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BOUNDARY MAPS IN BOUNDED COHOMOLOGY

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Appendix to: **Continuous bounded cohomology and applications to rigidity theory**
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In this appendix we show how, given a group homomorphism $\pi : G_1 \to G_2$, boundary maps can be used to implement contravariance in bounded continuous cohomology

$$\pi^\bullet : \mathrm{H}^\bullet_{\mathrm{cb}}(G_2) \to \mathrm{H}^\bullet_{\mathrm{cb}}(G_1)\,.$$

To illustrate the issues involved, let us consider for example the typical situation of the study of a representation of a discrete group Γ into, say, a semisimple Lie group G. On the one hand, associated to every representation $\pi : \Gamma \to G$, we have the natural pullback $\pi^\bullet : \mathrm{H}^\bullet_{\mathrm{cb}}(G) \to \mathrm{H}^\bullet_{\mathrm{b}}(\Gamma)$ in bounded cohomology which leads to useful invariants. On the other hand, the fundamental fact that bounded cohomology can be realized as L^∞-cocycles on a boundary ([BM1, §1]), suggests the following: let G/P be the maximal Furstenberg boundary of G, (B, ν) an amenable Γ-space and $\varphi : B \to G/P$ an equivariant measurable map; it is natural to use the resolution $L^\infty((G/P)^\bullet)$ by essentially bounded cocycles on $(G/P)^\bullet$ to represent the bounded cohomology of G, and to try to implement the pullback $\pi^\bullet$ by precomposition with $\varphi^\bullet : B^\bullet \to (G/P)^\bullet$. However, this does not provide a well defined map $L^\infty((G/P)^\bullet) \to L^\infty(B^\bullet)$, unless the pushforward measure $\varphi_*(\nu)$ on G/P is absolutely continuous with respect to the Lebesgue measure. The proof of this last property however is one of the difficult points in many rigidity questions, and therefore cannot be seriously used as an assumption. To circumvent this problem, we are guided by the fact that all bounded cohomology classes of "geometric" origin are represented by bounded Borel measurable *strict* invariant cocycles on flag manifolds, which can therefore be precomposed with $\varphi^\bullet$.

In this appendix we formalize this situation in general, and we prove that the resolution of bounded measurable functions on a measurable space has the necessary properties which allow us to implement in a very concrete way – via precomposition with $\varphi^\bullet$ though in a canonical way – the pullback of any class which can be represented by a bounded Borel measurable strict invariant cocycle. This leads in particular to geometrically meaningful formulae, representing bounded characteristic classes. These general results are being applied to rigidity theory, especially the study of group actions on complex hyperbolic spaces in [BI1] and [BI2], on hermitian symmetric spaces in [BI3], and are also used in the recent work of Monod and Shalom on orbit equivalence ([MoS1] and [MoS2]). We refer to [I] for an illustration of these techniques in a new proof of Milnor–Wood's inequality ([Mi], [W]) and Matsumoto's theorem [M] on the Euler number rigidity of actions of surface groups by homeomorphisms of the circle.

1 More on Contravariance

Let G_i, $i = 1, 2$, be groups which are either discrete or locally compact second countable. Some of the contravariance properties of the continuous bounded cohomology with respect to a continuous homomorphism $\pi : G_1 \to G_2$ have already been mentioned in [BM1, §1.5 (and §2.4)]; here we need to collect more results which we shall apply in §2 to specific situations of interest. For ease of reference, we start recalling the definition of the pullback map $\pi^\bullet : \mathrm{H}^\bullet_{\mathrm{cb}}(G_2, E) \to \mathrm{H}^\bullet_{\mathrm{cb}}(G_1, E)$ induced in cohomology. To avoid heavy notation, we use here $\pi^\bullet$ for the map that in [BM1, §1.5] was denoted by $\mathrm{H}^\bullet_{\mathrm{cb}}(\pi, E)$, where (ρ, E) is a coefficient G_2-module. Analogously, the corresponding map in degree n will be denoted by $\pi^{(n)}$. We start by recording the following obvious fact:

REMARK 1.1. Let G be any group and $E_\bullet$ be a complex of G-modules. For any subgroup $H < G$, the natural injection $i^\bullet : E_\bullet^G \hookrightarrow E_\bullet^H$ is a morphism of complexes which induces a map in cohomology

$$i^\bullet : \mathrm{H}^\bullet(E_\bullet^G) \to \mathrm{H}^\bullet(E_\bullet^H)\,.$$

