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Equivariant Embeddings of Trees into Hyperbolic Spaces

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1 Introduction

For every cardinal $\alpha \geq 2$ there are three complete constant curvature model manifolds of Hilbert dimension α: the sphere $\mathbf{S}^\alpha$, the Euclidean space $\mathbf{E}^\alpha$, and the hyperbolic space $\mathbf{H}^\alpha$. Studying isometric actions on these spaces corresponds in the first case to studying orthogonal representations and in the second case to studying cohomology in degree one with orthogonal representations as coefficients. In this paper, we address the third case and, in particular, we study isometric actions of automorphism groups of trees on $\mathbf{H}^\alpha$.

The goal of this paper is twofold. First we exhibit, for every tree $\mathcal{T}$, a one-parameter family of equivariant embeddings (with respect to appropriate representations of $\mathrm{Aut}(\mathcal{T})$) into an infinite-dimensional hyperbolic space which are, up to rescaling, asymptotically isometric and have convex cobounded image. Secondly, in the case in which the tree is regular of finite valence at least 3 and $G < \mathrm{Aut}(\mathcal{T})$ is a closed subgroup satisfying appropriate transitivity properties, we show that the representations constructed above give the unique irreducible nonelementary actions of G by isometries on a hyperbolic space of appropriate infinite dimension.

Quadratic forms of finite index, and in particular of index one, can be studied on real vector spaces of arbitrary dimension. A quadratic form of index one leads, *via* its cone of negative vectors, to a geodesic CAT(-1) space which is then complete if and only if the quadratic form satisfies a strong nondegeneracy condition. For any dimension α, there is one such space $\mathbf{H}^\alpha$ with ideal boundary $\partial\mathbf{H}^\alpha$ and bordification $\overline{\mathbf{H}}^\alpha = \mathbf{H}^\alpha \cup \partial\mathbf{H}^\alpha$. Then we have the following theorem.

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Theorem 1.1. Let V be the set of vertices of a tree $\mathcal{T}$ with $|V| = \alpha + 1$. Then for every $\lambda > 1$ there is an embedding $\Psi_\lambda : V \to \mathbf{H}^\alpha$ and a representation $\pi_\lambda : \mathrm{Aut}(\mathcal{T}) \to \mathrm{Isom}(\mathbf{H}^\alpha)$ such that

(i) the map Ψ_λ is π_λ-equivariant and extends equivariantly to a boundary map $\partial\Psi_\lambda : \partial\mathcal{T} \to \partial\mathbf{H}^\alpha$ which is a homeomorphism onto its image,

(ii) for any two vertices $x, y \in V$ there is a precise relation between the combinatorial distance $d_\mathcal{T}$ and the Riemannian distance $d_{\mathbf{H}^\alpha}$:

$$\lambda^{d_\mathcal{T}(x,y)} = \cosh d_{\mathbf{H}^\alpha}\big(\Psi_\lambda x, \Psi_\lambda y\big), \tag{1.1}$$

(iii) the set $\Psi_\lambda(V)$ has finite codiameter in the convex hull $\mathcal{C} \subseteq \mathbf{H}^\alpha$ of the image of $\partial\Psi_\lambda$. □

This result is the outcome of our attempt to understand certain claims of Gromov [6, Section 6.A], to the extent that nontrivial amalgams admit actions with unbounded orbits on infinite dimensional hyperbolic spaces.

Theorem 1.1 applies in particular to the automorphism group $\mathrm{Aut}(\mathcal{T}_r)$ of an r-regular tree $\mathcal{T}_r$. By taking products and by denoting by $\mathbf{H}^\infty$ the real hyperbolic space of countable dimension $\alpha = \aleph_0$, we obtain a family of metrically proper convex cobounded actions of $\mathrm{Aut}(\mathcal{T}_r \times \mathcal{T}_s)$ on $\mathbf{H}^\infty \times \mathbf{H}^\infty$. This leads immediately to the following.

Corollary 1.2. Any cocompact lattice $\Gamma < \mathrm{Aut}(\mathcal{T}_r \times \mathcal{T}_s)$ admits a metrically proper convex cobounded action on the product $\mathbf{H}^\infty \times \mathbf{H}^\infty$ of two hyperbolic spaces of countable dimension. □

Recall that in [3, 4] these types of lattices were studied systematically and examples of torsion-free simple groups Γ were obtained. These Γ's are then fundamental groups of finite aspherical (two-dimensional) complexes; on the other hand, the question to which extent there are compact aspherical manifolds (with or without boundary) with a simple fundamental group is open. Here we obtain metrically proper convex cobounded actions of Γ on $\mathbf{H}^\infty \times \mathbf{H}^\infty$; in particular, $\Gamma\backslash(\mathbf{H}^\infty \times \mathbf{H}^\infty)$ retracts to a convex bounded (infinite-dimensional) aspherical manifold with boundary. In contrast to the algebraic aspect of this situation, observe that if Λ is a group acting in a metrically proper convex cobounded way on $\mathbf{H}^\infty$, then Λ is a nonelementary Gromov hyperbolic group, and hence SQ-universal [10]; in particular, it admits many normal subgroups.

Turning to the classification of isometric actions, we recall that an action of a group G on $\mathbf{H}^\alpha$ by isometries is *elementary* if it preserves a point in $\overline{\mathbf{H}}^\alpha$ or a geodesic. Hence the study of elementary actions on $\mathbf{H}^\alpha$ reduces essentially to the study of isometric actions on the Euclidean space $\mathbf{E}^{\alpha-1}$ or on the sphere $\mathbf{S}^{\alpha-1}$. (Conversely, as observed by

Gromov [6, Section 7.A], any isometric action on $\mathbf{E}^{\alpha-1}$ can be extended to an *elementary* action on $\mathbf{H}^{\alpha}$ by realizing $\mathbf{E}^{\alpha-1}$ as a horosphere based at a fixed point at infinity; similarly, isometric actions on $\mathbf{S}^{\alpha-1}$ extend obviously to actions with a fixed point in $\mathbf{H}^{\alpha}$.)

Since any nonelementary action admits a unique minimal G-invariant hyperbolic subspace (Proposition 4.3), we say that a (nonelementary) action $G \to \mathrm{Isom}(\mathbf{H}^{\alpha})$ is *irreducible* if there is no G-invariant hyperbolic subspace other than $\mathbf{H}^{\alpha}$.

Let now $\mathcal{T}_r$ be the regular tree of finite valence $r \geq 3$, $G < \mathrm{Aut}(\mathcal{T}_r)$ a closed subgroup, and $\pi : G \to \mathrm{Isom}(\mathbf{H}^{\alpha})$ a nonelementary action of G on $\mathbf{H}^{\alpha}$. If G acts triply transitively on $\partial\mathcal{T}_r$, we will see that the image $\pi(g)$ of any hyperbolic automorphism $g \in G$ is a hyperbolic isometry of $\mathbf{H}^{\alpha}$ of translation length ℓ_π independent of g, provided g is of translation length one in $\mathcal{T}_r$. With this in mind, we can state the following.

Theorem 1.3. Let $G < \mathrm{Aut}(\mathcal{T}_r)$ be a closed subgroup which acts triply transitively on $\partial\mathcal{T}_r$. For every $\ell > 0$, there exists, up to equivalence, a unique irreducible nonelementary continuous homomorphism $\pi : G \to \mathrm{Isom}(\mathbf{H}^{\infty})$ with $\ell_\pi = \ell$. □

It follows that the representation π in Theorem 1.3 is exactly the irreducible component of $\pi_\lambda|_G$ for $\lambda = e^{\ell_\pi}$ in Theorem 1.1.

The structure and unitary representation theory of closed subgroups of $\mathrm{Aut}(\mathcal{T}_r)$ with some transitivity conditions on their action at infinity are the object of intensive study. We refer to [1, 5] and the references therein for a more comprehensive picture. A first notable set of examples to which Theorem 1.3 applies is given by the topological group $G = \mathrm{PGL}_2(k)$, where k is a non-Archimedean local field; indeed, if q is the cardinality of the residue field of k, then the action of $\mathrm{PGL}_2(k)$ on the associated Bruhat-Tits tree $\mathcal{T}_{q+1}$ identifies it with a closed subgroup of $\mathrm{Aut}(\mathcal{T}_{q+1})$ which acts triply transitively on $\partial\mathcal{T}_{q+1}$.

Another important class of examples of closed subgroups of $\mathrm{Aut}(\mathcal{T}_r)$ are the universal groups introduced in [3]. Recall that when $\mathcal{T}_r$ is an r-regular tree, one can label its edges in such a way that for every vertex, the edges issued from it are labelled $\{1, 2, \ldots, r\}$. Thus, for any $g \in \mathrm{Aut}(\mathcal{T}_r)$ and any vertex x, one obtains a permutation $c(g, x) \in S_r$ representing g "locally" at x. To a permutation group $F < S_r$, one can then associate $U(F)$, the closed subgroup of $\mathrm{Aut}(\mathcal{T}_r)$ consisting of all $g \in \mathrm{Aut}(\mathcal{T}_r)$, such that $c(g, x) \in F$ for every vertex x. Then $U(F)$ does not depend, up to conjugation, on the labelling of the edges. It acts transitively on the set of vertices and at every vertex x it induces the full permutation group F on the edges issued from x and is, by construction, maximal with respect to this property. The group $U(F)$ satisfies Tits' independence condition [13] and, in fact, all closed vertex transitive subgroups of $\mathrm{Aut}(T_r)$ satisfying Tits' independence condition are of the form $U(F)$.

Many properties of $U(F)$ can be read off the finite permutation group $F < S_r$. For example, for $n = 2$ and 3, $U(F)$ is n-transitive on $\partial\mathcal{T}_r$ if and only if F is n-transitive. In the case in which F is doubly transitive, the unitary dual of $U(F)$ has been determined by Amann [1]. When F is triply transitive, Theorem 1.3 applies to $U(F)$.

Remark 1.4. We have been informed by Valette that the algebraic part of the construction of Theorem 1.1 can also be derived elegantly from the "tree cocycles" that he proposed in [14].

The structure of the paper is as follows. In Section 2, modelling on the finite-dimensional case, we discuss basic properties of quadratic forms of finite index on a real vector space of arbitrary dimension, we single out the notion of strongly nondegenerate form, and show that strongly nondegenerate forms are determined by their signature. In Section 3, we associate a hyperbolic space to every nondegenerate quadratic form of index one; this is a geodesic CAT(-1) space which is complete if and only if the form is strongly nondegenerate. This leads to the existence and the uniqueness of $\mathbf{H}^\alpha$ for every cardinal α. In Section 4, we discuss the existence of irreducible hyperbolic subspaces (Proposition 4.3) and establish the description of elementary actions in terms of orthogonal representations and cocycles in degree one (Proposition 4.4). In Sections 5, 6, and 7 we turn more specifically to the study of actions on $\mathbf{H}^\alpha$ of automorphism groups of trees. In Sections 5 and 6, we study more closely actions on $\mathbf{H}^\alpha$ of certain locally compact groups occurring as stabilizers of ends of trees, that is, topological ascending HNN-extensions. These actions turn out to be elementary and hence a substantial use of Section 4 is made. In Section 7, the proof of Theorem 1.3 is completed by showing that the irreducible part of the action under consideration is determined by its restriction to any parabolic subgroup. In Section 8, we give the explicit construction in Theorem 1.1. Finally, the appendix contains some of the explicit matrix representations used throughout the paper.

2 Quadratic forms of finite index

A *quadratic space* is a pair $(\mathcal{H}, Q)$ consisting of a real vector space $\mathcal{H}$ and a quadratic form $Q : \mathcal{H} \to \mathbb{R}$. As usual, Q is *positive definite* if $Q(x) > 0$ for all $x \neq 0$ and *negative definite* if $-Q$ is positive definite; $\dim\mathcal{H}$ denotes the cardinal of any $\mathbb{R}$-basis of $\mathcal{H}$. Define

$$i_\pm(Q) = \sup\left\{\dim W : W \text{ is a subspace of } \mathcal{H} \text{ and } Q|_W \text{ is positive/negative definite}\right\} \tag{2.1}$$

and the *index* of Q as

$$i(Q) = \sup \{ \dim W : W \text{ is an isotropic subspace of } \mathcal{H} \}. \tag{2.2}$$

(Recall that W is said to be *isotropic* if $Q|_W = 0$.) Let $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be the bilinear (symmetric) form associated to Q. For any subset $S \subseteq \mathcal{H}$, write

$$^{\perp}S = \{ x \in \mathcal{H} : B(x, s) = 0 \; \forall s \in S \}. \tag{2.3}$$

We say that Q is *nondegenerate* if $^{\perp}\mathcal{H} = 0$ and that the quadratic space is of *finite index* if $i(Q) \in \mathbb{N}$ (we agree that $0 \in \mathbb{N}$).

Just like in the case of finite-dimensional quadratic spaces, we have the following proposition.

Proposition 2.1. Let $(\mathcal{H}, Q)$ be a nondegenerate quadratic space of finite index. Then

(i) $i(Q) = \min\{i_-(Q), i_+(Q)\}$.

Assume now $i(Q) = i_-(Q)$.

(ii) If $W_- \subseteq \mathcal{H}$ is a negative-definite subspace with $\dim W_- = i(Q)$, then $W_+ := {}^{\perp}W_-$ is positive definite and $\mathcal{H} = W_- \oplus W_+$.

(iii) If $\mathcal{H} = W'_- \oplus W'_+$ is an orthogonal direct sum with $W'_\pm$ positive/negative definite, then $\dim W'_- = i(Q)$. □

We precede the proof of the proposition by a couple of lemmas.

