g)^{-1}\sigma(gg')$, we must show

$$\int_{\mathcal{F}} \big\|\psi(\kappa(g, g'))\big\|^2 dm(g) < \infty \quad (\forall g' \in G)\,.$$

By the conclusion of section IX.3 in Margulis' book [M], Γ is finitely generated; we fix a finite generating set S and denote by ℓ the corresponding word length on Γ. Since for all $\gamma_0, \gamma_1 \in \Gamma$ we have

$$\big\|\psi(\gamma_0\gamma_1) - \psi(\gamma_0) - \pi(\gamma_0)\psi(\gamma_1)\big\| \leq \|\omega\|_\infty\,,$$

one can check by induction on the word length of $\gamma \in \Gamma$ that

$$\|\psi(\gamma)\| \leq C\ell(\gamma) \quad \text{for} \quad C = \max_{s \in S} \|\psi(s)\| + \|\omega\|_\infty\,.$$

Therefore the above integral is dominated by

$$C \int_{\mathcal{F}} \ell\big(\kappa(g, g')\big)^2 dm(g)\,.$$

In [Sh2] (cf. also [Sh1]), Y. Shalom shows how to use the work [LMR] of A. Lubotzky, S. Mozes and M.S. Raghunathan in order to deduce that this last integral is finite for lattices as those considered here. □

Let us begin with the

Proof of Theorem 21. Retain the notation of Theorem 21. We denote by $L^2_{\mathrm{loc}}\mathrm{Ind}_\Gamma^G\mathfrak{H}$ the Fréchet space defined as $L^2\mathrm{Ind}_\Gamma^G\mathfrak{H}$, except that the maps are only required to be *locally* square-summable. The Blanc–Borel–Wallach version of the Eckmann–Shapiro lemma (see [BoW]) states that cochain induction yields an isomorphism from $\mathrm{H}^\bullet(\Gamma, \mathfrak{H})$ onto $\mathrm{H}^\bullet_{\mathrm{c}}(G, L^2_{\mathrm{loc}}\mathrm{Ind}_\Gamma^G\mathfrak{H})$. However, in general, the induction map does not factor through $\mathrm{H}^\bullet_{\mathrm{c}}(G, L^2\mathrm{Ind}_\Gamma^G\mathfrak{H})$.

By contrast, the induction of a bounded cochain ranges in $L^2\mathrm{Ind}_\Gamma^G\mathfrak{H}$ since Γ has finite co-volume. This situation accounts for the missing arrow in the following commutative diagram, in which the space $\mathrm{H}^2_{\mathrm{cb}}(G, L^2_{\mathrm{loc}}\mathrm{Ind}_\Gamma^G\mathfrak{H})$ is only added for more symmetry; we define it ad hoc using cochains which are bounded for the canonical bornology of the Fréchet space $L^2_{\mathrm{loc}}\mathrm{Ind}_\Gamma^G\mathfrak{H}$.

$$\begin{tikzcd}
\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathfrak{H}) \arrow[rr] \arrow[dr] \arrow[ddd] & & \mathrm{H}^2_{\mathrm{cb}}(G, L^2_{\mathrm{loc}}\mathrm{Ind}_\Gamma^G\mathfrak{H}) \arrow[ddd] \\
& \mathrm{H}^2_{\mathrm{cb}}(G, L^2\mathrm{Ind}_\Gamma^G\mathfrak{H}) \arrow[ur] \arrow[d] & \\
& \mathrm{H}^2_{\mathrm{c}}(G, L^2\mathrm{Ind}_\Gamma^G\mathfrak{H}) \arrow[dr] & \\
\mathrm{H}^2(\Gamma,\mathfrak{H}) \arrow[rr] & & \mathrm{H}^2_{\mathrm{c}}(G, L^2_{\mathrm{loc}}\mathrm{Ind}_\Gamma^G\mathfrak{H})
\end{tikzcd}$$

Our proof consists of showing that the diagonal path from $\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathfrak{H})$ to $\mathrm{H}^2_{\mathrm{c}}(G, L^2_{\mathrm{loc}}\mathrm{Ind}_\Gamma^G\mathfrak{H})$ is injective. We have shown in [BuM1, Proposition 4.2] that the induction from $\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathfrak{H})$ to $\mathrm{H}^2_{\mathrm{cb}}(G, L^2\mathrm{Ind}_\Gamma^G\mathfrak{H})$ is injective (without co-compactness assumption). The injectivity of the map from $\mathrm{H}^2_{\mathrm{cb}}(G, L^2\mathrm{Ind}_\Gamma^G\mathfrak{H})$ to $\mathrm{H}^2_{\mathrm{c}}(G, L^2\mathrm{Ind}_\Gamma^G\mathfrak{H})$ is Proposition 6.2 in [BuM1]; notice here that the co-compactness assumption is not used in the proof of this Proposition 6.2.