Now recall that if $\pi : G_1 \to G_2$ is any homomorphism as above, any coefficient G_2-module (ρ, E) can be viewed as a coefficient G_1-module $(\pi^*\rho, E)$ via π: as such, we have an inclusion $\delta : \mathcal{C}_{G_2}E \hookrightarrow \mathcal{C}_{G_1}E$, which we can consider as an inclusion of G_1-modules. As the above observation holds for Banach G_2-modules in general, we can say analogously that, if $C_\bullet$ is any strong G_2-resolution of $\mathcal{C}_{G_2}E$, then $\mathcal{C}_{G_2}C_\bullet$ can be considered as a strong (in fact, even admissible) G_1-resolution of the G_1-module $\mathcal{C}_{G_2}E$. Now let $A_\bullet$ be a relatively injective resolution of the G_1-module $\mathcal{C}_{G_1}E$. By [BM1, Proposition 1.5.2] applied to the inclusion of G_1-modules $\delta : \mathcal{C}_{G_2}E \hookrightarrow \mathcal{C}_{G_1}E$, we obtain a G_1-morphism of resolutions $\mathcal{C}_{G_2}E_{2\bullet} \to A_\bullet$ which is unique up to homotopy and induces a map in cohomology $\delta^\bullet : \mathrm{H}^\bullet(C_\bullet^{\pi(G_1)}) \to \mathrm{H}^\bullet(A_\bullet^{G_1})$ (observe that obviously $\mathcal{C}_{G_2}C_\bullet^{\pi(G_1)} = C_\bullet^{G_1}$). However, because $C_\bullet$ is a G_2-resolution of $\mathcal{C}_{G_2}E$, as observed in Remark 1.1 we have a map in cohomology $i^\bullet : \mathrm{H}^\bullet(C_\bullet^{G_2}) \to \mathrm{H}^\bullet(C_\bullet^{\pi(G_1)})$. Hence we can define a map $\pi^\bullet$ by composition

$$\begin{tikzcd} \mathrm{H}^\bullet(A_\bullet^{G_1}) & \mathrm{H}^\bullet(C_\bullet^{\pi(G_1)}) \arrow[l, "\delta^\bullet"'] \\ & \mathrm{H}^\bullet(C_\bullet^{G_2}) . \arrow[u, "i^\bullet"'] \arrow[ul, "\pi^\bullet"] \end{tikzcd} \tag{1}$$

If now $A_\bullet$ and $C_\bullet$ are strong resolutions – of $\mathcal{C}_{G_1}E$ and $\mathcal{C}_{G_2}E$ respectively – via relatively injective modules, we have the usual canonical isomorphisms $\mathrm{H}^\bullet(A_\bullet^{G_1}) \simeq \mathrm{H}^\bullet_{\mathrm{cb}}(G_1, E)$ and $\mathrm{H}^\bullet(C_\bullet^{G_2}) \simeq \mathrm{H}^\bullet_{\mathrm{cb}}(G_2, E)$, so that we can define the pullback $\pi^\bullet$ as the composition

$$\begin{tikzcd} \mathrm{H}^\bullet_{\mathrm{cb}}(G_1, E) & \mathrm{H}^\bullet(A_\bullet^{G_1}) \arrow[l, "\simeq"'] & \mathrm{H}^\bullet(C_\bullet^{\pi(G_1)}) \arrow[l, "\delta^\bullet"'] \\ & & \mathrm{H}^\bullet(C_\bullet^{G_2}) \arrow[u, "i^\bullet"'] \\ & & \mathrm{H}^\bullet_{\mathrm{cb}}(G_2, E) . \arrow[u, "\simeq"'] \arrow[uull, "\pi^\bullet"] \end{tikzcd}$$

Proposition 1.2. *Let $\pi : G_1 \to G_2$ be a continuous homomorphism of either discrete or locally compact second countable groups, and let (ρ, E) be a coefficient G_2-module. Let $C_\bullet$ and $D_\bullet$ be strong resolutions of E by G_2-modules and let $\alpha^\bullet : \mathcal{C}_{G_2}D_\bullet \to \mathcal{C}_{G_2}C_\bullet$ be a G_2-morphism. Then, for any resolution $A_\bullet$ of $(\pi^*\rho, E)$ by relatively injective G_1-modules, the diagram in cohomology*

$$\begin{tikzcd} \mathrm{H}^\bullet(A_\bullet^{G_1}) & \mathrm{H}^\bullet(D_\bullet^{G_2}) \arrow[l, "\gamma^\bullet"'] \arrow[d, "\alpha^\bullet"] \\ & \mathrm{H}^\bullet(C_\bullet^{G_2}) \arrow[ul, "\pi^\bullet"] \end{tikzcd}$$

is commutative, where $\pi^\bullet$ is the map induced in cohomology by the homomorphism π, and $\gamma^\bullet$ is the map induced in cohomology by any G_1-morphism of complexes $\mathcal{C}_{G_2}D_\bullet \to A_\bullet$ extending the inclusion of G_1-morphisms $\mathcal{C}_{G_2}E \hookrightarrow E$.

Remark 1.3. Notice that it would have sufficed, in the statement of Proposition 1.2, to require that $C_\bullet$ and $D_\bullet$ are strong resolutions of $\mathcal{C}_{G_2}E$.

Moreover, the existence in Proposition 1.2 of the G_2-morphism $\alpha^\bullet$: $\mathcal{C}_{G_2}D_\bullet \to \mathcal{C}_{G_2}C_\bullet$ is automatically verified if $C_\bullet$ is a resolution by relatively injective modules (see also [BM1, Remark 1.4.3]).