Lemma 2.2. Let $(\mathcal{H}, Q)$ be a quadratic space and $W \subseteq \mathcal{H}$ a finite-dimensional subspace such that $Q|_W$ is nondegenerate. Then $\mathcal{H} = W \oplus {}^{\perp}W$. If moreover Q is nondegenerate, then $Q|_{^{\perp}W}$ is so too. □

Proof. The kernel of the linear map $\mathcal{H} \to W^*$ induced by B is $^{\perp}W$; since W has finite dimension, we have

$$\dim (\mathcal{H}/{}^{\perp}W) \leq \dim W^* = \dim W. \tag{2.4}$$

On the other hand, since $W \cap {}^{\perp}W = 0$, the canonical projection $W \to \mathcal{H}/{}^{\perp}W$ is injective; hence it is an isomorphism by (2.4). ■

Lemma 2.3. Let $(\mathcal{H}, Q)$ be a quadratic space with $Q(x) \geq 0$ for all $x \in \mathcal{H}$. Then ${}^{\perp}\mathcal{H} = \{x \in \mathcal{H} : Q(x) = 0\}$. □

Proof. If $Q(x) = 0$, then for all $y \in \mathcal{H}$ and all $\lambda \in \mathbb{R}$, we have

$$0 \leq Q(\lambda x + y) = 2\lambda B(x, y) + Q(y); \tag{2.5}$$

hence $B(x, y) = 0$ for all y. ■

Proof of Proposition 2.1. Let $A_\pm \subseteq \mathcal{H}$ be positive/negative definite subspaces of finite dimension and set $A = A_- + A_+$. Then $i(Q|_A) \leq i(Q)$ and the theory of finite-dimensional quadratic spaces implies

$$\min\left\{\dim A_-, \dim A_+\right\} \leq i\big(Q|_A\big) \leq i(Q), \tag{2.6}$$

whence $\min\{i_-(Q), i_+(Q)\} \leq i(Q)$. Assume without loss of generality that $i_-(Q) \leq i_+(Q)$, pick a negative-definite subspace W_- of dimension $i_-(Q)$, and let $W_+ := {}^{\perp}W_-$. Since $Q|_{W_-}$ is nondegenerate, $\mathcal{H} = W_- \oplus W_+$ and $Q|_{W_+}$ is nondegenerate (Lemma 2.2). Since $\dim W_- = i_-(Q)$, we have $Q(x) \geq 0$ for all $x \in W_+$ and hence, by Lemma 2.3, W_+ is positive definite. If now W is an isotropic subspace with $\dim W = i(Q)$, then $W \cap W_+ = 0$ and thus the canonical projection $W \to \mathcal{H}/W_+ \cong W_-$ is injective; hence $i(Q) \leq i_-(Q)$. This proves (1) and (2). As for (3), if W_- is negative definite with $\dim W_- = i(Q)$, then $W_- \to \mathcal{H}/W'_+ \cong W'_-$ is injective and hence an isomorphism since $\dim W'_- \leq i_-(Q) = i(Q)$. ■

In view of Proposition 2.1, we call any orthogonal direct sum decomposition $\mathcal{H} = W_- \oplus W_+$, where $W_\pm$ are positive/negative definite, a $\pm$-*decomposition* of $(\mathcal{H}, Q)$.

$$\langle x, y\rangle_\pm := B\big(x_+, y_+\big) - B\big(x_-, y_-\big), \tag{2.7}$$

where $x = x_- + x_+$, $y = y_- + y_+$ are the corresponding decompositions.

Lemma 2.4. Let $(\mathcal{H}, Q)$ be a nondegenerate quadratic space of finite index and $\mathcal{H} = W_- \oplus W_+ = W'_- \oplus W'_+$ two $\pm$-decompositions. Then $(\mathcal{H}, \langle\cdot,\cdot\rangle_\pm)$ is a Hilbert space if and only if $(\mathcal{H}, \langle\cdot,\cdot\rangle'_\pm)$ is a Hilbert space, in which case the two scalar products are equivalent. □

We need the following lemma.

Lemma 2.5. Let $\mathcal{H}$ be a real vector space, $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ two scalar products, and $\mathcal{H}' \subseteq \mathcal{H}$ a subspace such that

(1) $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ coincide on $\mathcal{H}'$ and $\mathcal{H}'$ is complete;

(2) $\mathcal{H}'$ is of finite codimension.

Then $\langle\cdot,\cdot\rangle_1$, $\langle\cdot,\cdot\rangle_2$ are equivalent and $\mathcal{H}$ is a Hilbert space. □

Proof. Let $\mathcal{H}'_1$ be the orthogonal of $\mathcal{H}'$ for $\langle\cdot,\cdot\rangle_1$. Since $\mathcal{H}'$ is complete, we have $\mathcal{H} = \mathcal{H}' \oplus \mathcal{H}'_1$. But $\mathcal{H}'_1$ is complete because it is finite dimensional and hence $(\mathcal{H}, \langle\cdot,\cdot\rangle_1)$ is a Hilbert space. For any $x \in \mathcal{H}$, write $x = x' + x'_1$ according to the above decomposition. Since $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent on $\mathcal{H}'_1$, there is $c > 0$ with $\|x'_1\|_1^2 \geq c\|x'_1\|_2^2$ for all x. We may choose $c \leq 1$ and now

$$\begin{aligned} \|x\|_1^2 &= \|x'\|_1^2 + \big\|x'_1\big\|_1^2 \geq \|x'\|_1^2 + c\big\|x'_1\big\|_2^2 \\ &\geq c\big(\|x'\|_1^2 + \big\|x'_1\big\|_2^2\big) \geq \frac{c}{2}\big(\|x'\|_1 + \big\|x'_1\big\|_2\big)^2 \geq \frac{c}{2}\|x\|_2^2. \end{aligned} \tag{2.8}$$

■

Proof of Lemma 2.4. Assume $i(Q) = i_-(Q)$. Since B is continuous with respect to both $\|\cdot\|_\pm$ and $\|\cdot\|'_\pm$, all subspaces considered are closed for both topologies and so is in particular $W_+ \cap W'_+$. Moreover, the latter is of codimension at most $2i(Q)$, hence we conclude by Lemma 2.5. ■

Definition 2.6. A nondegenerate quadratic space of finite index $(\mathcal{H}, Q)$ is *strongly nondegenerate* if for some (hence any) $\pm$-decomposition $\mathcal{H} = W_- \oplus W_+$, the space $(\mathcal{H}, \langle\cdot,\cdot\rangle_\pm)$ is a Hilbert space.

We denote by $d(\mathcal{L})$ the cardinal of any Hilbert basis of a Hilbert space $\mathcal{L}$. Observing that $d(\mathcal{L})$ depends only on the equivalence class of the scalar product, we deduce that the pair $(d(W_+), d(W_-))$ is independent of the choice of a $\pm$-decomposition $\mathcal{H} = W_- \oplus W_+$. We call $(d(W_+), d(W_-))$ the *signature* of the strongly nondegenerate quadratic space $(\mathcal{H}, Q)$.

Two quadratic spaces $(\mathcal{H}_1, Q_1)$ and $(\mathcal{H}_2, Q_2)$ are *isomorphic* if there is a vector space isomorphism $T : \mathcal{H}_1 \to \mathcal{H}_2$ with $Q_1 = Q_2 \circ T$. Observe that if $(\mathcal{H}_1, Q_1)$ is nondegenerate of finite index, then so is $(\mathcal{H}_2, Q_2)$. Since the image of a $\pm$-decomposition for Q_1 is a $\pm$-decomposition for Q_2, we see that Q_2 is strongly nondegenerate if Q_1 is so; in that case, T is automatically continuous. In particular, the orthogonal group $\mathbf{O}(Q)$ of a strongly nondegenerate form Q of finite index consists of bounded linear operators.

Proposition 2.7. For strongly nondegenerate forms of finite index, the signature is a complete invariant of isomorphisms. □

Proof. Using $\pm$-decompositions, this follows immediately from the fact that $d(\mathcal{L})$ determines completely the Hilbert spaces $\mathcal{L}$ up to isomorphisms. ■

Let $(\mathcal{H}, Q)$ be a strongly nondegenerate form of finite index, $\mathcal{H}^*$ the topological dual, and $A : \mathcal{H} \to \mathcal{H}^*$ the continuous morphism associated to B. Applying the Riesz representation theorem to the restrictions of B to $W_\pm$ for a $\pm$-decomposition $\mathcal{H} = W_- \oplus W_+$, we deduce that A is an isomorphism (of topological vector spaces).

Proposition 2.8. Let $(\mathcal{H}, Q)$ be a strongly nondegenerate form of finite index and $V \subseteq \mathcal{H}$ a closed subspace such that $Q|_V$ is nondegenerate. Then $(V, Q|_V)$ is strongly nondegenerate and $\mathcal{H} = V \oplus {}^\perp V$. □

Proof. Assume $i(Q) = i_-(Q)$ and let $V = U_- \oplus U_+$ be a $\pm$-decomposition of V with $U_\pm$ positive/negative definite (which exists by Proposition 2.1). Since V is closed and B continuous, $U_+ = {}^\perp U_- \cap V$ is closed. Let now $\mathcal{H} = W_- \oplus W_+$ be any $\pm$-decomposition of $\mathcal{H}$. Then $W_+ \cap U_+$ is closed, of finite codimension in V, and B coincides with $\langle \cdot, \cdot \rangle_\pm$ on it. By Lemma 2.5 with $\langle \cdot, \cdot \rangle_1 = \langle \cdot, \cdot \rangle_2$, we deduce that V is a Hilbert space and hence $(V, Q|_V)$ is strongly nondegenerate. To conclude, the nondegeneracy of $(V, Q|_V)$ implies, as observed above, that the morphism $A_V : V \to V^*$ associated to $B|_V$ is a topological isomorphism. In particular, for every $x \in \mathcal{H}$, there is $x_V = A_V^{-1} B(x, \cdot)|_V \in V^*$ such that $B(x, y) = B(x_V, y)$ for all $y \in V$. Thus $x \in {}^\perp V + x_V$ and the claim follows. ■

3 Real hyperbolic space

Let $(\mathcal{H}, Q)$ be a nondegenerate quadratic space of index one; we assume more specifically that $i(Q) = i_-(Q) = 1$. Let $C_- := \left\{x \in \mathcal{H} : Q(x) < 0\right\}$ be the cone of negative vectors and $\mathbf{H} := \mathbb{R}^* \backslash C_-$ the set of negative lines. One proves as usual the reverse Cauchy-Schwartz inequality

$$B(x, y)^2 \geq Q(x)Q(y) \quad \forall x, y \in C_-, \tag{3.1}$$

with equality if and only if $\mathbb{R}^* x = \mathbb{R}^* y$. This allows to define $\widetilde{d} : C_- \times C_- \to \mathbb{R}_+$ by

$$\cosh^2 \widetilde{d}(x, y) = \frac{B(x, y)^2}{Q(x)Q(y)}, \tag{3.2}$$

which descends to a well-defined function $d : \mathbf{H} \times \mathbf{H} \to \mathbb{R}_+$.

Remark 3.1. A useful geometric fact is that for any finite set S of negative lines, the restriction $Q|_{\mathcal{H}_S}$ of Q to the span $\mathcal{H}_S \subseteq \mathcal{H}$ of S is equivalent to the standard real quadratic form of signature $(\dim \mathcal{H}_S - 1, 1)$ and that, under this isomorphism, the restriction of d to the image $\mathbf{H}_S$ of $\mathcal{H}_S$ in $\mathbf{H}$ corresponds to the standard distance on the finite-dimensional real hyperbolic space of dimension $|S| - 1$.

As a consequence of the above remark and [2, Theorem 10.10], we have the following proposition.

Proposition 3.2. The function d is a distance function with respect to which $\mathbf{H}$ is a geodesic CAT(-1) space. □

Let ℓ_- be a vector of length -1 and let $\mathcal{H} = \mathcal{H}_+ \oplus \mathbb{R}\ell_-$ be the orthogonal decomposition, where $Q|_{\mathcal{H}_+}$ is positive definite (see **Proposition 2.1**). The exponential map $\exp : \mathcal{H}_+ \to \mathbf{H}$ is defined as follows. For every $v \in \mathcal{H}_+$, there is a unique $t > 0$ such that $x := v + t\ell_-$ has length -1; we define $\exp(v) := [x]$ to be the image of x in $\mathbf{H}$. A straightforward computation gives

$$\cosh d\big(\exp(v), \exp(w)\big) = \Big| - B(v, w) + \sqrt{1 + Q(v)}\sqrt{1 + Q(w)}\Big|, \tag{3.3}$$

and, in particular, $\cosh d(\exp(v), [\ell_-]) = \sqrt{1 + Q(v)}$.

Proposition 3.3. The CAT(-1) space $\mathbf{H}$ is complete if and only if $(\mathcal{H}, Q)$ is strongly nondegenerate. □

Proof. Using the above formulæ, on checks that exp is for all $R > 0$ a bi-Lipschitz bijection between the ball in $(\mathcal{H}_+, Q|_{\mathcal{H}_+})$ of radius $(\sinh R)^2$ centered at 0 and the ball in $(\mathbf{H}, d)$ of radius R centered at $[\ell_-]$. Thus $(\mathbf{H}, d)$ is complete if and only if $\mathcal{H}_+$ is complete, which in view of **Lemma 2.5** (with $\langle\cdot,\cdot\rangle_1 = \langle\cdot,\cdot\rangle_2$) is equivalent to the quadratic space $(\mathcal{H}, Q)$ being strongly nondegenerate. ■

Observe that any orthogonal transformation $T \in \mathbf{O}(Q)$ preserves C_- and descends to an isometry of $\mathbf{H}$. The group $\mathbf{O}(Q)$ is a direct product $\mathbf{O}_+(Q) \cdot \{\pm \mathrm{Id}\}$, where $\mathbf{O}_+(Q)$ is the subgroup preserving the (two) connected components of C_-. Let $\mathbf{PO}(Q) = \mathbf{O}(Q)/\pm \mathrm{Id}$. Then we have the following.

Proposition 3.4. The homomorphism $\mathbf{O}(Q) \to \mathrm{Isom}(\mathbf{H})$ induces isomorphisms

$$\mathbf{O}_+(Q) \longrightarrow \mathbf{PO}(Q) \longrightarrow \mathrm{Isom}(\mathbf{H}). \tag{3.4}$$

□

Remark 3.5. Let G be a topological group and $\pi : G \to \mathbf{O}(Q)$ a group homomorphism. We call π *continuous* if the action map $G \times \mathcal{H} \to \mathcal{H}$ is continuous; then, the resulting action $G \times \mathbf{H} \to \mathbf{H}$ is continuous. Conversely, given a continuous action $G \times \mathbf{H} \to \mathbf{H}$, one verifies that the resulting homomorphism $G \to \mathbf{O}_+(Q)$ deduced from Proposition 3.4 is continuous.

