Bounded cohomology and applications: a panorama

Marc Burger

Bounded cohomology for groups and spaces is related to usual cohomology and in fact enriches it by providing stronger invariants. The aim of this talk is to illustrate certain aspects of this philosophy. General references for bounded cohomology are [12, 21, 22, 3].

1. Definition, low degrees. The continuous cohomology $\mathrm{H}_{\mathrm{c}}^{\bullet}(G,E)$ of a topological group G acting continuously by linear isometries on a Banach space E is defined using the resolution of E by the complex of continuous E-valued cochains on G. Using the subcomplex of continuous cochains which are bounded in the supremum norm leads to the bounded continuous cohomology; it is equipped with a quotient seminorm and comes with a comparison map $\mathrm{H}_{\mathrm{cb}}^{\bullet}(G,E) \to \mathrm{H}_{\mathrm{c}}^{\bullet}(G,E)$. In degree zero both cohomology groups equal E^G. In degree one the comparison map is always injective; while $\mathrm{H}_{\mathrm{c}}^{1}(G,E)$ describes affine isometric G-actions on E with given linear part, $\mathrm{H}_{\mathrm{cb}}^{1}(G,E)$ classifies those with bounded orbits. Starting with degree two this theory really exhibits new phenomena. The kernel $\mathrm{EH}_{\mathrm{cb}}^{2}(G,E)$ of the comparison map admits a description in terms of quasiactions. In the case of trivial coefficients, where we denote the corresponding objects by $\mathrm{H}_{\mathrm{cb}}^{\bullet}(G)$ and $\mathrm{H}_{\mathrm{c}}^{\bullet}(G)$, this kernel $\mathrm{EH}_{\mathrm{cb}}^{2}(G)$ is the quotient of the space

$$\mathrm{QH}(G) := \big\{ f : G \to \mathbb{R} : f \text{ is continuous and } \sup_{x,y} \big| f(xy) - f(x) - f(y) \big| < \infty \big\}$$

of continuous quasimorphisms by the subspace $\mathrm{C}^{\mathrm{b}}(G) \oplus \mathrm{Hom}_{\mathrm{c}}(G,\mathbb{R})$, where $\mathrm{C}^{\mathrm{b}}(G)$ is the space of continuous bounded functions on G.

2. Examples. This interpretation, together with the exploitation of certain hyperbolicity phenomena, leads to nonvanishing results on $\mathrm{H}_{\mathrm{b}}^{2}(\Gamma)$; for instance, $\mathrm{H}_{\mathrm{b}}^{2}(\Gamma)$ is infinite dimensional when Γ is:

(1) a lattice in a real rank one Lie group [14]
(2) Gromov hyperbolic and nonelementary [13],
(3) a subgroup of the mapping class group $\mathcal{M}_g$ for $g \geq 2$ which is not virtually Abelian [2].

Many of these examples are fundamental groups of finite aspherical complexes; this illustrates the fact that there are no simple minded finiteness conditions on Γ ensuring that $\mathrm{H}_{\mathrm{b}}^{2}(\Gamma)$ is finite dimensional; indeed for the free group $\mathbb{F}_r$ on $r \geq 2$ generators, which is inherently a one-dimensional object, $\mathrm{H}_{\mathrm{b}}^{2}(\mathbb{F}_r)$ and $\mathrm{H}_{\mathrm{b}}^{3}(\mathbb{F}_r)$ are infinite dimensional. This seems to be the price to pay for the advantage that bounded cohomology is the receptacle of rather refined invariants as the next section illustrates. Let's however mention that if Γ is amenable $\mathrm{H}_{\mathrm{b}}^{n}(\Gamma) = 0$ for $n \geq 1$.

3. Two important examples.

(1) Bounded Euler class: The Euler class classifies the universal covering of the group $\mathrm{Homeo}^{+}(S^1)$ of orientation preserving homeomorphisms of the circle

S^1; it admits a natural bounded representative $e^{\mathrm{b}} \in \mathrm{H}^2_{\mathrm{b}}\big(\mathrm{Homeo}^+(S^1), \mathbb{Z}\big)$. For a minimal action $\rho : \Gamma \to \mathrm{Homeo}^+(S^1)$, its bounded Euler class $\rho^*(e^{\mathrm{b}}) \in \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{Z})$ is then a complete invariant of conjugacy [16]. Let $e^{\mathrm{b}}_{\mathbb{R}}$ be the bounded class obtained by changing coefficients from $\mathbb{Z}$ to $\mathbb{R}$. Recently the author showed [4] that if $\Gamma < G$ is a lattice in a locally compact (second countable) group G, then for a minimal action which is not conjugate into the group of rotations, $\rho^*(e^{\mathrm{b}}_{\mathbb{R}})$ is in the image of the restriction map $\mathrm{H}^2_{\mathrm{cb}}(G) \to \mathrm{H}^2_{\mathrm{b}}(\Gamma)$ if and only if ρ finitely covers an action which extends continuously to G.