Therefore, the Proposition 5.2.1 completes the proof that the diagonal path is injective. □

Proof of Theorem 20. Retain the notation of Theorem 20. In view of Proposition 5.2.1, it remains to show that the comparison map

$$\mathrm{H}^2_{\mathrm{cb}}(G, L^2\mathrm{Ind}_\Gamma^G\mathfrak{H}) \longrightarrow \mathrm{H}^2_{\mathrm{c}}(G, L^2\mathrm{Ind}_\Gamma^G\mathfrak{H}) \tag{13}$$

is injective. We split $L^2\mathrm{Ind}_\Gamma^G\mathfrak{H}$ as $\big(L^2\mathrm{Ind}_\Gamma^G\mathfrak{H}\big)^G \oplus \mathfrak{L}$, where $\mathfrak{L}$ is the orthogonal complement to the G-invariants, and handle the summands separately. The G-invariant part is dealt with by our Lemma 6.1 in [BuM1].

As for $\mathfrak{L}$, we write for all $\alpha \in \mathrm{A}$

$$\mathfrak{L}_\alpha = \mathfrak{L}^{\prod_{\beta\neq\alpha}\mathbf{G}_\beta(k_\beta)}\,.$$

According to Theorem 14, we have

$$\mathrm{H}^2_{\mathrm{cb}}(G,\mathfrak{L}) \cong \bigoplus_{\alpha\in\mathrm{A}} \mathrm{H}^2_{\mathrm{cb}}(G,\mathfrak{L}_\alpha)\,.$$

Since $\mathfrak{H}$ is non-degenerate, we have $\mathfrak{L}_\alpha = 0$ whenever the k_α-rank of $\mathbf{G}_\alpha$ is one. For the higher rank factors, one repeats exactly the arguments of our Proposition 6.2 in [BuM1] and concludes that the comparison map

$$\bigoplus_{\alpha \in \mathrm{A}} \mathrm{H}^2_{\mathrm{cb}}(G, \mathfrak{L}_\alpha) \longrightarrow \bigoplus_{\alpha \in \mathrm{A}} \mathrm{H}^2_{\mathrm{c}}(G, \mathfrak{L}_\alpha)$$

is injective. Now the injectivity of (13) follows readily. □

5.3 Corollaries.

Proof of Corollary 22. In this situation, the Theorem 16 implies

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma) \cong \bigoplus_{j=1}^{n} \mathrm{H}^2_{\mathrm{cb}}(G_j) = 0\,,$$

so that $e_{\pi,\mathbf{R}}$ vanishes in $\mathrm{H}^2_{\mathrm{cb}}(\Gamma, \mathbf{R})$. Considering the coefficient exact sequence pointed out by S. Gersten [Ge]

$$\cdots \longrightarrow \mathrm{H}^1(\Gamma, \mathbf{S}^1) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbf{Z}) \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbf{R}) \longrightarrow \cdots \tag{14}$$

we see that e_π must be in the image of $\mathrm{H}^1(\Gamma, \mathbf{S}^1)$, so that by É. Ghys' criterion the action is semi-conjugated to an action by rotations. □

Proof of Corollary 23. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^2_{\mathrm{b}}(\Gamma) \arrow[d] & \mathrm{H}^2_{\mathrm{cb}}(G) \arrow[l, "\mathrm{res}", "\cong"'] \arrow[d] \\
\mathrm{H}^2(\Gamma) & \mathrm{H}^2_{\mathrm{c}}(G) \arrow[l, hook, "\mathrm{res}"]
\end{tikzcd}$$

