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Surface group representations with maximal Toledo invariant

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*Dedicated to Domingo Toledo on his 60*th *birthday*

Abstract

We develop the theory of maximal representations of the fundamental group $\pi_1(\Sigma)$ of a compact connected oriented surface Σ (possibly with boundary) into Lie groups G of Hermitian type. For any homomorphism $\rho : \pi_1(\Sigma) \to G$, we define the *Toledo invariant* $\mathrm{T}(\Sigma, \rho)$, a numerical invariant which has both topological and analytical interpretations. We establish important properties of $\mathrm{T}(\Sigma, \rho)$, among which continuity, uniform boundedness on the representation variety, additivity under connected sum of surfaces and congruence relations mod $\mathbb{Z}$. We thus obtain information about the representation variety as well as striking geometric properties of *maximal* representations, that is representations whose Toledo invariant achieves the maximum value.

Moreover we establish properties of boundary maps associated to maximal representations which generalize naturally monotonicity properties of semiconjugations of the circle.

We define a rotation number function for general locally compact groups and study it in detail for groups of Hermitian type. Properties of the rotation number, together with the existence of boundary maps, lead to additional invariants for maximal representations and show that the subset of maximal representations is always real semialgebraic.

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1. **Introduction**

Let Σ be an oriented compact surface with boundary $\partial\Sigma$ and let G be a connected semisimple Lie group with finite center. The problem of understanding the representation variety $\mathrm{Hom}\big(\pi_1(\Sigma), G\big)$ has received considerable interest. A major theme is the problem of singling out special components of this representation variety which should generalize Teichmüller space and then studying the geometric significance of the representations belonging to such components.

If G is a split real group and $\partial\Sigma = \varnothing$, there is a component whose global properties were studied by Hitchin [33]; the geometric significance of these representations was recently brought into the open on the one hand by the work of Labourie [39] relating them to Anosov structures, and on the other hand by Fock and Goncharov [22], [23] studying them via the notion of positivity introduced by Lusztig [42].

When G is of Hermitian type and $\partial\Sigma = \varnothing$, one can define the *Toledo invariant* of a representation and hence the notion of *maximal* representation: these form a union of connected components of the representation variety. The global properties of these components were investigated by García-Prada, Bradlow, and Gothen using Higgs bundles [29], [4], and [5] and the geometric properties of maximal representations were investigated by the authors in [13], [12].

The purpose of this paper is to introduce and study the notion of Toledo invariant when $\partial\Sigma \neq \varnothing$ and investigate the structure of the corresponding maximal representations. The treatment includes the case in which $\partial\Sigma = \varnothing$ on which it sheds new light.

The main results are the structure theorem (Theorem 5), the regularity properties of boundary maps (Theorem 8) and the formula for the Toledo invariant in terms of rotations numbers (Theorem 12). For more background on the study of maximal representations we refer to [4], [5], [11], [12], [13], [26], [27], [28], [32], [49], [40], and [51].

1.1. *The Toledo invariant.* Let G be a group of Hermitian type (see §2.1.1), so that in particular the associated symmetric space $\mathscr{X}$ is Hermitian of noncompact type; then $\mathscr{X}$ carries a unique Hermitian (normalized) metric of minimal holomorphic sectional curvature -1. The Kähler form $\omega_{\mathscr{X}}$ of this metric gives rise, in the familiar way, to a continuous class $\kappa_G \in \mathrm{H}^2_{\mathrm{c}}(G, \mathbb{R})$ and, owing to the isomorphism between bounded continuous and continuous cohomology in degree two, to the *bounded Kähler class* $\kappa^{\mathrm{b}}_G \in \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$ (see §2.1). The bounded Kähler class is the source of new invariants for representations and has been considered in [10], [12], [13], [14], and [15].

Let Σ be a connected oriented compact surface with boundary $\partial\Sigma$, and $\rho : \pi_1(\Sigma) \to G$ a representation. When $\partial\Sigma = \varnothing$, the Toledo invariant is given by the evaluation of $\rho^*(\kappa_G)$ on the fundamental class $[\Sigma]$,

$$\mathrm{T}(\Sigma, \rho) = \langle \rho^*(\kappa_G), [\Sigma] \rangle .$$

In the general case we obtain, by pullback in bounded cohomology, a bounded class

$$\rho^*(\kappa^{\mathrm{b}}_G) \in \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(\Sigma), \mathbb{R}\big) \cong \mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbb{R}) .$$

The canonical map $j_{\partial\Sigma} : \mathrm{H}^2_{\mathrm{b}}(\Sigma, \partial\Sigma, \mathbb{R}) \to \mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbb{R})$ from singular bounded cohomology relative to $\partial\Sigma$ to singular bounded cohomology is an isomorphism (see (2.d) in §2.2), and we define

$$\mathrm{T}(\Sigma, \rho) = \langle j^{-1}_{\partial\Sigma} \rho^*(\kappa^{\mathrm{b}}_G), [\Sigma, \partial\Sigma] \rangle ,$$

where now $j^{-1}_{\partial\Sigma} \rho^*(\kappa^{\mathrm{b}}_G)$ is considered as an ordinary relative class and $[\Sigma, \partial\Sigma]$ is the relative fundamental class. The above construction applies to any class $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$, and we denote by $\mathrm{T}_\kappa(\Sigma, \rho)$ the resulting invariant. This generalization will be useful when we consider integral classes. This construction circumvents the fact that $\mathrm{H}^2(\Sigma,\mathbb{R})=0$ when $\partial\Sigma\neq\varnothing$; indeed in all cases $\mathrm{H}^2_{\mathrm{b}}(\Sigma,\mathbb{R})$ is infinite dimensional, provided $\chi(\Sigma)\leq -1$.

The basic properties of the Toledo invariant are summarized in the following

THEOREM 1. *Let G be a group of Hermitian type and $\rho:\pi_1(\Sigma)\to G$ a representation. Then*

(1) $|\mathrm{T}(\Sigma,\rho)|\leq|\chi(\Sigma)|\,\mathrm{r}_{\mathscr{X}}$, *where* $\mathrm{r}_{\mathscr{X}}$ *is the rank of* $\mathscr{X}$.

(2) *The map* $\mathrm{T}(\Sigma,\cdot)$ *is continuous on* $\mathrm{Hom}\big(\pi_1(\Sigma),G\big)$; *if* $\partial\Sigma=\varnothing$, *its range is finite*, *while if* $\partial\Sigma\neq\varnothing$ *its range is the interval*

$$\big[-|\chi(\Sigma)|\,\mathrm{r}_{\mathscr{X}},|\chi(\Sigma)|\,\mathrm{r}_{\mathscr{X}}\big].$$

(3) *If* Σ *is the connected sum of two* (*connected*) *surfaces* Σ_i *along a separating loop*, *then*

$$\mathrm{T}(\Sigma,\rho)=\mathrm{T}(\Sigma_1,\rho_1)+\mathrm{T}(\Sigma_2,\rho_2)\,,$$

where ρ_i *is the restriction of* ρ *to* $\pi_1(\Sigma_i)$.

[Theorem 1 follows from Corollary 3.4, Proposition 3.10, Corollary 3.12, and Proposition 3.2.]

Here and in the sequel, an essential role is played by Theorem 3.3, where we identify the Toledo invariant with an invariant defined in analytic terms, introduced and studied in [11]. In view of Theorem 1, we set the following

Definition 2. A representation $\rho:\pi_1(\Sigma)\to G$ is *maximal* if

$$\mathrm{T}(\Sigma,\rho)=|\chi(\Sigma)|\,\mathrm{r}_{\mathscr{X}}\;.$$

We denote by $\mathrm{Hom}_{\max}\big(\pi_1(\Sigma),G\big)$ the subspace of the representation variety consisting of maximal representations. Notice that when $\partial\Sigma=\varnothing$ this subspace is a union of components of the representation variety, while when $\partial\Sigma\neq\varnothing$ the whole representation variety is connected (see however Corollary 14).

1.2. *Geometric properties of maximal representations.* Before treating the case of a general group of Hermitian type, we state the structure theorem for maximal representation into $\mathrm{PU}(1,1)$ which generalizes to the case of surfaces with boundary Goldman's characterization of maximal representations when $\partial\Sigma=\varnothing$ [26], [27].