Proof. We have observed already that both $\mathcal{C}_{G_2}C_\bullet$ and $\mathcal{C}_{G_2}D_\bullet$ can be viewed as strong resolutions of the G_1-module $(\pi^*\rho, E)$. Applying twice [BM1, Proposition 1.5.2] with $G = G_1$, $F_\bullet = A_\bullet$ and with $E_\bullet = C_\bullet$ first, and then $E_\bullet = D_\bullet$, we obtain that there are G_1-morphisms of resolutions $\delta^\bullet$: $\mathcal{C}_{G_2}C_\bullet \to A_\bullet$ and $\beta^\bullet : \mathcal{C}_{G_2}D_\bullet \to A_\bullet$ which extend the inclusion $\mathcal{C}_{G_2}E \hookrightarrow E$ (of G_1-morphisms), are unique up to G_1-homotopy and induce canonical maps in cohomology

$$\mathrm{H}^\bullet(C_\bullet^{\pi(G_1)}) \xrightarrow{\delta^\bullet} \mathrm{H}^\bullet(A_\bullet^{G_1})$$

and

$$\mathrm{H}^\bullet(D_\bullet^{\pi(G_1)}) \xrightarrow{\gamma_1^\bullet} \mathrm{H}^\bullet(A_\bullet^{G_1})\,. \tag{2}$$

But now the map $\alpha^\bullet$: $\mathcal{C}_{G_2}D_\bullet \to \mathcal{C}_{G_2}C_\bullet$ can be considered as a G_1-morphism of G_1-resolutions (via π), hence giving a G_1-morphism of G_1-complexes

$$\mathcal{C}_{G_2}D_\bullet \xrightarrow{\alpha^\bullet} \mathcal{C}_{G_2}C_\bullet \xrightarrow{\delta^\bullet} A_\bullet$$

which induces in cohomology the map $\gamma_1^\bullet$ in (2). Hence we have a diagram of G_1-morphisms

$$\begin{tikzcd} A_\bullet & \mathcal{C}_{G_2}D_\bullet \arrow[l, "\gamma_1^\bullet"'] \arrow[d, "\alpha^\bullet"] \\ & \mathcal{C}_{G_2}C_\bullet \arrow[ul, "\delta^\bullet"] \end{tikzcd}\,.$$

so that, by [BM1, Proposition 1.5.2], the diagram in cohomology

$$\begin{tikzcd} \mathrm{H}^\bullet(A_\bullet^{G_1}) & \mathrm{H}^\bullet(D_\bullet^{\pi(G_1)}) \arrow[l, "\gamma_1^\bullet"'] \arrow[d, "\alpha^\bullet"] \\ & \mathrm{H}^\bullet(C_\bullet^{\pi(G_1)}) \arrow[ul, "\delta^\bullet"] \end{tikzcd} \tag{3}$$

commutes.

Applying now Remark 1.1 to $H = \pi(G_1)$ and $G = G_2$, we have that the diagram

$$\begin{tikzcd} D_\bullet^{\pi(G_1)} \arrow[d, "\alpha^\bullet"'] & D_\bullet^{G_2} \arrow[l, hook'] \arrow[d, "\alpha^\bullet"] \\ C_\bullet^{\pi(G_1)} & C_\bullet^{G_2} \arrow[l, hook'] \end{tikzcd}$$

commutes and hence induces a commutative diagram in cohomology. Putting this together with (1), and recalling the definition of $\pi^\bullet$ given in (1), we have the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{H}^\bullet(A_\bullet^{G_1}) & \mathrm{H}^\bullet(D_\bullet^{\pi(G_1)}) \arrow[l, "\gamma_1^\bullet"'] \arrow[d, "\alpha^\bullet"] & \\
 & \mathrm{H}^\bullet(C_\bullet^{\pi(G_1)}) \arrow[ul, "\delta^\bullet"] & \mathrm{H}^\bullet(D_\bullet^{G_2}) \arrow[ul, "i^\bullet"'] \arrow[d, "\alpha^\bullet"] \arrow[ull, bend right=50, "\gamma^\bullet"'] \\
 & & \mathrm{H}^\bullet(C_\bullet^{G_2})\,, \arrow[ul, "i^\bullet"'] \arrow[uull, bend left=40, "\pi^\bullet"]
\end{tikzcd}$$

from which the assertion follows with $\gamma^\bullet = \gamma_1^\bullet \circ i^\bullet$. □

2 Resolutions from Measurable Actions

Let X be a measurable space, that is a set with a σ-algebra of subsets, and let E be the dual of a separable Banach space $E^\flat$ with ground field $\mathbf{K}$. We say that a map $f : X^n \to E$ is weak-$*$-measurable, if the evaluation function $x \to \langle f(x), v\rangle$ from X^n to $\mathbf{K}$ is measurable for every $v \in E^\flat$. Define the vector space

$$\mathcal{B}(X^n, E) = \{f : X^n \to E :\ f \text{ is weak-}*\text{-measurable}\}\,.$$

It is straightforward to verify that if $\|f\| := \sup_{x\in X^n} \|f(x)\|_E$, then

$$\mathcal{B}^\infty(X^n, E) = \big\{f \in \mathcal{B}(X^n, E) : \|f\| < \infty\big\}$$

is a Banach space.

