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Linear Representations and Arithmeticity of Lattices in Products of Trees

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1.1. Prolegomenon

In this paper we continue our study of lattices in the automorphisms groups of products of trees initiated in [BM97], [BM00a], [BM00b], [Moz98] (see also [Gla03], [BG02], [Rat04]). We concentrate here on the interplay between the linear representation theory and the structure of these lattices. Before turning to the main results of this paper it may be worthwhile to put certain concepts and results from [BM00a], [BM00b] in perspective, and explain the motivation for our approach.

A lattice Γ in a locally compact group G is a discrete subgroup such that the quotient space G/Γ carries a finite G-invariant measure. If in addition G/Γ is compact the lattice is called cocompact or uniform.

In the framework of this paper the following classical setting will serve as a motivating example. Let $\mathbb{Q}_p$ denote the field of p-adic numbers and given two primes p, q define the locally compact group:

$$G_{p,q} = \mathrm{PSL}(2, \mathbb{Q}_p) \times \mathrm{PSL}(2, \mathbb{Q}_q).$$

Lattices in $G_{p,q}$ fall into two classes: The reducible ones and the irreducible ones. A lattice $\Gamma < G_{p,q}$ is called reducible if it has a subgroup of finite index which is a direct product $\Gamma_p \times \Gamma_q$ where $\Gamma_p < \mathrm{PSL}(2, \mathbb{Q}_p)$ and $\Gamma_q < \mathrm{PSL}(2, \mathbb{Q}_q)$ are lattices. A lattice is called irreducible if it is not reducible. Thus, for a reducible lattice, each of the projections is a closed, in fact discrete, subgroup of the corresponding factor. It is a remarkable fact that for an irreducible lattice $\Gamma < G_{p,q}$ both projections are dense in the corresponding factor. We recall at this point that the notions of reducibility and irreducibility pertain to the theory of lattices in products of semi-simple groups over local fields where analogous density properties hold for irreducible lattices (cf. [Mar91] II Thm. (6.7)). It should be noted that this density result makes essential use of the algebro-geometric structure of the ambient group and in particular of the Borel density theorem. Irreducible lattices in $G_{p,q}$ and more generally in a product G of semisimple groups over local fields of total rank at least 2 enjoy remarkable algebraic and geometric properties as the following results due to G.A. Margulis show:

- *Arithmeticity:* Any irreducible lattice in G is arithmetic. [Mar91, IX Thm (A)].
- *Superrigidity:* Any unbounded irreducible linear representation of an irreducible lattice comes from a rational representation of the ambient group G. [Mar91, VII Thm (5.6)].
- *Normal subgroup Theorem:* Any non-trivial normal subgroup of an irreducible lattice in G is either finite and central or of finite index. [Mar91, IV Thm (4.9)].

An important geometric object associated with $\mathrm{PGL}(2,\mathbb{Q}_p)$ is its Bruhat–Tits building $\mathcal{T}_{p+1}$ which is a regular tree of degree $p+1$ on which $\mathrm{PGL}(2,\mathbb{Q}_p)$ acts by automorphisms. In fact $\mathrm{PGL}(2,\mathbb{Q}_p)$ acts transitively on the set of vertices of $\mathcal{T}_{p+1}$ and any pair of adjacent vertices is a complete set of representatives for the $\mathrm{PSL}(2,\mathbb{Q}_p)$-orbits. The group $\mathrm{Aut}\mathcal{T}_{p+1}$, with the topology of pointwise convergence on the set of vertices, is a locally compact group containing $\mathrm{PGL}(2,\mathbb{Q}_p)$, and hence $\mathrm{PSL}(2,\mathbb{Q}_p)$, as a closed (cocompact) subgroup. In particular a cocompact lattice in $\mathrm{PGL}(2,\mathbb{Q}_p)$, or $\mathrm{PSL}(2,\mathbb{Q}_p)$, is also one in $\mathrm{Aut}\mathcal{T}_{p+1}$; this allows applying combinatorial and finite group theory techniques, such as the theory of graphs of groups, to the study of the structure of lattices in $\mathrm{PGL}(2,\mathbb{Q}_p)$. As an example we mention the result that a cocompact lattice in $\mathrm{Aut}\mathcal{T}_{p+1}$ is virtually free on finitely many generators (see for example [Ser03]). There are however certain more subtle finiteness properties shared by cocompact lattices in $\mathrm{PGL}(2,\mathbb{Q}_p)$ which fail in general for cocompact lattices in $\mathrm{Aut}\mathcal{T}_{p+1}$. For example one has a positive lower bound on the covolume of lattices in $\mathrm{PGL}(2,\mathbb{Q}_p)$ whereas there are lattices of arbitrarily small covolume in $\mathrm{Aut}\mathcal{T}_{p+1}$. The similarities and differences between the theory of lattices in $\mathrm{Aut}\mathcal{T}_{p+1}$ and the theory of lattices in Lie groups, and in $\mathrm{PGL}(2,\mathbb{Q}_p)$ in particular, are part of an extensive study and we refer to [BL01] for many results and references. Returning to the higher rank situation and in the same geometric vein, we observe that a cocompact lattice Γ in $\mathrm{PGL}(2,\mathbb{Q}_p)\times\mathrm{PGL}(2,\mathbb{Q}_q)$, or in $G_{p,q}$ for that matter, is also a cocompact lattice in $\mathrm{Aut}\mathcal{T}_{p+1}\times\mathrm{Aut}\mathcal{T}_{q+1}$. If Γ is moreover torsion free then it acts freely on the 2-dimensional square complex $\mathcal{T}_{p+1}\times\mathcal{T}_{q+1}$ and the quotient $\Gamma\backslash(\mathcal{T}_{p+1}\times\mathcal{T}_{q+1})$ is a finite square complex inheriting the particular link structure of its universal covering, namely that the link of each vertex is a complete bipartite graph. This puts the study of such lattices in a rich geometric and combinatorial context. In fact, it was observed by D. Wise ([Wis95, Thm 1.5]) that the universal covering space of a 2-dimensional square complex is a product of trees exactly when the link at each vertex is a complete bipartite graph. As for the similarities and differences between the theory of cocompact lattices in $\mathrm{Aut}\mathcal{T}_{p+1}\times\mathrm{Aut}\mathcal{T}_{q+1}$ and $G_{p,q}$, or $\mathrm{PGL}(2,\mathbb{Q}_p)\times\mathrm{PGL}(2,\mathbb{Q}_q)$, the first major difficulty seems to be the proper generalization of the notion of irreducible lattice. This issue will be discussed below after example 1.1.1. Let us indicate one striking phenomenon: while cocompact lattices in $\mathrm{PSL}(2,\mathbb{Q}_p)$, $\mathrm{Aut}\mathcal{T}_{p+1}$ and $\mathrm{PGL}(2,\mathbb{Q}_p)\times\mathrm{PGL}(2,\mathbb{Q}_q)$ are linear groups this need not be so for cocompact lattices in $\mathrm{Aut}\mathcal{T}_{p+1}\times\mathrm{Aut}\mathcal{T}_{q+1}$ and as a consequence of the result proven in this paper we will obtain an algorithmic criterion of non-linearity (see cor. 1.1.20) using the information contained in the finite quotient square complex.

Example 1.1.1. An interesting example of an irreducible lattice in $\mathrm{PGL}(2, \mathbb{Q}_p) \times \mathrm{PGL}(2, \mathbb{Q}_q)$ may be explicitly described as follows. Let p, q be two fixed distinct primes both congruent to 1 modulo 4. Let $\mathbb{H}_{\mathbb{Z}}$ denote the ring of integer quaternions. Fix $\epsilon_p \in \mathbb{Q}_p$ and $\epsilon_q \in \mathbb{Q}_q$, square roots of -1. Define a map $\varphi : \mathbb{H}_{\mathbb{Z}} \backslash \{0\} \to \mathrm{PGL}(2, \mathbb{Q}_p) \times \mathrm{PGL}(2, \mathbb{Q}_q)$ where for $x = x_0 + x_1 i + x_2 j + x_3 k$

$$\varphi(x) = \left(\begin{pmatrix} x_0 + x_1\epsilon_p & x_2 + x_3\epsilon_p \\ -x_2 + x_3\epsilon_p & x_0 - x_1\epsilon_p \end{pmatrix}, \quad \begin{pmatrix} x_0 + x_1\epsilon_q & x_2 + x_3\epsilon_q \\ -x_2 + x_3\epsilon_q & x_0 - x_1\epsilon_q \end{pmatrix} \right)$$

Let $\mathbb{H}_{\mathbb{Z}}(p, q) = \{x \in \mathbb{H}_{\mathbb{Z}} : |x|^2 = x\overline{x} = p^k q^l \text{ for some } k, l \in \mathbb{Z}^+\}$. Its image $\varphi(\mathbb{H}_{\mathbb{Z}}(p, q)) < \mathrm{PGL}(2, \mathbb{Q}_p) \times \mathrm{PGL}(2, \mathbb{Q}_q)$ is an irreducible arithmetic lattice. To avoid torsion we shall be interested in a certain congruence sublattice of it, namely the image of those quaternions in $\mathbb{H}_{\mathbb{Z}}(p, q)$ which are congruent to 1 modulo 2 (i.e., for which x_0 is odd and x_1, x_2 and x_3 are even). Let us denote this lattice by $\Gamma < \mathrm{PGL}(2, \mathbb{Q}_p) \times \mathrm{PGL}(2, \mathbb{Q}_q)$. Using Jacobi's theorem giving the number of ways of representing a number as sum of four squares, one can show that Γ acts freely transitively on the set of vertices of the square complex $\mathcal{T}_{p+1} \times \mathcal{T}_{q+1}$. The resulting quotient $\mathcal{X}_{p,q} = \Gamma \backslash (\mathcal{T}_{p+1} \times \mathcal{T}_{q+1})$ which is a square complex on one vertex admits an explicit description to which we now turn. Let us define the following two sets

$$L_p = \{x \in \mathbb{H}_{\mathbb{Z}} : |x|^2 = p, \quad x \equiv 1 (\mathrm{mod}\ 2) \quad x_0 > 0\} \tag{1.1.1}$$

$$L_q = \{x \in \mathbb{H}_{\mathbb{Z}} : |x|^2 = q, \quad x \equiv 1 (\mathrm{mod}\ 2) \quad x_0 > 0\} \tag{1.1.2}$$

Observe that L_p consists of $p + 1$, and L_q of $q + 1$, elements and if $x \in L_p$ (resp. L_q) then $\overline{x} \in L_p$ (resp. L_q). The 0-skeleton $\mathcal{X}_{p,q}^{(0)}$ consists of a single vertex v_0. The 1-skeleton of $\mathcal{X}_{p,q}$ consists of $\frac{p+1}{2} + \frac{q+1}{2}$ loops based at v_0. We shall partition these loops into two sets E_h and E_v of sizes $\frac{p+1}{2}$ and $\frac{q+1}{2}$ respectively where the elements of E_h will be thought of as horizontal loops and the elements of E_v as vertical loops. We shall label each oriented loop of E_h by an element of L_p where the two orientations of a geometric loop are labeled by conjugate quaternions. Similarly we shall label the oriented loops of E_v by the elements of L_q. Abusing notation we shall refer also to the elements of L_p, L_q as (oriented) loops. To describe the 2-skeleton $\mathcal{X}_{p,q}^{(2)}$, consider all the quadruples $(a, b, \overline{a'}, \overline{b'})$ where $a, a' \in L_p$, $b, b' \in L_q$ and

$$ab = \pm b'a'. \tag{1.1.3}$$

These quadruples come in equivalence classes:

$$\{(a, b, \overline{a'}, \overline{b'}), (\overline{a}, b', a', \overline{b}), (a', \overline{b}, \overline{a}, b'), (\overline{a'}, \overline{b'}, a, b)\}$$

Where the second quadruple is obtained from equation (1.1.3) by multiplying by $\overline{a}$ on the left and $\overline{a'}$ on the right. Similarly the third is obtained by multiplying by $\overline{b'}$ on the left and $\overline{b}$ on the right and finally the last one is obtained by applying quaternionic conjugation to equation (1.1.3).