THEOREM 3. *Let* Σ *be a connected oriented surface such that* $\chi(\Sigma)\leq -1$. *A representation* $\rho:\pi_1(\Sigma)\to\mathrm{PU}(1,1)$ *is maximal if and only if it is the holonomy representation of a complete hyperbolic metric on the interior* Σ° *of* Σ.

Remark 4. Theorem 3 is proved in Section 3.2; the "only if" part is a consequence of [11] together with the formula in Theorem 3.3; the "if" part, while being Gauss-Bonnet's theorem in the boundaryless case, requires a quite different argument when $\partial\Sigma\neq\varnothing$.

We observe that when $\partial\Sigma \neq \varnothing$, the invariant $\mathrm{T}(\Sigma, \rho)$ depends not only on $\pi_1(\Sigma)$ but also on Σ: in fact, if Σ_1 and Σ_2 are nondiffeomorphic surfaces with isomorphic fundamental groups, $i : \pi_1(\Sigma_1) \to \pi_1(\Sigma_2)$ is an isomorphism and $\rho : \pi_1(\Sigma_2) \to \mathrm{PU}(1, 1)$ is a maximal representation, then it follows from Theorem 3 that $\rho \circ i$ is not maximal.

The first result beyond the case $\mathrm{PU}(1, 1)$ was obtained by Toledo [49] who, in the boundaryless case, showed that a maximal representation into $\mathrm{PU}(1, m)$ stabilizes a complex geodesic. It turns out that the appropriate generalization of complex geodesic is, in this context, the notion of *maximal tube type subdomain*. Roughly speaking, *tube type domains* are bounded symmetric domains which admit a model which corresponds to the upper half plane model in the case of the Poincaré disk, and their significance for rigidity questions of isometric group actions already appeared in [10] and [14]. For a general Hermitian symmetric space $\mathscr{X}$, maximal tube type subdomains exist, are of rank equal to the rank of $\mathscr{X}$, and are G-conjugate. As alluded to above, the maximal tube type subdomains in complex hyperbolic n-space are the complex geodesics.

The main structure theorem for maximal representations is:

THEOREM 5. *Let* $\mathbf{G}$ *be a connected semisimple algebraic group defined over* $\mathbb{R}$ *such that* $G = \mathbf{G}(\mathbb{R})^\circ$ *is of Hermitian type. Let* Σ *be a compact connected oriented surface with* (*possibly empty*) *boundary and* $\chi(\Sigma) \leq -1$. *If* $\rho : \pi_1(\Sigma) \to G$ *is a maximal representation*, *then*

(1) ρ *is injective with discrete image*;

(2) *the Zariski closure* $\mathbf{H} < \mathbf{G}$ *of the image of* ρ *is reductive*;

(3) *the reductive Lie group* $H := \mathbf{H}(\mathbb{R})^\circ$ *has compact centralizer in* G, *and the symmetric space* $\mathscr{Y}$ *associated to* H *is Hermitian of tube type*;

(4) $\rho\big(\pi_1(\Sigma)\big)$ *stabilizes a maximal tube type subdomain* $\mathscr{T} \subset \mathscr{X}$.

[Theorem 5 is proved in Section 4 when ρ has Zariski dense image, while the general case is treated in Section 6.]

Remark 6. (1) In the case in which $\partial\Sigma = \varnothing$, Theorem 5 was announced in [13].

(2) When $\partial\Sigma = \varnothing$, Theorem 5(4) was obtained by Hernández for $G = \mathrm{PU}(2, m)$ [32]. Assuming that the representation is reductive, Bradlow, García-Prada, and Gothen also obtained Theorem 5(4) for $G = \mathrm{SU}(n, m)$ [4] and for $\mathrm{SO}^*(2n)$ [5]. In each of these works the Toledo invariant appears as the first Chern class of an appropriate complex line bundle over Σ.

(3) When $\partial\Sigma \neq \varnothing$ and $G = \mathrm{PU}(1, m)$, Koziarz and Maubon introduced [38] an invariant lying in the de Rham cohomology of Σ with compact support, whose evaluation on $[\Sigma, \partial\Sigma]$ can be shown to be equal to our notion of Toledo invariant; in this context, they obtained in [38] Theorem 5(4) as well as Theorem 3.

The symmetric space $\mathcal{Y}$ in Theorem 5 is the variety of maximal compact subgroups of H; since H has compact centralizer in G, there is a unique totally geodesic embedding $i : \mathcal{Y} \to \mathcal{X}$ which is not necessarily holomorphic but is *tight*. This latter notion, which is analytic in nature, stems from our approach via bounded cohomology; see [15].

A special case of Theorem 5 is when the homomorphism ρ has Zariski dense image. Then $\mathcal{Y} = \mathcal{X}$ and hence $\mathcal{X}$ is of tube type. This result is optimal in the sense that every tube type domain admits a maximal representation with Zariski dense image. More precisely, let $d : \mathbb{D} \to \mathcal{X}$ be a *diagonal disk*, also called *tight holomorphic disk* in [21] and [15] (see (2.b), §2.1.2 for the definition), and $\Delta : \mathrm{SU}(1,1) \to \mathbf{G}(\mathbb{R})^\circ$ a homomorphism associated to d.

THEOREM 7. *Assume that $\mathcal{X}$ is of tube type, that $\chi(\Sigma) \leq -2$, and let*

$$h : \pi_1(\Sigma) \to \mathrm{SU}(1,1)$$

be a complete hyperbolization of Σ°. If the surface is of type $(g,n) = (1,2)$ or $(0,4)$, we assume that h sends one, respectively two, boundary components of $\partial\Sigma$ to hyperbolic elements. Then $\rho_0 := \Delta \circ h : \pi_1(\Sigma) \to G$ admits a deformation $(\rho_t)_{t \geq 0}$ such that:

(1) *ρ_t is maximal for all $t \geq 0$, and*

(2) *ρ_t has Zariski dense image for all $t > 0$.*

[This theorem is proved in Section 9.]

1.3. *Boundary maps.* Maximal representations give rise to boundary maps with special regularity properties which in turn play an important role in the study of the set of maximal representations and in the construction of new invariants thereof. *Monotonicity (or positivity)* is one of these properties and in order to express it we need the notion of maximal triples of points in the Shilov boundary of a symmetric domain: those are the vertices of ideal geodesic triangles of maximal Kähler area in a sense made precise in [21] (see §2.1.3 for the definition).

THEOREM 8. *Let $h : \pi_1(\Sigma) \to \mathrm{PU}(1,1)$ be a complete hyperbolization of Σ° of finite area and $\rho : \pi_1(\Sigma) \to G$ a representation into a group of Hermitian type. Then ρ is maximal if and only if there exists a left continuous map $\varphi : \partial\mathbb{D} \to \check{S}$ with values in the Shilov boundary $\check{S}$ of the bounded symmetric domain associated to G such that*

(1) *φ is strictly $\rho \circ h^{-1}$-equivariant, and*

(2) *φ is monotone, that is, it maps positively oriented triples on $\partial\mathbb{D}$ to maximal triples on $\check{S}$.*

[Theorem 8 is proved in Section 5 in the case in which ρ has Zariski dense image and in Section 6 in the general case.]

Remark 9. The theorem holds true also if "left continuous" is replaced by "right continuous".

The characterization in Theorem 8 clarifies the relation between maximal representations and the Hitchin — respectively positive — representations into split real Lie groups which were recently studied by Labourie [39] and Guichard [31] — respectively Fock and Goncharov [22]. Indeed in the latter the authors established a similar characterization in terms of equivariant maps from $\partial\mathbb{D}$ into (full) flag varieties which send positively oriented triple in $\partial\mathbb{D}$ to positive triples of flags in the sense of Lusztig.

In the only case when G is of Hermitian type as well as real split, namely when G is locally isomorphic to a symplectic group $\mathrm{Sp}(V)$, the Shilov boundary can be identified with the space of Lagrangian subspaces in V and in this case the notion of maximality of triples in $\check{S}$ coincides with the notion of positivity of triples in partial flag varieties — such as the space of Lagrangians — defined by Lusztig in [43]. Thus Theorem 8 implies that the space of positive representations into $\mathrm{PSp}(V)$ defined by Fock and Goncharov [22, Def. 1.10], is a proper subset of the space of maximal representations. For the Hitchin component (when $\partial\Sigma = \varnothing$) this was observed in [12].

