



Group Theory/Geometry

Surface group representations with maximal Toledo invariant

Sur les représentations d'un groupe de surface compacte
avec invariant de Toledo maximal

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Abstract

We study representations of compact surface groups on Hermitian symmetric spaces and characterize those with maximal Toledo invariant. *To cite this article: M. Burger et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

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Résumé

Nous étudions les représentations d'un groupe de surface compacte sur un espace symétrique hermitien et caractérisons celles avec invariant de Toledo maximal. *Pour citer cet article: M. Burger et al., C. R. Acad. Sci. Paris, Ser. I 336 (2003).*

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1. Introduction

Let $\Gamma = \pi_1(\Sigma_g)$ be the fundamental group of a compact oriented surface Σ_g of genus $g \geq 2$, and X a Hermitian symmetric space of noncompact type, equipped with its Bergman metric. The Toledo invariant τ_ρ of a representation $\rho: \Gamma \rightarrow \text{Is}(X)^\circ$ is the integral over Σ_g of the pullback of the Kähler form ω_X of X by any smooth equivariant map $\widetilde{\Sigma}_g \rightarrow X$. Then $|\tau_\rho| \leq 2|\chi(\Sigma_g)|\pi r_X$, r_X being the rank of X [8,7]. The classical problem of characterizing representations with maximal Toledo invariant has been solved when X is of rank 1 [9,15] and partial results are available when X is associated to $\text{SU}(p, q)$ [10,2].

Theorem 1.1. *Let $\rho: \pi_1(\Sigma_g) \rightarrow \text{Is}(X)^\circ$ be a representation with maximal Toledo invariant. Then*

- (a) *the Zariski closure L of the image of ρ is reductive;*
- (b) *the symmetric subspace $Y \subset X$ associated to L is isometric to a tube type domain;*

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(c) the group $\pi_1(\Sigma_g)$ acts on Y properly discontinuously without fixed points.

We give in Section 5 examples where Y is not holomorphically embedded into X . The theorem is optimal also in the following sense:

Proposition 1.2. *For any Hermitian symmetric space X of tube type and any $g \geq 2$ there exist representations $\rho : \pi_1(\Sigma_g) \rightarrow \text{Is}^\circ(X)$ with maximal Toledo invariant and Zariski dense image.*

Surface group representations with maximal Toledo invariant provide therefore a class of geometrically meaningful Kleinian groups acting on higher rank Hermitian symmetric spaces.

The proof of the theorem relies heavily on [4,7,12,5] and [3]. For a comprehensive treatment of continuous bounded cohomology, we refer to [13].

2. Maximal representations with Zariski dense image

The Toledo invariant of a representation is the evaluation of a linear form on an appropriate bounded cohomology class. Namely, if $\rho : \pi_1(\Sigma_g) \rightarrow G$ is any homomorphism and $\kappa \in H_{cb}^2(G)$, we define $\tau(\rho, \kappa) := \langle \rho^*(\kappa), [\Sigma_g] \rangle$, where $\langle \cdot, \cdot \rangle$ is the natural pairing. Then $\tau_\rho = \tau(\rho, \kappa_X^b)$, where $\kappa_X^b \in H_{cb}^2(G)$ is the bounded Kähler class. Since $|\tau(\rho, \kappa)| \leq 2|\chi(\Sigma_g)|\|\rho^*(\kappa)\| \leq 2|\chi(\Sigma_g)|\|\kappa\|$, where $\chi(\Sigma_g)$ is the Euler characteristic of Σ_g and $\|\kappa\|$ is the Gromov norm of the class $\kappa \in H_{cb}^2(G)$, we say that ρ is κ -maximal if $\tau(\rho, \kappa) = 2|\chi(\Sigma_g)|\|\kappa\|$.

A totally geodesic embedding $t : \mathbb{D} \rightarrow X$ is tight if $t^*(\omega_X) = \|\kappa_X^b\|/\|\kappa_{\mathbb{D}}^b\|\omega_{\mathbb{D}}$. If t is holomorphic this is equivalent to saying that \mathbb{D} is mapped diagonally into a maximal polydisc in X .