Lemma 3.6. The $\mathbf{O}(Q)$-action on $\mathbf{H}$ is transitive. □

Proof of the lemma. Let $L, L' \subseteq \mathcal{H}$ be two negative lines. Then $L \oplus {}^{\perp}L$ and $L' \oplus {}^{\perp}L'$ are two $\pm$-decompositions, and ${}^{\perp}L$, ${}^{\perp}L'$ are isomorphic Hilbert spaces. Hence there is (compare also Proposition 2.7) an isomorphism of $(\mathcal{H}, Q)$ bringing L to L'. ■

Proof of Proposition 3.4. Let $T \in \mathrm{Isom}(\mathbf{H})$. By Lemma 3.6, we may assume that T fixes $[\ell_-]$. Define a map $U : \mathcal{H}_+ \to \mathcal{H}_+$ by $\exp(U(v)) = T(\exp(v))$. It follows from the above formulæ that U is a bijection, fixes 0 and preserves $B|_{\mathcal{H}_+}$. Hence U is a linear orthogonal transformation of $\mathcal{H}_+$. Defining $S := U \oplus \mathrm{Id}$, one verifies that $S \in \mathbf{O}_+(Q)$ corresponds to T *via* $\mathbf{O}(Q) \to \mathrm{Isom}(\mathbf{H})$ and the statement follows. ■

One proves the following similarly.

Proposition 3.7. Let $(\mathcal{H}_i, Q_i)$ be strongly nondegenerate quadratic spaces of signature $(\alpha_i, 1)$ and let $\mathbf{H}_i$ be the associated hyperbolic spaces (for $i = 1, 2$). The following are equivalent:

(i) $(\mathcal{H}_1, Q_1)$ is isomorphic to $(\mathcal{H}_2, Q_2)$;
(ii) $\alpha_1 = \alpha_2$;
(iii) $\mathbf{H}_1$ is isometric to $\mathbf{H}_2$. □

Thus we obtain for each cardinal α "the" real hyperbolic space $\mathbf{H}^\alpha$.

3.1 Bordification

Let again $\mathbf{H} = \mathbb{R}^*\backslash C_-$ be the real hyperbolic space associated to a strongly nondegenerate quadratic space $(\mathcal{H}, Q)$ of signature $(\alpha, 1)$. Let $\partial\mathbf{H}$ be the boundary of the CAT(-1) space $(\mathbf{H}, d)$ defined as usual as classes of asymptotic rays. Set

$$C_0 := \{x \in \mathcal{H} : Q(x) = 0, x \neq 0\}, \qquad C_{\leq 0} := \{x \in \mathcal{H} : Q(x) \leq 0, x \neq 0\}. \tag{3.5}$$

Using that any configuration of finitely many geodesics in $\mathbf{H}$ is contained in a finite-dimensional hyperbolic subspace (see Remark 3.1), we obtain a bijection identifying the bordification $\overline{\mathbf{H}} = \mathbf{H} \sqcup \partial\mathbf{H}$ with the set $\mathbb{R}^*\backslash C_{\leq 0}$. We relate this to the description of $\overline{\mathbf{H}}$ in terms of Busemann cocycles: for every $x \in C_{\leq 0}$, define $\widetilde{b}_x : C_- \times C_- \to \mathbb{R}$ by

$$\widetilde{b}_x(y,z) := \begin{cases} \widetilde{d}(x,y) - \widetilde{d}(x,z) & \text{if } x \in C_-, \\ \dfrac{1}{2} \ln \dfrac{B(x,y)^2 Q(z)}{B(x,z)^2 Q(y)} & \text{if } x \in C_0. \end{cases} \tag{3.6}$$

Then, for every $x \in C_{\leq 0}$, $\widetilde{b}_x$ satisfies the cocycle identity

$$\widetilde{b}_x(y_2, y_3) - \widetilde{b}_x(y_1, y_3) + \widetilde{b}_x(y_1, y_2) = 0. \tag{3.7}$$

Moreover, $\widetilde{b}_x$ gives a well-defined function $b : \mathbb{R}^*\backslash C_{\leq 0} \times \mathbf{H} \times \mathbf{H} \to \mathbb{R}$ which coincides on $\mathbf{H} \times \mathbf{H} \times \mathbf{H}$ with $(x, y, z) \mapsto d(x, y) - d(x, z)$.

For every $x \in \mathbb{R}^*\backslash C_{\leq 0}$ and $z_1, z_2 \in \mathbf{H}$, the cocycle property implies that the continuous functions $y \mapsto b_x(y, z_1)$ and $y \mapsto b_x(y, z_2)$, defined on $\mathbf{H}$, differ by a constant. Thus we obtain, for every $x \in \mathbb{R}^*\backslash C_{\leq 0}$, a well-defined class $B(x) \in C(\mathbf{H})/\mathbb{R}$, where $C(\mathbf{H})$ is the space of continuous functions on $\mathbf{H}$. Endowing $C(\mathbf{H})/\mathbb{R}$ with the topology coming from the topology on $C(\mathbf{H})$ of uniform convergence on bounded sets, and denoting by $\overline{B(\mathbf{H})}$ the closure of $B(\mathbf{H})$ in $C(\mathbf{H})/\mathbb{R}$, we have the following proposition.

Proposition 3.8. The map $B : \mathbb{R}^*\backslash C_{\leq 0} \to \overline{B(\mathbf{H})}$ is a homeomorphism when $\mathbb{R}^*\backslash C_{\leq 0}$ is endowed with the quotient of the norm topology. □

One can verify that in this topology, $\overline{\mathbf{H}}$ is compact if and only if $\mathbf{H}$ is finite dimensional.

By a slight abuse of terminology, we call the level sets of $\widetilde{b}_x(\cdot, z)$ (resp., $B(\xi)$) *horospheres* in $\mathcal{H}$ (resp., in $\mathbf{H}$) centered at $x \in C_0$ (resp., at $\xi \in \partial\mathbf{H}$).

4 Nonelementary and elementary actions

In this section, we study basic properties of group actions on hyperbolic spaces. First we establish that any nonelementary action has a unique minimal invariant hyperbolic subspace, and then we turn to the description of elementary actions in terms of orthogonal representations, characters, and continuous cocycles.

Let X be a metric space. Recall that a *semicontraction* is a map $T : X \to X$ such that $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$. Recall the following.

Proposition 4.1. Let X be a complete CAT(-1) space and $T : X \to X$ a semicontraction. Then one of the following holds:

(i) the set $\{T^n x : n \geq 1\}$ is bounded for some (hence any) $x \in X$ and the set $X^T \subseteq X$ of T-fixed points is not empty,

(ii) the set $\{T^n x : n \geq 1\}$ is unbounded for some (hence any) $x \in X$ and there exist a subsequence $\{n_k\}_{k\geq 1}$ and $\xi \in \partial X$ with $\lim_{k\to\infty} T^{n_k} x = \xi$ and $T\xi = \xi$. Moreover, $|(\partial X)^T| = 1$ or 2. □

(A general semicontraction need not extend to infinity; the notation $T\xi = \xi$ means that $T : X \to X$ extends by continuity to $X \cup \{\xi\}$ endowed with the topology induced by $\overline{X}$.)

Proof of the proposition. The case (ii) follows from the argument given by Karlsson ([7, Proof of Proposition 5.1]); see [8, Section 3] for the additional statement on $|(\partial X)^T|$. If on the other hand, the T-orbits are bounded, then it is known that X^T is nonempty; indeed, one verifies that for any $x \in X$, the circumcentre of the set $\{T^k(x) : k \geq n\}$ converges to a T-fixed point as $n \to \infty$. ■

This result applies in particular to the case where T is an isometry and is the basis for the classification of isometries.

Definition 4.2. An isometry T is called

(i) *elliptic* if $\{T^n x : n \geq 1\}$ is bounded;

(ii) *parabolic* if $\{T^n x : n \geq 1\}$ is unbounded and $|(\partial X)^T| = 1$;

(iii) *hyperbolic* if $\{T^n x : n \geq 1\}$ is unbounded and $|(\partial X)^T| = 2$.

If T is hyperbolic, then $(\partial X)^T = \{\xi_-, \xi_+\}$ with $\lim_{n\to\pm\infty} T^n x = \xi_\pm$ for all $x \in X$ (see again [8, Section 3]). However, when X is not proper and T is parabolic, the sequence $T^n x$ might not converge in $\overline{X}$.

As usual a group action $G \times X \to X$ by isometries is called *elementary* if G preserves a nonempty finite subset of $\overline{X}$. This is equivalent to saying that either G fixes a point in $\overline{X}$ or it preserves a geodesic.

Let now $\mathbf{H}$ be the hyperbolic space associated to a strongly nondegenerate quadratic space $(\mathcal{H}, Q)$ of signature $(\alpha, 1)$. In the sequel, we will study nonelementary and elementary actions and we will prove the following proposition.

Proposition 4.3. Let $\pi : G \to \mathbf{O}(Q)$ be a homomorphism. Then, one of the following holds:

(i) G preserves an isotropic line and all horospheres in $\mathcal{H}$ centered at it;

(ii) G preserves a negative line;

(iii) there is a unique minimal nondegenerate closed G-invariant subspace $\mathcal{H}_1 \subseteq \mathcal{H}$ of index one. Any nondegenerate closed G-invariant subspace of index one contains $\mathcal{H}_1$. □

First we prove the proposition in the case where the associated action on $\mathbf{H}$ is nonelementary, which excludes of course (i) and (ii). The remaining will be a consequence of a closer analysis of elementary actions.

Proof in the nonelementary case. We need to show that (iii) holds. Let $\mathcal{P}$ be the set of G-invariant closed positive-definite subspaces of $\mathcal{H}$, ordered by inclusion, let $\mathcal{C} \subseteq \mathcal{P}$ be a maximal chain, and let $L := \overline{\bigcup \mathcal{C}}$. Then L is closed, G-invariant, and $Q|_L \geq 0$. By **Lemma 2.3** applied to L, we have that $\{x \in L : Q(x) = 0\} = L \cap {}^{\perp}L$; if the latter were not zero, it would be a G-invariant isotropic line, contradicting the assumption that the action is nonelementary. Thus L is a G-invariant closed maximal positive-definite subspace of $\mathcal{H}$. Set $\mathcal{H}_0 := L$, $\mathcal{H}_1 := {}^{\perp}L$. Then, by **Proposition 2.8**, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ and $\mathcal{H}_1$ is a G-invariant closed nondegenerate subspace of index one which is minimal with respect to these properties.

Let now $\mathcal{H} = \mathcal{H}'_0 \oplus \mathcal{H}'_1$ be any other orthogonal decomposition into G-invariant closed subspaces where $\mathcal{H}'_0$ is positive definite and $\mathcal{H}'_1$ of index one. We need to show that $\mathcal{H}'_1 \supseteq \mathcal{H}_1$. Consider $J := \mathcal{H}_1 \cap \mathcal{H}'_1$. Again, since the G-action is not elementary, J is nondegenerate. There are two cases.

(i) J *is indefinite*. Since $J \subseteq \mathcal{H}_1$, we have either $J = \mathcal{H}_1$, whence $\mathcal{H}'_1 \supseteq \mathcal{H}_1$ and we are done, or $J = 0$.

(ii) J *is positive definite*. Then $\mathcal{H}_0 \oplus J$ would be a G-invariant closed positive-definite subspace and, by maximality, we would have $\mathcal{H}_0 \oplus J \subseteq \mathcal{H}_0$ and hence, once again, $J = 0$.

Thus, we may assume (for a contradiction) $\mathcal{H}_1 \cap \mathcal{H}'_1 = 0$. Let $\mathbf{H}_1, \mathbf{H}'_1 \subseteq \mathbf{H}$ be the corresponding hyperbolic subspaces and consider the orthogonal projections $p : \mathbf{H} \to \mathbf{H}_1$ and $p' : \mathbf{H} \to \mathbf{H}'_1$ given by the nearest point retraction. Since $\mathbf{H}_1 \cap \mathbf{H}'_1 = \varnothing$ and $\mathbf{H}$ is CAT(-1), both $p|_{\mathbf{H}'_1}$ and $p'|_{\mathbf{H}_1}$ are contractions. Hence the map $f : \mathbf{H}_1 \to \mathbf{H}_1$, defined by $f := p|_{\mathbf{H}'_1} \circ p'|_{\mathbf{H}_1}$, is a G-equivariant contraction, that is, $d(f(x), f(y)) < d(x, y)$ for all distinct $x, y \in \mathbf{H}_1$. If for some $x \in \mathbf{H}_1$, the set $\{d(f^n(x), x) : n \geq 1\}$ were bounded, **Proposition 4.1**(i) would imply the existence of an f-fixed point in $\mathbf{H}$. Since f is G-equivariant, the set of its fixed points $\mathbf{H}_1^f$ is G-invariant. However, since f is a contraction, $\mathbf{H}_1^f$ consists of one point, which is hence G-fixed, contradicting the assumption that the G-action is nonelementary.

Thus, the f-orbits are unbounded and by **Proposition 4.1**(ii), there is a subsequence $\{n_k\}$ and $\xi \in \partial \mathbf{H}_1$ with $\lim_{k\to\infty} f^{n_k}(x) = \xi$. However, for every given $g \in G$ and for $x \in \mathbf{H}_1$,

$$d\big(gf^n(x), f^n(x)\big) = d\big(f^n(gx), f^n(x)\big) < d(gx, x) \tag{4.1}$$

is bounded independently of n, thus, by passing to subsequences, the sequences $gf^n(x)$

and $f^n(x)$ are at bounded distance and hence define the same point at infinity, namely $g\xi = \xi$. Since this contradicts again the assumption that the action is nonelementary, the proof in this case is complete. ∎

4.1 Elementary actions

As before, let $(\mathcal{H}, Q)$ be a strongly nondegenerate quadratic space of signature $(\alpha, 1)$. We will study, for a topological group G, the elementary actions on $\mathbf{H}^\alpha$, and more specifically, the actions fixing a point in $\partial\mathbf{H}^\alpha$. Thus, fix L_+ an isotropic line in $\mathcal{H}$, let $\mathbf{O}_{L_+}(Q)$ be its stabilizer in $\mathbf{O}(Q)$, and let $\mathrm{Rep}(G, \mathbf{O}_{L_+}(Q))$ be the set of continuous representations (see Remark 3.5). Then G acts on L_+ by multiplication by a continuous character $\chi : G \to \mathbb{R}^*$. The bilinear form B induces a Hilbert space structure of Hilbert dimension $\alpha - 1$ on ${}^\perp L_+/L_+$ and we may thus fix a real Hilbert space E of Hilbert dimension $\alpha - 1$ and an isomorphism $i : {}^\perp L_+/L_+ \to E$. Since $\pi : G \to \mathbf{O}_{L_+}$ is continuous and G preserves L_+ and ${}^\perp L_+$, it induces on ${}^\perp L_+/L_+$ a continuous orthogonal representation which we transport to E *via* i, obtaining $\rho : G \to \mathbf{O}(E)$. The space $\mathcal{H}/{}^\perp L_+$ is one dimensional; since G preserves B, it acts on that space by multiplication by χ^{-1}.