The issue of continuity of the boundary map φ presents itself naturally. When $\partial\Sigma = \varnothing$, the continuity of φ was established in [12] in the case in which $G = \mathrm{Sp}(V)$ is a symplectic group, as a byproduct of the construction of an Anosov system; the case of a general group of Hermitian type will be treated in a forthcoming paper. When $\partial\Sigma \neq \varnothing$, then already in the case $G = \mathrm{PU}(1,1)$ the map φ will not be in general continuous as the case in which ρ is an infinite area hyperbolization indicates. In fact, if $G = \mathrm{PU}(1,1)$, the map φ is a semiconjugacy in the sense of Ghys [25] and this will be used in Section 8.2 to define a canonical integral bounded class

$$\kappa^{\mathrm{b}}_{\Sigma,\mathbb{Z}} \in \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(\Sigma), \mathbb{Z}\big)$$

which, when $\partial\Sigma = \varnothing$, corresponds to the fundamental class under the comparison map. Letting $\kappa^{\mathrm{b}}_{\Sigma} \in \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(\Sigma), \mathbb{R}\big)$ denote the corresponding real class, we will establish in Section 8.2 (see Corollary 8.6) the following

COROLLARY 10. *For any homomorphism $\rho : \pi_1(\Sigma) \to G$, the following are equivalent*:

(1) *ρ is maximal, and*

(2) $\rho^*(\kappa) = \lambda_G(\kappa)\kappa^{\mathrm{b}}_{\Sigma}$, *for all $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$, where λ_G is a certain explicit linear form on* $\mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$.

The extent to which Corollary 10 does not hold for integral classes will be the source of new invariants of maximal representations which will be given an explicit form once rotation numbers are introduced in the next section.

1.4. *Toledo invariant and rotation numbers.* In order to define a notion of integral class in continuous bounded cohomology we consider, for G a locally compact second countable group and $A = \mathbb{Z}$ or $\mathbb{R}$, the cohomology $\widehat{\mathrm{H}}^\bullet_{\mathrm{cb}}(G, A)$ of the complex of bounded Borel cochains on G which turns out to coincide with bounded continuous cohomology if $A = \mathbb{R}$ (see Section 2.3). Given $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G, \mathbb{Z})$, we introduce in Section 7 the rotation number

$$\mathrm{Rot}_\kappa : G \to \mathbb{R}/\mathbb{Z}\,,$$

which is a class function whose restriction to any amenable closed subgroup is a homomorphism and we show its continuity (Corollary 7.6).

The rotation number Rot_κ generalizes the classical rotation number of an orientation preserving homeomorphism of the circle as well as the symplectic rotation number introduced by Barge and Ghys in [1] and the construction of Clerc and Koufany in [18]. The exact relations are discussed in Section 7.

When G if of Hermitian type and $K < G$ is a maximal compact subgroup, the basic properties of the rotation number Rot_κ are summarized in the following

THEOREM 11. (1) *The map*

$$\begin{aligned}\widehat{\mathrm{H}}^2_{\mathrm{cb}}(G, \mathbb{Z}) &\to \mathrm{Hom}_{\mathrm{c}}(K, \mathbb{R}/\mathbb{Z})\\ \kappa \quad &\mapsto \quad \mathrm{Rot}_\kappa|_K\end{aligned}$$

is an isomorphism.

(2) *The change of coefficients* $\widehat{\mathrm{H}}^2_{\mathrm{cb}}(G, \mathbb{Z}) \to \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$ *is injective with image a lattice.*

(3) *For every* $g \in G$,

$$\mathrm{Rot}_\kappa(g) = \mathrm{Rot}_\kappa(k)\,,$$

where $k \in K$ *is conjugate to the elliptic component* g_e *in the refined Jordan decomposition* $g = g_e g_h g_u$ *of* g.

(4) *The unique continuous lift*

$$\widetilde{\mathrm{Rot}}_\kappa : \widetilde{G} \to \mathbb{R}$$

vanishing at e is a homogeneous quasimorphism.

[Theorem 11 is proved in Propositions 7.7, 7.8, and Theorem 7.9; for the refined Jordan decomposition see [3, §2].]

We turn now to the formula of the Toledo invariant $\mathrm{T}_\kappa(\Sigma, \rho)$ when κ is an integral bounded class. For this we assume that $\partial\Sigma \neq \varnothing$ and let

$$\pi_1(\Sigma) = \left\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n : \prod_{i=1}^{g} [a_i, b_i] \prod_{j=1}^{n} c_j = e \right\rangle \tag{1.1}$$

be a presentation where the elements c_i represent loops which are freely homotopic to the corresponding boundary components of $\partial\Sigma$ with positive orientation. Given a homomorphism $\rho : \pi_1(\Sigma) \to G$, let $\widetilde{\rho} : \pi_1(\Sigma) \to \widetilde{G}$ be a lift of ρ to the universal covering $\widetilde{G}$, taking into account that $\pi_1(\Sigma)$ is free.

THEOREM 12. *Let $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G, \mathbb{Z})$. Then*

$$\mathrm{T}_\kappa(\Sigma, \rho) = -\sum_{j=1}^{n} \widetilde{\mathrm{Rot}_\kappa}\big(\widetilde{\rho}(c_j)\big)\,.$$

[Theorem 12 is proved in Section 8.1.]

When the boundary of Σ is empty, a formula for T_κ (see Theorem 8.3) can be obtained by cutting Σ along a separating loop and using Theorem 12 together with the additivity property of the Toledo invariant in Theorem 1(3).

In conjunction with Corollary 10, rotation numbers give rise to nontrivial invariants of maximal representations; recalling that the Shilov boundary $\check{S}$ of the bounded symmetric domain $\mathcal{D}$ associated to G is a homogeneous space with typical stabilizer Q and letting e_G denote the exponent of the finite group Q/Q°, we have:

THEOREM 13. *Let $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G, \mathbb{Z})$ and $\rho_0 : \pi_1(\Sigma) \to G$ a maximal representation.*

(1) *For every maximal representation $\rho : \pi_1(\Sigma) \to G$ the map*

$$\begin{aligned} \mathrm{R}^{\rho_0}_\kappa(\rho) : \pi_1(\Sigma) &\longrightarrow \mathbb{R}/\mathbb{Z} \\ \gamma &\mapsto \mathrm{Rot}_\kappa\big(\rho(\gamma)\big) - \mathrm{Rot}_\kappa\big(\rho_0(\gamma)\big) \end{aligned}$$

is a homomorphism.

(2) *If $\mathcal{D}$ is of tube type, then $\mathrm{R}^{\rho_0}_\kappa(\rho)$ takes values in $e_G^{-1}\mathbb{Z}/\mathbb{Z}$ and*

$$\mathrm{Hom}_{\max}\big(\pi_1(\Sigma), G\big) \to \mathrm{Hom}\big(\pi_1(\Sigma), \mathbb{R}/\mathbb{Z}\big)$$

is constant on connected components.

[Theorem 13 is proved in Section 8.2.]

In Example 8.7 we describe for $G = \mathrm{Sp}(V)$, $\dim(V) = 4m$ and $\partial\Sigma = \varnothing$ that already $\mathrm{Rot}_\kappa(\rho) : \pi_1(\Sigma) \to \mathbb{Z}/2\mathbb{Z}$ is a homomorphism and discuss its relationship with the first Stiefel-Withney class of a certain real vector bundle constructed using the boundary map from Theorem 8.

We turn now to our final application to representation varieties. For this we assume again that $\partial\Sigma \neq \varnothing$ and use the familiar presentation of $\pi_1(\Sigma)$ given in (1.1). Then

$$\mathrm{Hom}^{\check{S}}\big(\pi_1(\Sigma), G\big) := \big\{\rho \in \mathrm{Hom}\big(\pi_1(\Sigma), G\big) : \ \rho(c_i) \text{ has at least one fixed point in } \check{S},\ 1 \le i \le n\big\}$$

is a semialgebraic set if G is real algebraic and we have as a consequence of Theorem 8, that

$$\mathrm{Hom}_{\max}\big(\pi_1(\Sigma), G\big) \subset \mathrm{Hom}^{\check{S}}\big(\pi_1(\Sigma), G\big)\,. \tag{1.2}$$

COROLLARY 14. *Let* $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G, \mathbb{Z})$ *and assume that* $\mathscr{D}$ *is of tube type. Then*:

(1) $\mathrm{T}_\kappa(\Sigma, \rho) \in e_G^{-1}\mathbb{Z}$ *for every* $\rho \in \mathrm{Hom}^{\breve{S}}\big(\pi_1(\Sigma), G\big)$, *and*

(2) $\mathrm{Hom}_{\max}\big(\pi_1(\Sigma), G\big)$ *is a union of connected components of the set*

$$\mathrm{Hom}^{\breve{S}}\big(\pi_1(\Sigma), G\big).$$

[Corollary 14 is proved in Section 8.3.]