The upper restriction map is an isomorphism by Theorem 16. It is well known that the lower restriction map is injective because Γ is co-compact. If now $f : \Gamma \to \mathbf{C}$ is a quasimorphism, it follows from this diagram that there is a continuous quasimorphism $F : G \to \mathbf{C}$ and $h \in \ell^\infty(\Gamma)$ such that $\delta(f + h) = \delta F|_{\Gamma \times \Gamma}$. In particular, $\chi = f + h - F$ is in $\mathrm{Hom}(\Gamma, \mathbf{C})$. By Y. Shalom's result (Theorem 0.8 in [Sh2]), χ extends to a continuous homomorphism $\mathrm{X} : G \to \mathbf{C}$. Pick now any $H \in \mathrm{C}_{\mathrm{b}}(G)$ with $H|_\Gamma = h$. Then $f_{\mathrm{ext}} = F - H + \mathrm{X}$ is the desired extension. □

Proof of Corollary 24. We have shown in [BuM1, Lemma 6.1] that the natural map $\mathrm{H}^2_{\mathrm{cb}}(\mathbf{G}_\alpha(k_\alpha)) \to \mathrm{H}^2_{\mathrm{c}}(\mathbf{G}_\alpha(k_\alpha))$ is injective for any α. On the other hand, the right-hand side is known: it vanishes unless the associated symmetric space is Hermitian, in which case it is one dimensional and generated by a *bounded* cocycle, see [GuW]. This determines $\mathrm{H}^2_{\mathrm{cb}}(\mathbf{G}_\alpha(k_\alpha))$, and hence, by Theorem 16, it determines also at once $\mathrm{H}^2_{\mathrm{cb}}(G)$ and $\mathrm{H}^2_{\mathrm{b}}(\Gamma)$. □

Proof of Corollary 26. Let $G_j = \overline{\mathrm{pr}_j(\Gamma)}$. Using a Cartan decomposition for G_j, we have shown in [BuM1, Lemma 7.1] that the natural map $\mathrm{H}^2_{\mathrm{cb}}(G_j) \to \mathrm{H}^2_{\mathrm{c}}(G_j)$ is injective. However, the right hand side vanishes because G_j acts properly on the tree $\mathcal{T}_j$ (this vanishing is a particular case of Lemma 1.12 in [BoW, chap. X]). Thus $\mathrm{H}^2_{\mathrm{cb}}(G_j) = 0$ for all j and we conclude with Theorem 16. □

Proof of Corollary 33. Take first Γ arbitrary. Denoting by $\mathrm{EH}^\bullet_{\mathrm{b}}$ the kernel of the maps $\mathrm{H}^\bullet_{\mathrm{b}} \to \mathrm{H}^\bullet$, the long exact sequence sketched at (14) yields by a diagram chase the exact sequence

$$0 \longrightarrow \mathrm{H}^1(\Gamma, \mathbf{R})/\mathrm{H}^1(\Gamma, \mathbf{Z}) \longrightarrow \mathrm{EH}^2_{\mathrm{b}}(\Gamma, \mathbf{Z}) \longrightarrow \mathrm{EH}^2_{\mathrm{b}}(\Gamma, \mathbf{R}) \longrightarrow 0\,.$$

Since moreover $\mathrm{EH}^\bullet_{\mathrm{cb}}(\Gamma) \cong \mathrm{EH}^\bullet_{\mathrm{cb}}(\Gamma, \mathbf{R})^2$, the equivalence of (a) and (b) preceding Corollary 33 follows from the fact that Γ_{Ab} is torsion if and only if the map $\mathrm{H}^1(\Gamma, \mathbf{Z}) \to \mathrm{H}^1(\Gamma, \mathbf{R})$ is surjective. Thus, turning back to our particular Γ and in view of Theorem 21, Corollary 24, Corollary 26 and Theorem 28, it remains only to justify that Γ_{Ab} is torsion. In the first and third settings, this is a result of Margulis, while in the second the additional co-compactness assumption allow us to apply Shalom's result [Sh2, Theorem 0.8] to the same end. (We have taken Theorem 28 for granted since its proof below is independent of Corollary 33.) □

5.4 Proof of Theorem 28. We may and do suppose that $\mathbf{G}$ is K-almost simple by applying the product formula of Theorem 14.

We write $\mathcal{V}$ for the set of places of K and $\mathbf{A}_K$ for the ring of adèles. We denote by $\mathcal{V}_\infty$ the finite set of Archimedean places and by $\mathcal{A}$ the finite set of places at which $\mathbf{G}$ is anisotropic; we put $\mathcal{I} = \mathcal{V} \setminus \mathcal{A}$. For any $\mathcal{U} \subset \mathcal{V}$ let $G_{\mathcal{U}}$ be the group of all elements of $\mathbf{G}(\mathbf{A}_K)$ which are trivial outside $\mathcal{U}$. Thus for instance $G_{\mathcal{A}}$ is compact and $\mathbf{G}(\mathbf{A}_K) \cong G_{\mathcal{I}} \times G_{\mathcal{A}}$.