Now let G be either a discrete or a locally compact second countable group acting measurably on the space X, that is assume that the action $a : G \times X \to X$ is measurable when G is equipped with the σ-algebra of the Haar measurable sets. We assume that E is a coefficient G-module so that the space $\mathcal{B}^\infty(X^n, E)$ is itself a Banach G-module (see [BM1, §1.1]). Let $d_n : \mathcal{B}^\infty(X^n, E) \to \mathcal{B}^\infty(X^{n+1}, E)$, $n \geq 1$, be the standard homogeneous coboundary operator $d_n f(x_0, \dots, x_n) = \sum_{i=0}^n (-1)^i f(x_0, \dots, \hat{x}_i, \dots, x_n)$, and let $d_0 : E \to \mathcal{B}^\infty(X, E)$ be the inclusion.

Our goal is to show that the complex $\mathcal{B}^\infty(X^\bullet, E)$ is a strong resolution of E. In order to do this we need to define homotopy operators; if μ is a probability measure on X, and $f \in \mathcal{B}^\infty(X^{n+1}, E)$, for $n \geq 0$, then the map $h_n f : X^n \to E$ defined by

$$h_n f : (x_1, \dots, x_n) \mapsto \int_X f(x_0, x_1, \dots, x_n) d\mu(x_0) \tag{4}$$

is weak-$*$-measurable and $\|h_n f\| \leq \|f\|$, so that h_n defines an operator $h_n : \mathcal{B}^\infty(X^{n+1}, E) \to \mathcal{B}^\infty(X^n, E)$. It is also straightforward to verify that for $n \geq 0$, $d_n h_n + h_{n+1} d_{n+1} = Id_{\mathcal{B}^\infty(X^{n+1},E)}$. For an appropriate choice of the measure μ on X, we have the desired:

Proposition 2.1. *The complex $\mathcal{B}^\infty(X^\bullet, E)$ is a strong resolution of E with homotopy operators defined in* (4) *with respect to the measure $\mu := a_*(\nu \otimes \delta_p)$, where $\nu \in \mathcal{M}^1(G)$ is a probability measure which is absolutely continuous with respect to the left Haar measure, δ_p is the Dirac mass of a base point $p \in X$, and a_* denotes the pushforward of measures via the action map a.*

Proof. Let λ_ρ denote, as usual, the action of G on $\mathcal{B}^\infty(X^n, E)$, namely $\lambda_\rho(g) f(x_1, \dots, x_n) = \rho(g) f(g^{-1}x_1, \dots, g^{-1}x_n)$ for $f \in \mathcal{B}^\infty(X^n, E)$, (see [BM1, §1.3]). It remains to be verified that, for $n \geq 0$, the homotopy operator h_n sends continuous vectors in $\mathcal{B}^\infty(X^{n+1}, E)$ to continuous vectors in $\mathcal{B}^\infty(X^n, E)$. Let $d\nu(h) = \psi(h)dh$, where dh is the left Haar measure on G, $\psi \in L^1(G)$, $\psi \geq 0$, and $\int_G \psi(h)dh = 1$. Let $f \in \mathcal{CB}^\infty(X^{n+1}, E)$ be a continuous vector. For every $v \in E^\flat$ we compute

$$\begin{aligned}
&\langle \lambda_\rho(g)^{-1} h_n f(x_1, \dots, x_n), v\rangle - \langle h_n f(x_1, \dots, x_n), v\rangle \\
&\quad = \int_G \langle \pi(g)^{-1} f(hp, gx_1, \dots, gx_n), v\rangle \psi(h) dh \\
&\qquad\qquad - \int_G \langle f(hp, x_1, \dots, x_n), v\rangle \psi(h) dh \\
&\quad = \int_G \langle \pi(g)^{-1} f(ghp, gx_1, \dots, gx_n), v\rangle \psi(gh) dh \\
&\qquad\qquad - \int_G \langle f(hp, x_1, \dots, x_n), v\rangle \psi(h) dh \\
&\quad = \int_G \langle \pi(g)^{-1} f(ghp, gx_1, \dots, gx_n) - f(hp, x_1, \dots, x_n), v\rangle \psi(gh) dh \\
&\qquad\qquad + \int_G \langle f(hp, x_1, \dots, x_n), v\rangle (\psi(gh) - \psi(h)) dh\,.
\end{aligned}$$

so that

$$\begin{aligned}
&|\langle \lambda_\rho(g)^{-1} h_n f(x_1, \dots, x_n), v\rangle - \langle h_n f(x_1, \dots, x_n), v\rangle| \\
&\qquad \leq \|\lambda_\rho(g)^{-1} f - f\| \, \|v| + \|f\| \, \|v\| \int_G |\psi(gh) - \psi(h)| dh\,,
\end{aligned}$$

and hence

$$\|\lambda_\rho(g)^{-1}h_n f - h_n f\| \leq \|\lambda_\rho(g)^{-1}f - f\| + \|f\| \int_G |\psi(gh) - \psi(h)|dh\,.$$