An alternative boundary condition might be imposed by fixing instead a set $\mathscr{C} = \{\mathscr{C}_1, \dots, \mathscr{C}_n\}$ of conjugacy classes in G and defining

$$\mathrm{Hom}^{\mathscr{C}}\big(\pi_1(\Sigma), G\big) := \big\{\rho \in \mathrm{Hom}\big(\pi_1(\Sigma), G\big) : \rho(c_i) \in \mathscr{C}_i,\ 1 \le i \le n\big\}.$$

Then $\mathrm{Hom}^{\mathscr{C}}\big(\pi_1(\Sigma), G\big)$ is also a semialgebraic set, and it follows immediately from Theorem 12 that T_κ is constant on its connected components.

Notice however that Corollary 14 implies that for many choices of conjugacy classes the intersection $\mathrm{Hom}_{\max}\big(\pi_1(\Sigma), G\big) \cap \mathrm{Hom}^{\mathscr{C}}\big(\pi_1(\Sigma), G\big)$ considered above is actually empty. For example in the case when Σ has precisely one boundary component Theorem 12 readily implies that for any maximal representation the rotation number of the conjugacy class $\mathscr{C} = \{\mathscr{C}_1\}$ has to be zero. From a different point of view, fixing a conjugacy class $\mathscr{C}$ with nonzero rotation number gives a modified Milnor-Wood type inequality as in Theorem 1(1) for the Toledo invariant restricted to $\mathrm{Hom}^{\mathscr{C}}\big(\pi_1(\Sigma), G\big)$. Goldman showed in [28] that for the one-punctured torus, representations into $\mathrm{PSL}(2, \mathbb{R})$ maximizing the Toledo invariant with respect to this modified Milnor-Wood type inequality correspond to singular hyperbolic structures on the torus with cone type singularities in the puncture.

2. **Preliminaries**

2.1. *Hermitian symmetric spaces, bounded continuous cohomology.*

2.1.1. A Lie group G is of *Hermitian type* if it is connected, semisimple with finite center and no compact factors, and if the associated symmetric space is Hermitian. A Lie group G is of type (RH) if it is connected reductive with compact center and the quotient G/G_c by the largest connected compact normal subgroup G_c is of Hermitian type.

If G is a locally compact group, $\mathrm{H}^\bullet_{\mathrm{c}}(G, \mathbb{R})$ denotes the continuous cohomology with $\mathbb{R}$-trivial coefficients, while $\mathrm{H}^\bullet_{\mathrm{cb}}(G, \mathbb{R})$ is the bounded continuous cohomology; for the general theory concerning the latter and its relation to the former, we refer to [46], [7], [8], and [9].

When G is of type (RH) and $\mathscr{X}$ is its associated symmetric space, we have isomorphisms

$$\Omega^2(\mathscr{X})^G \longrightarrow \mathrm{H}^2_{\mathrm{c}}(G, \mathbb{R}) \longleftarrow \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R}),$$

where the first is the van Est isomorphism between the complex $\Omega^\bullet(\mathscr{X})^G$ of G-invariant differential forms on $\mathscr{X}$ and $\mathrm{H}^\bullet_{\mathrm{c}}(G, \mathbb{R})$ [50], while the second is the comparison map which in degree two is an isomorphism [7]. Given $\omega \in \Omega^2(\mathscr{X})^G$ and $x \in \mathscr{X}$ a basepoint, the function

$$c_\omega(g_0, g_1, g_2) := \frac{1}{2\pi} \int_{\Delta(g_0x, g_1x, g_2x)} \omega\,,$$

where $\Delta(g_0x, g_1x, g_2x)$ denotes a smooth triangle with geodesic sides, defines a homogeneous G-invariant cocycle which is moreover bounded; when $\omega = \omega_\mathscr{X}$ is the Kähler form for the unique G-invariant Hermitian metric of minimal holomorphic sectional curvature -1 (*normalized metric*), we let $\kappa_G \in \mathrm{H}^2_\mathrm{c}(G, \mathbb{R})$ and $\kappa_G^\mathrm{b} \in \mathrm{H}^2_\mathrm{cb}(G, \mathbb{R})$ denote the corresponding classes and refer to κ_G (respectively κ_G^b) as the Kähler class (respectively bounded Kähler class). For the Gromov norm of κ_G^b, we have that

$$\|\kappa_G^\mathrm{b}\| = \frac{\mathrm{r}_\mathscr{X}}{2}\,, \tag{2.1}$$

where $\mathrm{r}_\mathscr{X}$ is the rank of $\mathscr{X}$.

2.1.2. Let G be of Hermitian type and $\mathscr{X}$ be the associated symmetric space. Then:

(2.a) A *maximal polydisk* in $\mathscr{X}$ is the image of a totally geodesic and holomorphic embedding $t : \mathbb{D}^{\mathrm{r}_\mathscr{X}} \to \mathscr{X}$ of a product of $\mathrm{r}_\mathscr{X}$ Poincaré disks.

(2.b) A *diagonal disk* (or *tight holomorphic disk*) in $\mathscr{X}$, is the image of the diagonal $\mathbb{D} \subset \mathbb{D}^{\mathrm{r}_\mathscr{X}}$ under t; we will denote by $d : \mathbb{D} \to \mathscr{X}$ the resulting totally geodesic and holomorphic embedding.

To the above objects are associated a connected finite covering L of $\mathrm{PU}(1,1)$ and homomorphisms $\tau : L^{\mathrm{r}_\mathscr{X}} \to G$ and $\Delta : L \to G$ with respect to which t and d are equivariant. We have moreover that

$$\tau^*(\kappa_G^\mathrm{b}) = \kappa_{L^{\mathrm{r}_\mathscr{X}}}^\mathrm{b} \quad \text{and} \quad \Delta^*(\kappa_G^\mathrm{b}) = \mathrm{r}_\mathscr{X}\kappa_L^\mathrm{b}\,. \tag{2.2}$$

2.1.3. Let G be of type (RH), $\mathscr{X}$ the associated symmetric space, $\mathscr{D}$ the bounded domain realization, and $\check{S}$ its Shilov boundary. Then $\check{S}$ is a homogeneous G-space of the form G/Q, where Q is a specific parabolic subgroup (which is maximal if $\mathscr{X}$ is irreducible). Two points $x, y \in \check{S}$ are *transversal* if (x, y) lies in the open G-orbit in $\check{S}^2$. Let $\check{S}^{(3)}$ denote the set of triples of pairwise transversal points. It was shown by Clerc and Ørsted ([20], [21]) that the map

$$\begin{aligned} \mathscr{D}^3 &\longrightarrow \qquad \mathbb{R} \\ (x, y, z) &\mapsto \frac{1}{2\pi}\int_{\Delta(x,y,z)} \omega_\mathscr{D}\,, \end{aligned}$$

where $\omega_\mathscr{D}$ is the Kähler form for the normalized metric on $\mathscr{D}$, extends continuously to $\check{S}^{(3)}$; Clerc showed then that by taking appropriate tangential limits one obtains a well defined G-invariant Borel cocycle

$$\beta_{\check{S}} : \check{S}^3 \to \mathbb{R}$$

extending the previous one, which satisfies

$$|\beta_{\check{S}}(x,y,z)| \leq \frac{\mathrm{r}_{\mathscr{D}}}{2}\,,$$

where $\mathrm{r}_{\mathscr{D}} = \mathrm{r}_{\mathscr{X}}$ (see [17, Th. 5.3]). In the following $\beta_{\check{S}}$ will be referred to as the *generalized Maslov cocycle* and a triple $x, y, z \in \check{S}$ for which $\beta_{\check{S}}(x,y,z) = \frac{\mathrm{r}_{\mathscr{D}}}{2}$ will be called *maximal*.