We define a new G-module structure on $\mathcal{H}$ by means of the continuous representation $\chi \otimes \pi$. Then we have a short exact sequence of G-modules

$$0 \longrightarrow {}^\perp L_+/L_+ \longrightarrow \mathcal{H}/L_+ \longrightarrow \mathcal{H}/{}^\perp L_+ \longrightarrow 0 \tag{4.2}$$

in which the last term is a trivial G-module of dimension one. Thus, applying the corresponding transgression map in degree zero, the image of the trivial module $\mathcal{H}/{}^\perp L_+$ in the continuous cohomology $H^1_c(G, {}^\perp L_+/L_+)$ yields *via* i a subspace

$$\mathbb{R} \cdot \eta \subseteq H^1_c(G, \chi \otimes \rho). \tag{4.3}$$

Thus, with i fixed, we associated to every continuous homomorphism $\pi : G \to \mathbf{O}_{L_+}(Q)$ the following data:

(i) a continuous homomorphism $\chi_\pi \in \mathrm{Hom}_c(G, \mathbb{R}^*)$,

(ii) a continuous orthogonal representation $\rho_\pi \in \mathrm{Hom}_c(G, \mathbf{O}(E))$,

(iii) a continuous class $\eta_\pi \in H^1_c(G, \chi \otimes \rho)$, well defined up to scalar multiplication.

Denoting by $\mathcal{Z}(G, E)$ the set of all triples (χ, ρ, η) with $\chi \in \mathrm{Hom}_c(G, \mathbb{R}^*)$, $\rho \in \mathrm{Hom}_c(G, \mathbf{O}(E))$, and $\eta \in H^1_c(G, \chi \otimes \rho)$, we have that $\mathbf{O}(E) \times \mathbb{R}^*$ acts on $\mathcal{Z}(G, E)$ by $(T, \lambda)(\chi, \rho, \eta) = (\chi, T\rho T^{-1}, \lambda T\eta)$.

Proposition 4.4. (i) The map $\mathrm{Rep}(G, \mathbf{O}_{L_+}(Q)) \to \mathcal{Z}(G, E)$, $\pi \mapsto (\chi_\pi, \rho_\pi, \eta_\pi)$, induces a bijection

$$\mathrm{Rep}\left(G, \mathbf{O}_{L_+}(Q)\right)/\mathbf{O}_{L_+}(Q) \xrightarrow{\cong} \left[\mathbf{O}(E) \times \mathbb{R}^*\right]\backslash \mathcal{Z}(G, E). \tag{4.4}$$

(ii) The representation π leaves all horospheres centered at L_+ invariant if and only if $|\chi_\pi| = 1$. □

Proof. Given $\chi : G \to \mathbb{R}^*$, $\rho : G \to \mathbf{O}(E)$, and $\eta \in H^1_c(G, \chi \otimes \rho)$, we indicate how to reconstruct $\pi \in \mathrm{Rep}(G, \mathbf{O}_{L_+}(Q))$. Fix an isotropic line $L_- \neq L_+$ and let $f : G \to E$ be a continuous cocycle representing η. Set $F = {}^\perp(L_+ \oplus L_-)$, and denote by j the isomorphism of Hilbert spaces obtained by composing $F \to {}^\perp L_+ \to E$. Fix $\ell_\pm \in L_\pm$ with $B(\ell_-, \ell_+) = 1$. Using the notation of Appendix A, we define

$$\pi(g) = \begin{pmatrix} \pi(g)_1 & & \pi(g)_2^+ \\ & & 0 \\ 0 & \pi(g)_3^- & \pi(g)_4 \end{pmatrix} \tag{4.5}$$

by

$$\pi(g)_1 = \begin{pmatrix} \chi(g) & a(g) \\ 0 & \chi(g)^{-1} \end{pmatrix}, \tag{4.6}$$

where

$$\begin{aligned} a(g) &= -\frac{1}{2}\chi(g)\left\|f(g)\right\|_E^2, \\ \pi(g)_2^+(v) &= -\left\langle \rho(g)j(v), f(g)\right\rangle_E \quad \forall v \in F, \\ \pi(g)_3^- &= \chi(g)^{-1} j^{-1}\left(f(g)\right), \\ \pi(g)_4 &= j^{-1}\rho(g)j. \end{aligned} \tag{4.7}$$

The rest of the proposition is now a verification left to the reader and uses the fact that $|\chi_\pi|$ is the exponential of the Busemann character associated to the fixed point L_+. ■

We turn now to continuous representations $\pi : G \to \mathbf{O}_{L_+}(Q)$ for which $|\chi|$ is not identically 1 (write $\chi = \chi_\pi$). We fix once and for all χ and $a \in G$ such that $|\chi(a)| \neq 1$.

Definition 4.5. A continuous cocycle $f : G \to E$ for $\chi \otimes \rho$ is *standard* if $f(a) = 0$.

Lemma 4.6. Let $\rho : G \to \mathbf{O}(E)$ be a continuous orthogonal representation and $\chi : G \to \mathbb{R}^\times$ a continuous homomorphism with $|\chi(a)| \neq 1$.

(i) Every class in $H^1_c(G, \chi \otimes \rho)$ admits a unique standard representative.

(ii) If $M < G$ is a compact subgroup normalised by a, then any standard cocycle vanishes on M. □

Proof. (i) Set $\tau = \chi \otimes \rho$. Recall that $|\chi(a)| \neq 1$ implies that $1 - \tau(a)$ is invertible and hence $H^1(\langle a \rangle, \tau)$ vanishes. Therefore, for any cocycle $f' : G \to E$, there exists $v \in E$ such that $f'(a^n) = \tau(a^n)v - v$. Now $f(g) := f'(g) + v - \tau(g)v$ defines a standard cocycle. If f_1 and f_2 are any two cohomologous cocycles, there exists $v \in E$ such that $f_1(g) = f_2(g) + \tau(g)v - v$. If in addition f_1 and f_2 are standard, then $\tau(a)v = v$, which implies, since $|\chi(a)| \neq 1$, that $v = 0$ and hence $f_1 = f_2$.

(ii) Let f be a standard cocycle. Since M is compact, $C := \sup_{k \in M} \|f(g)\|$ is finite. We have for all $k \in M$,

$$\begin{aligned} f(k) &= \tau(k)f(a) + f(k) = f(ka) = f\big(aa^{-1}ka\big) \\ &= \tau(a)f\big(a^{-1}ka\big) + f(a) = \tau(a)f\big(a^{-1}ka\big), \end{aligned} \tag{4.8}$$

which implies that $C = |\chi(a)|C$ and hence $C = 0$. ■

Let $\eta \in H^1_c(G, \chi \otimes \rho)$ and let $\pi : G \to \mathbf{O}_{L_+}(Q)$ be the homomorphism associated to (χ, ρ, η) by the above construction. Then $\pi(a)$ is hyperbolic with fixed points L_- and L_+. If $f : G \to E$ is the standard cocycle representing the class η, define

$$I_\eta := \overline{\langle f(g) : g \in G \rangle}, \tag{4.9}$$

which is a closed G-invariant subspace of E. With these definitions, we have the following proposition.

Proposition 4.7. There is a G-invariant orthogonal decomposition

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_0, \tag{4.10}$$

where

$$\mathcal{H}_1 = \big(L_+ \oplus L_-\big) \oplus j^{-1}\big(I_\eta\big), \quad \mathcal{H}_0 = {}^{\perp}\mathcal{H}_1. \tag{4.11}$$

Moreover, the subspace $\mathcal{H}_1$ is nondegenerate of index one, $\mathcal{H}_0$ is positive definite, and if $\mathcal{H}_1' \subseteq \mathcal{H}$ is any closed nondegenerate G-invariant subspace of index one, then $\mathcal{H}_1' \supset \mathcal{H}_1$. □

Proof. We verify the last assertion: let $\mathbf{H}_1' \subseteq \mathbf{H}_1$ be the hyperbolic subspace associated to $\mathcal{H}_1'$. Without loss of generality, $|\chi(a)| > 1$. Then $\lim_{n\to\pm\infty} \pi(a^n)x = L_\pm$ for all $x \in \mathbf{H}_1'$, hence $L_\pm \in \partial\mathbf{H}_1'$. If $L := L_+ \oplus L_-$ and $\mathrm{Pr}_L : \mathcal{H} \to L$ is the orthogonal projection, this implies that $\pi(g)\ell_- - \mathrm{Pr}_L(\pi(g)\ell_-)$ is in $\mathcal{H}_1'$ for all $g \in G$; therefore, using the formulæ in (4.7) and Appendix A, $\mathcal{H}_1' \supseteq j^{-1}(I_\eta)$. ■

End of proof of Proposition 4.3. We know already that the alternative (iii) of the proposition holds in the nonelementary case. If the G-action is elementary, then there are the following possibilities.

(1) G fixes a point in $\mathbf{H}$. This corresponds to (ii).

(2) G leaves a geodesic line in $\mathbf{H}$ invariant. Denote this line by $\mathbf{H}_1 \subseteq \mathbf{H}$. This means that there is a G-invariant two-dimensional subspace $\mathcal{H}_1 \subseteq \mathcal{H}$ of index one. In fact, $\mathcal{H}_1 = L_+ \oplus L_-$, where $\partial\mathbf{H}_1 = \{L_-, L_+\}$. We may assume that G has no fixed point in $\mathbf{H}_1$, so there is $g \in G$ acting hyperbolically: say, $\lim_{n\to\pm\infty} \pi(g)^n x = L_\pm$. But then, if $\mathcal{H}_1'$ is any G-invariant closed nondegenerate subspace of index one and $\mathbf{H}_1' \subseteq \mathbf{H}$ the associated hyperbolic subspace, we get $\partial\mathbf{H}_1' \ni L_\pm$, hence $\mathcal{H}_1' \supseteq \mathcal{H}_1$.

(3) G fixes an isotropic line L_+. Let then $\chi : G \to \mathbb{R}^*$ be the associated character. Either $|\chi| = 1$ and we are in alternative (i), or $|\chi| \neq 1$ and we can apply Proposition 4.7, so that alternative (iii) holds once again. ■

5 Actions of certain HNN-extensions

Let $P = \langle a \rangle \ltimes N$ be a locally compact group, semidirect product of an infinite cyclic subgroup $\langle a \rangle$ with generator a and a closed normal subgroup $N = \bigcup_{n\in\mathbb{Z}} K_n$ which is the increasing union of compact open subgroups $K_n < K_{n+1}$ such that $aK_na^{-1} = K_{n+1}$ for all $n \in \mathbb{Z}$. In other words, P is the topological ascending HNN-extension $P = K_0 *_a$.

In this section, we study more in detail elementary actions of P on a hyperbolic space $\mathbf{H}$, using the results in Section 4, in particular Proposition 4.4. We begin with the following general fact.

Lemma 5.1. Let X be a complete CAT(-1) space and $P \times X \to X$ a continuous action by isometries. Then there is a P-fixed point in $\overline{X}$ and, in fact, one of the following holds:

(i) a is elliptic and $X^P \neq \varnothing$;

(ii) a is parabolic and $|(\partial X)^P| = 1$;

(iii) a is hyperbolic and the attracting fixed point of a is P-fixed. □

Proof. For (i), we have for all $x \in X$,

$$\sup_{g\in N} d(gx, x) = \sup_{k\in K_0, n\geq 0} d(ka^{-n}x, a^{-n}x). \tag{5.1}$$

Since by assumption a has bounded orbits and K_0 is compact, the latter quantity is bounded and hence N has bounded orbits; the setwise decomposition $P = \langle a\rangle \cdot N$ implies that P itself has bounded orbits and hence admits a fixed point in X.

In cases (ii) and (iii), the set $\{a^n x : n \geq 0\}$ is unbounded for all $x \in X$. Pick $\xi \in \partial X$ and a subsequence $\{n_k\}$ of $\mathbb{N}$ with $\lim_{k\to\infty} a^{n_k}x = \xi$ for all $x \in X$ (see **Proposition 4.1**). It is enough to show that ξ is P-fixed. Setting $F_j := \overline{X}^{K_j}$, we have

$$\begin{aligned} &F_j \cap X \neq \varnothing \quad \forall j \in \mathbb{Z},\\ &F_j \supseteq F_\ell \quad \forall j \leq \ell, \\ &a^n F_j = F_{j+n} \quad \forall j, n \in \mathbb{Z}. \end{aligned} \tag{5.2}$$

Picking $x \in F_j$, we have $a^{n_k}x \in F_{j+n_k} \subseteq F_j$ for all $k \geq 0$ and hence $\xi = \lim_{k\to\infty} a^{n_k}x \in F_j$ since F_j is closed in $\overline{X}$. Thus $\xi \in \bigcap_{j\in\mathbb{Z}} F_j = \overline{X}^N$. It follows that ξ is indeed P-fixed. ■

We now turn back to the particular case where $X = \mathbf{H}$ is the hyperbolic space attached to a strongly nondegenerate quadratic space $(\mathcal{H}, Q)$ of signature $(\alpha, 1)$. We will focus on the study of continuous elementary representations $\pi : P \to \mathbf{O}_{L_+}(Q)$ for which $\pi(a)$ is hyperbolic with attracting fixed point L_+; that is, $|\chi(a)| > 1$.

Proposition 5.2. Let $\chi : P \to \mathbb{R}^*$ be a continuous homomorphism with $|\chi(a)| > 1$, let $\rho : P \to \mathbf{O}(E)$ be a continuous orthogonal representation, and $\tau := \chi \otimes \rho$.