Since f is a continuous vector and G acts continuously on $L^1(G)$, we conclude that $h_n f$ is a continuous vector. □

Corollary 2.2. *There is a canonical map*

$$\omega^\bullet : \mathrm{H}^\bullet(\mathcal{B}^\infty(X^\bullet, E)^G) \longrightarrow \mathrm{H}^\bullet_{\mathrm{cb}}(G, E)\ .$$

That is, every bounded, measurable G-invariant cocycle $c : X^{n+1} \to E$ determines canonically a class $[c] \in \mathrm{H}^\bullet_{\mathrm{cb}}(G, E)$.

Proof. This follows from [BM1, Proposition 1.5.2] with $F = E$, $\alpha : \mathcal{C}E \to E$ the inclusion, $E_\bullet = \mathcal{B}^\infty(X^\bullet, E)$, and $F_\bullet$ any strong resolution of E by relatively injective G-modules. □

We draw one more consequence. Let X be a measurable space with a measurable G-action and let $Z \subset X$ be a non-empty measurable G-invariant subset; we consider Z endowed with the σ-algebra of X restricted to Z. The restriction map

$$R^\bullet : \mathcal{B}^\infty(X^\bullet, E) \to \mathcal{B}^\infty(Z^\bullet, E)$$

is a norm-decreasing, G-morphism of complexes extending the identity. Then Proposition 1.2 with $\pi = Id$, $D_\bullet = \mathcal{B}^\infty(X^\bullet, E)$ and $A_\bullet$ any strong resolution of E by relatively injective modules, implies, together with Proposition 2.1 and Corollary 2.2, the following:

Corollary 2.3. *The diagram in cohomology*

$$\begin{tikzcd}
\mathrm{H}^\bullet_{\mathrm{cb}}(G, E) & \mathrm{H}^\bullet(\mathcal{B}^\infty(X^\bullet, E)^G) \arrow[l] \arrow[d, "R^\bullet"] \\
 & \mathrm{H}^\bullet(\mathcal{B}^\infty(Z^\bullet, E)^G) \arrow[ul, "Id^\bullet"]
\end{tikzcd}$$

is commutative. □

We need to introduce now one more morphism of complexes, the existence of which does requires some additional structure. Namely, if Y is any topological space, Proposition 2.1 implies that the complex $\mathcal{B}^\infty(Y^\bullet, E)$ is a strong resolution of E, once Y is equipped with its σ-algebra of Borel sets. Let Y be a compact metrizable space on which G acts continuously, and let $\mathcal{M}^1(Y)$ be the space of probability measures with the weak-$*$ topology; then $\mathcal{M}^1(Y)$ is a compact metrizable space on which G acts continuously. Our next goal is to construct a natural morphism of G-complexes $\mathcal{B}^\infty(Y^\bullet, E) \longrightarrow \mathcal{B}^\infty(\mathcal{M}^1(Y)^\bullet, E)$ extending the identity $E \to E$. For this, the following lemma is crucial:

LEMMA 2.4. *Let Y be a compact metrizable space. Then, for every $f \in \mathcal{B}^\infty(Y, \mathbf{K})$, the evaluation map*

$$\begin{aligned} ev(f) : \mathcal{M}^1(Y) &\to \mathbf{K} \\ \mu &\mapsto \mu(f)\,, \end{aligned}$$

is a Borel measurable function.

Proof. It is enough to consider the case in which $\mathbf{K} = \mathbf{R}$. Let $\mathcal{B}^\infty(Y, \mathbf{R}) = \bigcup_{N \geq 1} \mathcal{B}(Y, (-N, N))$. Fix $N \in \mathbf{N}$ and consider the class

$$\mathcal{B}_N = \left\{ f \in \mathcal{B}(Y, (-N, N)) : \ ev(f) \text{ is Borel measurable} \right\}.$$

This class contains all continuous functions and, by the dominated convergence theorem, is closed under pointwise convergence of sequences. Hence $\mathcal{B}_N$ contains all Baire functions. Since $(-N, N)$ is homeomorphic to $\mathbf{R}$ and Y is metrizable, the Lebesgue–Hausdorff theorem [S, Theorem 3.1.36] implies that all Borel functions $Y \to (-N, N)$ are Baire functions and hence $\mathcal{B}_N = \mathcal{B}(Y, (-N, N))$, which proves the lemma. □

Now let $f \in \mathcal{B}^\infty(Y^n, E)$ and, for $\mu_1, \dots, \mu_n \in \mathcal{M}^1(Y)$ define

$$e_n(f)(\mu_1, \dots, \mu_n) = \int_{Y^n} f(y_1, \dots, y_n) d\mu(y_1) \dots d\mu(y_n)\,.$$

Evaluating on vectors in $E^\flat$, the preceding lemma implies that the map $e_n(f) : \mathcal{M}^1(Y)^n \to E$ is weak-$*$-measurable. Observe also that $\|e_n(f)\| = \|f\|$. The following is then a straightforward verification.