The main additional ingredient that we need for Theorem 28 is the following exhaustion principle, which makes use of the Kolmogorov zero-one law:

PROPOSITION 5.4.1. *Let $\mathcal{B} \subset \mathcal{I}$ be a set of places with $\mathcal{B} \cap \mathcal{V}_\infty = \varnothing$. Then $\mathrm{H}^2_{\mathrm{cb}}(G_{\mathcal{B}}) = 0$.*

The point here is of course that $\mathcal{B}$ needs not be finite.

Proof of the proposition. For every $v \in \mathcal{V}$, fix a minimal parabolic subgroup P_v of $\mathbf{G}(K_v)$. Define for $\mathcal{U} \subset \mathcal{V}$ the direct product

$$S_{\mathcal{U}} = \prod_{v \in \mathcal{U}} \mathbf{G}(K_v)/P_v\,,$$

where $\mathbf{G}(K_v)/P_v$ are considered as measure spaces and $S_{\mathcal{U}}$ is endowed with the product measure. The $G_{\mathcal{B}}$-action on $S_{\mathcal{B}}$ is transitive because $G_{\mathcal{B}}$ contains the unrestricted product of a choice of maximal compact subgroups in each of the $\mathbf{G}(K_v)$ as v ranges over $\mathcal{B}$. Thus $S_{\mathcal{B}}$ is a homogeneous $G_{\mathcal{B}}$-space with amenable isotropy groups, and hence the action is amenable. A class $[\omega]$ in $\mathrm{H}^2_{\mathrm{cb}}(G_{\mathcal{B}})$ is therefore (Theorem 2) given by an alternating measurable bounded $G_{\mathcal{B}}$-invariant cocycle

$$\omega:\ S_{\mathcal{B}}\times S_{\mathcal{B}}\times S_{\mathcal{B}}\longrightarrow \mathbf{R}\,.$$

For every *finite* subset $\mathcal{F}\subset\mathcal{B}$, we have $\mathrm{H}^2_{\mathrm{cb}}(G_{\mathcal{F}})=0$. Indeed, by Theorem 14 this space is the direct sum of the local terms $\mathrm{H}^2_{\mathrm{cb}}(\mathbf{G}(K_v))$ over $v\in\mathcal{F}$, and we have shown in [BuM1, Lemma 7.1] that the latter space injects into $\mathrm{H}^2_{\mathrm{c}}(\mathbf{G}(K_v))$, which vanishes since v is non-Archimedean by the assumption on $\mathcal{B}$.

We may realize the restriction map

$$\mathrm{H}^2_{\mathrm{cb}}(G_{\mathcal{B}})\longrightarrow \mathrm{H}^2_{\mathrm{cb}}(G_{\mathcal{F}})=0$$

associated to $G_{\mathcal{F}}\to G_{\mathcal{B}}$ by the inclusion

$$L^\infty_{\mathrm{alt}}(S^3_{\mathcal{B}})^{G_{\mathcal{B}}}\longrightarrow L^\infty_{\mathrm{alt}}(S^3_{\mathcal{B}})^{G_{\mathcal{F}}}\,,$$

so that there is a bounded $G_{\mathcal{F}}$-invariant measurable function

$$\alpha_{\mathcal{F}}: S_{\mathcal{B}}\times S_{\mathcal{B}}\longrightarrow \mathbf{R}$$

with $d\alpha_{\mathcal{F}}=\omega$. We claim that $\alpha_{\mathcal{F}}$ does not depend on the first factor in the decomposition

$$S^2_{\mathcal{B}}\cong S^2_{\mathcal{F}}\times S^2_{\mathcal{B}\setminus F}\,.$$

Indeed, the diagonal $G_{\mathcal{F}}$-action on $S^2_{\mathcal{F}}$ is ergodic because each $\mathbf{G}(K_v)$ has an orbit of full measure in $(\mathbf{G}(K_v)/P_v)^2$. We conclude that whenever $\mathcal{F}\subset\mathcal{B}$ is finite, ω is independent of the factor $S^3_{\mathcal{F}}$ of $S^3_{\mathcal{B}}$. In other words, ω is invariant under the cofinality equivalence relation. The Kolmogorov zero-one law states that this equivalence relation is ergodic; therefore, the cocycle ω must be constant and hence $\omega=0$ by alternation. □

We can now complete the proof of Theorem 28. The diagonal embedding $K\subset\mathbf{A}_K$ realizes $\mathbf{G}(K)$ as a lattice in $\mathbf{G}(\mathbf{A}_K)$ (see e.g. Theorem 3.2.1 in [M, chap. I]) and thus also in $G_{\mathcal{I}}$.