(i) The orthogonal complement $E^{K_{-1}} \ominus E^{K_0}$ of E^{K_0} in $E^{K_{-1}}$ is isomorphic to $H^1_c(P, \tau)$, with isomorphism given by $v \mapsto f_{\sigma(v)}$, where for $v \in E^{K_{-1}} \ominus E^{K_0}$,

$$\sigma(v) := \sum_{n\leq -1} \tau(a)^{n+1} v \tag{5.3}$$

and $f_{\sigma(v)}$ is the standard cocycle uniquely determined by

$$f_{\sigma(v)}(k) = \tau(k)\sigma(v) - \sigma(v) \quad \bigl(\forall\, k \in K_0\bigr). \tag{5.4}$$

(ii) For $\eta \in H^1_c(P, \tau)$, let $f_{\sigma(\nu)}$ be the standard cocycle representing η (with $\nu \in E^{K_{-1}} \ominus E^{K_0}$). Then the subspace $I_\eta = \overline{\langle f_{\sigma(\nu)}(p) : p \in P\rangle}$ coincides with the closed cyclic subspace generated by ν and, in fact,

$$I_\eta = \overline{\left\langle f_{\sigma(\nu)}(n) : n \in N \setminus \bigcap_{j\in\mathbb{Z}} K_j \right\rangle}. \tag{5.5}$$

□

Proof. (i) The proof consists of two steps.

Claim 5.3. There is an isomorphism of topological vector spaces

$$\begin{aligned} &\{\nu \in E : (\mathrm{Id} - \tau(a))\nu \in E^{K_0}\}/E^{K_0} \longrightarrow H^1_c(P, \tau), \\ &\nu \longmapsto [f_\nu], \end{aligned} \tag{5.6}$$

where f_ν is the standard cocycle determined uniquely by

$$f_\nu(k) := \tau(k)\nu - \nu \quad \forall k \in K_0. \tag{5.7}$$

□

Proof. Though this follows immediately from the Mayer-Vietoris sequence associated to topological HNN-extensions in continuous cohomology, we give an explicit proof.

Let f be a standard cocycle. Then repeated applications of the cocycle identity imply that for all $n \in \mathbb{Z}$ and $g \in P$, one has

$$f(a^n g a^{-n}) = \tau(a)^n f(g), \tag{5.8}$$

which, applied to $n \geq 0$ and $g \in K_0$, shows that f is determined by its restriction to K_0. Since K_0 is compact, there is $\nu \in E$, uniquely determined modulo E^{K_0}, such that, for all $k \in K_0$,

$$f(k) = \tau(k)\nu - \nu. \tag{5.9}$$

Substituting (5.9) into (5.8) with $n = -1$ and $g \in K_0$, we get

$$(\mathrm{Id} - \tau(a))\nu \in E^{K_0}. \tag{5.10}$$

Conversely, any cocycle f on K_0 defined by (5.9) with ν satisfying (5.10) extends uniquely to a standard cocycle on P, which hence completes the proof of Claim 5.3. ■

Claim 5.4. The map

$$\begin{aligned} \sigma : E^{K_{-1}} \ominus E^{K_0} &\longrightarrow \{v \in E : (\mathrm{Id} - \tau(a))v \in E^{K_0}\}/E^{K_0}, \\ v &\longmapsto \sum_{n \le -1} \tau(a)^{n+1} v \mod E^{K_0}, \end{aligned} \tag{5.11}$$

is an isomorphism of topological vector spaces. □

Proof. To start the proof of the claim, observe that, since $|\chi(a)| > 1$, the operator $S := \mathrm{Id} - \tau(a) = \tau(a)(\tau(a)^{-1} - \mathrm{Id})$ has a (bounded) inverse given by $S^{-1} = -\sum_{n=0}^{\infty} \tau(a)^{-(n+1)}$. Since $\tau(a)^{\ell} E^{K_0} = E^{K_\ell}$, this implies that

$$J := \overline{\bigcup_{\ell \le -1} E^{K_\ell}} \supseteq \{v \in E : Sv \in E^{K_0}\}. \tag{5.12}$$

Defining $E_j := E^{K_j} \ominus E^{K_{j+1}}$, we have an orthogonal decomposition

$$J = E^{K_0} \oplus \widehat{\bigoplus_{j \le -1}} E_j. \tag{5.13}$$

Let $v = v_0 + \sum_{j \le -1} v_j$ with $Sv \in E^{K_0}$; since we need to determine v mod E^{K_0}, we may assume $v_0 = 0$. Then

$$Sv = -\tau(a)v_{-1} - \sum_{j=1}^{\infty} \left(\tau(a)v_{-(j+1)} - v_{-j}\right), \tag{5.14}$$

where $\tau(a)v_{-1} \in E^{K_0}$ and $\tau(a)v_{-(j+1)} - v_{-j} \in E_{-j}$. In view of (5.12) and (5.13), saying that $Sv \in E^{K_0}$ is equivalent to saying that

$$\tau(a)v_{-(j+1)} = v_{-j}, \quad j \ge 1, \tag{5.15}$$

which implies $v = \sum_{n \le -1} \tau(a)^{n+1} v_{-1}$ and hence yields Claim 5.4. ■

(ii) It is clear that I_η is contained in $\overline{\langle \tau(p)v : p \in P \rangle}$. Conversely, since for $k \in K_0$ we have $f_{\sigma(v)}(k) = \tau(k)\sigma(v) - \sigma(v)$, it follows that

$$\begin{aligned} &-\int_{K_0} f_{\sigma(v)}(k)\,dk = \sigma(v) - \mathrm{Pr}_{E^{K_0}}\left(\sigma(v)\right) = \sigma(v), \\ &-\int_{K_{-1}} \tau(u) \int_{K_0} f_{\sigma(v)}(k)\,dk\,du = v. \end{aligned} \tag{5.16}$$

Thus $v \in I_\eta$, which is a closed invariant subspace, and hence contains the closure of its orbit $\overline{\langle \tau(p)v : p \in P \rangle}$. The additional formula for I_η follows since $f_{\sigma(v)}$ vanishes on $M = \bigcap_{j \in \mathbb{Z}} K_j$ by Lemma 4.6(ii). ■

6 Representations of certain parabolic subgroups

6.1 Gelfand pairs

We start this section recalling some definitions and facts about Gelfand pairs which will be essential in the sequel. We refer to [12, Section 24], for example, for a complete discussion and proofs. Whilst this theory is generally presented for unitary representations, it carries over without changes to orthogonal representations; this will be our viewpoint here.

Let G be a locally compact group, $K < G$ a compact subgroup, and let $C_c(G)^{\natural_K}$ be the convolution algebra of bi-K-invariant functions on G with compact support. Then (G, K) is a Gelfand pair if $C_c(G)^{\natural_K}$ is commutative. It is easy to see that the condition $x^{-1} \in KxK$ for all $x \in G$ is sufficient for (G, K) to be a Gelfand pair.

If (G, K) is a Gelfand pair, a continuous bi-K-invariant function $\varphi \in C(G)^{\natural_K}$ is a spherical function if

(1) $\varphi(e) = 1$,

(2) for all $f \in C_c(G)^{\natural_K}$, there exists a constant c_f such that $\varphi * f = c_f \varphi$.

An irreducible orthogonal representation of G is K-spherical if there exists a nonzero K-invariant vector. If (G, K) is a Gelfand pair and $(\pi, \mathcal{H})$ is any irreducible orthogonal representation of G, then $\dim \mathcal{H}^K \leq 1$, and hence $(\pi, \mathcal{H})$ is K-spherical if and only if $\dim \mathcal{H}^K = 1$. Moreover, (equivalence classes of) K-spherical representations of a Gelfand pair (G, K) are in bijective correspondence with positive-definite spherical functions, with the correspondence given by $\varphi(g) = \langle \pi(g)v, v \rangle$, where $v \in \mathcal{H}$ is a K-invariant vector of norm one and $\langle \cdot, \cdot \rangle$ is the inner product in $\mathcal{H}$.

6.2

In the remainder of this section, we will consider a closed subgroup $P < \mathrm{Aut}(\mathcal{T})$ of the automorphism group of a locally finite tree $\mathcal{T}$ satisfying the following conditions:

(1) P fixes a point $\xi \in \partial\mathcal{T}$;

(2) P acts doubly transitively on $\partial\mathcal{T} \setminus \{\xi\}$.

We will assume throughout that the vertices of $\mathcal{T}$ have valence at least three; observe that under these hypotheses it follows that P acts transitively on the vertices of $\mathcal{T}$, which is therefore a regular tree (and ξ is uniquely determined). Fix a geodesic line $c : \mathbb{Z} \to \mathcal{T}$ with $c(+\infty) = \xi$; by (2) there is a hyperbolic element $a \in P$ with axis c, translation length one, and attracting fixed point ξ. We denote by K_j the stabilizer of $c(j)$ in P for $j \in \mathbb{Z}$. Then P has the structure $P = \langle a \rangle \ltimes N$ as in Sections 5.

Let now $(\mathcal{H}, Q)$ be strongly nondegenerate of signature $(\aleph_0, 1)$ and let L be an isotropic line. The main objective of this section is to prove the following.

Theorem 6.1. For every continuous homomorphism $\chi : P \to \mathbb{R}^*$ with $|\chi(a)| > 1$, there is up to conjugation a unique continuous representation $\lambda : P \to \mathbf{O}_L(Q)$ such that

(i) P acts on L by multiplication by χ;

(ii) there is no proper closed P-invariant subspace of $\mathcal{H}$ which is nondegenerate of index one. □

This result is based on Sections 4 and 5 and on the following generalization of [9].

Proposition 6.2. Let P_x be the stabilizer in P of a vertex $x \in \mathcal{T}$. Then there is a unique (up to equivalence) irreducible orthogonal representation of P having a P_x-fixed vector and whose restriction to N is nontrivial. □

We precede the proof with some intermediate results. For $j \in \mathbb{Z}$, we denote by H_j the horosphere centered at ξ passing through $c(j)$; furthermore, for $\ell \geq 0$, let $H_j(2\ell)$ be the intersection of H_j with the sphere of radius 2ℓ centered at $c(j)$. Notice that for $r \leq j$, the group K_r acts on $H_j(2\ell)$ for every $\ell > 0$, and moreover we have the following.

Lemma 6.3. The group K_r acts transitively on $H_j(2\ell)$ for all $r \leq j$ and $\ell \geq 0$. □

Proof. Write $\xi_\pm := c(\pm\infty)$ and pick $a, b \in H_j(2\ell)$. Complete the geodesic segments $[c(j+\ell), a]$ and $[c(j+\ell), b]$ to infinite rays $[c(j+\ell), \alpha]$ and $[c(j+\ell), \beta]$, respectively, where $\alpha, \beta \in \partial\mathcal{T}$. We may assume $\ell \neq 0$, thus $\alpha, \beta \neq \xi_-$. By double transitivity, there is $g \in P$ with $g(\alpha) = \beta$ and $g(\xi_-) = \xi_-$. The center of the tripods (ξ_+, α, ξ_-) and (ξ_+, β, ξ_-) is $c(j+\ell)$, hence g fixes that point. But since g fixes $\xi_\pm$ and hence preserves $c(\mathbb{Z})$, it follows that $g \in \bigcap_{n\in\mathbb{Z}} K_n$. ■

Corollary 6.4. (N, K_j) is a Gelfand pair for all $j \in \mathbb{Z}$. □

Proof. As mentioned in Section 6.1, it is enough to prove that $n^{-1} \in K_j n K_j$ for all $n \in N$ and $j \in \mathbb{Z}$. We may assume that $n \notin K_j$. Then $n \in K_\ell \setminus K_{\ell-1}$ for some $\ell \geq j+1$ and $n(c(j)), n^{-1}(c(j)) \in H_j(2(\ell - j))$. By Lemma 6.3, this implies the existence of $k \in K_j$ with $kn(c(j)) = n^{-1}(c(j))$ and hence $n^{-1} \in K_j n K_j$. ■

Let $\widehat{N}$ be the set of (equivalence classes of) irreducible orthogonal representations of N; the group $\langle a \rangle$ acts on $\widehat{N}$ by $a_*\pi(g) = \pi(aga^{-1})$, thus preserving the subset $\widehat{N}^1$ of representations that have a K_j-fixed vector for some $j \in \mathbb{Z}$. In fact, if we set

$$\widehat{N}^1_j := \{(\pi, \mathcal{H}) \in \widehat{N}^1 : \mathcal{H}^{K_j} \neq 0, \mathcal{H}^{K_{j+1}} = 0\}, \tag{6.1}$$

then $\widehat{N}^1 = \bigsqcup_{j\in\mathbb{Z}} \widehat{N}^1_j$ and $a_*\widehat{N}^1_j = \widehat{N}^1_{j-1}$, so that any $\widehat{N}^1_j$ is a fundamental domain for the action of $\langle a \rangle$ on $\widehat{N}^1$.

Lemma 6.5. $|\widehat{N}_j^1| = 1$ for all $j \in \mathbb{Z}$. □

Proof. Since $K_\ell \leq K_j$ for $\ell \leq j$, for any K_j-spherical representation $(\pi, \mathcal{H})$, we have that $\mathcal{H}^{K_\ell} \supset \mathcal{H}^{K_j}$; since (N, K_j) is a Gelfand pair, these spaces are of dimension one and hence $\mathcal{H}^{K_\ell} = \mathcal{H}^{K_j}$. Thus, in order to show that $|\widehat{N}_j^1| = 1$, it is sufficient to show that there is a unique positive-definite K_j-spherical function φ with

$$\int_{K_{j+1}} \varphi(kg)dk = 0 \quad \forall g \in N. \tag{6.2}$$

Since it suffices to show that $|\widehat{N}_0^1| = 1$, we start by showing that the space

$$S_0 := \left\{ \varphi \in \left(C(N)^\natural\right)^{K_0} : \int_{K_1} \varphi(kg)dk = 0 \, \forall g \in N \right\} \tag{6.3}$$

is of dimension one. By identifying N/K_0 with the horosphere H_0, we can identify S_0 with the space of K_0-invariant functions on $H_0 = \bigsqcup_{\ell \geq 0} H_0(2\ell)$, and by applying **Lemma 6.3**, we deduce that any function $\varphi \in S_0$ can be written as

$$\varphi = \sum_{\ell=0}^{\infty} \kappa_\ell \mathbf{1}_{H_0(2\ell)}, \tag{6.4}$$

where $\kappa_\ell \in \mathbb{R}$. The condition defining S_0 means that the sum of the values of φ over any K_1-orbit in H_0 is zero; denoting by q the valence of $\mathcal{T}$, this implies that $\kappa_\ell = 0$ for all $\ell \geq 2$ and $\kappa_0 + (q-2)\kappa_1 = 0$, thus proving the claim.