For each equivalence class we will have a geometric square in $\mathcal{X}_{p,q}^{(2)}$ attached to the 1-skeleton of $\mathcal{X}_{p,q}$ so that when we read the labels of the oriented loops along its boundary in one of the four possible ways starting with an horizontal one we see an element of the corresponding equivalence class. The following arithmetic fact (cf. [Dic22]) is crucial. For each $a \in L_p$ and $b \in L_q$ there are unique $a' \in L_p$ and $b' \in L_q$ so that

$$ab = \pm b'a'.$$

Hence there are $\left(\frac{p+1}{2}\right) \cdot \left(\frac{q+1}{2}\right)$ geometric squares and the link of the vertex v_0 is a complete bipartite graph. Thus, indeed, the universal covering space $\widetilde{\mathcal{X}}_{p,q}$ is a product of a $(p+1)$-regular tree with a $(q+1)$-regular tree. One can then identify $\widetilde{\mathcal{X}}_{p,q}$ with the Bruhat–Tits building $\mathcal{T}_{p+1} \times \mathcal{T}_{q+1}$ associated with $\mathrm{PGL}(2, \mathbb{Q}_p) \times \mathrm{PGL}(2, \mathbb{Q}_q)$ so that $\pi_1(\mathcal{X}_{p,q})$ is identified with Γ.

Let T_1, T_2 be regular trees of finite degrees at least three. We turn to the study of cocompact lattices in a product $\mathrm{Aut}T_1 \times \mathrm{Aut}T_2$. As above we shall call a lattice $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ reducible if it has a subgroup of finite index which is a product of lattices in the factors, and irreducible otherwise. In particular any reducible cocompact lattice is virtually a product of two free groups. Let pr_i denote the projection on $\mathrm{Aut}T_i$. Then it is straightforward to verify that for a lattice $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$, the following are equivalent:

(1) Γ is reducible.
(2) Both subgroups $\mathrm{pr}_1(\Gamma)$ and $\mathrm{pr}_2(\Gamma)$ are closed (hence discrete) in $\mathrm{Aut}T_1$ and $\mathrm{Aut}T_2$.

Given a cocompact lattice Γ we will thus introduce G_i, the closure of $\mathrm{pr}_i(\Gamma)$ in $\mathrm{Aut}T_i$, and for a vertex x_i of T_i let $G_i(x_i)$ denote the stabilizer in G_i of x_i. It is a compact open subgroup of G_i. While $\mathrm{Aut}T_i/G_i$ is compact, the following finiteness result shows that from another point of view, G_i is "far away" from $\mathrm{Aut}T_i$, namely:

Proposition 1.1.2. *Assume that $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ is a cocompact lattice. Then the compact group $G_i(x_i)$ is topologically finitely generated.*

Proof. Since $\mathrm{pr}_1(\Gamma)$ is dense in G_1 by definition and $G_1(x_1)$ is open, the subgroup $\mathrm{pr}_1(\Gamma) \cap G_1(x_1)$ is dense in $G_1(x_1)$. On the other hand, we have $\mathrm{pr}_1(\Gamma) \cap G_1(x_1) = \mathrm{pr}_1(\Gamma \cap (G_1(x_1) \times G_2))$. We now proceed to show that $\Gamma \cap (G_1(x_1) \times G_2)$ is finitely generated which will conclude the proof. Let $V := G_1(x_1) \times G_2$ and $G = G_1 \times G_2$. Then V is an open subgroup of G and we claim that $V \cap \Gamma$ is a cocompact lattice in V. Indeed every V-orbit in G/Γ being open is also closed and hence compact. In particular, considering the V-orbit of $e\Gamma$ we obtain a continuous injective map

$$\begin{aligned} V/(V \cap \Gamma) &\to G/\Gamma \\ g(V \cap \Gamma) &\mapsto g\Gamma \end{aligned}$$

which is a homeomorphism on its image, since the image $V\Gamma$ is closed (in fact compact) in G/Γ. Thus $V/(V \cap \Gamma)$ is compact which shows the claim. Finally, $\mathrm{Aut}T_2$ is compactly generated and $\mathrm{Aut}T_2/G_2$ is compact, hence G_2 is compactly generated and hence so is $V := G_1(x_1) \times G_2$. Since $\Gamma \cap V$ is a cocompact lattice in V we deduce that $\Gamma \cap (G_1(x) \times G_2)$ is finitely generated. □

If now T is a regular tree and x a vertex, we have the homomorphism

$$\tau : \mathrm{Aut}T(x) \to (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$$

which associate to each element $g \in \mathrm{Aut}T(x)$ the sequence of signs of the permutations induced by g on all the spheres with center x. If the degree of T is at least 3 it is clear that an arbitrary sequence of signs can specified and obtained by an element $g \in \mathrm{Aut}T(x)$. Hence the homomorphism τ is surjective and it follows in particular that $\mathrm{Aut}T(x)$ is not topologically finitely generated. In particular, cocompact lattices in $\mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ never have dense projections. It would be of interest to understand what happens in the case of non-uniform lattices.

The above discussion leads naturally to the following:

Basic Question. Which groups arise as closures of projections of cocompact lattices in $\mathrm{Aut}T_1 \times \mathrm{Aut}T_2$?

The interest in this question is partly due to a certain number of results showing that there are strong connections between the topological and algebraic properties of the groups $G_i < \mathrm{Aut}T_i$ and the algebraic and geometric properties of Γ; in other words, Γ is as "rigid" as its ambient group $G_1 \times G_2$. In particular, results of this type played a role in [BM00a], [BM00b]. More recently, such connections have been established in a remarkable degree of generality. Monod [Mon05], [Mon06] and Monod–Shalom [MS02] have proved superrigidity results for the actions on CAT(0)-spaces of cocompact lattices $\Gamma < G_1 \times G_2$ in a product of locally compact groups. These results in particular imply the following theorems which we will use in the sequel. In Theorems 1.1.3, 1.1.4, 1.1.5, the groups G_1, G_2 are assumed to be locally compact and compactly generated.

Theorem 1.1.3 ([Mon06, Cor. 4]). *Let $\Gamma < G_1 \times G_2$ be a cocompact lattice with dense projections. Let $\mathbb{H}$ be a connected, k-simple, adjoint k-group where k is a local field and let $\pi : \Gamma \to \mathbb{H}(k)$ be a homomorphism with unbounded Zariski dense image. Then π extends continuously to $G_1 \times G_2$ factoring via one of the factors G_i.*

In certain situations this result will imply that the linear representation theory of Γ over any field is controlled by the continuous representation theory of $G_1 \times G_2$ over local fields.

The next result shows that the various decompositions of Γ as non-trivial amalgams are completely described by the decompositions of G_i, $i = 1, 2$ as non-trivial amalgams. See also [Sha00, Theorem 0.7].

Theorem 1.1.4 ([MS02, Thm. 1.5]). *Let $\Gamma < G = G_1 \times G_2$ be a lattice with dense projections. Let Γ act non-elementarily on a countable tree T. Then there exists an invariant subtree on which the Γ-action extends continuously to a G-action factoring via one of the G_i's.*

Finally the following result of Bader and Shalom (generalizing Margulis' Normal Subgroup Theorem for the case of products) shows that normal subgroups of Γ are in a sense controlled by the closed normal subgroups of $G_1 \times G_2$.

Theorem 1.1.5 ([BS03, Thm. 1.1]). *Let $\Gamma < G = G_1 \times G_2$ be a cocompact lattice with dense projections. Assume that* $\mathrm{Hom}_c(G, \mathbb{R}) = (0)$. *Let $N \triangleleft \Gamma$ be a normal subgroup. The group Γ/N is finite if and only if the groups $G_i/\overline{\mathrm{pr}_i(N)}$ are compact.*

Remark 1.1.6. The idea of extending the scope of the rigidity phenomena of lattices in higher rank Lie groups to the general locally compact setting by considering lattices in general products goes at least back to [BnK70] and is present in the work of G. Margulis in the seventies. When G_1 and G_2 are Lie groups over local fields, Theorems 1.1.3, 1.1.4 and 1.1.5 are special cases of results of G. Margulis proven respectively in [Mar74], [Mar81] and [Mar79]. Superrigidity theorems of varying degrees of generality can be found in the book [Mar91] (and references therein), [Bur95], [Mar94], [BM96], [Gao97], [Sha00], [MS02], [Bon], [Mon06] and [KM08].

If Γ is a cocompact lattice in $\mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ and $G_i = \overline{\mathrm{pr}_i(\Gamma)}$ are, as before, the closures of projections, assuming that Γ is torsion free, we consider the finite square complex $X := \Gamma\backslash(T_1 \times T_2)$. The finite permutation group induced by $G_i(x_i)$ on the set of vertices adjacent to x_i can be explicitly calculated from the combinatorics of X. (We will see an illustration of this later in the paper in the case where X has a single vertex.) It turns out that when Γ is irreducible and this finite permutation group is sufficiently transitive, one obtains rather strong information on the global structure of the non-discrete group G_i. With this in mind we introduced in [BM00a] various classes of closed subgroups defined via "local conditions" and showed how these local conditions imply certain global structure results. We now recall some salient features of this theory.

Let $T = (V, E)$ be a locally finite tree, V its set of vertices and E its set of edges. Given a subgroup $H < \mathrm{Aut}T$ and a vertex $x \in V$, its stabilizer $H(x)$ acts as a finite permutation group on the set $E(x) \subset E$ of edges whose origin is the vertex x, and we say that H is locally quasiprimitive (respectively, locally primitive, locally 2-transitive, etc.) if this finite permutation group is quasiprimitive (respectively primitive, 2-transitive, etc.) for every vertex x. Recall that a finite permutation group $F < \mathrm{Sym}\Omega$ acting on a finite set Ω is called *quasiprimitive* if every non-trivial normal subgroup of F act transitively on Ω. It is called *primitive* if any equivariant factor of Ω is trivial (i.e. either a point or Ω). The action is called 2-transitive if F acts transitively on $\Omega^2\backslash\Delta_\Omega$ where Δ_Ω is the diagonal. Quasiprimitive and primitive groups have a rich structure theory, exemplified by results such as the O'Nan–Scott theorem (cf., [DM96], [Pra97]). Following the classification of the finite simple groups one has also a classification of 2-transitive groups (cf. the survey paper by P. Cameron [Cam95]).

Given a totally disconnected group H let

$$H^{(\infty)} = \cap_{L<H} L$$

where the intersection is taken over all open finite index subgroups. Let

$$\mathrm{QZ}(H) = \{h \in H : Z_H(h) \text{ is open}\}$$

be the quasi-center of H. Both are topologically characteristic subgroups of H. The subgroup $H^{(\infty)}$ is closed, and any normal discrete subgroup of H is contained in

$\mathrm{QZ}(H)$. To motivate these definitions observe that when $H = \mathbb{G}(\mathbb{Q}_p)$, where $\mathbb{G}$ is a connected semisimple $\mathbb{Q}_p$-group, then $H^{(\infty)}$ coincides with the subgroup $\mathbb{G}^+$ generated by all one parameter unipotent subgroups of $\mathbb{G}(\mathbb{Q}_p)$, and $\mathrm{QZ}(H)$ is the kernel of the adjoint representation of the Lie group $\mathbb{G}(\mathbb{Q}_p)$. We recall next a few basic results established in [BM00a] concerning the structure of these subgroups.

Theorem 1.1.7 ([BM00a] Prop. 1.2.1). *Let T be a locally finite tree. Let $H < \mathrm{Aut}T$ be a closed non-discrete locally quasiprimitive group. Then*

(1) *$H/H^{(\infty)}$ is compact.*
(2) *$\mathrm{QZ}(H)$ is a discrete subgroup of H and $H/\mathrm{QZ}(H)$ is not compact.*
(3) *Any closed normal subgroup of H either contains $H^{(\infty)}$ or is contained in $\mathrm{QZ}(H)$.*

The subgroup $\mathrm{QZ}(H)$ acts freely on the set of vertices of T. In particular if $\mathrm{QZ}(H)$ is non-trivial then the subgroup (of index at most 2) of $\mathrm{QZ}(H)$ consisting of those elements whose translation length is even is a free group on infinitely many generators. In analogy with semisimple Lie groups one would expect the poset of normal subgroups of $H^{(\infty)}$ to have a relatively simple structure; this is indeed the case as shown by theorem 1.1.8, the difficulty being however that local transitivity properties of H are not inherited by $H^{(\infty)}$. A guiding principle in the study of the structure of H is the analogy with the O'Nan-Scot structure theory of finite primitive permutation groups of which theorem 1.1.8 is an analogue. Indeed we have:

Theorem 1.1.8 ([BM00a] Prop. 1.2.1, 1.5.1). *Let $H < \mathrm{Aut}T$ be a closed non-discrete subgroup.*

(1) *If H is locally quasiprimitive, then $H^{(\infty)}/\mathrm{QZ}(H^{(\infty)})$ admits minimal closed normal subgroups. These are finitely many, topologically simple, H-conjugate and their product is dense.*
(2) *If H is locally primitive, $H^{(\infty)}/\mathrm{QZ}(H^{(\infty)})$ is a finite product of topologically simple groups.*
(3) *If H is locally 2-transitive, $H^{(\infty)}/\mathrm{QZ}(H^{(\infty)})$ is topologically simple.*

In all the above cases we have that $\mathrm{QZ}(H^{(\infty)}) = \mathrm{QZ}(H) \cap H^{(\infty)}$ and it acts freely without inversions on T.