The main point in the proof of the theorem is the following

Proposition 2.1. *Let X be an irreducible Hermitian symmetric space and $\rho : \Gamma \rightarrow \text{Is}(X)^\circ$ a representation with maximal Toledo invariant and Zariski dense image. Then X is a symmetric space of tube type, on which $\rho(\Gamma)$ acts properly discontinuously without fixed points.*

We outline the main steps of the proof using results of [7] and following the methods developed in [4] and [12]. Let $\mathcal{D} \subset \mathbb{C}^n$ be the Harish-Chandra realization of X as a bounded symmetric domain with normalized Bergman kernel k . We have $k(x, y) = h(x, y)^{-2}$, where h is a polynomial in x, \bar{y} . Following [7] and [4] we say that $x, y \in \bar{\mathcal{D}}$ are transverse if $h(x, y) \neq 0$; then there is a unique continuous determination of the argument of $k(x, y)$ on the set of pairwise transverse points in $\bar{\mathcal{D}}$. Denoting by $\check{S}^{(3)}$ the set of triples of pairwise transverse points in the Shilov boundary $\check{S} \subset \partial\mathcal{D}$, the function $\check{\beta}_{\mathcal{D}}(x, y, z) := -(\arg k(x, y) + \arg k(y, z) + \arg k(z, x))$ is a well defined continuous alternating G -invariant cocycle on $\check{S}^{(3)}$, where $G = \text{Is}(X)^\circ$. Define as in [4, §5], $Z_n := \{(x_1, x_2, \dots, x_n) \in \check{S} : x_i, x_j \text{ are transverse for all } i \neq j\}$ and let $(B_{alt}^\infty(Z_n), d_n)$ be the complex of bounded alternating Borel functions on Z_n , endowed with the supremum norm. Using the formula for the symplectic area of a geodesic triangle in \mathcal{D} given in [8,7], and arguing as in [4, Lemmas 5.1 and 5.2], the class $[\check{\beta}_{\mathcal{D}}]$ corresponds to κ_X^b under the canonical map $H^\bullet(B_{alt}^\infty(Z_\bullet)^G) \rightarrow H_{cb}^\bullet(G)$. Next, realize Γ as a cocompact lattice in $\text{PSU}(1, 1)$; by using that $\rho(\Gamma)$ is Zariski dense and that transversality in \check{S} is given by a polynomial condition, we deduce as in [4, Proposition 6.2] the existence of a Γ -equivariant measurable map $\varphi : S^1 \rightarrow \check{S}$ such that for almost every $x, y \in S^1$, the points $\varphi(x), \varphi(y) \in \check{S}$ are transverse. As a consequence, $\varphi^*\check{\beta}_{\mathcal{D}}(x, y, z) := \check{\beta}_{\mathcal{D}}(\varphi(x), \varphi(y), \varphi(z))$ is a well defined measurable alternating Γ -invariant bounded cocycle on $(S^1)^3$, which corresponds [4, §7] to $\rho^*(\kappa_X^b)$ under the isomorphism $H_b^2(\Gamma, \mathbb{R}) \simeq \mathcal{Z}L_{alt}^\infty((S^1)^3)^\Gamma$.

As in [12, §3], we get that for almost every $x, y, z \in S^1$

$$\int_{\Gamma \backslash \text{PSU}(1,1)} \check{\beta}_{\mathcal{D}}(\varphi(hx), \varphi(hy), \varphi(hz)) dh = \frac{\tau_\rho}{2|\chi(\Sigma_g)|} \check{\beta}_{\mathbb{D}}(x, y, z). \tag{1}$$

If ρ is maximal, $\tau_\rho = 2|\chi(\Sigma_g)|\|\kappa_X^b\|$, which together with $\|\check{\beta}_D\|_\infty = \|\kappa_X^b\|$ and (1) implies that for almost every $x, y, z \in S^1$, $\check{\beta}_D(\varphi(x), \varphi(y), \varphi(z)) = \|\kappa_X^b\|/\|\kappa_D^b\| \check{\beta}_D(x, y, z)$. To conclude the proof of Proposition 2.1, fix x and y : by Fubini’s theorem, for almost every z , $\check{\beta}_D(\varphi(x), \varphi(y), \varphi(z)) = \pm\pi r_X$, hence by [7, Proof of Theorem 4.7] the essential image $\text{Ess Im } \varphi$ of φ lies in the Shilov boundary of the tube type domain Y of X determined by $\varphi(x)$ and $\varphi(y)$. Since any two transverse points in $\text{Ess Im } \varphi$ determine a tube type subdomain, which hence coincides with Y , it follows that Y is $\rho(\Gamma)$ -invariant and, by Zariski density, also $G = \text{Is}(X)^\circ$ -invariant. Hence $X = Y$.