LEMMA 2.5. *The map $e_n : \mathcal{B}^\infty(Y^n, E) \to \mathcal{B}^\infty(\mathcal{M}^1(Y)^n, E)$ gives an isometric morphism of G-complexes extending the identity which, in particular, restricts to $e_n : \mathcal{CB}^\infty(Y^n, E) \to \mathcal{CB}^\infty(\mathcal{M}^1(Y)^n, E)$.* □

Now we apply the results in §1 to the specific resolutions we just studied. Let $\pi : G_1 \to G_2$ be a continuous homomorphism as above, (B, ν) a G_1-measure space and X a G_2-measurable space. We say that a measurable map $\varphi : B \to X$ is a.e.-G_1-equivariant if $\varphi(gx) = \pi(g)\varphi(x)$ for all $g \in G_1$ and ν-almost every $x \in B$. It is plain that any such map induces a norm decreasing morphism of G_1-complexes by precomposition

$$L^\infty_{\mathrm{w}*}(B^\bullet, E) \xleftarrow{\Phi^\bullet} \mathcal{B}^\infty(X^\bullet, E)\,.$$

COROLLARY 2.6. *Let π, φ, E and X be as above, and assume that (B, ν) is an amenable regular G_1-measure space. Then any a.e.-G_1-equivariant* *measurable map $\varphi : B \to X$ induces a commutative diagram in cohomology*

$$\begin{tikzcd}
\mathrm{H}^\bullet_{\mathrm{cb}}(G_1,E) & \mathrm{H}^\bullet(\mathcal{B}^\infty(X^\bullet,E)^{G_2}) \arrow[l, "\Phi^\bullet"'] \arrow[d] \\
& \mathrm{H}^\bullet_{\mathrm{cb}}(G_2,E) \arrow[ul, "\pi^\bullet"]
\end{tikzcd}$$

Proof. This is immediate from Proposition 1.2 and [BM1, Theorems 1 and 2] with $D_\bullet = \mathcal{B}^\infty(X^\bullet, E)$, $A_\bullet = L^\infty_{\mathrm{w}*}(B^\bullet, E)$ and $C_\bullet$ any strong resolution of (ρ, E) by relatively injective G_2-modules (see Remark 1.3). $\square$

Finally:

Corollary 2.7. *Let π be a continuous homomorphism of discrete or locally compact second countable groups, (ρ, E) a coefficient G_2-module, Y a separable compact metrizable continuous G_2-space, (B,ν) an amenable regular G_1-space, and $\varphi : B \to \mathcal{M}^1(Y)$ a measurable a.e.-G_1-equivariant map. Let $c : Y^{n+1} \to E$ be a Borel measurable G_2-invariant bounded cocycle, and $[c] \in \mathrm{H}^n_{\mathrm{cb}}(G_2, E)$ the associated cohomology class. Then*

$$(b_1, \dots, b_{n+1}) \to \varphi(b_1) \otimes \cdots \otimes \varphi(b_{n+1})(c)$$

defines an element in $L^\infty_{\mathrm{w}*}(B^{n+1}, E)$ which represents the class $\pi^{(n)}([c]) \in \mathrm{H}^n_{\mathrm{cb}}(G_1, E)$.

Proof. According to Corollary 2.2, there is a canonical map

$$\omega^\bullet : \mathrm{H}^\bullet(\mathcal{B}^\infty(Y^\bullet, E)^{G_2}) \to \mathrm{H}^\bullet_{\mathrm{cb}}(G_2, E).$$

The assertion will then follow from the commutativity of the following diagram:

$$\begin{tikzcd}
\mathrm{H}^\bullet_{\mathrm{cb}}(G_1,E) & \mathrm{H}^\bullet(\mathcal{B}^\infty(\mathcal{M}^1(Y)^\bullet,E)^{G_2}) \arrow[l, "\Phi^\bullet"'] \arrow[d, "Id^\bullet"] & \mathrm{H}^\bullet(\mathcal{B}^\infty(Y^\bullet,E)^{G_2}) \arrow[l, "e_\bullet"'] \arrow[dl, "\omega^\bullet"] \\
& \mathrm{H}^\bullet_{\mathrm{cb}}(G_2,E) \arrow[ul, "\pi^\bullet"] &
\end{tikzcd}$$

The commutativity of the diagram on the left follows from Corollary 2.6 with $X = \mathcal{M}^1(Y)$. The commutativity of the diagram on the right follows from Proposition 1.2 with $\pi = Id$, $G_1 = G_2$, $C_\bullet = \mathcal{B}^\infty(\mathcal{M}^1(Y)^\bullet, E)$, $D_\bullet = \mathcal{B}^\infty(Y^\bullet, E)$ and, finally, $\alpha^\bullet = e_\bullet$ as defined in Lemma 2.5. $\square$

Remark 2.8. Just like in §1.7, one can replace the complex $\mathcal{B}^\infty(X^\bullet, E)$ with the subcomplex $\mathcal{B}^\infty_{\mathrm{alt}}(X^\bullet, E)$ of alternating measurable bounded cochains, and all of the above results hold true verbatim.