Let now $\big(\mathscr{B}^{\infty}_{\mathrm{alt}}(\check{S}^{\bullet})\big)$ denote the complex of bounded alternating Borel cocycles on $\check{S}$. Then, under the canonical map

$$\mathrm{H}^{\bullet}\big(\mathscr{B}^{\infty}_{\mathrm{alt}}(\check{S}^{\bullet})^{G}\big) \to \mathrm{H}^{\bullet}_{\mathrm{cb}}(G,\mathbb{R}) \tag{2.3}$$

(see [14, §4.2.], [9, Cor. 2.2.]) the class defined by $\beta_{\check{S}}$ corresponds to κ^{b}_{G}.

2.2. *Bounded singular and bounded group cohomology.* A "space" will always refer to a countable CW-complex and A will be one of the coefficients $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{R}/\mathbb{Z}$. For a pair of spaces $Y \subset X$, $\mathrm{H}^{\bullet}(X,Y,A)$ and $\mathrm{H}^{\bullet}_{\mathrm{b}}(X,Y,A)$ denote respectively the singular relative cohomology with coefficients in A and its bounded counterpart; observe that $\mathrm{H}^{\bullet}(X,Y,\mathbb{R}/\mathbb{Z}) = \mathrm{H}^{\bullet}_{\mathrm{b}}(X,Y,\mathbb{R}/\mathbb{Z})$. Also, $\mathrm{H}^{\bullet}\big(\pi_1(X),A\big)$ and $\mathrm{H}^{\bullet}_{\mathrm{b}}\big(\pi_1(X),A\big)$ denote respectively the group cohomology and the bounded group cohomology of $\pi_1(X)$ with A-coefficients, and $\mathrm{H}^{\bullet}\big(\pi_1(X),\mathbb{R}/\mathbb{Z}\big)=\mathrm{H}^{\bullet}_{\mathrm{b}}\big(\pi_1(X),\mathbb{R}/\mathbb{Z}\big)$. These cohomology theories come with the following natural comparison maps

$$\begin{aligned}
\mathrm{H}^{\bullet}_{\mathrm{b}}(X,Y,A) &\to \mathrm{H}^{\bullet}(X,Y,A)\,,\\
\mathrm{H}^{\bullet}_{\mathrm{b}}\big(\pi_1(X),A\big) &\to \mathrm{H}^{\bullet}\big(\pi_1(X),A\big)\,,\\
\mathrm{H}^{\bullet}\big(\pi_1(X),A\big) &\to \mathrm{H}^{\bullet}\big(X,A\big)\,,\\
\mathrm{H}^{\bullet}_{\mathrm{b}}\big(\pi_1(X),A\big) &\to \mathrm{H}^{\bullet}_{\mathrm{b}}\big(X,A\big)\,,
\end{aligned}$$

where the last two are induced by the classifying map $X \to B\pi_1(X)$.

We recall the following facts:

(2.c) the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0 \tag{2.4}$$

gives rise to long exact sequences in each of the four cohomology theories; these sequences are natural with respect to the four comparison maps;

(2.d) the inclusion of spaces $Z_1 \subset Z_2 \subset X$ induces a long exact sequence in singular relative and bounded singular relative cohomology which fits into the long exact sequences coming from the coefficient sequence in (2.c);

(2.e) in general the comparison map

$$g_X : \mathrm{H}^{\bullet}_{\mathrm{b}}\big(\pi_1(X),\mathbb{R}\big) \to \mathrm{H}^{\bullet}_{\mathrm{b}}(X,\mathbb{R})$$

is an isomorphism [30], [6], [35], referred to as *Gromov isomorphism*; as a consequence, if each connected component of Z_1 and Z_2 has amenable fundamental group, then the map

$$j_{Z_1,Z_2} : \mathrm{H}^\bullet_{\mathrm{b}}(X, Z_2, \mathbb{R}) \to \mathrm{H}^\bullet_{\mathrm{b}}(X, Z_1, \mathbb{R})$$

is an isomorphism; when $Z_1 = \varnothing$ we set $j_{\varnothing,Z_2} =: j_{Z_2}$ for ease of notation. We will only need the above isomorphism when all spaces involved are $K(\pi, 1)$'s, in which case we have for all coefficients A a commutative diagram

$$\begin{tikzcd} \mathrm{H}^\bullet_{\mathrm{b}}\big(\pi_1(X), A\big) \arrow[r, "g_X"] \arrow[d] & \mathrm{H}^\bullet_{\mathrm{b}}(X, A) \arrow[d] \\ \mathrm{H}^\bullet\big(\pi_1(X), A\big) \arrow[r, "g_X"] & \mathrm{H}^\bullet(X, A)\,, \end{tikzcd}$$

where the horizontal maps are isomorphisms.

2.3. (*Bounded*) *Borel cohomology versus* (*bounded*) *continuous cohomology.* Given a locally compact group G and $A = \mathbb{Z}, \mathbb{R}, \mathbb{R}/\mathbb{Z}$, we have the complexes $\big(\mathrm{C}(G^\bullet, A)\big)$, $\big(\mathrm{C_b}(G^\bullet, A)\big)$, $\big(\mathscr{B}(G^\bullet, A)\big)$, and $\big(\mathscr{B}_{\mathrm{b}}(G^\bullet, A)\big)$, of A-valued continuous, bounded continuous, Borel and bounded Borel cochains on G which lead, by taking the cohomology of the G-invariants, to the A-valued continuous $\mathrm{H}^\bullet_{\mathrm{c}}(G, A)$, bounded continuous $\mathrm{H}^\bullet_{\mathrm{cb}}(G, A)$, Borel $\widehat{\mathrm{H}}^\bullet_{\mathrm{c}}(G, A)$ and bounded Borel $\widehat{\mathrm{H}}^\bullet_{\mathrm{cb}}(G, A)$ cohomology. Of course when $A = \mathbb{Z}$ the first two cohomology theories are not of much use and their Borel version is a natural substitute. We have at any rate comparison maps coming from the obvious inclusions of complexes

$$\begin{tikzcd} \mathrm{H}^\bullet_{\mathrm{cb}}(G, A) \arrow[r] \arrow[d] & \mathrm{H}^\bullet_{\mathrm{c}}(G, A) \arrow[d] \\ \widehat{\mathrm{H}}^\bullet_{\mathrm{cb}}(G, A) \arrow[r] & \widehat{\mathrm{H}}^\bullet_{\mathrm{c}}(G, A)\,. \end{tikzcd}$$

For us the following facts will be of importance:

(2.f) the short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z} \longrightarrow 0$$

gives rise to long exact sequences in Borel and bounded Borel cohomology which are compatible with respect to the comparison map;

(2.g) $\mathrm{H}^\bullet_{\mathrm{cb}}(G, \mathbb{R}/\mathbb{Z}) = \mathrm{H}^\bullet_{\mathrm{c}}(G, \mathbb{R}/\mathbb{Z})$ and $\widehat{\mathrm{H}}^\bullet_{\mathrm{cb}}(G, \mathbb{R}/\mathbb{Z}) = \widehat{\mathrm{H}}^\bullet_{\mathrm{c}}(G, \mathbb{R}/\mathbb{Z})$;

(2.h) if $A = \mathbb{R}, \mathbb{Z}$, then $\widehat{\mathrm{H}}^1_{\mathrm{c}}(G, A) = \mathrm{Hom_c}(G, A)$ and $\mathrm{H}^1_{\mathrm{cb}}(G, A) = 0$;

(2.i) $\mathrm{H}^\bullet_{\mathrm{cb}}(G, \mathbb{R}) = \widehat{\mathrm{H}}^\bullet_{\mathrm{cb}}(G, \mathbb{R})$, which can be checked by using the regularization operators defined in [2, §4];

(2.l) if G is a Lie group, the comparison map $\mathrm{H}^\bullet_{\mathrm{c}}(G, \mathbb{R}) \to \widehat{\mathrm{H}}^\bullet_{\mathrm{c}}(G, \mathbb{R})$ is an isomorphism [52, Th. 3].