We observe that $H^{(\infty)}/\mathrm{QZ}(H^{(\infty)})$ may have an arbitrary large number of simple factors. This is exemplified by Proposition 1.1.15 below and the subsequent discussion. We conclude from Theorem 1.1.8.1 that if H is locally quasiprimitive and $\mathrm{QZ}(H^{(\infty)}) = \{e\}$, then $H^{(\infty)}$ is topologically simple. Indeed, assume $\mathrm{QZ}(H^{(\infty)})$ is trivial; we need to show that there is only one minimal closed normal subgroup in $H^{(\infty)}$. Assume that there were two or more such groups $M_1, M_2, \ldots$; then as all are conjugate and their product is dense they are unbounded and hence M_1 has a non-void limit set in the boundary $T(\infty)$ of the tree. As M_1 is a normal subgroup of $H^{(\infty)}$ and the latter acts cofinitely on the tree, it follows that this limit set must be the whole boundary of T. Since M_1 and M_2 commute this limit set must be pointwise fixed by M_2 which is impossible. We conclude that the unique minimal closed normal subgroup of $H^{(\infty)}$ coincides with $H^{(\infty)}$ showing it is topologically simple. This observation together with Theorem 1.1.5 leads to:

Corollary 1.1.9. *Let $H_i < \mathrm{Aut}T_i$ be closed non-discrete locally quasiprimitive groups and assume $\mathrm{QZ}(H_i) = \{e\}$. Let $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ be a cocompact lattice satisfying $H_i^{(\infty)} \subset \overline{\mathrm{pr}_i(\Gamma)} \subset H_i$, $i = 1, 2$. Then any non-trivial normal subgroup of Γ is of finite index.*

Proof. Let $G_i := \overline{\mathrm{pr}_i(\Gamma)}$ and $N \lhd \Gamma$ a normal subgroup. We now verify the hypothesis of Theorem 1.1.5. We will use repeatedly the fact established above that ${H_i}^{(\infty)}$ is topologically simple. We have $\overline{[G_i, G_i]} \supset \overline{[{H_i}^{(\infty)}, {H_i}^{(\infty)}]} = {H_i}^{(\infty)}$, where the last equality follows from the topologically semisimplicity of ${H_i}^{(\infty)}$. Thus, since $H_i/{H_i}^{(\infty)}$ is compact (by theorem 1.1.7 1.) we deduce that $G_i/\overline{[G_i, G_i]}$ is compact as well and hence $\mathrm{Hom}_c(G_i, \mathbb{R}) = (0)$. The subgroup $\overline{\mathrm{pr}_i(N)}$ is normalized by ${H_i}^{(\infty)}$ and hence we have either

(1) $\overline{\mathrm{pr}_i(N)} \cap {H_i}^{(\infty)} = (e)$ or
(2) $\overline{\mathrm{pr}_i(N)} \supset {H_i}^{(\infty)}$.

Assume for instance that (1) occurs for $i = 1$. Then $\overline{\mathrm{pr}_1(N)}$ commutes with ${H_1}^{(\infty)}$ but since ${H_1}^{(\infty)} \backslash T_1$ is finite this implies that $\overline{\mathrm{pr}_1(N)} = (e)$. This implies that $\mathrm{pr}_2(N)$ is a discrete subgroup of $\mathrm{Aut}T_2$, normalized by ${H_2}^{(\infty)}$ and thus $\mathrm{pr}_2(N) \cap {H_2}^{(\infty)} \subset QZ({H_2}^{(\infty)}) = (e)$. Thus again, $\mathrm{pr}_2(N) = (e)$ which implies $N = (e)$. Thus if $N \neq (e)$ then $\overline{\mathrm{pr}_i(N)} \supset {H_i}^{(\infty)}$ for $i = 1, 2$ and hence $G_i/\overline{\mathrm{pr}_i(N)}$ is compact which together with Theorem 1.1.5 implies that Γ/N is finite. □

Remark 1.1.10. If Γ satisfies the conditions of Corollary 1.1.9 then so does also any finite index subgroup of Γ. See the beginning of §1.3.

The quasi-center plays, in several ways, an important role for the structure of Γ. In [BM00b], Theorem 1.1.7 was applied to obtain the following criterion for non-residually finiteness of Γ. Let $\Lambda_1 = \Gamma \cap (\mathrm{Aut}T_1 \times e)$, $\Lambda_2 = \Gamma \cap (e \times \mathrm{Aut}T_2)$. We have:

Proposition 1.1.11 ([BM00b] Prop. 2.1, 2.2). *Let $H_i < \mathrm{Aut}T_i$ be closed, non-discrete, locally quasiprimitive groups and $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ be a lattice with $H_i^{(\infty)} \subset \overline{\mathrm{pr}_i(\Gamma)} \subset H_i$. Then*

(1) *If $\Lambda_1 \cdot \Lambda_2 \neq \{e\}$ then Γ is not residually finite.*
(2) *If $H_i = \overline{\mathrm{pr}_i(\Gamma)}$ then Λ_i is a normal subgroup of $\mathrm{QZ}(H_i)$ and the quotients $\mathrm{QZ}(H_i)/\Lambda_i$ are locally finite groups.*

In the setting of Proposition 1.1.11.2, we have that Λ_i is non-trivial if and only if the quasi center $\mathrm{QZ}(H_i)$ is non-trivial. Theorems 1.1.5, 1.1.7 and Proposition 1.1.11.2 imply (see 1.3.2):

Theorem 1.1.12. *Let $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ be a cocompact lattice such that $H_i = \overline{\mathrm{pr}_i(\Gamma)}$ are non-discrete, locally quasiprimitive. Let $N \lhd \Gamma$ be a normal subgroup. Then either*

1. *$N \subset \mathrm{QZ}(H_1) \times \mathrm{QZ}(H_2)$, or*
2. *N is of finite index in Γ.*

Theorem 1.1.11 together with a predecessor (see [BM00b] §4) of the Normal Subgroup Theorem of Bader–Shalom (Theorem 1.1.5) enabled us to construct a new class of finitely presented simple groups.

Theorem 1.1.13 ([BM97], [BM00b]). *For all sufficiently large $n, m \in \mathbb{N}$ there is a torsion-free cocompact lattice in* $\mathrm{Aut}T_{2n} \times \mathrm{Aut}T_{2m}$ *which is a simple group.*

Remark 1.1.14. In [Rat04] Diego Rattaggi has constructed many interesting lattices $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$. In particular he has constructed a $(4, 12)$-complex ([Rat04, Example 2.26]) whose fundamental group is not residually finite (using Proposition 1.1.11). Using this example as well as his construction of an (A_6, A_6)-square complex described in Example 1.1.19 and following the same strategy as in [BM00b] he has established the existence of simple groups as above for all $n \geq 9$ and $m \geq 13$, see [Rat04, Prop. 2.29 (1)].

An important ingredient in the proof of Theorem 1.1.13 consists of the identification of the closures of the projections of lattices constructed via geometric methods. This brings us back to the basic question above of which groups arise as closures of projections of uniform lattices in $\mathrm{Aut}T_1 \times \mathrm{Aut}T_2$.

The finitely generated simple groups of Theorem 1.1.13 are of course non-linear.

In the rest of the paper we will be interested in studying lattices which admit infinite linear images. We will show that such lattices are essentially extensions of arithmetic lattices in semisimple Lie groups over appropriate local fields.

In view of the superrigidity theorems mentioned above, locally quasiprimitive groups which admit a p-adic analytic structure will play a central role in the discussion of arithmeticity. We have:

Proposition 1.1.15. *Let $H < \mathrm{Aut}T$ be a closed non-discrete locally quasiprimitive group. Assume that it admits a $\mathbb{Q}_p$-analytic structure. Let $\mathfrak{H}$ denote the Lie algebra of H, let $\mathbb{G} = \mathrm{Aut}(\mathfrak{H} \otimes \overline{\mathbb{Q}_p})$ a linear algebraic group defined over $\mathbb{Q}_p$ and let $\mathrm{Ad} : H \to \mathbb{G}(\mathbb{Q}_p)$ be the adjoint representation. Then*

(1) $\mathbb{G}$ *is adjoint and semisimple.*
(2) $\ker \mathrm{Ad} = \mathrm{QZ}(H)$.
(3) $\mathrm{Ad}(H) \supset \mathbb{G}^+$.

As mentioned above $\mathbb{G}^+$ denotes for a $\mathbb{Q}_p$-group $\mathbb{G}$, the subgroup of $\mathbb{G}(\mathbb{Q}_p)$ generated by all the one parameter unipotent $\mathbb{Q}_p$-subgroups.

Thus the group H is an extension by $\mathrm{QZ}(H)$ of the group $\mathrm{Ad}(H)$ which lies between $\mathbb{G}^+$ and $\mathbb{G}(\mathbb{Q}_p)$. Recall that $\mathrm{QZ}(H)$ is either trivial or virtually a free group on infinitely many generators. We remark that even though H, being a subgroup of $\mathrm{Aut}T$, is a "rank one object" the group $\mathbb{G}$ may be of arbitrarily large rank. Consider a connected simple algebraic group $\mathbb{G}$ defined over $\mathbb{Q}_p$ and let Δ be its affine Bruhat–Tits building. The action of $\mathbb{G}$ on Δ induces an action on any graph $\mathcal{G}$ which is equivariantly drawn on Δ and this leads to the extension of $\mathbb{G}$ by $\pi_1(\mathcal{G})$ acting on the universal covering tree $\tilde{\mathcal{G}}$. In particular if one takes $\mathcal{G}$ to be the subgraph of the 1-skeleton of Δ obtained by considering all edges of a given colour, then the resulting extension H, $1 \to \pi_1(\mathcal{G}) \to H \to \mathbb{G}(\mathbb{Q}_p) \to 1$ is a locally primitive group.

Observe that in the setting of Proposition 1.1.15 the homomorphism $\mathrm{Ad} : H \to \mathbb{G}(\mathbb{Q}_p)$ sends every vertex stabilizer $H(x)$, x a vertex of T, isomorphically to a compact subgroup of $\mathbb{G}(\mathbb{Q}_p)$. In particular $H(x)$ is a virtually pro-p-group, i.e., $H(x)$ contains an open pro-p-subgroup of finite index. However in general it is not clear to which extent the homomorphism is implemented by a geometric mapping of the tree into the Bruhat–Tits building associated to $\mathbb{G}(\mathbb{Q}_p)$, and thus we have no real way of comparing the "congruence" structure on $H(x)$ induced by the tree structure of T and the congruence structure induced by that of a maximal compact subgroup of $\mathbb{G}(\mathbb{Q}_p)$. There is however a powerful tool going back to Thompson and Wielandt:

Proposition 1.1.16 ([BM00a], Prop. 2.1.2). *Let $T = (V, E)$ be a locally finite tree. Let $H < \mathrm{Aut}T$ be a closed locally primitive subgroup and when H is non-discrete assume that $H(x)$ is virtually pro-p. Then*

(1) *There is a vertex $y \in V$ such that $H_2(y)$ is pro-p.*
(2) *If H acts transitively on V, then for any two adjacent vertices x, y the subgroup $H_1(x, y) = H_1(x) \cap H_1(y)$ is pro-p.*

Where as in [BM00a] $H_i(x)$ denotes the pointwise stabilizer of the ball of radius i centered at x.