The image $\rho(\Gamma)$ is discrete: for $r_X = 1$ this follows from [9] (or by [12]); for $r_X \geq 2$, G has at least three open orbits in \check{S}^3 since X is of tube type [6, Theorem 4.3, Lemma 5.3], while $\text{Ess Im}(\varphi^3)$ is contained in the closure of two open orbits in \check{S}^3 , namely $\{(x, y, z) \in \check{S}^{(3)} : \check{\beta}_D(x, y, z) = \pm\pi r_X\}$. Hence $\text{Ess Im}(\varphi^3) \neq \check{S}^3$, which implies that $\rho(\Gamma)$ is not dense, and thus discrete.

The cocycle $\check{\beta}_D$ can be used to equip the essential graph F of φ with a cyclic ordering. Arguing as in [12, Lemma 5.6] we conclude that if $(x_1, \eta), (x_2, \eta) \in F$ then $x_1 = x_2$, hence ρ is faithful.

3. Proof of Theorem 1.1

Let $L := \overline{\rho(\Gamma)}^Z(\mathbb{R})$ be the real points of the Zariski closure of $\rho(\Gamma)$. By passing to a finite index subgroup of Γ we may assume that L is connected. Since the radical of L is amenable, the projection $p : L \rightarrow M$ of L to its semisimple part M induces a canonical isometric isomorphism in bounded cohomology, $H_{\text{cb}}^2(L) \simeq H_{\text{cb}}^2(M)$ (see [5, Corollary 4.2.4]), with respect to which the class $\kappa_X^b \in H_{\text{cb}}^2(L)$ defines a class $k \in H_{\text{cb}}^2(M)$. Let $M' = M_1 \times \cdots \times M_\ell$ be the product of the simple factors of M such that $k_i := k|_{M_i} \neq 0$ and let $\rho_i := \text{pr}_i \circ p \circ \rho : \Gamma \rightarrow M_i$, where $\text{pr}_i : M \rightarrow M_i$ is the projection, $i = 1, \dots, \ell$. From $2|\chi(\Sigma_g)|\|k\| = \tau(\rho, k) = \sum_{i=1}^\ell \tau(\rho_i, k_i) \leq 2|\chi(\Sigma_g)| \sum_{i=1}^\ell \|k_i\|$ and $\|k\| = \sum_{i=1}^\ell \|k_i\|$, it follows that for all i , the representations ρ_i are k_i -maximal. Hence Proposition 2.1 implies that the Hermitian symmetric space Y associated to M' is of tube type.

Let $H < L$ be a connected semisimple subgroup which is isogenous to M' via p , and let $Z \subset X$ be a subsymmetric space associated to H , such that the induced equivariant map $\psi : Y \rightarrow Z$ satisfies $\psi^*(\kappa_X^b|_Z) = \sum k_i$. For any triple of points in the Shilov boundary of Y for which $\check{\beta}_Y$ is defined and maximal, we get by [7, Theorem 4.7] a holomorphic tight embedding $t : \mathbb{D} \rightarrow Y$. The map $T = \psi \circ t : \mathbb{D} \rightarrow X$ associated to a homomorphism $\pi : \text{SU}(1, 1) \rightarrow G$ satisfies $T^*(\omega_X|_Z) = \|\kappa_X^b\|/\|\kappa_D^b\| \omega_D$. Since up to scaling T is an isometry and the Euclidean metric on X as a bounded symmetric domain is dominated by the Riemannian metric and T is tight, the map T extends to a π -equivariant map of the boundary $T : \partial\mathbb{D} \rightarrow \partial\mathcal{D}$ with $T(\partial\mathbb{D}) \subset \check{S}$. Let C be the centralizer of $\pi(\text{SU}(1, 1))$ in G .