Thus, let $\varphi_0 \in S_0$ be the unique function such that $\varphi(e) = 1$. To complete the proof we need to show that for all $f \in C_c(G)^{\natural K}$, there exists a constant c_f such that $\varphi * f = c_f \varphi$. To this end, observe that any $f \in C_c(G)^{\natural K}$ is a linear combination of characteristic functions $\mathbf{1}_{K_0}$ and $\mathbf{1}_{K_n \setminus K_{n-1}} := \chi_n$ for $n > 0$. In this notation, we have that

$$\varphi_0 = \mathbf{1}_{K_0} - \frac{1}{q-2} \mathbf{1}_{K_1 \setminus K_0}. \tag{6.5}$$

Moreover, a direct computation shows that

$$\chi_n * \chi_m = \begin{cases} \mu(K_n \setminus K_{n-1})\chi_m & \text{if } n < m, \\ \mu(K_n \setminus K_{n-1})\mathbf{1}_{K_n} - \mu(K_{n-1})\chi_n & \text{if } n = m, \end{cases} \tag{6.6}$$

wherein μ is the Haar measure implicit in the chosen convolution structure on $C_c(G)$. Now it follows that

$$\begin{aligned} \varphi_0 * \chi_m &= 0 \quad \text{if } m > 1, \\ \varphi_0 * \chi_1 &= -\mu(K_0)\varphi_0, \\ \varphi_0 * \mathbf{1}_{K_0} &= \mu(K_0)\varphi_0, \end{aligned} \tag{6.7}$$

and hence shows that φ_0 is spherical.

Finally, observe that since φ_0 is compactly supported, it follows from the identity

$$\varphi_0 * \check{\varphi}_0 = c_{\check{\varphi}_0}\varphi_0, \tag{6.8}$$

where as usual $\check{\varphi}_0(x) := \varphi_0(x^{-1})$, that φ_0 is positive definite. ∎

Proof of **Proposition 6.2**. There is no loss of generality in assuming $x = c(0)$, so that $P_x = K_0$. Let π_j be a representative of the unique equivalence class in $\widehat{N}^1_j$ and observe that $a_*\pi_j \cong \pi_{j-1}$. Thus the N-representation $\pi := \oplus_{j\in\mathbb{Z}}\pi_j$ extends canonically to a P-representation. To verify that π is irreducible, observe that if σ is any sub-P-representation of π, then $\sigma|_N$ is a direct sum of sub-N-representations σ_j of π_j, because the π_j are irreducible and pairwise inequivalent. Therefore, each σ_j is either zero or irreducible. The a-invariance of σ shows that either σ is zero or it coincides with π, hence π is irreducible as a P-representation. The uniqueness of π follows from **Lemma 6.5** and the existence of a K_0-fixed vector by construction. ∎

Proof of **Theorem 6.1**

Existence. Let $\pi : P \to \mathbf{O}(E)$ be the continuous orthogonal representation of P constructed in **Proposition 6.2** with underlying Hilbert space E. Let $\rho := |\chi| \otimes \pi \otimes \chi^{-1}$ and endow E with the P-action defined by $\chi \otimes \rho$. Since we have by construction that $E^{K_{-1}} \ominus E^{K_0} \neq 0$, then **Proposition 5.2**(i) implies that $H^1_c(P, \chi \otimes \rho) \neq 0$ and hence **Proposition 4.4** (with $G = P$) gives us a representation $\lambda : P \to \mathbf{O}_L(Q) \subseteq \mathbf{O}(Q)$, so that $(\chi \otimes \rho)|_N = \pi|_N$ acting on L by χ. By **Proposition 4.3**(iii), one can extract the corresponding "irreducible" part and hence the existence is proved.

Uniqueness. We set $L_+ = L$. Since $|\chi(a)| \neq 1$, $\lambda(a)$ is hyperbolic; let L_- be the isotropic line representing the repelling fixed point and let $\rho : P \to \mathbf{O}(E)$ be the orthogonal representation obtained as in **Section 4.1** *via* the identification $E \to {}^{\perp}L_+/L_+$ and $\eta \in H^1_c(P, \chi \otimes \rho)$ the cohomology class defined by the above action. The irreducibility hypothesis and

Propositions 4.7 and 5.2 then imply that

$$E = \overline{\langle f_{\sigma(v)}(p) : p \in P \rangle}, \tag{6.9}$$

where $f_{\sigma(v)}$ is the standard cocycle representing η and moreover E is the cyclic subspace generated by $v \in E^{K_{-1}} \ominus E^{K_0}$.

Consider the orthogonal representation $\psi := |\chi|^{-1} \otimes \chi \otimes \rho$ on E and let π denote the orthogonal representation of P given by **Proposition 6.2**. Then

$$\psi = m\pi \oplus \pi_1 \oplus \pi_2, \tag{6.10}$$

where $\pi_1|_N = \mathbf{1}$ and π_2 does not have any nonzero K_n-invariant vectors. Since $v \in E^{K_{-1}} \ominus E^{K_0}$, the projections of v to the components of π_1 and π_2 are zero. Being a cyclic vector, this implies $\psi = m\pi$ and therefore also $m = 1$. It now follows that $\rho = |\chi| \otimes \chi^{-1} \otimes \pi$; in the notation of **Proposition 5.2**, $\tau = |\chi| \otimes \pi$. By **Proposition 5.2**, we have $H^1_c(P, \tau) \cong E^{K_{-1}} \ominus E^{K_0}$. Let now $(\pi_n, \mathcal{H}_n)$ be the unique K_n-spherical representation of N that is not K_{n+1}-spherical. Then it follows from **Lemma 6.5** that $\tau|_N = \pi|_N = \oplus_{n\in\mathbb{Z}}\pi_n$. Hence,

$$E^{K_{-1}} = \widehat{\bigoplus_{n\geq -1}} \mathcal{H}_n^{K_{-1}}, \qquad E^{K_0} = \widehat{\bigoplus_{n\geq 0}} \mathcal{H}_n^{K_0}. \tag{6.11}$$

Observing that $\mathcal{H}_n^{K_{-1}} = \mathcal{H}_n^{K_0}$ for all $n \geq 0$, we deduce that $E^{K_{-1}} \ominus E^{K_0}$ has dimension one. Hence $\dim H^1_c(P, \tau) = 1$, which implies now by **Proposition 4.4** that, up to conjugation, $\lambda : P \to \mathbf{O}_L(Q)$ is completely determined by χ. ■

7 Representations of G into $\mathbf{O}(Q)$

7.1

In this section, $\mathcal{T}$ denotes a regular or biregular tree of finite bivalency (r, s) with $r, s \geq 3$. As usual, we endow $\mathrm{Aut}(\mathcal{T})$ with the locally compact topology of uniform convergence on finite sets of vertices. A subgroup $G < \mathrm{Aut}(\mathcal{T})$ satisfies the property T_2^+ if for every $\xi_1 \neq \xi_2$ in $\partial\mathcal{T}$ and $\eta_1, \eta_2 \in \partial\mathcal{T} \setminus \{\xi_1, \xi_2\}$ such that the distance between the projections of η_1 and η_2 on the geodesic $[\xi_1, \xi_2]$ is even, there exists $h \in G$ fixing ξ_1, ξ_2 and $h(\eta_1) = \eta_2$.

This property is implied by triple transitivity of the G-action on $\partial\mathcal{T}$ and implies double transitivity. The main result of this section is the following.

Theorem 7.1. Let $G < \mathrm{Aut}(\mathcal{T})$ be a closed subgroup satisfying property T_2^+, $\xi \in \partial\mathcal{T}$ and P the stabilizer of ξ in G. Let $(\mathcal{H}, Q)$ be a strongly nondegenerate quadratic space of index 1 and $\pi : G \to \mathbf{O}(Q)$ a continuous, nonelementary representation. Then $\pi|_P$ has an irreducible indefinite component $\mathcal{H}_{1,P}$ and the canonical orthogonal decomposition $\mathcal{H} = \mathcal{H}_{1,P} \oplus \mathcal{H}_{0,P}$ is G-invariant.

Let $\pi_1, \pi_2 : G \to \mathbf{O}(Q)$ be nonelementary continuous representations such that $\pi_1|_P = \pi_2|_P$. Then the restriction of π_1 and π_2 to the indefinite irreducible components of P coincide. □

7.2

Let G be a locally compact group boundedly generated by $\{s\} \cup P$, where $s \in G$ and $P < G$ is a closed subgroup with the structure considered in Sections 5; assume further that

(1) $\langle s \rangle$ is relatively compact,

(2) $\{a^n s a^n s^{-1} : n \geq 1\}$ is relatively compact.

(We recall that a group G is said to be *boundedly generated* by a subset $X \subseteq G$ if there is $n \in \mathbb{N}$ with $X^n = G$.)

Proposition 7.2. Let X be a complete CAT(-1) space and $G \times X \to X$ a continuous, nonelementary isometric action. Then a acts as a hyperbolic element, s exchanges both fixed points of a, and $(\partial X)^P$ is the attracting fixed point of a.

Proof. We use Lemma 5.1 and we distinguish the three cases for a.

a is elliptic. Then $X^P \neq \varnothing$ and since $\langle s \rangle$ is relatively compact and G is boundedly generated by $\langle s \rangle$ and P, the G-orbits in X are bounded. Hence $X^G \neq \varnothing$, contradicting nonelementarity.

a is parabolic. Then $\partial X^P = \{\xi\}$. Let $\{n_i\}_{i \geq 1}$ be a sequence such that $a^{n_i} x \to \xi$; then $a^{-n_i} x \to \xi$, and since $\{a^n s a^n s^{-1} : n \geq 1\}$ is bounded, we have that $s a^{n_i} s^{-1} x \to \xi$ which, in view of the fact that $a^{n_i} s^{-1} x \to \xi$, implies that $s(\xi) = \xi$ and hence $G\xi = \xi$, a contradiction, thus leading to a being hyperbolic.

a is hyperbolic. . Let now ξ_-, ξ_+ be, respectively, the repelling and the attracting fixed points of a on ∂X. Since $a^n x \to \xi_+$, $a^{-n} x \to \xi_-$, and $\{a^n s a^n s^{-1} : n \geq 1\}$ is bounded, we deduce that $s a^n s^{-1} x \to \xi_-$ and hence $s(\xi_+) = \xi_-$ and $s(\xi_-) = \xi_+$.

Finally, we know $(\partial X)^P \ni \xi_+$ from Lemma 5.1; since $(\partial X)^{\langle a \rangle} = \{\xi_\pm\}$, it remains only to observe that ξ_- is not P-fixed, since otherwise the set $\{\xi_\pm\}$ would be preserved by G. ■

7.3

Let now $G < \mathrm{Aut}(\mathcal{T})$ be any closed subgroup which acts doubly transitively on $\partial\mathcal{T}$. Then (see [4, Sections 4.1 and 0.4])

(i) for every $\xi \in \partial\mathcal{T}$, the Busemann character $\chi_\xi : G_\xi \to \mathbb{Z}$ has image $\mathbb{Z}$ or $2\mathbb{Z}$ depending on whether G is vertex transitive or not;

(ii) for every geodesic $c : \mathbb{Z} \to \mathcal{T}$, there is $s \in G$ and $n_0 \in \mathbb{Z}$ with $s(c(\pm\infty)) = c(\mp\infty)$ and $s(c(n_0)) = c(n_0)$;

(iii) for any $\xi \neq \eta$ in $\partial\mathcal{T}$ and $s \in G$ exchanging ξ and η, we have $G = G_\xi \cup G_\xi s G_\xi$.

Lemma 7.3. Let $G < \mathrm{Aut}(\mathcal{T})$ be a closed subgroup satisfying T_2^+, $c : \mathbb{Z} \to \mathcal{T}$ a geodesic, $\xi_\pm = c(\pm\infty)$, and $S = \{s \in G : s(\xi_\pm) = (\xi_\mp), s(c(0)) = c(0)\}$. Let also $K_j = G_\xi \cap G(c(j))$. Then, given $j \in \mathbb{Z}$, $n \in K_j \setminus K_{j-1}$, and $s \in S$, there exists $h \in G$ such that

(i) for all $q \in \mathbb{Z}$, $hc(q) = c(q - 2j)$;

(ii) for all $s', s'' \in S$, $s'nshns'' \in K_{-j} \setminus K_{-j-1}$. □

Proof. Since $n \in K_j \setminus K_{j-1}$, then $\mathrm{Pr}_{[\xi_+,\xi_-]}(n\xi_-) = c(j)$ and $\mathrm{Pr}_{[\xi_+,\xi_-]}(s^{-1}n^{-1}\xi_-) = c(-j)$. Thus property T_2^+ implies that there exists $h \in G_{\xi_+} \cap G_{\xi_-}$ such that $hn\xi_- = s^{-1}n^{-1}\xi_-$ so that (i) follows. Let now $s', s'' \in S$ and set $g = s'nshns''$. Then we have

(i) $g\xi_+ = s'nshns''\xi_+ = s'nshn\xi_- = s'nss^{-1}n^{-1}\xi_- = s'\xi_- = \xi_+$.

(ii) $gc(-j) = s'nshns''c(-j) = s'nshnc(j) = s'nshc(j) = s'nsc(-j) = s'nc(j) = s'c(j) = c(-j)$. Thus $g \in K_{-j}$.

(iii) $gc(-j-1) = s'nshns''c(-j-1) = s'nshnc(j+1) = s'nshc(j+1) = s'nsc(-j+1) = s'nc(j-1)$. But since $n \in K_j$, we have that $nc(j-1) \neq c(j+1)$ and hence $s'nc(j-1) \neq c(-j-1)$. Thus $g \notin K_{-j-1}$. ■

7.4

We are finally ready to give the following proof.

Proof of Theorem 7.1. Let $\pi : G \to \mathbf{O}(Q) \to \mathrm{Isom}(\mathbf{H})$ be a continuous nonelementary action. Being the stabilizer of $\xi \in \partial\mathcal{T}$, P has the structure of Section 5; we will use the corresponding notation for a. In view of Section 7.3, we can choose s and a parametrisation $c : \mathbb{Z} \to \mathcal{T}$ of the axis of a such that $sc(0) = c(0)$ and $sc(\pm\infty) = c(\mp\infty)$. Observe further that these notations also put us in the setting of Section 7.2. By Proposition 7.2, $\pi(a)$ is hyperbolic and hence cannot fix a point in $\mathbf{H}$ or preserve any horosphere. It follows from Proposition 4.3 applied to P that $\pi|_P$ has an irreducible indefinite component $\mathcal{H}_{1,P}$.