Our main result is:

Theorem 1.1.17. *Let T_1, T_2 be locally finite trees and $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ a cocompact lattice. Assume*

(1) $H^{(\infty)} < \overline{\mathrm{pr}_i(\Gamma)} < H_i$, *where $H_i < \mathrm{Aut}T_i$ is a closed non-discrete, locally quasi-primitive subgroup.*
(2) *There is a linear representation $\pi : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ with infinite image.*

Then there are prime numbers p_1, p_2 such that for each $i = 1, 2$ H_i is $\mathbb{Q}_{p_i}$-analytic. Let $\mathbb{G}_i$ be the adjoint semisimple algebraic group defined over $\mathbb{Q}_{p_i}$ given by Proposition 1.1.15 and $\mathrm{Ad}_i : H_i \to \mathbb{G}_i(\mathbb{Q}_{p_i})$ the adjoint representation. Then

(1) $(\mathrm{Ad}_1 \times \mathrm{Ad}_2)(\Gamma)$ *is an arithmetic lattice in* $\mathbb{G}_1(\mathbb{Q}_{p_1}) \times \mathbb{G}_2(\mathbb{Q}_{p_2})$.
(2) $\Lambda_i := \Gamma \cap H_i$ *is of finite index in* $\mathrm{QZ}(H_i) := \ker \mathrm{Ad}_i$.

and Γ fits into the exact sequence

$$1 \to \Lambda_1 \times \Lambda_2 \to \Gamma \to (\mathrm{Ad}_1 \times \mathrm{Ad}_2)(\Gamma) \to 1$$

Using the above result we can now characterize the "classical" situation:

Corollary 1.1.18. *Let T_1, T_2 be locally finite trees. Let $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ be a cocompact lattice. Assume*

(1) ${H_i}^{(\infty)} < \overline{\mathrm{pr}_i(\Gamma)} < H_i$, *where $H_i < \mathrm{Aut}T_i$ is a closed non-discrete, locally primitive subgroup.*
(2) *There is a linear representation $\pi : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ with infinite image.*

Then the following are equivalent:

(1) Γ *is linear over* $\mathbb{C}$.
(2) Γ *is residually finite.*
(3) $\mathrm{rank}_{\mathbb{Q}_{p_i}}(\mathbb{G}_i) = 1$ *for both* $i = 1, 2$.

In this case the geometric realization $|T_i|$ *is isometric to the Bruhat–Tits tree associated to* $\mathbb{G}_i$.

We turn next to discuss some consequences of the above results formulated in geometric terms, that is expressed in terms of finite square complexes $\Gamma\backslash(T_1 \times T_2)$ rather than in terms of lattices $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$. We will concentrate here on the special case of a square complex with a single vertex. Such square complexes correspond to lattices acting freely transitively on the set of vertices of the product of trees $T_1 \times T_2$. A square complex with one vertex whose universal covering space is a product of two trees is given by the following data (see Example 1.1.1):

– Two finite sets A_1, A_2 each endowed with a fixed point free involution $x \mapsto \overline{x}$.
– A set of "squares" $S \subset A_1 \times A_2 \times A_1 \times A_2$ satisfying the following two conditions:
  (1) The elements of S come in equivalence classes:
  $$\{(a, b, a', b'), (\overline{a}, \overline{b'}, \overline{a'}, \overline{b}), (\overline{a'}, \overline{b}, \overline{a}, \overline{b'}), (a', b', a, b)\} \subset S$$
  (2) *(Link condition)* For each pair $(a, b) \in A_1 \times A_2$ there is a unique pair $(a', b') \in A_1 \times A_2$ such that $(a, b, a', b') \in S$.

As in Example 1.1.1 this data determines a 2-dimensional 1-vertex square complex X with a single vertex denoted x_0, $n_1 = \frac{1}{2}|A_1|$ horizontal geometric loops, $n_2 = \frac{1}{2}|A_2|$ vertical geometric loops and $n_1 \cdot n_2 = \frac{1}{4}|S|$ geometric squares. The complex X is the quotient $\Gamma\backslash(T_1 \times T_2)$ where each T_i is a $2n_i$ regular tree and $\Gamma = \pi_1(X, x_0)$. The complex X provides also a presentation for Γ, namely:

$$\Gamma = \langle A_1 \cup A_2 \mid aba'b',\ (a, b, a', b') \in S,\ a\overline{a}, a \in A_1,\ b\overline{b},\ b \in A_2\rangle.$$

The local transitivity properties of Γ on each factor can be easily read off the complex X by considering the holonomy action of elements corresponding to oriented horizontal loops on vertical edges and vice versa. Fixing a vertex $(x_1, x_2) \in T_1 \times T_2$ the group $\langle A_1\rangle$, which is a free group on $\frac{1}{2}|A_1|$ generators, may be identified with the stabilizer $\mathrm{Stab}_\Gamma(x_2)$. The covering map $T_1 \times T_2 \to X$ induces a labeling of the oriented 1-skeleton of $T_1 \times T_2$ by the elements of $A_1 \cup A_2$. We may identify T_1 with the tree consisting of the connected component of the "horizontal 1-skeleton" of $T_1 \times T_2$ containing the chosen vertex (x_1, x_2). This induces a labeling of the oriented edges of T_1 by the elements of A_1 and similarly we have a labeling of the oriented edges of T_2 by the elements of A_2. Observe that paths without backtracking of length k starting at the vertex $x_2 \in T_2$ correspond to $A_2^{(k)}$ which is the set of irreducible words of length k over A_2. The action of $\langle A_1\rangle$ on $A_2^{(k)}$, that is the permutation representation $\rho_{12}^{(k)} : \langle A_1\rangle \to \mathrm{Sym}A_2^{(k)}$ can be read directly from the complex by observing that for $a \in A_1$ and $w \in A_2^{(k)}$ there exist a unique pair: $a' \in A_1$, $w' \in A_2^{(k)}$ such that $a \cdot w' = w \cdot a'$. We have $\rho_{12}^{(k)}(a)(w) = w'$. Exchanging the roles of the trees and the generating sets we obtain: $\rho_{21}^{(k)} : \langle A_2 \rangle \to \mathrm{Sym} A_1^{(k)}$. Let us define

$$\begin{aligned} P_2^{(k)} &= \mathrm{Im}\, \rho_{12}^{(k)} < \mathrm{Sym} A_2^{(k)} \\ P_1^{(k)} &= \mathrm{Im}\, \rho_{21}^{(k)} < \mathrm{Sym} A_1^{(k)} \end{aligned}$$

We shall omit the superscript (k) when $k = 1$. We will say that X is a (P_1, P_2)-complex when we want to emphasize the local permutation groups $P_1 < \mathrm{Sym} A_1$ and $P_2 < \mathrm{Sym} A_2$.

Example 1.1.19 ([Rat04] Theorem 2.3). The (A_6, A_6)-complex X given by:

$$\{a_1 b_1 \overline{a_1} \overline{b_1}, a_1 b_2 \overline{a_1} \overline{b_3}, a_1 b_3 a_2 \overline{b_2}, a_1 \overline{b_3} \overline{a_3} b_2, a_2 b_1 \overline{a_3} \overline{b_2},$$
$$a_2 b_2 \overline{a_3} \overline{b_3}, a_2 b_3 \overline{a_3} b_1, a_2 \overline{b_3} a_3 b_2, a_2 \overline{b_1} \overline{a_3} \overline{b_1}\}$$

Satisfies the following:

(1) Any non-trivial normal subgroup of $\pi_1(X)$ is of finite index.
(2) Any linear representation of $\pi_1(X)$ in characteristic zero has finite image.
(3) $\mathrm{Out}(\pi_1(X)) \equiv \mathbb{Z}/2\mathbb{Z}$.

We will say that a complex X is irreducible if the lattice $\Gamma = \pi_1(X)$ is irreducible, namely when no finite covering of X is a product of two graphs. It is a fundamental problem whether there is an algorithm for deciding if a given X is irreducible. There is however a sufficient condition based on the Thompson-Wielandt Theorem, see Proposition 1.1.16, which we describe next. Assume that $P_i < \mathrm{Sym} A_i$ is transitive, $i = 1, 2$. Fix an edge e_i in T_i and let $B_r(e_i) \subset T_i$ denote the neighbourhood of radius r around e_i, let Q_i denote the restriction to $B_2(e_i)$ of the subgroup of $P_i^{(3)}$ consisting of elements pointwise fixing $B_1(e_i)$. It follows from the Thompson-Wielandt theorem 1.1.16 that if for some $i = 1, 2$ $Q_i \neq \{e\}$ and is not a p-group then the lattice is irreducible. In fact as a corollary of the arithmeticity theorem a much stronger assertion holds:

Corollary 1.1.20. *Let X be a (P_1, P_2)-complex where $P_i < \mathrm{Sym} A_i$ are primitive permutation groups, $i = 1, 2$, and assume that $Q_1 \neq \{e\}$ and is not a p-group. Then any linear representation of $\pi_1(X)$ over a field of characteristic zero has finite image.*

In applying the criterion for non-residually finiteness (Proposition 1.1.11) one needs to detect whether the projection of Γ on one of the factor, say $\mathrm{Aut} T_2$, has a non-trivial kernel. In other words whether there exists an element $w \in \langle A_1 \rangle$ whose action on T_2 is trivial. Observe that while verifying whether a given element $w \in \langle A_1 \rangle$ acts trivially on T_2 is a (easily) computable question, we do not know whether the existence of such an element is decidable. We also do not know an algorithm for deciding whether the quasi-center of $H_1 = \overline{\mathrm{pr}_1(\Gamma)}$ is trivial or not. There is however, a construction – called fibered product, see [BM00b] §2.2, 2.3 – which starting with a 1-vertex square complex X produces a new 1-vertex square complex $X \boxtimes X$ which in suitable settings has a non-residually finite fundamental group whose closures of projections have non-trivial quasi-centers. Let X be given by the data: A_1, A_2 and $S \subset A_1 \times A_2 \times A_1 \times A_2$ then $X \boxtimes X$ is defined by the data: $A_1 \times A_1$, $A_2 \times A_2$ and $R = \{((a_1, a_2), (b_1, b_2), (a'_1, a'_2), (b'_1, b'_2)) : (a_i, b_i, a'_i, b'_i) \in S,\ i = 1, 2\}$. If X is a (P_1, P_2)-complex then $X \boxtimes X$ is a $(P_1 \times P_1, P_2 \times P_2)$-complex and hence $\pi_1(X \boxtimes X)$ can never have locally primitive projections. However, see [BM00b] §2.2, it is contained as an index 2 subgroup in a group which is locally primitive when P_1 and P_2 are primitive. The natural homomorphism

$$\pi_1(X \boxtimes X) \to \pi_1(X) \times \pi_1(X) \tag{1.1.4}$$

has image of index 1, 2 or 4 and its kernel defines an infinite covering which is of the form $\mathcal{D}_1 \times \mathcal{D}_2$ where each $\mathcal{D}_i$ is a graph whose fundamental group is an infinitely generated free group. Using [BM00b] Prop. 2.1 we showed:

Proposition 1.1.21. *If X is an irreducible (P_1, P_2)-square complex where each P_i is a 2-transitive permutation group with 2-transitive socle*[1] *then $\pi_1(X \boxtimes X)$ is not residually finite; in fact $\pi_1(X \boxtimes X)^{(\infty)} \supset \pi_1(\mathcal{D}_1) \times \pi_1(\mathcal{D}_2)$.*

Examples of 2-transitive permutation groups with 2-transitive socle include S_n, A_n ($n \geq 5$), $\mathrm{PSL}(2, \mathbb{F}_q)$. See [Cam95, Table 5.1(b)] and [DM96, 7.7] for a complete list of such groups. Composing the homomorphism in (1.1.4) with the projection on either factor we obtain a surjective homomorphism, $\pi_1(X \boxtimes X) \to \pi_1(X)$ showing that rigidity fails in a strong way since not even the universal covering spaces of $X \boxtimes X$ and X are isomorphic. However in §1.4 we will show:

Corollary 1.1.22. *Let X be an irreducible (P_1, P_2)-complex and Y a (Q_1, Q_2)-complex. Assume that P_1 and P_2 are primitive permutation groups. Then any surjective homomorphism $\varphi : \pi_1(X) \to \pi_1(Y)$ is induced by an isomorphism $f : X \to Y$. In particular φ is an isomorphism and each Q_i is permutation isomorphic to P_i (up to reindexing).*

1.2. Locally quasiprimitive groups and *p*-adic structure

The basic objective of this chapter is to show that a locally quasiprimitive group H which admits a continuous representation into $\mathrm{GL}(n, \mathbb{Q}_p)$ with unbounded image is p-adic analytic, and to investigate this p-adic structure more closely. First we establish a general fact about continuous representations of totally disconnected groups. This will rest on the following consequence of the Howe-Moore Theorem [HM79]. We have:

Theorem 1.2.1. *Let $\mathbb{G}$ be a $\mathbb{Q}_p$-almost simple group and $O < \mathbb{G}(\mathbb{Q}_p)$ an open unbounded subgroup. Then $O \supset \mathbb{G}^+$*

Proof. We may clearly assume that $\mathbb{G}$ is connected. Then $\delta_O \in \ell^2(\mathbb{G}(\mathbb{Q}_p)/O)$ is a vector which is invariant under the unbounded group O and hence by the Howe-Moore theorem is $\mathbb{G}^+$-invariant. □

[1] The *socle* is the subgroup generated by all minimal normal subgroups.