(2) Bounded Kähler class: The integral of the Kähler form on triangles with geodesic sides in a hermitian symmetric space (of noncompact type) gives the bounded Kähler class $\kappa^{\mathrm{b}}_{\mathcal{X}} \in \mathrm{H}^2_{\mathrm{bc}}(G)$, where $G = (\mathrm{Aut}(\mathcal{X}))^\circ$. The bounded Kähler class of a representation $\rho : \Gamma \to G$ is then $\rho^*(\kappa^{\mathrm{b}}_{\mathcal{X}}) \in \mathrm{H}^2_{\mathrm{b}}(\Gamma)$. When $\mathcal{X}$ is irreducible and not of tube type, it is a complete conjugacy invariant for representations with Zariski dense image [5, 10]. This invariant has served to define new types of embeddings between Hermitian symmetric spaces [8] and enters as well in the higher Teichmüller theory developed in [9, 7].

4. The comparison map. It encodes subtle information of algebraic and geometric nature.

(1) Commutator length [1]*:* The stable length ℓ_{st} on the commutator subgroup Γ' of Γ is $\ell_{st}(\gamma) := \lim_{n\to\infty} \|\gamma^n\|/n$, where $\|\cdot\|$ denotes the commutator length. Then ℓ_{st} vanishes identically if and only if $\mathrm{EH}^2_{\mathrm{b}}(\Gamma) = 0$, that is the comparison map in degree two is injective.

(2) Hyperbolicity [19]*:* A finitely generated group is (nonelementary) Gromov hyperbolic if and only if the comparison map $\mathrm{H}^n_{\mathrm{b}}(\Gamma, E) \to \mathrm{H}^n(\Gamma, E)$ is surjective for every Banach Γ-module E and $n = 2$ (or equivalently $n \geq 2$).

(3) Measure equivalence [24, 20]*:* It preserves the property that $\mathrm{H}^2_{\mathrm{b}}\big(\Gamma, ell^2(\Gamma)\big) \neq 0$. The latter holds for all (nonelementary) Gromov hyperbolic groups.

(4) Higher rank [11, 23]*:* The comparison map is injective with image the G-invariant classes if $\Gamma < G$ is an irreducible lattice in a connected semisimple Lie group G with finite center and rank $\mathrm{rk}_G \geq 2$. This is also implied by the recent result [23] that $\mathrm{H}^2_{\mathrm{cb}}(G) \to \mathrm{H}^2_{\mathrm{b}}(\Gamma)$ is an isomorphism for $n < 2\,\mathrm{rk}_G$. Together with 2(3), 3(1) and 4(1) we obtain that:

- any homomorphism $\Gamma \to \mathcal{M}_g$ has finite image [18];
- any Γ-action by homeomorphisms of S^1 has a finite orbit [17];
- the stable length on commutators vanishes.

(5) Geometry of central extensions [15]*:* If a class α is in the image of the comparison map $\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{Z}) \to \mathrm{H}^2(\Gamma, \mathbb{Z})$ then the associated central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_\alpha \longrightarrow \Gamma \longrightarrow e$$

is quasiisometric to $\Gamma \times \mathbb{Z}$; here Γ is a finitely generated group.

(6) Differential forms [6]*:* For a symmetric space of noncompact type $\mathcal{X}$ and a discrete subgroup $\Gamma < \mathrm{Iso}(\mathcal{X})$. there is a factorization

$$\begin{array}{ccc} \mathrm{H}^\bullet_{\mathrm{b}}(\Gamma) & \longrightarrow & \mathrm{H}^\bullet(\Gamma) = \mathrm{H}^\bullet_{\mathrm{dR}}(\Gamma\backslash\mathcal{X}) \\ & \searrow \quad \nearrow & \\ & \mathrm{H}_{(\infty)}(\Gamma\backslash\mathcal{X}) & \end{array}$$

of the comparison map by a geometrically defined map to the cohomology $\mathrm{H}_{(\infty)}(\Gamma\backslash\mathcal{X})$ of the complex of smooth Γ-invariant differential forms on $\mathcal{X}$ which are bounded and d-bounded. In case Γ is a lattice, $\mathrm{H}_{(\infty)}(\Gamma\backslash\mathcal{X})$ can be replaced by L^2-cohomology.

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