We need to show that $\mathcal{H}_{1,P}$ is G-invariant and that, on $\mathcal{H}_{1,P}$, the representation π is determined by $\pi|_P$. Let $L_\pm$ be the attracting/repelling fixed points of $\pi(a)$ and let

$L = L_+ \oplus L_-$, $F = {}^{\perp}L$. We are in the setting of Proposition 4.3 for P (instead of G) and we adopt its notation. By Proposition 4.7, $\mathcal{H}_{1,P} = L \oplus j^{-1}(I_\eta)$. By Proposition 7.2, $\pi(s)$ exchanges $L_\pm$. It is enough to show that $\pi(s)$ preserves $j^{-1}(I_\eta)$ and that its restriction to $\mathcal{H}_{1,P}$ is determined by $\pi|_P$.

We adopt the notation of Appendix A with $\ell_\pm$. Then

$$\pi(s) = \begin{pmatrix} 0 & \mu & 0 \\ \mu^{-1} & 0 & 0 \\ 0 & 0 & \pi_0(s) \end{pmatrix}, \tag{7.1}$$

where $\pi_0(s)$ is orthogonal and $\mu \in \mathbb{R}^*$.

Fix $j \in \mathbb{Z}$, $n \in K_j \setminus K_{j-1}$, and h as in Lemma 7.3 such that $g := snshns \in K_{-j} \setminus K_{-j-1}$. We write

$$\pi(n) = \begin{pmatrix} \chi(n) & \alpha(n) & N_2^+ \\ 0 & \chi(n)^{-1} & 0 \\ 0 & N_3^- & \pi_0(n) \end{pmatrix}, \tag{7.2}$$

and likewise

$$\pi(g) = \begin{pmatrix} \chi(g) & \alpha(g) & M_2^+ \\ 0 & \chi(g)^{-1} & 0 \\ 0 & M_3^- & \pi_0(g) \end{pmatrix}. \tag{7.3}$$

As to $\pi(h)$, since it fixes both $L_\pm$, it is of the form

$$\pi(h) = \begin{pmatrix} \chi(h) & 0 & 0 \\ 0 & \chi(h)^{-1} & 0 \\ 0 & 0 & \pi_0(h) \end{pmatrix}. \tag{7.4}$$

Computing $g = snshns$, we find

$$\chi(g) = \mu^{-1}\chi\big(n^{-1}h\big)\alpha(n), \tag{7.5}$$

$$M_3^- = \chi(hn)\pi_0(s)N_3^-. \tag{7.6}$$

Equation (7.5) shows that μ is determined in terms of $\pi|_P$. We are left to determine $\pi_0(s)$. Write f for the standard cocycle associated to η; then (7.6) gives

$$\pi_0(s)f(n) = \chi(gh)^{-1}f(g). \tag{7.7}$$

We obtain such a formula for every $n \in N \setminus \cap_{j\in\mathbb{Z}}K_j$. Thus we are done since by Proposition 5.2(ii), I_η is spanned by these $f(n)$. ■

Proof of Theorem 1.3. By Proposition 3.4, we are reduced to study homomorphisms from G into $\mathbf{O}_+(Q)$ and can thus apply the results obtained so far. The existence statement in Theorem 1.3 follows from Theorem 1.1 and Proposition 4.3(iii).

For the uniqueness part, let $\xi \in \partial\mathcal{T}_r$, P the stabilizer of ξ in G, and $a \in P$ a hyperbolic element with attracting fixed point ξ and translation length 1. Let $\pi : G \to \mathbf{O}_+(Q)$ be a nonelementary continuous representation. Let $c : \mathbb{Z} \to \mathcal{T}_r$ be a parametrization of the axis of a, and let K_n be, as usual, the stabilizer in P of $c(n)$. Using that G is doubly transitive on $\partial\mathcal{T}_r$, we see that any other hyperbolic element b with translation length 1 is conjugate to an element of the form $a \cdot k$, where $k \in \cap_{n\in\mathbb{Z}}K_n$. Since $\pi(a)$ is hyperbolic (Proposition 7.2), and using that a normalises $\cap_{n\in\mathbb{Z}}K_n$, one sees that $\pi(k)$ fixes pointwise the axis of $\pi(a)$ and hence $\pi(a)$ and $\pi(b)$ have the same translation length, say ℓ_π. Let L_+ be the attractive fixed point of $\pi(a)$ and χ the character by which P acts on L_+. Then, since π takes values in $\mathbf{O}_+(Q)$, we have that χ takes values in $\mathbb{R}_+$ which implies first that $\chi(a) = e^{\ell_\pi}$ and then that χ is trivial on N, since N is an increasing union of compact groups. This shows that χ is completely determined by ℓ_π.

Assume now that π is irreducible. Then Theorem 7.1 implies that $\pi|_P$ is irreducible, Theorem 6.1 that it is completely determined by χ (and hence by ℓ_π), and Theorem 7.1 again that π is completely determined by ℓ_π. ■

8 Explicit constructions

8.1

Let $\mathcal{T}$ be any simplicial tree. Let $\alpha+1$ be the cardinal of the vertex set V of $\mathcal{T}$ and let $(\mathcal{H}, Q)$ be a strongly nondegenerate quadratic space of signature $(\alpha, 1)$; let $\mathbf{H}$ be the corresponding hyperbolic space. Denote by G the (abstract) group $G = \mathrm{Aut}(\mathcal{T})$. We denote by d both the metric on $\mathbf{H}$ and the metric on (the geometric realization of) $\mathcal{T}$ that gives unit-length edges.

Theorem 8.1. For every $\lambda > 1$, there is an embedding $\Psi : \mathcal{T} \to \mathbf{H}$ and a representation $\pi : G \to \mathbf{O}(Q) \to \mathrm{Isom}(\mathbf{H})$ such that

(i) the map Ψ is G-equivariant for π,

(ii) $\lambda^{d(x,y)} = \cosh d(\Psi x, \Psi y)$ for any two vertices x, y of $\mathcal{T}$,

(iii) Ψ extends to an equivariant boundary map $\partial\Psi : \partial T \to \partial\mathbf{H}$ which is a homeomorphism onto its image,

(iv) $\Psi(V)$ is cobounded in the convex hull $\mathcal{C} \subseteq \mathbf{H}$ of the image of $\partial\Psi$. □

Remark 8.2. The formula in (ii) shows in particular that $d(\Psi x, \Psi y)$ is asymptotically proportional to $d(x, y)$. If we denote by $b^{\mathcal{T}}$ the Busemann cocycle for $\mathcal{T}$ and b is the one for $\mathbf{H}$ as in Section 3, we have

$$b_{\Psi\xi}(\Psi x, \Psi y) = b^{\mathcal{T}}_{\xi}(x, y) \ln\lambda \quad \forall \xi \in \partial\mathcal{T},\ \forall x, y \in V. \tag{8.1}$$

We can give right away the construction of Ψ; the remainder of the section will be devoted to proving the properties stated in Theorem 8.1.

Fix a vertex $w \in V$. By Proposition 2.7, we may identify $\mathcal{H}$ with $\ell^2(V)$ in such a way that the bilinear form B associated to Q reads

$$B(f, g) = \sum_{v \in V, v \neq w} f(v)g(v) - f(w)g(w). \tag{8.2}$$

We define a map $V \to \mathcal{H}$, $v \mapsto f_v$ as follows. Denote for $u \in V$ by δ_u the unit function supported on u; then

$$f_v := \lambda^{d(w,v)}\delta_w + \sqrt{\lambda^2 - 1} \sum_{k=1}^{d(w,v)} \lambda^{d(w,v)-k}\delta_{u_k}, \tag{8.3}$$

where $w, u_1, u_2, \ldots, u_{d(w,v)} = v$ is the geodesic from w to v (it is understood that the right-hand side summation is zero when $v = w$). A computation gives $Q(f_v) = -1$ so that f_v is in the negative cone C_-; now Ψ is the resulting map $V \to C_- \to \mathbf{H}$ extended to $\mathcal{T}$ by sending each edge to a geodesic segment. Each element $\xi \in \partial\mathcal{T}$ can be realized by a unique geodesic ray of vertices $\{v_k\}_{k=0}^{\infty}$ with $v_0 = w$; we define $(\partial\Psi)(\xi)$ by considering the element

f_ξ of the isotropic cone C_0 given by

$$f_\xi := \delta_w + \sqrt{\lambda^2 - 1}\sum_{k=1}^{\infty} \lambda^{-k}\delta_{v_k}. \tag{8.4}$$

Observe that one obtains a multiple of f_{v_k}, hence the same point Ψv_k, by truncating the above sum at k. It follows that the resulting map $\overline{\mathcal{T}} \to \overline{\mathbf{H}}$ is continuous; claim (iii) now follows from Remark 8.2 and claim (i). The formula of claim (ii) can be verified by inspection; however, we will see that it can be reduced to the obvious case $u = w$.

The strategy for the proof of Theorem 8.1 is first to construct π in the case where $\mathcal{T}$ is *regular*, that is, G acts transitively on V. The general case will follow by the naturality of our construction with respect to the pointed tree $(\mathcal{T}, w)$.

8.2 Regular case

We assume that $G = \mathrm{Aut}(\mathcal{T})$ acts transitively on V. Fix a neighbour z of w and denote by $K < G$ the stabilizer of w, by L the subgroup of G preserving the set $\{w, z\}$, and by $E = K \cap L$ the pointwise stabilizer of $\{w, z\}$. The K-action on V preserves B and hence induces a representation $\pi : K \to \mathbf{O}(Q)$ by $\pi(g)\delta_u = \delta_{gu}$ $(g \in K)$. We extend now π by defining $\pi : L = E \sqcup (L \setminus E) \to \mathbf{O}(Q)$ as follows: the map is already defined on $E = K \cap L$; for every $g \in L \setminus E$ and every $u \in V$, set

$$\pi(g)\delta_u := \begin{cases} \lambda\delta_w + \sqrt{\lambda^2 - 1}\delta_z & \text{if } u = w, \\ -\sqrt{\lambda^2 - 1}\delta_w - \lambda\delta_z & \text{if } u = z, \\ \delta_{gu} & \text{otherwise.} \end{cases} \tag{8.5}$$

It is a matter of computation to verify that $\pi(g)$ is in $\mathbf{O}(Q)$.

Proposition 8.3. The map $\pi : L \to \mathbf{O}(Q)$ extends uniquely to a homomorphism $\pi : G \to \mathbf{O}(Q)$. □

Proof. We start by showing that the map $\pi : L \to \mathbf{O}(Q)$ is a homomorphism; that is, we need to verify that $\pi(g)\pi(g') = \pi(gg')$ holds on L, which we do by discussing the cases according to where g, g' are in the coset decomposition $L = E \sqcup (L \setminus E)$. There is nothing to do if g, g' are both in E since π is a homomorphism on K. We will write out the verification in the case $g, g' \in L \setminus E$; the two remaining cases are simpler and similar. Let thus $g, g' \in L \setminus E$. Then $\pi(gg')\delta_u = \delta_{gg'u}$ for all $u \in V$ since $gg' \in E$ (as E has index two in L). On the other hand, we have the following cases.

Case 8.4 ($u = w$).

$$\begin{aligned}\pi(g)\pi(g')\delta_w &= \pi(g)\big(\lambda\delta_w + \sqrt{\lambda^2-1}\delta_z\big)\\ &= \lambda\big(\lambda\delta_w + \sqrt{\lambda^2-1}\delta_z\big) + \sqrt{\lambda^2-1}\big(-\sqrt{\lambda^2-1}\delta_w - \lambda\delta_z\big)\\ &= \delta_w,\end{aligned} \tag{8.6}$$

which is indeed $\delta_{gg'u}$ since E fixes w.

Case 8.5 ($u = z$).

$$\begin{aligned}\pi(g)\pi(g')\delta_z &= \pi(g)\big(-\sqrt{\lambda^2-1}\delta_w - \lambda\delta_z\big)\\ &= -\sqrt{\lambda^2-1}\big(\lambda\delta_w + \sqrt{\lambda^2-1}\delta_z\big) - \lambda\big(-\sqrt{\lambda^2-1}\delta_w - \lambda\delta_z\big)\\ &= \delta_z,\end{aligned} \tag{8.7}$$

which is indeed $\delta_{gg'u}$ since E fixes z.

Case 8.6 ($u \neq w, z$). Then we have also $g'u \neq w, z$ and hence

$$\pi(g)\pi(g')\delta_u = \pi(g)\delta_{g'u} = \delta_{gg'u} = \pi(gg')\delta_u. \tag{8.8}$$

To show that the map π defined on L extends uniquely to a homomorphism on G, observe that the G-action on (the first barycentric subdivision of) $\mathcal{T}$ determines a splitting of G into an amalgamation $G = K \ast_E L$. Therefore, the statement follows from the universal property of amalgamations. ■

Remark 8.7. The definition of π on G is independent of the choice of z: indeed, observe first that K acts transitively on the set of neighbours of w. If k is any element of K, we obtain another neighbour kz of w and another amalgamation $G = K \ast_{kEk^{-1}} kLk^{-1}$. With π defined as before using L, one checks immediately that for $g \in kLk^{-1} \setminus kEk^{-1}$, the formula (8.5) for $\pi(g)$ remains valid upon replacing z with kz.