We recall that the Strong Approximation Theorem for simply connected K-almost simple linear groups states that given $\mathcal{U}\subset\mathcal{V}$, the image of $\mathbf{G}(K)$ in $G_{\mathcal{V}\setminus\mathcal{U}}$ is dense as soon as $\mathcal{U}$ is not contained in $\mathcal{A}$ (see section II.6.8 in [M]).

Therefore, according to the definition of irreducibility given in the Introduction, we see that for any non-empty finite set $\mathcal{U}\subset\mathcal{I}$, the group $\mathbf{G}(K)$

is an irreducible lattice in the product

$$\prod_{v\in\mathcal{U}}\mathbf{G}(K_v)\times G_{\mathcal{I}\setminus\mathcal{U}}\cong G_{\mathcal{I}}\,.$$

Now we would like to apply the Theorem 16 and deduce

$$\mathrm{H}^2_{\mathrm{b}}(\mathbf{G}(K))\cong\bigoplus_{v\in\mathcal{U}}\mathrm{H}^2_{\mathrm{cb}}(\mathbf{G}(K_v))\oplus\mathrm{H}^2_{\mathrm{cb}}(G_{\mathcal{I}\setminus\mathcal{U}})\,,\tag{15}$$

except that $G_{\mathcal{I}}$ might not be compactly generated. However, as pointed out in Remark 4.4.4, we may still apply the Theorem 14 and get as in section 5.1 the isomorphism

$$\mathrm{H}^2_{\mathrm{cb}}\big(G_{\mathcal{I}},L^2(\mathbf{G}(K)\backslash G_{\mathcal{I}})\big)\cong\ \mathrm{H}^2_{\mathrm{cb}}(G_{\mathcal{I}})\,.$$

Since the restriction $\mathrm{H}^2_{\mathrm{cb}}(G_{\mathcal{I}})\to\mathrm{H}^2_{\mathrm{b}}\big(\mathbf{G}(K)\big)$ is injective (Proposition 2.4.1), it remains only to see that the L^2 induction

$$\mathrm{H}^2_{\mathrm{b}}(\mathbf{G}(K))\longrightarrow\mathrm{H}^2_{\mathrm{cb}}\big(G_{\mathcal{I}},L^2(\mathbf{G}(K)\backslash G_{\mathcal{I}})\big)$$

is injective. As we see in the proof of Proposition 3.6.1, it is enough to find a doubly ergodic (i.e. just doubly $\mathbf{C}$-ergodic) amenable $G_{\mathcal{I}}$-space S. We claim that $S=S_{\mathcal{I}}$ is such a space. We have already seen that it is amenable, and double ergodicity follows from the double ergodicity of $G_{\mathcal{I}'}$ on $S_{\mathcal{I}'}$ for every finite $\mathcal{I}'\subset\mathcal{I}$, which is a consequence of Proposition 3.2.2. Thus (15) is established (this corrects an omission in [Mo]).

Take now $\mathcal{U}$ big enough to include $\mathcal{V}_\infty\cap\mathcal{I}$; then Proposition 5.4.1 shows that the last term in (15) vanishes. Moreover, as explained above in the proof of Corollary 24, we have

$$\mathrm{H}^2_{\mathrm{cb}}(\mathbf{G}(K_v))\cong\mathrm{H}^2_{\mathrm{c}}(\mathbf{G}(K_v))$$

and the latter vanishes if $v\notin\mathcal{V}_\infty$. This proves Theorem 28 up to terms $\mathrm{H}^2_{\mathrm{cb}}\big(\mathbf{G}(K_v)\big)$ associated to places $v\in\mathcal{V}_\infty\cap\mathcal{A}$. However, for such places, $\mathbf{G}(K_v)$ is compact and hence both $\mathrm{H}^2_{\mathrm{cb}}$ and $\mathrm{H}^2_{\mathrm{c}}$ vanish.

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