3. Toledo numbers, basic properties and first consequences

3.1. *Definitions and basic properties.* Let Σ be a compact oriented surface with (possibly empty) boundary $\partial\Sigma$, G a locally compact group, and $\rho:\pi_1(\Sigma)\to G$ a homomorphism. Using the diagram

$$\begin{tikzcd} \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R}) \arrow[r, "\rho^*"] & \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(\Sigma),\mathbb{R}\big) \arrow[r, "g_\Sigma"] & \mathrm{H}^2_{\mathrm{b}}(\Sigma,\mathbb{R}) \\ & & \mathrm{H}^2_{\mathrm{b}}(\Sigma,\partial\Sigma,\mathbb{R}) \arrow[u, "j_{\partial\Sigma}"'] \end{tikzcd}$$

where ρ^* is the pullback in bounded cohomology, g_Σ the Gromov isomorphism in (2.e), and $j_{\partial\Sigma}$ is the isomorphism in (2.e) in bounded singular cohomology induced by the inclusion $(\Sigma,\varnothing)\to(\Sigma,\partial\Sigma)$, we make the following

Definition 3.1. The *Toledo number of ρ relative to a class $\kappa\in\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ is*

$$\mathrm{T}_\kappa(\Sigma,\rho):=\big\langle (j_{\partial\Sigma})^{-1}g_\Sigma\rho^*(\kappa),[\Sigma,\partial\Sigma]\big\rangle\,.$$

Here $[\Sigma,\partial\Sigma]\in\mathrm{H}^2(\Sigma,\partial\Sigma,\mathbb{R})$ denotes the relative fundamental class and $(j_{\partial\Sigma})^{-1}g_\Sigma\rho^*(\kappa)$ is considered as an ordinary relative cohomology class.

When G is of type (RH) the *Toledo number* $\mathrm{T}(\Sigma,\rho)$ *of* ρ is defined as $\mathrm{T}_\kappa(\Sigma,\rho)$ where $\kappa=\kappa_G^{\mathrm{b}}$ is the bounded Kähler class (see §2.1.1). The following two properties are immediate:

- if ρ_1 and ρ_2 are G-conjugate, then $\mathrm{T}_\kappa(\Sigma,\rho_1)=\mathrm{T}_\kappa(\Sigma,\rho_2)$;
- if $f:\Sigma_1\to\Sigma_2$ is a continuous map of degree $d\geq 1$ and $\rho_i:\pi_1(\Sigma_i)\to G$ are homomorphisms related by $\rho_1=\rho_2 f_*$, where f_* is the morphism induced on the fundamental groups, then $\mathrm{T}_\kappa(\Sigma_1,\rho_1)=d\cdot\mathrm{T}_\kappa(\Sigma_2,\rho_2)$.

The next results describe the behavior of the Toledo numbers under natural topological operations on surfaces.

PROPOSITION 3.2. *Let Σ be a surface and $\rho:\pi_1(\Sigma)\to G$ a homomorphism.*

(1) (*Additivity*) *If $\Sigma=\Sigma_1\cup_C\Sigma_2$ is the connected sum of two subsurfaces Σ_i along a separating loop C, then*

$$\mathrm{T}_\kappa(\Sigma,\rho)=\mathrm{T}_\kappa(\Sigma_1,\rho_1)+\mathrm{T}_\kappa(\Sigma_2,\rho_2)\,,$$

where ρ_i is the restriction of ρ to $\pi_1(\Sigma_i)$.

(2) (*Invariance under gluing*) *If Σ' is the surface obtained by cutting Σ along a nonseparating loop C and $i:\Sigma'\to\Sigma$ is the canonical map, then*

$$\mathrm{T}_\kappa(\Sigma',\rho i_*)=\mathrm{T}_\kappa(\Sigma,\rho)\,.$$

Proof. Here we prove the additivity property, the proof of the invariance under gluing proceeds along similar lines. Let $\alpha \in \mathrm{H}^2_{\mathrm{b}}(\Sigma, \partial\Sigma)$ and let

$$j := j_{\partial\Sigma\cup C, \partial\Sigma} : \mathrm{H}^2_{\mathrm{b}}(\Sigma, \partial\Sigma \cup C) \to \mathrm{H}^2_{\mathrm{b}}(\Sigma, \partial\Sigma)$$

be the morphism given by the inclusion $(\Sigma, \partial\Sigma) \to (\Sigma, \partial\Sigma \cup C)$; notice that j is an isomorphism since every connected component of $\partial\Sigma \cup C$ has amenable fundamental group. Then:

$$\begin{aligned}\big\langle \alpha, [\Sigma, \partial\Sigma]\big\rangle &= \big\langle j^{-1}(\alpha), [\Sigma_1, \partial\Sigma_1] + [\Sigma_2, \partial\Sigma_2]\big\rangle \\ &= \big\langle j^{-1}(\alpha)|_{\Sigma_1}, [\Sigma_1, \partial\Sigma_1]\big\rangle + \big\langle j^{-1}(\alpha)|_{\Sigma_2}, [\Sigma_2, \partial\Sigma_2]\big\rangle .\end{aligned}$$

Using that $j^{-1}(\alpha)|_{\Sigma_i} = (j_i)^{-1}(\alpha|_{\Sigma_i})$, where $j_i := j_{\partial\Sigma_i - C, \partial\Sigma_i}$, we get

$$\big\langle \alpha, [\Sigma, \partial\Sigma]\big\rangle = \big\langle (j_1)^{-1}(\alpha|_{\Sigma_1}), [\Sigma_1, \partial\Sigma_1]\big\rangle + \big\langle (j_2)^{-1}(\alpha|_{\Sigma_2}), [\Sigma_2, \partial\Sigma_2]\big\rangle .$$

Specializing to $j_{\partial\Sigma}(\alpha) = g_\Sigma \rho^*(\kappa)$ and observing that

$$(j_i)^{-1}(\alpha|_{\Sigma_i}) = (j_i)^{-1}(\alpha)|_{\Sigma_i} = (j_{\partial\Sigma_i})^{-1} g_{\Sigma_i} \rho_i^*(\kappa)$$

concludes the proof. □

3.2. *The analytic formula.* In this section we relate the Toledo numbers introduced in Section 3.1 to invariants introduced and studied in [11]. Let G be a locally compact group. Let L be a finite connected covering of $\mathrm{PU}(1,1)$, $\Gamma < L$ a lattice, and $\rho : \Gamma \to G$ a homomorphism. Composing the transfer map

$$\mathrm{T_b} : \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R}) \to \mathrm{H}^2_{\mathrm{cb}}(L, \mathbb{R})$$

with the pullback ρ^*, we obtain the bounded Toledo map

$$\mathrm{T_b}(\rho) : \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R}) \to \mathrm{H}^2_{\mathrm{cb}}(L, \mathbb{R}) = \mathbb{R}\kappa_L^{\mathrm{b}}$$

(see [11]) which leads to an invariant $\mathrm{t_b}(\rho, \kappa) \in \mathbb{R}$ given by

$$\mathrm{T_b}(\rho)(\kappa) = \mathrm{t_b}(\rho, \kappa)\kappa_L^{\mathrm{b}}$$

for $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$. When G is of type (RH) we set, in analogy with Section 3.1,

$$\mathrm{t_b}(\rho) = \mathrm{t_b}(\rho, \kappa_G^{\mathrm{b}}) .$$

THEOREM 3.3. *Let $h : \pi_1(\Sigma) \to \Gamma$ be an isomorphism whose composition with the projection to* $\mathrm{PU}(1,1)$ *is the developing homomorphism of a complete hyperbolic structure on Σ° with finite area. Then*

$$\mathrm{T}_\kappa(\Sigma, \rho) = |\chi(\Sigma)|\, \mathrm{t_b}(\rho \circ h^{-1}, \kappa)$$

for any homomorphism $\rho : \pi_1(\Sigma) \to G$ and $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$.

We defer the proof of this theorem until Section 3.3 and collect here a few important consequences.

COROLLARY 3.4. *We have the following Milnor-Wood type bounds*:

(1) $|\mathrm{T}_\kappa(\Sigma,\rho)| \leq 2|\chi(\Sigma)|\,\|\kappa\|$;

(2) *if G is of type* (RH), *then* $|\mathrm{T}(\Sigma,\rho)| \leq |\chi(\Sigma)|\,\mathrm{r}_{\mathscr{X}}$.

Proof. The first assertion follows from Theorem 3.3 and the fact that the transfer T_{b} and the pullback are both norm decreasing. The second assertion follows from the first one and the equality $\|\kappa_G^{\mathrm{b}}\| = \frac{\mathrm{r}_{\mathscr{X}}}{2}$ (see §2.1). $\square$

The following are then the two main concepts of this paper:

Definition 3.5. Let G be a group of type (RH).