Remark 1.2.2. To prove Theorem 1.2.1 one needs in fact only the local field analogue of Mautner phenomenon — as proven by C. Moore for semisimple real Lie groups in [Moo66, Theorem 1]. The arguments of Howe–Moore in [HM79, Proposition 5.5] can be used to give a direct proof of the local field analogue of [Moo66, Theorem 1] from which Theorem 1.2.1 follows.

Lemma 1.2.3. *Let H be a locally compact totally disconnected group, $K < H$ an open compact subgroup and $\mathbb{G}$ a $\mathbb{Q}_p$-almost simple group. Let $\pi : H \to \mathbb{G}(\mathbb{Q}_p)$ be a continuous homomorphism with Zariski dense image. Then one of the following holds:*

(1) *$\pi(K)$ is finite.*
(2) *$\pi(K)$ is open and $\pi(H)$ is compact.*
(3) *$\pi(K)$ is open and $\pi(H) \supset \mathbb{G}^+$.*

Proof. Denote by $\mathfrak{G}$ the Lie algebra of $\mathbb{G}$. If the closed subgroup $\pi(K)$ is infinite, its Lie algebra $\mathcal{K} \subset \mathfrak{G}(\mathbb{Q}_p)$ is of positive dimension. As $\pi(H)$ commensurates $\pi(K)$ it follows that $\mathrm{Ad}\,\pi(H)$ leaves $\mathcal{K}$ invariant. Hence by the Zariski density of $\pi(H)$ it follows that $\mathcal{K} = \mathfrak{G}(\mathbb{Q}_p)$, implying that $\pi(K)$ is open. If $\pi(H)$ is not compact it is open and unbounded and hence by Theorem 1.2.1 we have $\pi(H) \supset \mathbb{G}^+$. □

Before proceeding we need a few more results from [BM00a] concerning locally quasiprimitive groups.

Proposition 1.2.4 (see [BM00a, Prop. 1.2.1]). *Let T be a tree and $H <$ Aut$T$ a closed non-discrete locally quasiprimitive subgroup. we have:*

(1) *$\mathrm{QZ}(H^{(\infty)})$ is discrete, in fact, $\mathrm{QZ}(H^{(\infty)}) = H^{(\infty)} \cap \mathrm{QZ}(H)$.*
(2) *for any open normal subgroup $NH^{(\infty)}$ we have $N = H^{(\infty)}$.*
(3) *$H^{(\infty)} = \overline{[H^{(\infty)}, H^{(\infty)}]}$.*

Lemma 1.2.3 applied to locally quasiprimitive groups implies the following:

Lemma 1.2.5. *Let $H <$ Aut$T$ be a closed non-discrete locally quasiprimitive group. Let L be a closed subgroup, $H^{(\infty)} < L < H$, $\mathbb{G}$ an almost $\mathbb{Q}_p$-simple group and $\pi : L \to \mathbb{G}(\mathbb{Q}_p)$ a continuous homomorphism with Zariski dense image. Then either*

(1) *$\pi(L)$ is compact and hence $\pi(H^{(\infty)}) = \{e\}$. or*

(2) (a) *$\pi(H^{(\infty)}) = \mathbb{G}^+$,*
 (b) *$\pi(\mathrm{QZ}(H^{(\infty)})) \subset Z(\mathbb{G}(\mathbb{Q}_p))$,*
 (c) *$\pi(K)$ is open for any compact open subgroup $K < H^{(\infty)}$.*

Proof. Assume that $\pi(L)$ is not compact, then $\pi(H^{(\infty)})$ is not finite (Theorem 1.1.7.1) in particular not central and being normal in a Zariski dense subgroup of an almost $\mathbb{Q}_p$-simple group it is itself Zariski dense. Now we apply the trichotomy of Lemma 1.2.3 to π, $H^{(\infty)}$ and K. We have:

(1) If $\pi(K)$ is finite then $\ker \pi \lhd H^{(\infty)}$ is open and hence (by Prop. 1.2.4.2) we have $\ker \pi = H^{(\infty)}$ and (by Theorem 1.1.7.1) hence $\pi(L)$ is compact, contradiction.
(2) If $\pi(K)$ is open and $\pi(H^{(\infty)})$ is compact then $\pi(H^{(\infty)})$ is profinite; by Proposition 1.2.4.2 any profinite quotient of $H^{(\infty)}$ is trivial, hence $\pi(H^{(\infty)}) = \{e\}$, thus $\pi(L)$ is compact, contradiction.

(3) We have that $\pi(K)$ is open and $\pi(H^{(\infty)}) \supset \mathbb{G}^+$. Since $\mathbb{G}/\mathbb{G}^+$ is finite one has $\pi(H^{(\infty)}) = \mathbb{G}^+$. Finally $\pi(\mathrm{QZ}(H^{(\infty)}))$ is a countable subgroup of $\mathbb{G}(\mathbb{Q}_p)$ normalized by $\mathbb{G}^+$ hence contained in $Z(\mathbb{G}(\mathbb{Q}_p))$. $\square$

The preceding lemma gives useful information about homomorphisms of locally quasiprimitive groups into almost simple groups. The next lemma says that in a continuous representation there is always a semisimple part.

Lemma 1.2.6. *Let H be a non-discrete, locally quasiprimitive group. Let $\pi : H^{(\infty)} \to \mathrm{GL}(n, \mathbb{Q}_p)$ be a continuous non-trivial representation and $\mathbb{L}$ be the Zariski closure of $\pi(H^{(\infty)})$. Then*

(1) $\mathbb{L}$ *is connected.*
(2) $\mathbb{L}/\operatorname{Rad}(\mathbb{L})$ *is of positive dimension.*

Proof. Since $H^{(\infty)}$ does not have non-trivial continuous finite images it follows that $\mathbb{L}^\circ = \mathbb{L}$ and $\mathbb{L}$ is connected. Assertion 2 follows from the fact that $\mathbb{L}$ is connected and $H^{(\infty)}$ is topologically perfect (see Prop. 1.2.4.3). $\square$

It follows that if there exists a non-trivial continuous representation $\pi : H^{(\infty)} \to \mathrm{GL}(n, \mathbb{Q}_p)$ then there exists a $\mathbb{Q}_p$-simple, connected, adjoint group $\mathbb{G}$ of positive dimension and a continuous homomorphism $\rho_0 : H^{(\infty)} \to \mathbb{G}(\mathbb{Q}_p)$ with Zariski dense image. Thus we can use Lemma 1.2.5 with $L = H^{(\infty)}$. Observe that $\rho_0(H^{(\infty)})$ cannot be trivial, hence

- $\rho_0(H^{(\infty)}) = \mathbb{G}^+$
- $\mathrm{QZ}(H^{(\infty)}) \subset \ker \rho_0$

We recall a few results from [BM00a] concerning normal subgroups of $H^{(\infty)}$. Let $\mathcal{M}$ be the set of closed normal subgroups M of $H^{(\infty)}$, which are minimal with respect to the property $M \supsetneq \mathrm{QZ}(H^{(\infty)})$. Then (see Theorem 1.1.8.1) the set $\mathcal{M}$ is non-void and contains finitely many elements. These elements are H-conjugate and their product is dense in $H^{(\infty)}$.

In our setting notice that since $\mathrm{QZ}(H^{(\infty)}) \subset \ker \rho_0 \subsetneq H^{(\infty)}$ it follows from Theorem 1.1.8.1 that there exists $M_1 \in \mathcal{M}$, such that $M_1 \not\subset \ker \rho_0$. Let also $h_1 = e, h_2, \dots, h_n \in H$ be such that

$$\mathcal{M} = \{h_i M_1 h_i^{-1} : 1 \le i \le n\}$$

and define $\rho_i : H^{(\infty)} \to \mathbb{G}(\mathbb{Q}_p)$ by $\rho_i(h) = \rho_0(h_i^{-1} h h_i)$. Then the following holds:

Lemma 1.2.7. *The homomorphism $\rho : H^{(\infty)} \to (\mathbb{G}^+)^n$ given by $\rho(h) = (\rho_i(h))_{1 \le i \le n}$ induces a topological isomorphism*

$$H^{(\infty)}/\mathrm{QZ}(H^{(\infty)}) \to (\mathbb{G}^+)^n.$$

Proof. Since $\mathbb{G}^+$ is simple and $\rho_0(M_1)$ is a non-trivial normal subgroup of $\rho_0(H^{(\infty)}) = \mathbb{G}^+$ we have that $\rho_0(M_1) = \mathbb{G}^+$ and hence $M_1 \ker \rho_0 = H^{(\infty)}$. We claim that $M_2 \cdots M_n \subset \ker \rho_0$. Since $M_1 \cap M_i = \mathrm{QZ}(H^{(\infty)})$ (by minimality), it follows that for $2 \le i \le n$, $[M_1, M_i] \subset \mathrm{QZ}(H^{(\infty)})$ and hence $\rho_0(M_1) = \mathbb{G}^+$ commutes with

$\rho_0(M_i)$ which implies $\rho_0(M_i) = e$, $2 \leq i \leq n$, and the claim is proved. We conclude that $\rho(M_1 \cdots M_n) = (\mathbb{G}^+)^n$. Note also $\ker \rho \supset \mathrm{QZ}(H^{(\infty)})$ and in fact one has equality since otherwise (see [BM00a] Prop. 1.5.1(3)) $\ker \rho$ would contain one of the M_i's. $\square$

The following is one of the main results of this chapter:

Theorem 1.2.8. *Let $H < \mathrm{Aut}T$ be a closed non-discrete locally quasiprimitive subgroup, and let $H^{(\infty)} < L < H$ be a closed subgroup. Assume that there exists a continuous, unbounded representation:*

$$\pi : L \to \mathrm{GL}(m, \mathbb{Q}_p).$$

Then

(1) *H is a $\mathbb{Q}_p$-analytic group and $H^{(\infty)}$ is of finite index in H.*
(2) *Its Lie algebra is a product $\mathrm{Lie}(H) = \mathfrak{G}^n$ where $\mathfrak{G}$ is a $\mathbb{Q}_p$-simple Lie algebra.*
(3) *The adjoint representation $\mathrm{Ad} : H \to \mathrm{Aut}(\mathfrak{G}^n)$ has kernel equal $\mathrm{QZ}(H)$ and induces an isomorphism: $H^{(\infty)}/\,\mathrm{QZ}(H^{(\infty)}) \to (\mathbb{G}^+)^n$ where $\mathbb{G}$ is the connected $\mathbb{Q}_p$-simple adjoint group attached to $\mathfrak{G}$.*

Proof. Applying Lemmas 1.2.5, 1.2.6 we are in the setting of Lemma 1.2.7. As both $H^{(\infty)}$ and $\mathrm{QZ}(H^{(\infty)})$ are normalized by H, H acts by conjugation on $H^{(\infty)}/\,\mathrm{QZ}(H^{(\infty)})$ and we obtain via ρ a continuous action of H on $(\mathbb{G}^+)^n$, which we denote $\epsilon : H \to \mathrm{Aut}((\mathbb{G}^+)^n)$, with $\epsilon(H^{(\infty)}) = \mathrm{Inn}((\mathbb{G}^+)^n)$. The group $H' = \epsilon^{-1}(\mathrm{Inn}((\mathbb{G}^+)^n))$ is a closed subgroup of finite index in H. Then $H^{(\infty)} \cdot \ker \epsilon = H'$, but since $\ker \epsilon$ is a closed normal subgroup which is not cocompact, we deduce (using Prop. 1.1.7.1) that $\ker \epsilon \subset \mathrm{QZ}(H)$, in particular $H^{(\infty)}$ is of countable index in H and hence (by Baire category) of finite index. Assertion (1) is proved. The remaining statements follow by repeated application of Lemma 1.2.7. $\square$

We finish this section with a result which gives information on homomorphisms from a totally disconnected group into a semisimple group over $\mathbb{Q}_p$.