3 An Illustration

Let X be a proper CAT(–1)-space, $G_2 < \mathrm{Iso}(X)$ a closed subgroup, E a coefficient G_2-module, and

$$c : X(\infty)^3 \to E$$

a Borel measurable, alternating, bounded, G_2-invariant cocycle. Let $\pi : G_1 \to G_2$ be a continuous homomorphism, where G_1 is locally compact second countable or discrete. Our objective is to give some natural sufficient conditions implying that the class $\pi^{(2)}([c]) \in \mathrm{H}^2_{\mathrm{cb}}(G_1, E)$ does not vanish. Given any set S, we denote by $\mathcal{C}_3(S)$ the subset of S^3 consisting of distinct triples.

Proposition 3.1. *Assume that E is separable, c is weak-$*$-continuous on $\mathcal{C}_3(X(\infty))$, and let $\mathcal{L}_{\pi(G_1)} \subset X(\infty)$ be the limit set of $\pi(G_1)$.*

(1) *If $c|_{(\mathcal{L}_{\pi(G_1)})^3}$ is not identically zero, then $\pi^{(2)}([c]) \neq 0$;*

(2) *Assume that G_1 is compactly generated. Then, for the Gromov norm of $\pi^{(2)}([c])$ we have*

$$\left\|\pi^{(2)}([c])\right\| = \max_{\xi_1,\xi_2,\xi_3 \in \mathcal{L}_\pi} \left\|c(\xi_1, \xi_2, \xi_3)\right\| .$$

Proof. We first prove (2). We distinguish two cases:

(a) Assume that $\pi(G_1)$ is elementary. Set $L := \overline{\pi(G_1)}$. Either L is compact, and hence $\mathrm{H}^2_{\mathrm{cb}}(L, E) = 0$, which implies in particular that the restriction of c to L vanishes in $\mathrm{H}^2_{\mathrm{cb}}(L, E)$, so that $\pi^{(2)}([c]) = 0$; since $\mathcal{L}_{\pi(G_1)} = \varnothing$, this proves the equality. Or $\left|\mathcal{L}_{\pi(G_1)}\right| \neq \varnothing$ and it consists of at most two points; since c is alternating, its restriction to $(\mathcal{L}_{\pi(G_1)})^3$ is identically zero; Corollary 2.3 applied to $Z = \mathcal{L}_{\pi(G_1)}$ implies then that the restriction of c to L vanishes, hence $\pi^{(2)}([c]) = 0$, which proves the equality.

(b) Assume that $\pi(G_1)$ is not elementary. Let $G_1^* \lhd G_1$ be the finite index subgroup given by [BM1, Theorem 6], π_r the restriction of π to G_1^*, and $\mathcal{L}_{\pi_r(G_1^*)}$ the limit set of $\pi_r(G_1^*)$. Since G_1^* is of finite index in G_1, we have $\mathcal{L}_{\pi(G_1^*)} = \mathcal{L}_{\pi(G_1)}$. Moreover, since the restriction map gives an isometric embedding $\mathrm{H}^\bullet_{\mathrm{cb}}(G_1, E) \to \mathrm{H}^\bullet_{\mathrm{cb}}(G_1^*, E)$ (see [BM1, Proposition 2.4.1]), we have that $\|\pi_r^{(2)}([c])\| = \|\pi^{(2)}([c])\|$. Now let (B, ν) be a doubly $\mathfrak{X}^{\mathrm{sep}}$-ergodic, regular, amenable G_1^*-space (see [BM1, Theorem 6]). Since $\pi_r(G_1^*)$ is non-elementary, there is an equivariant measurable map $\varphi : B \to \mathcal{L}_{\pi(G_1^*)}$, [BMo]; it follows from Corollary 2.7 that the map $(b_1, b_2, b_3) \mapsto c(\varphi(b_1), \varphi(b_2), \varphi(b_3))$ is a representative of $\pi_r^{(2)}([c])$ and, from double $\mathfrak{X}^{\mathrm{sep}}$-ergodicity,

$$\|\pi_r^{(2)}([c])\| = \operatorname{ess\,sup}_{b_i \in B} \left\|c(\varphi(b_1), \varphi(b_2), \varphi(b_3))\right\|$$

$$= \operatorname{ess\,sup}_{\xi_i \in (\mathcal{L}_{\pi(G_1^*)})^3} \left\| c(\xi_1, \xi_2, \xi_3) \right\|,$$