(1) A homomorphism $\rho : \pi_1(\Sigma) \to G$ of a surface group $\pi_1(\Sigma)$ is *maximal* if $\mathrm{T}(\Sigma,\rho) = |\chi(\Sigma)|\mathrm{r}_{\mathscr{X}}$.

(2) A homomorphism $\rho : \Gamma \to G$ of a lattice $\Gamma < L$ is *maximal* if $\mathrm{t}_{\mathrm{b}}(\rho) = \mathrm{r}_{\mathscr{X}}$.

Observe that the first definition generalizes the concept of maximal representation given in the introduction and puts it in the context of groups of type (RH) which will turn out to be the right one for the proofs. The second concept of maximality is equivalent to the one introduced in [11], the equivalence being given by [11, Lemma 5.3]. The relationship between the above definitions is given by Theorem 3.3.

Proof of Theorem 3. Let $h : \pi_1(\Sigma) \to \mathrm{PU}(1,1)$ be a hyperbolization of Σ° with finite area and image Γ; in particular h is induced by a diffeomorphism $f : \Sigma^\circ \to \Gamma\backslash\mathbb{D}$. Let $\rho : \pi_1(\Sigma) \to \mathrm{PU}(1,1)$ be a homomorphism. If ρ is maximal, then, by Theorem 3.3, $\rho \circ h^{-1} : \Gamma \to G$ is maximal as a representation of the lattice Γ, and it follows then from [11, Lemma 5.2 and Cor. 11] that $\rho \circ h^{-1}$ is induced by a diffeomorphism

$$f_\rho : \Gamma\backslash\mathbb{D} \to \rho\big(\pi_1(\Sigma)\big)\backslash\mathbb{D}$$

which implies that ρ itself is induced by the diffeomorphism

$$f_\rho \circ f : \Sigma^\circ \to \rho\big(\pi_1(\Sigma)\big)\backslash\mathbb{D}\,.$$

Conversely, if ρ is induced by a complete hyperbolic metric on Σ°, there exists a semiconjugation $F : \partial\mathbb{D} \to \partial\mathbb{D}$ in the sense of Ghys [25] with $\rho(\gamma)F = Fh(\gamma)$. Since $\kappa^{\mathrm{b}}_{\mathrm{PU}(1,1)}$ is the bounded real Euler class [25], we have that

$$\rho^*(\kappa^{\mathrm{b}}_{\mathrm{PU}(1,1)}) = h^*(\kappa^{\mathrm{b}}_{\mathrm{PU}(1,1)})$$

and hence

$$(\rho \circ h^{-1})^*(\kappa^{\mathrm{b}}_{\mathrm{PU}(1,1)}) = \kappa^{\mathrm{b}}_{\mathrm{PU}(1,1)}|_\Gamma,$$

which, applying the transfer map, implies that $\mathrm{t}_{\mathrm{b}}(\rho \circ h^{-1}) = 1$, and thus ρ is maximal by Theorem 3.3. $\square$

3.3. *Proof of Theorem* 3.3. The statement of Theorem 3.3 can be reformulated as follows. Let $\Gamma < L$ be a torsionfree lattice; we consider the finite area surface $S = \Gamma\backslash\mathbb{D}$ as interior of a compact surface $\overline{S}$ with boundary $\partial\overline{S}$, which is a union of circles. Given a homomorphism $\rho : \Gamma \to G$ and identifying Γ with $\pi_1(S) = \pi_1(\overline{S})$, the assertion is that

$$\mathrm{T}_\kappa(\overline{S}, \rho) = |\chi(S)| \mathrm{t_b}(\kappa, \rho) .$$

For $T \geq 0$ large enough, let $S_{\geq T}$ denote the union of the convex cusp neighborhoods bounded by horocycles of length $1/T$. It is easy to verify that if $\beta \in \mathrm{H}^2_\mathrm{b}(\Gamma, \mathbb{R})$, we have

$$\big\langle (j_{\partial\overline{S}})^{-1} g_{\overline{S}}(\beta), [\overline{S}, \partial\overline{S}] \big\rangle = \big\langle (j_T)^{-1} g_S(\beta), [S, S_{\geq T}] \big\rangle \tag{3.1}$$

where, for ease of notation, j_T refers to the canonical isomorphism

$$\mathrm{H}^2_\mathrm{b}(S, S_{\geq T}, \mathbb{R}) \to \mathrm{H}^2_\mathrm{b}(S, \mathbb{R}) .$$

Introducing the notation

$$\mathrm{T_b}(\beta) = \tau(\beta)\kappa^\mathrm{b}_L ,$$

where $\mathrm{T_b}$ is the transfer operator, the theorem will then follow from (3.1) and the proposition below applied to $\beta = \rho^*(\kappa)$.

PROPOSITION 3.6. *With the above notation,*

$$\big\langle (j_T)^{-1} g_S(\beta), [S, S_{\geq T}] \big\rangle = \tau(\beta)|\chi(S)| .$$

The rest of this subsection is devoted to the proof of Proposition 3.6 for which we will need the three lemmas below. We fix the following notation: if Y is any topological space, let $S_m(Y)$ denote the set of singular m-simplices and $F_\mathrm{b}(Y, \mathbb{R})$ the space of bounded m-cochains.

LEMMA 3.7 (Loeh-Strohm, [41, Th. 2.37]). *Let $U \subset \mathbb{D}$ be a convex subset and $\Lambda < L$ be a discrete torsionfree subgroup preserving U. The canonical isomorphism*

$$\mathrm{H}^m_\mathrm{b}(\Lambda, \mathbb{R}) \xrightarrow{\cong} \mathrm{H}^m_\mathrm{b}(\Lambda\backslash U, \mathbb{R})$$

can be implemented by the map

$$\begin{array}{ccc} \mathrm{C}_{\mathrm{b,alt}}(U^{m+1}, \mathbb{R})^\Lambda & \to & F^m_\mathrm{b}(\Lambda\backslash U, \mathbb{R}) \\ f & \longmapsto & \overline{f} , \end{array}$$

defined by $\overline{f}(\sigma) := f(\tilde{\sigma}_0, \ldots, \tilde{\sigma}_m)$, where $\sigma : \Delta^m \to \Lambda\backslash U$ is an m-simplex and $\tilde{\sigma} : \Delta^m \to U$ is a lift with vertices $\tilde{\sigma}_0, \ldots, \tilde{\sigma}_m$.

The next lemma follows from standard properties of the transfer map and will also be useful later on. Let $A(x, y, z)$ denote the area of a geodesic triangle in $\mathbb{D}$ with vertices x, y, and z, and let μ be the L-invariant probability measure on $\Gamma\backslash L$.

LEMMA 3.8. *Let $a \in \mathrm{C}_{\mathrm{b,alt}}(\mathbb{D}^3, \mathbb{R})^\Gamma$ be a representative of the class $\beta \in \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$. Then*

$$\int_{\Gamma\backslash L} a(gx, gy, gz) d\mu(g) = \frac{\tau(\beta)}{2\pi} A(x, y, z)\,.$$

We will also need to represent the relative cycle $[S, S_{\geq T}]$ using smearing in the context of relative measure homology. For $\delta \in S_2(\mathbb{D})$ we consider as usual the continuous map

$$\begin{aligned} m_\delta : \Gamma\backslash L &\to S_2(S) \\ \Gamma g \quad &\mapsto p(g\delta) \end{aligned}$$

where $p : \mathbb{D} \to S$ is the canonical projection and define

$$Sm_T(\delta) = (m_\delta)_*\big(\mu|_{(\Gamma\backslash L)_T}\big)\,,$$

where $(\Gamma\backslash L)_T = \{\Gamma g \in \Gamma\backslash L : p(g0) \in S_{\leq T}\}$ and $S_{\leq T}$ is the complement of $S_{\geq T}$. The following is then a verification proceeding along standard arguments.

LEMMA 3.9. *Let $\sigma : \Delta^2 \to \mathbb{D}$ be a geodesic simplex and σ' its reflection along one side. Then there is $C > 1$ such that the boundary of the measured chain*

$$\mu_{CT} := Sm_{CT}(\sigma) - Sm_{CT}(\sigma')$$

has its support in $S_1(S_{\geq T})$ and μ_{CT} represents the relative cycle

$$\frac{2A(\sigma)}{A(S)}[S, S_{\geq T}]$$

where A refers to the hyperbolic area.