Proposition 1.2.9. *Let H be a locally compact, totally disconnected second countable group, $\mathbb{G}_1$, $\mathbb{G}_2$ connected adjoint semisimple $\mathbb{Q}_p$-groups such that $\mathbb{G}_2$ is $\mathbb{Q}_p$-simple and every $\mathbb{Q}_p$-simple factor of $\mathbb{G}_1$ has positive $\mathbb{Q}_p$-rank. Let*

$$\pi : H \to \mathbb{G}_1(\mathbb{Q}_p) \times \mathbb{G}_2(\mathbb{Q}_p)$$

be a continuous homomorphism such that $\mathrm{pr}_1(\pi(H)) \supset \mathbb{G}_1^+$ and $\mathrm{pr}_2(\pi(H)) \supset \mathbb{G}_2^+$. Then one of the following holds:

(1) *$\pi(H) \supset \mathbb{G}_1^+ \times \mathbb{G}_2^+$.*
(2) *There exists a morphism $\omega : \mathbb{G}_1 \to \mathbb{G}_2$ of algebraic groups, defined over $\mathbb{Q}_p$ such that $\pi(H) \subset Graph_\omega(\mathbb{Q}_p) = \{(g, \omega(g)) : g \in \mathbb{G}_1(\mathbb{Q}_p)\}$.*

Proof. The intersection $\pi(H) \cap (e \times \mathbb{G}_2(\mathbb{Q}_p))$ is normalized by $e \times \mathbb{G}_2^+$ and hence either contains $e \times \mathbb{G}_2^+$ or is trivial. In the first case we deduce that $\pi(H) \supset \mathbb{G}_1^+ \times \mathbb{G}_2^+$. Assume that the second case occurs. Let $K < H$ be a compact open subgroup. Since

H/K is countable it follows that $\mathrm{pr}_i(\pi(K))$ is a compact subgroup of $\mathbb{G}_i(\mathbb{Q}_p)$ which is of countable index in $\mathrm{pr}_i(\pi(H)) \supset \mathbb{G}_i^+$ and hence (as can be seen for example using the Baire category theorem) is open in $\mathbb{G}_i(\mathbb{Q}_p)$. Let $\mathfrak{G}_i = \mathrm{Lie}(\mathbb{G}_i)$ and since $\pi(K)$ is a closed subgroup of the p-adic Lie group $\mathbb{G}_1(\mathbb{Q}_p) \times \mathbb{G}_2(\mathbb{Q}_p)$, denote by $\mathcal{K}$ its Lie algebra. Since $\mathrm{pr}_i(\pi(K))$ is open in $\mathbb{G}_i(\mathbb{Q}_p)$ we have $\mathrm{pr}_i(\mathcal{K}) = \mathfrak{G}_i(\mathbb{Q}_p)$. Moreover since $\pi(K) \cap (e \times \mathbb{G}_2(\mathbb{Q}_p)) = \{e\}$, we have that

$$\mathcal{K} \cap (0 \oplus \mathfrak{G}_2(\mathbb{Q}_p)) = \{0\}. \tag{1.2.1}$$

Let then $\mathbb{L} \subset \mathbb{G}_1 \times \mathbb{G}_2$ denote the Zariski closure of $\pi(H)$. Since $\pi(H)$ commensurates $\pi(K)$, this implies that $\pi(H)$ acts on $\mathcal{K}$ by adjoint action and thus $\mathbb{L}(\mathbb{Q}_p)$ preserves $\mathcal{K}$. Together with (1.2.1) this implies that $\mathbb{L} \neq \mathbb{G}_1 \times \mathbb{G}_2$ and since $\mathrm{pr}_i(\mathbb{L}) = \mathbb{G}_i$, this implies that $\mathbb{L} \cap e \times \mathbb{G}_2 = \{e\}$ and hence $\mathbb{L}$ is the graph of a morphism of algebraic groups $\omega : \mathbb{G}_1 \to \mathbb{G}_2$ defined over $\mathbb{Q}_p$. □

A simple inductive argument gives:

Corollary 1.2.10. *Assume that $\mathbb{G}_1, \ldots, \mathbb{G}_n$ are connected $\mathbb{Q}_p$-almost simple of positive rank and $\pi : H \to \prod_{i=1}^n \mathbb{G}_i(\mathbb{Q}_p)$ is a continuous homomorphism so that*

(1) $\mathrm{pr}_i(H) \supset \mathbb{G}_i^+$.
(2) *$\pi(H)$ is Zariski dense.*

Then $\pi(H) \supset \prod_{i=1}^n \mathbb{G}_i^+$.

1.3. Arithmeticity

Our goal in this section is to prove Theorem 1.1.17. A cocompact lattice $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ is said to be *LQP-sandwiched* if there exist closed non-discrete, locally quasiprimitive subgroups $H_i < \mathrm{Aut}T_i$ such that

$$H_i{}^{(\infty)} < \overline{\mathrm{pr}_i(\Gamma)} < H_i$$

Observe that since $H_i{}^{(\infty)}$ has no proper closed subgroup of finite index it follows that every finite index subgroup of Γ is also LQP-sandwiched.

Recall

Proposition 1.3.1 ([BM00b, Cor. 3.3]). *Let $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ be a cocompact lattice which is LQP-sandwiched. Then*

(1) *Its abelianization Γ_{ab} is finite.*
(2) *For any unitary representation $\omega : \Gamma \to \mathrm{U}(n)$, $H^1(\Gamma, \omega) = 0$.*

Combining Theorem 1.1.3 and Proposition 1.3.1 we obtain:

Corollary 1.3.2. *Let $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ be a LQP-sandwiched lattice. Then:*

(1) *Any homomorphism $\pi : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ has bounded image.*
(2) *Let k be a field of characteristic zero and $\pi : \Gamma \to \mathrm{GL}(n, k)$ a homomorphism, then the Zariski closure $\overline{\pi(\Gamma)}^Z$ is semisimple.*

Proof. In (2) we may replace the field k by $\mathbb{C}$ and we will prove both (1) and (2) simultaneously. Let $\mathbb{L}$ be the connected component of the Zariski closure of the image of Γ and let $\Gamma' < \Gamma$ be a subgroup of finite index with $\overline{\pi(\Gamma')}^Z = \mathbb{L}$. Since Γ'_{ab} is finite and its image in $\mathbb{L}/[\mathbb{L},\mathbb{L}]$ is Zariski dense, the latter is trivial and hence the radical $\mathrm{Rad}(\mathbb{L})$ coincides with the unipotent radical $\mathrm{Rad}_u(\mathbb{L})$. Let $\mathbb{S} = \mathbb{L}/\mathrm{Rad}(\mathbb{L})$ and $\overline{\pi} : \Gamma' \to \mathbb{S}$ be the composition of π with the canonical projection $\mathbb{L} \to \mathbb{S}$. If $\overline{\pi}(\Gamma)$ were unbounded then there would be a simple quotient $\mathbb{S} \to \mathbb{S}_1$ such that composing with $\overline{\pi}$ we would have a homomorphism $\pi_1 : \Gamma' \to \mathbb{S}_1$ with Zariski dense unbounded image. Setting $P_i = \overline{\mathrm{pr}_i(\Gamma')}$, we are in the setting of Theorem 1.1.3 and π_1 extends to a continuous homomorphism factoring via, say, P_1. Let $\pi_{1,ext} : P_1 \to \mathbb{S}_1$ be this continuous homomorphism. Since P_1 is totally disconnected, $\ker \pi_{1,ext} \lhd P_1$ is open. Since $P_1 \supset {H_1}^{(\infty)}$, we have $\ker \pi_{1,ext} \supset {H_1}^{(\infty)}$ (Proposition 1.2.4.2) and hence the image of $\pi_{1,ext}$ is compact, implying that $\pi_1(\Gamma')$ is bounded, a contradiction. Consider now $\mathbb{L}/[\mathrm{Rad}_u(\mathbb{L}), \mathrm{Rad}_u(\mathbb{L})] = \mathbb{S} \cdot V$, the semidirect product of $\mathbb{S}$ with a vector space V. The corresponding homomorphism $\Gamma' \to \mathbb{S} \cdot V$ has the form $\gamma \mapsto \overline{\pi}(\gamma) \cdot c(\gamma)$, where $\overline{\pi}(\Gamma')$ is bounded and $c : \Gamma \to V$ is a 1-cocycle with values in a bounded representation of Γ'. It follows from Prop 1.3.1.2 that this cocycle is trivial and hence that the image of Γ' in $\mathbb{S} \cdot V$ is contained in a conjugate of $\mathbb{S}$. Since the image is Zariski dense it follows that $V = 0$. I.e., $\mathrm{Rad}_u(\mathbb{L}) = [\mathrm{Rad}_u(\mathbb{L}), \mathrm{Rad}_u(\mathbb{L})]$ and hence we have $\mathrm{Rad}_u(\mathbb{L}) = 0$. □

We introduce the following terminology: Let k be a local field. A *k-triple* is a triple $(\Gamma', \pi, \mathbb{H})$ where $\Gamma' < \Gamma$ is a subgroup of finite index, $\mathbb{H}$ is a connected k-simple adjoint group of positive dimension and $\pi : \Gamma' \to \mathbb{H}(k)$ is a homomorphism with Zariski dense image. We say that it is of *unbounded type* if $\pi(\Gamma')$ is unbounded in $\mathbb{H}(k)$.

Lemma 1.3.3. *Let* $(\Gamma', \pi, \mathbb{H})$ *be a k-triple. Then there exist a finite extension* $\mathbb{Q} \subset K \subset k$ *of* $\mathbb{Q}$ *and a K-structure on* $\mathbb{H}$ *so that* $\pi(\Gamma') \subset \mathbb{H}(K)$.

Proof. Consider $\mathbb{H}$ as a subgroup of $\mathrm{GL}(n,k)$ via the adjoint representation, fix an embedding $\imath : k \to \mathbb{C}$ and let us abuse notation and denote by $\imath$ also the induced group homomorphism $\imath : \mathrm{GL}(n,k) \to \mathrm{GL}(n,\mathbb{C})$. The homomorphism $\imath \circ \pi : \Gamma' \to \mathrm{GL}(n,\mathbb{C})$ as well as any twist of it by an automorphism of $\mathbb{C}$ have bounded image (Corollary 1.3.2). It follows that for any $\gamma \in \Gamma'$ $\mathrm{tr}(\imath\circ\pi(\gamma)) = \imath(\mathrm{tr}\,\pi(\gamma))$ is an algebraic number. Since $\imath$ is the identity on $\mathbb{Q}$, it follows that $\mathrm{tr}\,\pi(\gamma)$ is in $\overline{\mathbb{Q}} \cap k$ for any $\gamma \in \Gamma'$. Since Γ' is finitely generated it follows that the field generated by all these traces is a finite extension of $\mathbb{Q}$ which we denote by K. It follows now by a standard argument using the Zariski density of $\pi(\Gamma')$ that there is a faithful k-rational representation $\rho : \mathbb{H} \to \mathrm{GL}(V)$ such that $\rho(\mathbb{H})$ is defined over K and $\rho(\pi(\Gamma')) \subset \rho(\mathbb{H})(K)$. □

Lemma 1.3.4. *Assume that there is a representation* $\rho : \Gamma \to \mathrm{GL}(n,\mathbb{C})$ *with infinite image. Then there exists a* $\mathbb{Q}_p$*-triple of unbounded type.*

Proof. By Corollary 1.3.2.2 the Zariski closure $\overline{\rho(\Gamma)}^Z$ is semisimple. Passing to a simple quotient we get a $\mathbb{C}$-triple $(\Gamma', \pi, \mathbb{H})$. Endow $\mathbb{H}$ with the K-structure given by Lemma 1.3.3. Then in particular $\pi(\Gamma') \subset \mathbb{H}(K)$. Let $\mathbb{L} = \mathrm{Res}_{K/\mathbb{Q}}\mathbb{H}$ and let