where now $(\mathcal{L}_{\pi(G_1^*)})^3$ is equipped with the measure $\varphi_*(\nu)^3 = \varphi_*(\nu) \otimes \varphi_*(\nu) \otimes \varphi_*(\nu)$. Since by hypothesis c is continuous on $\mathcal{C}_3(\mathcal{L}_{\pi(G_1^*)})$ and vanishes on its complement, we have that

$$\operatorname{ess\,sup}_{\xi_i \in (\mathcal{L}_{\pi(G_1^*)})^3} \left\| c(\xi_1, \xi_2, \xi_3) \right\| \leq \sup_{(\xi_1,\xi_2,\xi_3) \in \mathcal{C}_3(\mathcal{L}_{\pi(G_1^*)})} \left\| c(\xi_1, \xi_2, \xi_3) \right\| := b\,,$$

and we may assume that $b > 0$. On the other hand, let $\varepsilon > 0$ be such that $b - \varepsilon > 0$, and let $(\xi_1, \xi_2, \xi_3) \in \mathcal{C}_3(\mathcal{L}_{\pi(G_1^*)})$ and $v \in E^\flat$ with $\|v\| = 1$ be such that $\langle c(\xi_1, \xi_2, \xi_3), v\rangle > b - \varepsilon$. Then the set $\mathcal{S}_\varepsilon$ of triples (η_1, η_2, η_3) with $\langle c(\eta_1, \eta_2, \eta_3), v\rangle > b - \varepsilon$ is an open nonvoid set, and hence of positive $\varphi_*(\nu)^3$-measure, since $\operatorname{supp}(\varphi_*(\nu)^3) = (\mathcal{L}_{\pi(G_1^*)})^3$. Hence we also have that $\|c(\eta_1, \eta_2, \eta_3)\| > b - \varepsilon$ on $\mathcal{S}_\varepsilon$, which implies that $\operatorname{ess\,sup} \|c(\xi_1, \xi_2, \xi_3)\| \geq b - \varepsilon$ and hence is equal to b.

We now prove (1). Since c is alternating, it vanishes on $(\mathcal{L}_{\pi(G_1^*)})^3 \smallsetminus \mathcal{C}_3(\mathcal{L}_{\pi(G_1^*)})$, hence the set $\mathcal{V} = \{(\xi_1, \xi_2, \xi_3) \in (\mathcal{L}_{\pi(G_1^*)})^3 : c(\xi_1, \xi_2, \xi_3) \neq 0\}$ is open and, by hypothesis, nonvoid. Write G_1 as the union $\bigcup_{Q \in \mathcal{F}} Q$, where Q ranges in the family $\mathcal{F}$ of all compactly generated subgroups of G_1. It is plain that the union $\bigcup_{Q \in \mathcal{F}} \mathcal{L}_{\pi(Q)}$ of the limit sets of $\pi(Q)$ is dense in $\mathcal{L}_{\pi(G_1^*)}$ and hence there is $Q \in \mathcal{F}$ with $(\mathcal{L}_{\pi(Q)})^3 \cap \mathcal{V} \neq \varnothing$. Part (2) of the proposition allows us to conclude. $\square$

In order to illustrate Proposition 3.1, we present an immediate application to groups acting non-elementarily on the real hyperbolic plane $\mathbb{H}^2_{\mathbf{R}}$. Recall that (see [BM2]) in degree two, if $\mathcal{H}$ is a continuous irreducible unitary representation of $\mathrm{PSL}(2, \mathbf{R})$, we have

$$\dim \mathrm{H}^2_{\mathrm{cb}}(\mathrm{PSL}(2, \mathbf{R}), \mathcal{H}) = \begin{cases} 1 & \text{if } \mathcal{H} \text{ is spherical} \\ 0 & \text{otherwise}. \end{cases}$$

Corollary 3.2. *Let $\pi : \Gamma \to \mathrm{PSL}(2, \mathbf{R})$ be a homomorphism with non-elementary image. Then for any spherical representation $\mathcal{H}$, the map*

$$\pi^{(2)} : \mathrm{H}^2_{\mathrm{cb}}\big(\mathrm{PSL}(2, \mathbf{R}), \mathcal{H}\big) \to \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathcal{H})$$

is injective.

Proof. It is shown in [BM2] that a generator of $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{PSL}(2, \mathbf{R}), \mathcal{H})$ can be explicitly described by an alternating, weak-$*$-continuous $\mathrm{PSL}(2, \mathbf{R})$-invariant cocycle

$$\omega : \mathbb{H}^2_{\mathbf{R}}(\infty)^3 \to \mathcal{H}\,,$$

such that for every distinct triple $(x, y, z) \in \mathcal{C}_3(\mathbb{H}^2_{\mathbf{R}}(\infty))$, $\omega(x, y, z) \neq 0$. Since by hypothesis the limit set of $\pi(\Gamma)$ contains at least 3 points, Proposition 3.1 enables us to conclude. $\square$

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