Lemma 3.1. *Let γ be a geodesic in $T(\mathbb{D})$ connecting two points $x, y \in T(\partial\mathbb{D}) \subset \check{S}$. Then for all $g \in C$ the geodesic $g\gamma$ connects the same points x, y .*

Proof. One can realize the Shilov boundary, which is represented as G/Q , as the equivalence classes of asymptotic maximal singular Weyl chamber walls of type Q . There are natural projections $G/Q' \rightarrow G/Q$ for all parabolic subgroups $Q' \subset Q \subset G$, where G/Q' can also be realized as the equivalence classes of asymptotic Weyl chamber (walls) of type Q' . The geodesic γ connects $x, y \in \check{S}$ and hence lies in a Weyl chamber (wall) of type Q' for some $Q' \subset Q$. The geodesic $g\gamma$ lies in a Weyl chamber (wall) of the same type. Since $g \in C$, the distance between γ and $g\gamma$ is uniformly bounded, it follows that they determine the same point in G/Q' , hence in $G/Q = \check{S}$. \square

By the above lemma, any three distinct points $x, y, z \in T(\partial\mathbb{D})$ are fixed by C , hence C fixes the barycenter of x, y, z and is therefore compact.

If L were not reductive, by [1] it would be contained in a proper parabolic subgroup P of G . But then the center of an appropriate Levi component of P would be contained in C and noncompact, which is a contradiction since C is compact. Therefore L is reductive and hence $\rho(\Gamma)$ acts on Y . By Proposition 2.1 the action is properly discontinuous without fixed points.

4. Maximal Zariski dense representations into a tube type domain

For the construction of a representation as in Proposition 1.2, realize the fundamental group as an amalgamated product over a separating geodesic, $\Gamma = A *_{\langle \gamma \rangle} B$. Choose a hyperbolization of Γ , $\pi : \Gamma \rightarrow \mathrm{PSU}(1, 1)$ and use the diagonal embedding $\Delta : \mathrm{PSU}(1, 1) \rightarrow \mathrm{PSU}(1, 1)^r$ to define hyperbolizations $\rho_i := pr_i \circ \Delta \circ \pi|_A : A \rightarrow \mathrm{PSU}(1, 1)$ and $\omega_i := pr_i \circ \Delta \circ \pi|_B : B \rightarrow \mathrm{PSU}(1, 1)$. Let $t_A, t_B : \mathrm{PSU}(1, 1)^r \rightarrow G$ be two different embeddings, which coincide on $\Delta(\mathrm{PSU}(1, 1))$. Choose now two one-parameter families of deformations ρ_i^t, ω_i^t , such that the ρ_i^t 's, $i = 1, \dots, r$, respectively the ω_i^t 's, are pairwise not conjugated for all t and $\rho_i^t(\gamma) = \rho_i(\gamma)$, respectively $\omega_i^t(\gamma) = \omega_i(\gamma)$, for all t . The representations of A , respectively B , given by $\rho^t(a) = t_A(\rho_1^t(a), \dots, \rho_r^t(a))$, respectively $\omega^t = t_B(\omega_1^t(a), \dots, \omega_r^t(a))$, have Zariski dense image in $t_A(\mathrm{PSU}(1, 1)^r)$, respectively $t_B(\mathrm{PSU}(1, 1)^r)$, and define a representation $\pi^t : \Gamma \rightarrow G$ by the universal property of amalgamated products. By construction π^t has maximal Toledo invariant, hence the Zariski closure of its image is reductive and of maximal rank, since it contains the image of t_A . The symmetric space corresponding to its semisimple part is of tube type and holomorphically embedded into X . Using the characterizations of holomorphic embeddings in [14,11], one can choose t_A, t_B in such a way that the group generated by its images coincides with G .

5. Nonholomorphic tight embeddings

The complex irreducible representation π_p of $\mathrm{SU}(1, 1)$ of dimension $2p$ admits an invariant hermitian form unique up to scaling, which is of signature (p, p) . The corresponding homomorphism $\pi_p : \mathrm{SU}(1, 1) \rightarrow \mathrm{SU}(p, p)$ gives rise to a tight embedding $\mathbb{D} \rightarrow X_{p,p}$ into the Hermitian symmetric space associated to $\mathrm{SU}(p, p)$, which is holomorphic if and only if $p = 1$. For $p \geq 2$ this gives rise to representations of surface groups on $X_{p,p}$ with maximal Toledo invariant, and preserving a nonholomorphically tight embedded disc.

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