$\Delta : \mathbb{H}(K) \to \mathbb{L}(\mathbb{Q})$ be the "diagonal" isomorphism. Let S be the set of primes p for which $\Delta(\pi(\Gamma'))$ is unbounded in $\mathbb{L}(\mathbb{Q}_p)$. Using Corollary 1.3.2.2 as above it suffices to show that S is non-empty. Assume that $S = \emptyset$; then $\Delta(\pi(\Gamma'))$ is up to finite index, contained in $\mathbb{L}(\mathbb{Z})$, hence discrete. On the other hand $\Delta(\pi(\Gamma')) \subset \mathbb{L}(\mathbb{R})$ is bounded (Corollary 1.3.2.1) which implies that $\Delta(\pi(\Gamma'))$ is finite, contradicting the fact that $\pi(\Gamma')$ is Zariski dense in the non-trivial connected group $\mathbb{H}$. □

1.3.1. *Proof of the arithmeticity theorem 1.1.17*

Let P_Γ denote the set of primes p for which there exists a $\mathbb{Q}_p$-triple of unbounded type. According to Lemma 1.3.4, P_Γ is non-void. On the other hand let $(\Gamma', \pi, \mathbb{H})$ be a $\mathbb{Q}_p$-triple of unbounded type and denote $L_i = \overline{\mathrm{pr}_i(\Gamma')}$. Theorem 1.1.3 implies that π extends continuously to $L_1 \times L_2$ factoring via, say, L_1. Thus we obtain a continuous unbounded linear representation $L_1 \to \mathbb{H}(\mathbb{Q}_p)$. By Theorem 1.2.8 it follows that L_1 is $\mathbb{Q}_p$-analytic. This implies that $|P_\Gamma| \in \{1, 2\}$. Since the case where $|P_\Gamma| = 1$ is slightly more involved we shall describe the argument for it (leaving the other case to the reader) and assume henceforth that $|P_\Gamma| = 1$.

Claim 1.3.5. Given any $\mathbb{Q}_p$-triple $(\Gamma', \pi, \mathbb{H})$ of unbounded type. There are $\mathbb{Q}_p$-triples $(\Gamma'', \pi_i, \mathbb{H}_i)$, $1 \le i \le n$, of unbounded type such that $\Gamma'' < \Gamma'$ is of finite index, $\mathbb{H}_1 = \mathbb{H}$, $\pi_1 = \pi_{|_{\Gamma''}}$ and the product homomorphism $\prod \pi_i : \Gamma'' \to \prod_{i=1}^n \mathbb{H}_i(\mathbb{Q}_p)$ has Zariski dense discrete image.

Endow $\mathbb{H}$ with the K-structure given by lemma 1.3.3; $\mathbb{Q} \subset K \subset \mathbb{Q}_p$. Let $\Delta : \mathbb{H}(K) \to \mathrm{Res}_{K/\mathbb{Q}}\mathbb{H}(\mathbb{Q})$ be the diagonal isomorphism, $\mathbb{L}$ the connected component of the Zariski closure of the image of Γ' and $\Gamma'' = (\Delta \circ \pi)^{-1}(\mathbb{L}(\mathbb{Q}))$. Then $\mathbb{L}$ is defined over $\mathbb{Q}$, semisimple (see Corollary 1.3.2) and Γ'' is of finite index in Γ'. Since p is the only prime for which $\Delta \circ \pi(\Gamma'')$ is unbounded in $\mathbb{L}(\mathbb{Q}_p)$, we deduce that, up to a subgroup of finite index, $\Delta\pi(\Gamma'')$ is contained in $\mathbb{L}(\mathbb{Z}[1/p])$; since moreover the image in $\mathbb{L}(\mathbb{R})$ if $\Delta(\pi(\Gamma''))$ is bounded (Corollary 1.3.2) we conclude that $\Delta(\pi(\Gamma''))$ is discrete in $\mathbb{L}(\mathbb{Q}_p)$. If one lets now $\mathbb{H}_1, \ldots, \mathbb{H}_n$ be the $\mathbb{Q}_p$-simple adjoint quotients $p_i : \mathbb{L} \to \mathbb{H}_i$ of $\mathbb{L}$ such that $(\Gamma'', \pi_i, \mathbb{H}_i)$, with $\pi_i = p_i \circ \Delta \circ \pi$, is of unbounded type, then it is an easy verification that those fulfill the claim.

Let now $(\Gamma', \pi, \mathbb{H})$ be a $\mathbb{Q}_p$-triple of unbounded type and let $(\Gamma'', \pi, \mathbb{H}_i)$ be the triples given by the above claim. Each $\pi_i : \Gamma'' \to \mathbb{H}_i(\mathbb{Q}_p)$ extends continuously to $L_1 \times L_2$ (where $L_i = \overline{\mathrm{pr}_i(\Gamma'')}$) factoring via L_1 or L_2. Without loss of generality, assume that for $1 \le i \le r$, the extension factors via L_1 and let $\overline{\pi}_i : L_1 \to \mathbb{H}_i(\mathbb{Q}_p)$ be this continuous homomorphism and for $r + 1 \le i \le n$ it factors via L_2 giving rise to a $\overline{\pi}_i : L_2 \to \mathbb{H}_i(\mathbb{Q}_p)$. Let $\alpha_1 = \prod_{i=1}^r \overline{\pi}_i$, $\alpha_2 = \prod_{i=r+1}^n \overline{\pi}_i$. It follows from Lemma 1.2.5.2 and Lemma 1.2.10 that $\alpha(H_1{}^{(\infty)}) = \prod_{i=1}^r \mathbb{H}_i^+$ and $\alpha_2(H_2{}^{(\infty)}) = \prod_{i=r+1}^n \mathbb{H}_i^+$.

Claim 1.3.6. $r < n$

Indeed if $r = n$ then the continuous extension of $\prod_{i=1}^n \pi_i : \Gamma'' \to \prod_{i=1}^n \mathbb{H}_i(\mathbb{Q}_p)$ to $L_1 \times L_2$ factors via the projection to L_1, the extension being given by $(\ell_1, \ell_2) \mapsto$

$\alpha_1(\ell_1)$. Using Claim 1.3.5 this implies that $\alpha_1(\mathrm{pr}_1(\Gamma''))$ is discrete, in particular closed and hence contains $\alpha_1(\overline{\mathrm{pr}_1(\Gamma'')}) = \alpha_1(L_1) \supset \prod_{i=1}^{n} \mathbb{H}_i^+$ which is impossible.

Thus $r < n$ and it follows from Theorem 1.2.8 that both H_1 and H_2 are p-adic analytic groups. Let $\mathfrak{G}_i$, $\mathbb{G}_i$ and $\mathrm{Ad}_i : H_i \to \mathrm{Aut}(\mathfrak{G}_i^{n_i})$ be as in Theorem 1.2.8. Let us switch notation and denote $\mathfrak{F}_i = \mathfrak{G}_i$, $\mathbb{F}_i = \mathbb{G}_i$. We have and

$$\mathrm{Ad} = \mathrm{Ad}_1 \times \mathrm{Ad}_2 : H_1 \times H_2 \to \mathrm{Aut}(\mathfrak{G}_1^{n_1}) \times \mathrm{Aut}(\mathfrak{G}_2^{n_2})$$
$$H_1{}^{(\infty)} \times H_2{}^{(\infty)} \twoheadrightarrow (\mathbb{G}_1^+)^{n_1} \times (\mathbb{G}_2^+)^{n_2}$$

Claim 1.3.7. $\mathrm{Ad}(\Gamma)$ is discrete.

Since $H_i{}^{(\infty)}$ is of finite index in H_i (Theorem 1.2.8.1) we may replace Γ by $\Gamma' = \Gamma \cap (H_1{}^{(\infty)} \times H_2{}^{(\infty)})$. Let $\mathbb{H}$ be any factor of $\mathbb{G}_1^{n_1}$, and let π be the composition of Ad with the projection on $\mathbb{H}$. Then $(\Gamma', \pi, \mathbb{H})$ is a $\mathbb{Q}_p$-triple of unbounded type to which Claim 1.3.5 and the subsequent construction applies. In particular we obtain

$$\alpha_1 : H_1{}^{(\infty)} \to \prod_{i=1}^{r} \mathbb{H}_i, \quad \alpha_1(H_1{}^{(\infty)}) = \prod_{i=1}^{r} \mathbb{H}_i^+,$$
$$\alpha_2 : H_2{}^{(\infty)} \to \prod_{i=r+1}^{n} \mathbb{H}_i, \quad \alpha_2(H_2{}^{(\infty)}) = \prod_{i=r+1}^{n} \mathbb{H}_i^+.$$

Observe that $\mathrm{QZ}(H_i{}^{(\infty)}) \subset \ker \alpha_i$. This follows, for example, from the fact that each $\mathbb{H}_i^+$ is an adjoint group. Using Theorem 1.2.8.3 we conclude that the maps α_i factor via $\mathrm{Ad}_i : H_i{}^{(\infty)} \to G_i^{n-i}$. I.e., the re are quotient maps $q_1 : \mathbb{G}_1^{n_1} \to \prod_{i=1}^{r} \mathbb{H}_i$, $q_2 : \mathbb{G}_2^{n_2} \to \prod_{i=r+1}^{n} \mathbb{H}_i$ such that

$$\begin{tikzcd} H_1^{(\infty)} \arrow[r, "\mathrm{Ad}_1"] \arrow[dr, "\alpha_1"'] & \mathbb{G}_1^{n_1} \arrow[d, "q_1"] \\ & \prod_{i=1}^{r} \mathbb{H}_i \end{tikzcd}$$

and

$$\begin{tikzcd} H_2^{(\infty)} \arrow[r, "\mathrm{Ad}_2"] \arrow[dr, "\alpha_2"'] & \mathbb{G}_2^{n_2} \arrow[d, "q_2"] \\ & \prod_{i=r+1}^{n} \mathbb{H}_i \end{tikzcd}$$

commute. In particular $(\alpha_1 \times \alpha_2)(\Gamma') = (q_1 \times q_2)(\mathrm{Ad}(\Gamma'))$ is discrete.

Since this applies to any factor of $\mathbb{G}_1^{n_1}$ and similarly for $\mathbb{G}_2^{n_2}$, we deduce that $\mathrm{Ad}(\Gamma)$ is discrete. We deduce from the facts that $\mathrm{Ad} : H_1{}^{(\infty)} \times H_2{}^{(\infty)} \to (\mathbb{G}_1^+)^{n_1} \times (\mathbb{G}_2^+)^{n_2}$ is surjective, Γ' is a cocompact lattice and $\mathrm{Ad}(\Gamma')$ is discrete that $\mathrm{Ad}(\Gamma')$ is a cocompact lattice in $\mathbb{G}_1(\mathbb{Q}_p)^{n_1} \times \mathbb{G}_2(\mathbb{Q}_p)^{n_2}$ for which the closure of the projection on the i'th factor contains $(\mathbb{G}_i^+)^{n_i}$, for $i = 1, 2$. This implies that Γ is virtually a product of irreducible lattices in semisimple (not simple) Lie groups of rank at least 2 This implies that $\mathrm{Ad}(\Gamma')$ is an arithmetic lattice and completes the proof in the case $P_\Gamma = \{p\}$. $\square$

Proof of Corollary 1.1.18. $1 \Rightarrow 2$ is just the well-known fact that finitely generated linear groups are residually finite.

$2 \Rightarrow 3$. Since Γ is residually finite, Proposition 1.1.11.1 implies that $\Lambda_i = \{e\}$, $i = 1, 2$ and hence (by Theorem 1.1.17) we have $\mathrm{QZ}(H_i) = \ker \mathrm{Ad}_i = \{e\}$. Thus $\mathrm{Ad}_i : H_i^{(\infty)} \to \mathbb{G}_i^+$ is a topological isomorphism. By considering the action on T_i (via $(\mathrm{Ad}_i)^{-1}$) of a split Cartan subgroup of $\mathbb{G}_i$, one deduces that $\mathbb{G}_i$ has $\mathbb{Q}_{p_i}$-rank 1.