We turn to Theorem 8.1(i). Pick a vertex $v \in V$ and let $n = d(v, w)$. There is a hyperbolic element a of translation length one admitting an axis $\{u_k\}_{k\in\mathbb{Z}}$ such that $u_0 = w$, $u_n = v$, and $au_k = u_{k+1}$ for all $k \in \mathbb{Z}$. Notice that K and a generate G; since moreover Ψ is K-invariant by its construction, we need only verify that $\Psi(av) = \pi(a)\Psi(v)$. By Remark 8.7, there is no loss of generality in assuming that $z = u_1$. Let s be an element of $L \setminus E$ preserving $\{u_k\}$; that is, $su_k = u_{1-k}$ for all $k \in \mathbb{Z}$. Then $a = st$ for $t \in K$ such that $tu_k = u_{-k}$ for all $k \in \mathbb{Z}$, and thus an immediate computation using (8.5) for $\pi(s)$ shows that we have for all $k \in \mathbb{Z}$,

$$\pi(a)\delta_{u_k} = \pi(s)\pi(t)\delta_{u_k} = \begin{cases} \lambda\delta_w + \sqrt{\lambda^2-1}\delta_{u_1} & \text{if } k = 0, \\ -\sqrt{\lambda^2-1}\delta_w - \lambda\delta_{u_1} & \text{if } k = -1, \\ \delta_{u_{k+1}} & \text{otherwise.} \end{cases} \tag{8.9}$$

Now we can compute

$$\begin{aligned} \pi(a)f_v &= \pi(a)\left(\lambda^n\delta_w + \sqrt{\lambda^2-1}\sum_{k=1}^{n}\lambda^{n-k}\delta_{u_k}\right) \\ &= \lambda^n\left(\lambda\delta_w + \sqrt{\lambda^2-1}\delta_{u_1}\right) + \sqrt{\lambda^2-1}\sum_{k=1}^{n}\lambda^{n-k}\delta_{u_{k+1}} \\ &= \lambda^{n+1}\delta_w + \sqrt{\lambda^2-1}\sum_{k=1}^{n+1}\lambda^{n+1-k}\delta_{u_k} \\ &= f_{u_{k+1}} = f_{av}, \end{aligned} \tag{8.10}$$

and claim (i) is proved. Now the transitivity of G reduces (ii) to the case where one of the vertices is w, which is an immediate computation. As (iii) was addressed before, we are left with proving (iv).

Proposition 8.8. Every point $x \in \mathcal{C}$ is at distance at most $\cosh^{-1}\sqrt{1+\lambda}$ of some element of $\Psi(V)$. □

Proof. Every element of $\mathbf{H}$ is represented by a unique function in $\ell^2(V)$ with value one on w. In fact, if D' denotes the unit ball in $\ell^2(V\backslash\{w\})$, the set of such functions is $D := \delta_w + D'$. This gives the *Klein model* in finite dimensions, and thus it follows that geodesics in $\mathbf{H}$ correspond to affine lines in D. It is therefore enough to show the claim for any finite convex combination $x = \sum_{\xi\in S} s_\xi f_\xi$, where $S \subseteq \partial\mathcal{T}$ is a finite set and $s : S \to (0,1)$ is any function $\xi \mapsto s_\xi$ of sum one (observe that S must contain at least two points since $0 < s_\xi < 1$).

We write $\beta = b(\cdot, \delta_w)$ for Busemann function on $\mathbf{H}$ normalised at δ_w and $\beta^{\mathcal{T}} = b^{\mathcal{T}}(\cdot, w)$ for the analogous Busemann function on $\mathcal{T}$. Consider the function $\psi_s : V \to \mathbb{R}_+^*$ defined by

$$\psi_s(v) := \sum_{\xi\in S} s_\xi \lambda^{\beta_\xi^{\mathcal{T}}(v)}. \tag{8.11}$$

This function admits a minimum $v_0 \in V$; indeed, outside a finite subset of V determined by the configuration of S, it increases monotonically with the distance to this subset. We will see that $\Psi(v_0)$ is at distance at most $\cosh^{-1}\sqrt{1+\lambda}$ of x.

Pick $g \in G$ such that $gv_0 = w$; then w is a minimum of the function $v \mapsto \psi_s(g^{-1}v)$, which in view of $\beta^{\mathcal{T}}_{\xi}(g^{-1}v) = \beta^{\mathcal{T}}_{g\xi}(v) - \beta^{\mathcal{T}}_{g\xi}(gw)$ reads

$$\psi_s\big(g^{-1}v\big) = \sum_{\eta \in gS} s_{g^{-1}\eta} \lambda^{\beta^{\mathcal{T}}_{\eta}(v) - \beta^{\mathcal{T}}_{\eta}(gw)}. \tag{8.12}$$

Thus, setting $(g_\star s)_\xi := \lambda^{-\beta^{\mathcal{T}}_{\xi}(gw)} s_{g^{-1}\xi}$, it follows that w is also a minimum of the function $\psi_{s'}$ for $s' := g_\star s / \sum_\xi (g_\star s)_\xi$. Setting

$$\sigma_v := \sum \big\{ s'_\xi : v \text{ is in the ray } [w, \xi] \big\} \quad \forall v \in V, \tag{8.13}$$

the minimality implies for every neighbour v of w,

$$\begin{aligned} 1 &= \psi_{s'}(w) \le \psi_{s'}(v) \\ &= \lambda^{-1} \sum \big\{ s'_\xi : v \in [w, \xi] \big\} + \lambda \sum \big\{ s'_\xi : v \notin [w, \xi] \big\} \\ &= \lambda^{-1} \sigma_v + \lambda\big(1 - \sigma_v\big), \end{aligned} \tag{8.14}$$

and hence $\sigma_v \le \lambda/(1+\lambda)$. This in turn implies

$$\sigma_v \le \frac{\lambda}{1+\lambda} \quad \forall v \in V,\ v \ne w. \tag{8.15}$$

Formula (3.6) shows that for every $g \in G$ and $h \in C_0$, we have

$$\beta_{gh}\big(g\delta_w\big) = \ln \frac{B\big(gh, g\delta_w\big)\sqrt{-Q\big(\delta_w\big)}}{B\big(gh, \delta_w\big)\sqrt{-Q\big(g\delta_w\big)}} = \ln \frac{B\big(h, \delta_w\big)}{B\big(gh, \delta_w\big)} = \ln \frac{h(w)}{(gh)(w)}. \tag{8.16}$$

Thus we have for all $\xi \in \partial\mathcal{T}$,

$$\big(gf_\xi\big)(w) f_\xi(w) e^{-\beta_{gf_\xi}(g\delta_w)} = f_\xi(w) \lambda^{-\beta^{\mathcal{T}}_{g\xi}(gw)} \tag{8.17}$$

and since gf_ξ is proportional to $f_{g\xi}$, we deduce $gf_\xi = \lambda^{-\beta^{\mathcal{T}}_{g\xi}(gw)} f_{g\xi}$. We conclude that gx is represented in $\mathbf{H}$ by the element $y := \sum_{\xi \in gS} s'_\xi f_\xi$ of D. We proceed now to compute

$$\cosh d\big(\delta_w, y\big) = -\frac{B\big(\delta_w, y\big)}{\sqrt{Q\big(\delta_w\big) Q(y)}} = \frac{1}{\sqrt{-Q(y)}}. \tag{8.18}$$

If S_k is the sphere of radius k around w, we deduce from the definition of f_ξ that

$$y = \delta_w + \sqrt{\lambda^2 - 1} \sum_{k=1}^{\infty} \sum_{v \in S_k} \lambda^{-k} \sigma_v \delta_v \tag{8.19}$$

and therefore

$$Q(y) = -1 + \big(\lambda^2 - 1\big) \sum_{k=1}^{\infty} \lambda^{-2k} \sum_{v \in S_k} \sigma_v^2 \le -1 + \big(\lambda^2 - 1\big) \frac{\lambda}{1+\lambda} \sum_{k=1}^{\infty} \lambda^{-2k} \sum_{v \in S_k} \sigma_v. \tag{8.20}$$

Using $\sum_{v \in S_k} \sigma_v = 1$, one finally gets $Q(y) \le -1/(1+\lambda)$. It follows that

$$d\big(\Psi v_0, x\big) = d(\Psi w, gx) = d\big(\delta_w, y\big) \le \cosh^{-1} \sqrt{1+\lambda}. \tag{8.21}$$

This proposition completes the proof of **Theorem 8.1** in the regular case. ■

8.3 General case

Suppose now that $\mathcal{T}$ is a general tree with vertex set V of cardinal $\alpha + 1$ and $G = \mathrm{Aut}(\mathcal{T})$. Complete $\mathcal{T}$ to a regular tree $\mathcal{T}'$ with vertex set $V' \supseteq V$ of cardinal $\alpha' + 1$. We keep the notation of **Section 8.1** and define likewise $\mathcal{H}' = \ell^2(V')$, $\mathbf{H}'$, B', $G' = \mathrm{Aut}(\mathcal{T}')$, and so forth; take $w' := w \in V \subseteq V'$ and observe that $\mathbf{H}$ is a hyperbolic subspace of $\mathbf{H}'$. Let Ψ', π' be the maps associated to $\mathcal{T}'$ by the proof for the regular case.

Denote by $L_0 < L < G'$ the pointwise stabilizer, respectively, stabilizer, of $\mathcal{T}$; since any automorphism of $\mathcal{T}$ can be extended to some automorphism(s) of $\mathcal{T}'$, we have a natural identification $G = L/L_0$. It follows at once from the definition of Ψ, Ψ' and π, π' that the restriction of Ψ' gives Ψ, while π' descends to π. In fact, as all definitions for $\mathcal{T}'$ vanish on $V' \setminus V$ when restricted to $\mathcal{T}$, the only part of **Theorem 8.1** for $\mathcal{T}$ that does not follow immediately from the case of $\mathcal{T}'$ is point (iv). But the proof given above, when applied to $\mathcal{T}'$ and to the corresponding convex hull $\mathcal{C}'$, shows in fact that whenever $x \in \mathcal{C} \subseteq \mathcal{C}'$ is a finite affine convex combination of elements in $\Psi(\partial\mathcal{T})$, then the vertex $v_0 \in V'$ is actually in V. This follows indeed from the definition of the function ψ_s and thus concludes the proof of Theorem 8.1.

Appendix

A Matrix representations

Let $(\mathcal{H}, Q)$ a strongly nondegenerate quadratic space of signature $(\alpha, 1)$. If L_-, L_+ are two distinct isotropic lines, define $L := L_+ \oplus L_-$ and $F = {}^{\perp}L$ so that

$$\mathcal{H} = L \oplus F. \tag{A.1}$$

On L and F, we consider the restrictions $B|_{L\times L}$ and $B|_{F\times F}$, and if A is a continuous linear operator between any of these spaces, A^* denotes the adjoint with respect to these strongly nondegenerate bilinear forms. For any continuous linear operator $T : \mathcal{H} \to \mathcal{H}$, define

$$\begin{aligned} A_1 &= p_L T|_L, & A_2 &= P_L T|_F, \\ A_3 &= P_F T|_L, & A_4 &= p_F T|_F. \end{aligned} \tag{A.2}$$

Then $T \in \mathbf{O}(Q)$ if and only if the following conditions are satisfied:

(a) $A_1^* A_1 + A_3^* A_3 = \mathrm{Id}_L$;
(b) $A_2^* A_2 + A_4^* A_4 = \mathrm{Id}_F$;
(c) $A_1^* A_2 + A_3^* A_4 = 0$;
(d) $A_2^* A_1 + A_4^* A_3 = 0$.

Observe that by taking adjoints, the last two conditions are equivalent.

We will look more closely at $\mathbf{O}_{L_+}(Q)$, the stabilizer in $\mathbf{O}(Q)$ of L_+. For this, let $L_\pm = \mathbb{R}\ell_\pm$, with $B(\ell_+, \ell_-) = 1$. We represent A_1 by a two-by-two real matrix; $A_2 : F \to L$ will be represented by two linear forms A_2^+ and A_2^- given by $A_2(e) = A_2^+(e)\ell_+ + A_2^-(e)\ell_-$; $A_3 : L \to F$ will be represented by two vectors $A_3^+ = A_3(\ell_+)$, $A_3^- = A_3(\ell_-)$, and hence

$$T = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A_1}} & A_2^+ \\ & & A_2^- \\ A_3^+ & A_3^- & A_4 \end{pmatrix}. \tag{A.3}$$

Then $T \in \mathbf{O}_{L_+}(Q)$ if and only if it has the form

$$\begin{pmatrix} \lambda & \alpha & A_2^+ \\ 0 & \lambda^{-1} & 0 \\ 0 & A_3^- & A_4 \end{pmatrix} \tag{A.4}$$

with $\lambda \in \mathbb{R}^\times$, $A_3^- \in F$, $A_4 \in \mathbf{O}(F)$, and α, A_2^+ are determined by

$$\begin{aligned} \alpha &= -\frac{\lambda}{2} Q\big(A_3^-\big), \\ A_2^+(v) &= -\lambda B\big(A_4(v), A_3^-\big) \quad \forall v \in F. \end{aligned} \tag{A.5}$$

The inverse of

$$S = \begin{pmatrix} \mu & \beta & B_2^+ \\ 0 & \mu^{-1} & 0 \\ 0 & B_3^- & B_4 \end{pmatrix} \in \mathbf{O}_{L_+}(Q) \tag{A.6}$$

is given by

$$S^{-1} = \begin{pmatrix} \mu^{-1} & \beta & -\mu^{-1} B_2^+ B_4^{-1} \\ 0 & \mu & 0 \\ 0 & -\mu B_4^{-1}\big(B_3^-\big) & B_4^{-1} \end{pmatrix} \tag{A.7}$$

and the conjugate STS^{-1} of T by S has the following entries:

$$\begin{aligned} \big(STS^{-1}\big)_{1,1} &= \lambda = \big(STS^{-1}\big)_{2,2}^{-1}, \\ \big(STS^{-1}\big)_{2,1} &= \big(STS^{-1}\big)_{2,3} = \big(STS^{-1}\big)_{3,1} = 0, \\ \big(STS^{-1}\big)_{1,2} &= \lambda\mu\beta + \mu^2\alpha - \mu^2 A_2^+ B_4^{-1} B_3^- + \lambda^{-1}\mu\beta - \mu B_2^+ A_3^- - \mu B_2^+ A_4 B_4^{-1} B_3^-, \\ \big(STS^{-1}\big)_{1,3} &= -\lambda B_2^+ B_4^{-1} + \mu A_2^+ B_4^{-1} + B_2^+ A_4 B_4^{-1}, \\ \big(STS^{-1}\big)_{3,2} &= \lambda^{-1}\mu B_3 + \mu B_4^{-1} A_3^- - \mu B_4 A_4 B_4^{-1} B_3^-, \\ \big(STS^{-1}\big)_{3,3} &= B_4 A_4 B_4^{-1}. \end{aligned} \tag{A.8}$$

In particular, if $|\lambda| \neq 1$, by choosing $\mu = 1$, $B_4 = \mathrm{Id}$, there exists B_3^- such that

$$A_3^- + \big(\lambda^{-1} - A_4\big)\big(B_3^-\big) = 0. \tag{A.9}$$

Then, using the relations (i) and (ii), one can see that T is conjugate to

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda^{-1} & 0 \\ 0 & 0 & A_4 \end{pmatrix} \tag{A.10}$$

and is hence hyperbolic; conversely, one can show that if $|\lambda| \neq 1$, T is hyperbolic.

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