$3 \Rightarrow 1$. The locally primitive group H_i acts via the adjoint map Ad_i on the Bruhat–Tits tree Δ_i associated with $\mathbb{G}_i$. By Lemma 1.4.7 it follows that this action is implemented via an isometry between the geometric realizations $|T_i|$ and $|\Delta_i|$. This shows that Ad_i is injective and hence Γ is linear. □

1.3.2. *Proof of theorem 1.1.12*

Theorem 1.1.12 follows from Theorem 1.1.5 and the following:

Lemma 1.3.8. *Let* $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ *be a cocompact lattice such that* $H_i = \overline{\mathrm{pr}_i(\Gamma)}$ *are non-discrete, locally quasiprimitive. Let* $N \triangleleft \Gamma$ *be a normal subgroup. Then either*

1. $N \subset \mathrm{QZ}(H_1) \times \mathrm{QZ}(H_2)$, *or*
2. $\overline{\mathrm{pr}_i(N)} \supset H_i^{(\infty)}$.

Proof. The closure $\overline{\mathrm{pr}_i(N)}$ is a normal subgroup of H_i and hence (by Theorem 1.1.7) either contains $H_i^{(\infty)}$ or is contained in $\mathrm{QZ}(H_i)$. Hence assume

$$\mathrm{pr}_1(N) \subset \mathrm{QZ}(H_1); \tag{1.3.1}$$

we have to show

$$\mathrm{pr}_2(N) \subset \mathrm{QZ}(H_2). \tag{1.3.2}$$

Assume by contradiction that (1.3.2) does not hold. Let $\Lambda_i = \Gamma \cap \mathrm{Aut}T_i$ (where we slightly abuse notation). Since $\overline{\mathrm{pr}_2(N)}$ is normal in $\overline{\mathrm{pr}_2(\Gamma)} = H_2$ we have by Theorem 1.1.7 that

$$\overline{\mathrm{pr}_2(N)} \supset H_2^{(\infty)} \tag{1.3.3}$$

But then (using 1.3.2 in [BM00a]) there is a finitely generated group $L \subset N$ such that $\mathrm{pr}_2(L)$ acts co-finitely on the tree T_2. Observe that $(\mathrm{pr}_i(L) \cap \Lambda_i) \triangleleft \mathrm{pr}_i(L)$ and that $\mathrm{pr}_1(L)/(\mathrm{pr}_1(L) \cap \Lambda_1)$ is isomorphic to $\mathrm{pr}_2(L)/(\mathrm{pr}_2(L) \cap \Lambda_2)$, indeed both are isomorphic to $L/((\ker \mathrm{pr}_1 |_L) \cdot (\ker \mathrm{pr}_2 |_L))$. Since $\mathrm{pr}_1(L)/(\mathrm{pr}_1(L) \cap \Lambda_1)$ is isomorphic to a subgroup of $\mathrm{QZ}(H_1)/\Lambda_1$ it is locally finite (Proposition 1.1.11.2). Hence also the finitely generated group $\mathrm{pr}_2(L)/(\mathrm{pr}_2(L) \cap \Lambda_2)$ is locally finite and hence finite. But then $\mathrm{pr}_2(L) \cap \Lambda_2$ would act on T_2 with a finite quotient. Hence Λ_2 and $\mathrm{QZ}(H_2)$ will be cocompact in H_2, which contradicts Theorem 1.1.7, we conclude that $\mathrm{pr}_2(N) \subset \mathrm{QZ}(H_2)$. □

1.4. Geometric rigidity

1.4.1.

Terminology: our trees will be without leaves. Given a tree T let us denote by $|T|$ its geometric realization which is a CAT(-1) space, endowed with the structure of a

1-dimensional simplicial complex. Let $|T|^0$ denote the set of 0-cells (corresponding to the vertices of T) and every 1-cell of $|T|$ is isometric to [0, 1]. A group action on a tree T is called c-minimal if there is no proper invariant subtree.

Theorem 1.4.1. *Let $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ be a cocompact lattice such that $H_i = \overline{\mathrm{pr}_i(\Gamma)}$ are locally primitive and non-discrete. Let $\Gamma' < \mathrm{Aut}T_1' \times \mathrm{Aut}T_2'$ be such that $\Gamma' \backslash (T_1' \times T_2')$ is finite and let $\pi : \Gamma \to \Gamma'$ be a surjective homomorphism. Then up to re-scaling metrics and exchanging the factors there are isometries $t_i : |T_i| \to |T_i'|$ such that $t := t_1 \times t_2$ induces the homomorphism π, in particular π is an isomorphism.*

Remark 1.4.2. If we assume that π is an isomorphism we do not need to impose the condition that H_i is non-discrete.

Corollary 1.4.3. *Let $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ be a cocompact lattice such that each $H_i = \overline{\mathrm{pr}_i(\Gamma)}$ is locally primitive and $X_\Gamma = \Gamma \backslash (T_1 \times T_2)$ be the quotient square complex. Then*

$$\mathrm{Out}(\Gamma) \cong \mathrm{Aut}X_\Gamma$$

and hence is finite.

The above results follow from a general result describing all non-elementary actions of Γ on a tree:

Theorem 1.4.4. *Let $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ be a uniform lattice such that each $H_i = \overline{\mathrm{pr}_i(\Gamma)}$ is locally primitive and $\pi : \Gamma \to \mathrm{Aut}T$ be an action of Γ on a countable tree T such that it is non-elementary and c-minimal. Then π extends continuously to $H_1 \times H_2$, factoring via one H_i and the continuous homomorphism $\pi : H_i \to \mathrm{Aut}T$ obtained is realized by an isometry $|T_i| \to |T|$.*

Let us mention one more corollary:

Corollary 1.4.5. *Let $\Gamma < \mathrm{Aut}T_1 \times \mathrm{Aut}T_2$ be a cocompact lattice and assume that $H_i = \overline{\mathrm{pr}_i(\Gamma)}$ is locally primitive. Let $\Lambda_1 = \Gamma \cap (\mathrm{Aut}T_1 \times e)$ and $\Lambda_2 = \Gamma \cap (e \times \mathrm{Aut}T_2)$. If both $\Lambda_1 \neq \{e\}$ and $\Lambda_2 \neq \{e\}$, then $\Gamma/(\Lambda_1 \cdot \Lambda_2)$ has property FA. Namely any action of $\Gamma/(\Lambda_1 \cdot \Lambda_2)$ on a tree has a fixed point.*

1.4.2.

In this subsection we shall prove Theorem 1.4.4. We will need the following lemmas:

Lemma 1.4.6. *Let $T = (V, E)$ be a locally finite tree, $H < \mathrm{Aut}T$ a closed locally primitive subgroup, X a complete CAT(0)-space and $H \times X \to X$ a continuous action with unbounded orbits. Then after re-scaling of the metric on X there is a continuous H-equivariant map $\alpha : |T| \to X$ whose restriction to each 1-cell is isometric and whose restriction to the star of each 0-cell of $|T|$ is injective.*

Proof. Subdividing T once we may assume that H does not contain inversions and hence $H = \langle H_\alpha \cup H_\beta \rangle$ where α, β are any pair of adjacent vertices. Since for every vertex v the subgroup $\pi(H_v)$ has bounded orbits in X, it follows that $X^{\pi(H_v)} \neq \emptyset$

and we may pick $x_\alpha \in X^{\pi(H_\alpha)}$, $x_\beta \in X^{\pi(H_\beta)}$ and define $\varphi : V \to X$ by $\varphi(h\alpha) = \pi(h)x_\alpha$, $\varphi(h\beta) = \pi(h)x_\beta$, $h \in H$, thus obtaining an H-equivariant map from V to X. Pick $v \in V$ and let $\{x_1, \dots, x_k\}$ be the set of vertices adjacent to v. We show that $\varphi : \{v, x_1, \dots, x_k\} \to X$ is injective. If $\varphi(v) = \varphi(x_1)$, say, then $\pi(H_v)$ and $\pi(H_{x_1})$ both fix $b := \varphi(v) = \varphi(x_1)$ and hence $\pi(H)$ fixes b, a contradiction. The map $\varphi : \{x_1, \dots, x_k\} \to \{\varphi(x_1), \dots, \varphi(x_k)\}$ is H_v-equivariant and hence since the H_v action is primitive, it is either injective or constant. In the latter case, let $b \in X$ denote this constant value of the image. Then each of the subgroups $\pi(H_{x_i})$, $1 \le i \le k$ fixes b and so does the subgroup $N = \langle \cup H_{x_i} \rangle$ generated by them. Notice that $\pi(H_v)$ normalizes N and acts with bounded orbits on the subset $X^N \subset X$ of points fixed by N. Since $X^N \ni b$ is non-empty it follows that there is a point fixed by $\pi(H_v \cdot N) = \pi(H)$, a contradiction. In the sequel we pick x_α and x_β (and hence $\varphi()$ such that $[x_\alpha, x_\beta] \cap X^{\pi(H_\alpha)} = \{x_\alpha\}$ and $[x_\alpha, x_\beta] \cap X^{\pi(H_\beta)} = \{x_\beta\}$. Note that such a choice is clearly possible. Let $|\varphi| : |T| \to X$ be the geodesic extension of φ to $|T|$. Let $T_v = \cup_{i=1}^k [v, x_i]$; we have to show that $|\varphi|$ is injective on T_v. We say that $x_i \sim x_j$ if $[\varphi(v), \varphi(x_i)] \cap [\varphi(v), \varphi(x_j)] = [\varphi(v), q]$ with $q \ne \varphi(v)$. Clearly this is an H_v-invariant equivalence relation and hence is either separating points, i.e., $[\varphi(v), \varphi(x_i)] \cap [\varphi(v), \varphi(x_j)] = \{\varphi(v)\}$, $\forall i \ne j$, or consists of one equivalence class, in which case we have $\cap_{i=1}^k [\varphi(v), \varphi(x_i)] = [\varphi(v), q]$ with $q \ne \varphi(v)$. In this case however the point q is $\pi(H_v)$-fixed, contradicting the construction of φ. It follows that $|\varphi|_{|T_v}$ is injective on T_v. □

Lemma 1.4.7. *Let T and H be as in Lemma 1.4.6. Let T' be a countable tree, with a continuous H-action $\pi : H \to \mathrm{Aut}T'$ which is c-minimal and has unbounded orbits. Then $\pi : H \to \mathrm{Aut}T'$ is realized by an isometry $|T| \to |T'|$ (up to possible re-scaling of the distance on T').*

Proof. Apply Lemma 1.4.6 to $X = |T'|$ and let $\alpha : |T| \to |T'|$ be the H-equivariant map given by Lemma 1.4.6. We may assume that all the vertices of T have degree at least 3. Together with Lemma 1.4.6 this implies that $\alpha(|T|^0) \subset |T'|^0$ hence α is locally distance preserving. Observe that $\alpha(|T|) \subset |T'|$ is an H-invariant subtree and hence using the c-minimality of the action $\alpha(|T|) = |T'|$. A surjective map which is uniformly locally isometric is necessarily a covering map. Hence α is an isometry. □

Theorem 1.4.4 is an immediate consequence of Lemma 1.4.7 and Theorem 1.1.4 of Monod and Shalom.

1.4.3.

Here we complete the remaining assertions. To establish Theorem 1.4.1 consider $\pi_i := \mathrm{pr}_i \circ \pi : \Gamma \to \mathrm{Aut}T_i'$. These are non-elementary, c-minimal actions hence by Theorem 1.4.4 extend continuously to $\tilde{\pi_i} : H_1 \times H_2 \to \mathrm{Aut}T_i'$ factoring via one of the factors. Let us denote $\tilde{\pi} = \tilde{\pi_1} \times \tilde{\pi_2}$. If both $\tilde{\pi_1}$ and $\tilde{\pi_2}$ factored via, say, H_1 then we will have (abusing notation) that $\tilde{\pi}(H_1 \times H_2) = \tilde{\pi}(H_1) \subset \overline{\tilde{\pi}(\mathrm{pr}_1(\Gamma)} = \Gamma'$. It follows that the image of H_1 is countable and hence must be finite (since the kernel would be an open normal subgroup of a locally primitive group), which is impossible. We conclude that, after possible exchanging the indexes, the map $\tilde{\pi_i}$ factors via H_i, and the resulting homomorphism $\overline{\pi_i} : H_i \to \mathrm{Aut}T_i'$ is realized (see Lemma 1.4.7) by an isometry $t_i : |T_i| \to |T_i'|$ and hence the homomorphism π is realized by the isometry $t = t_1 \times t_2$. □

The remaining Corollaries 1.4.3 and 1.4.5 follow easily.

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