Proof of Proposition 3.6. In the notation of Lemma 3.7, let $a \in \mathrm{C}_{\mathrm{b,alt}}(D^3, \mathbb{R})^\Gamma$ be such that $\overline{a}$ is a representative of $g_S(\beta)$. Then for T_0 large enough $\overline{a}$ restricted to $S_{\geq T_0}$ is trivial in bounded cohomology, and using Lemma 3.7 applied to appropriate cusp neighborhoods we get a continuous bounded function

$$\overline{f} : S_1(S_{\geq T_0}) \to \mathbb{R}$$

with $\overline{a}|_{S_2(S_{\geq T_0})} = d\,\overline{f}$.

For $T \geq T_0$ define $f_T : S_1(S) \to \mathbb{R}$ as being equal to $\overline{f}$ on simplices in $S_{\geq T}$ and zero otherwise, and let $a_T := \overline{a} - df_T$. Then a_T is a bounded Borel function on $S_2(S)$ and $\|a_T\|_\infty \leq C$ for some constant $C > 0$. Moreover a_T is a representative of $(j_T)^{-1} g_S(\beta)$. For $T_2 \geq T_1 \geq T_0$ we clearly have

$$\big\langle (j_{T_1})^{-1} g_S(\beta), [S, S_{\geq T_1}]\big\rangle = \big\langle (j_{T_1})^{-1} g_S(\beta), [S, S_{\geq T_2}]\big\rangle\,. \tag{3.2}$$

According to Lemma 3.9 the right-hand side equals

$$\left\langle a_{T_1}, \frac{A(S)}{2A(\sigma)} \mu_{CT_2} \right\rangle$$

which, letting $T_2 \to \infty$, gives

$$\frac{A(S)}{2A(\sigma)} \int_{\Gamma \backslash L} \big(a_{T_1}(pg(\sigma)) - a_{T_1}(pg(\sigma'))\big) d\mu(g)\,.$$

Since however the left-hand side of (3.2) is independent of T_1, we let $T_1 \to \infty$ and, using the dominated convergence theorem, obtain

$$\begin{aligned}&\big\langle (j_T)^{-1} g_S(\beta), [S, S_{\geq T}]\big\rangle \\ &\qquad = \frac{A(S)}{2A(\sigma)} \int_{\Gamma \backslash L} \big(a(g\sigma_0, g\sigma_1, g\sigma_2) - a(g\sigma_0', g\sigma_1', g\sigma_2')\big) d\mu(g)\end{aligned}$$

which, together with Lemma 3.8, proves the proposition. □

3.4. *Continuity.* We will now use Theorem 3.3 to show the following

PROPOSITION 3.10. *Let G be a group of Hermitian type and let $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$. Then the map*

$$\begin{aligned} \mathrm{T}_\kappa(\Sigma, \cdot\,) : \mathrm{Hom}\big(\pi_1(\Sigma), G\big) &\longrightarrow \quad \mathbb{R} \\ \rho \qquad &\mapsto \mathrm{T}_\kappa(\Sigma, \rho) \end{aligned} \tag{3.3}$$

is continuous.

Together with the following basic example of maximal representation, the continuity of $\rho \mapsto \mathrm{T}_\kappa(\Sigma, \rho)$ allows us to determine the range of the map in (3.3) when $\kappa = \kappa_G^{\mathrm{b}}$ and $\partial\Sigma \neq \varnothing$.

Example 3.11. Let G be of Hermitian type with associated symmetric space $\mathscr{X}$, $d : \mathbb{D} \to \mathscr{X}$ a diagonal disk (see (2.b)), and $\Delta : L \to G$ the corresponding homomorphism, where L is an appropriate finite covering of $\mathrm{PU}(1,1)$. Then if $h : \pi_1(\Sigma) \to \Gamma < L$ is a finite area hyperbolization of Σ°, the homomorphism $\rho := \Delta \circ h$ is maximal. Indeed, we have that

$$\Delta^*(\kappa_G^{\mathrm{b}}) = \mathrm{r}_{\mathscr{X}} \kappa_L^{\mathrm{b}}$$

and hence (Theorem 3.3)

$$\mathrm{T}(\Sigma, \Delta \circ h) = |\chi(\Sigma)| \mathrm{t_b}(\Delta|_\Gamma) = |\chi(\Sigma)| \mathrm{r}_{\mathscr{X}}\,.$$

Observe that if d' is the composition of d with an antiholomorphic isometry of $\mathbb{D}$ and $\Delta' : L \to G$ is the corresponding homomorphism, then $\mathrm{T}(\Sigma, \Delta' \circ h) = -|\chi(\Sigma)| \mathrm{r}_{\mathscr{X}}$.

COROLLARY 3.12. *Assume that $\partial\Sigma \neq \varnothing$. Then the range of the map $T(\Sigma, \cdot\,)$ is the interval $\big[-\mathrm{r}_{\mathscr{X}}|\chi(\Sigma)|, \mathrm{r}_{\mathscr{X}}|\chi(\Sigma)|\big]$.*

Proof. By Corollary 3.4 the range is contained in the above interval, and, by Example 3.11 it contains the endpoints. Since $\partial\Sigma \neq \varnothing$, $\pi_1(\Sigma)$ is a free group and hence $\mathrm{Hom}\big(\pi_1(\Sigma), G\big)$ is connected. The corollary then follows from Proposition 3.10. □

Turning now to the proof of Proposition 3.10 we will need the following:

LEMMA 3.13. *Let G be a connected semisimple Lie group with associated symmetric space $\mathcal{X}$, let $\mathrm{C}(\mathbb{D},\mathcal{X})$ be the space of continuous maps from the Poincaré disk $\mathbb{D}$ into $\mathcal{X}$, with the topology of uniform convergence on compact sets, and let Γ be a torsionfree lattice in $\mathrm{PU}(1,1)$. Then there is a continuous map*

$$\begin{aligned}\mathrm{Hom}(\Gamma,G) &\to \mathrm{C}(\mathbb{D},\mathcal{X})\\ \rho \quad &\mapsto \quad F_\rho\end{aligned}$$

such that F_ρ is equivariant with respect to $\rho:\Gamma\to G$.

Proof. Let $\mathcal{K}$ be a simplicial complex such that $|\mathcal{K}|$ is homeomorphic to $\Gamma\backslash\mathbb{D}$. Let $\rho:\Gamma\to G$ be a homomorphism and $F:\widetilde{\mathcal{K}}^{(0)}\to\mathcal{X}$ a ρ-equivariant map defined on the 0-skeleton of the universal covering of $\mathcal{K}$. Using barycentric coordinates on the simplices of $\widetilde{\mathcal{K}}$ and the center of mass in $\mathcal{X}$, one obtains a canonical continuous extension $F^{\mathrm{ext}}:\widetilde{\mathcal{K}}\to\mathcal{X}$ which is thus ρ-equivariant and depends continuously on F. Fix $Y\subset\widetilde{\mathcal{K}}^{(0)}$ a complete set of representatives of Γ-orbits in $\widetilde{\mathcal{K}}^{(0)}$ and fix any map $f:Y\to\mathcal{X}$. Then given $\rho:\Gamma\to G$, we define $f_\rho:\widetilde{\mathcal{K}}^{(0)}\to\mathcal{X}$ as the unique ρ-equivariant extension of f and $F_\rho:=(f_\rho)^{\mathrm{ext}}$. The assertion that $\rho\mapsto F_\rho$ is continuous follows from the continuity of the center of mass construction in $\mathcal{X}$. □

Proof of Proposition 3.10. We realize, as we may, the bounded continuous cohomology of G on the complex $\big(\mathrm{C}_{\mathrm{b,alt}}(\mathcal{X}^\bullet,\mathbb{R})\big)$ of bounded continuous alternating cochains on $\mathcal{X}$ and similarly for L and Γ on $\big(\mathrm{C}_{\mathrm{b,alt}}(\mathbb{D}^\bullet,\mathbb{R})\big)$. Given a homomorphism $\rho:\Gamma\to G$, the continuous ρ-equivariant map $F_\rho\in\mathrm{C}(\mathbb{D},\mathcal{X})$ in the previous lemma induces by precomposition a map of complexes

$$\big(\mathrm{C}_{\mathrm{b,alt}}(\mathcal{X}^\bullet,\mathbb{R})\big)^G\to\big(\mathrm{C}_{\mathrm{b,alt}}(\mathbb{D}^\bullet,\mathbb{R})\big)^\Gamma$$

which, according to [9], represents the pullback $\rho^*:\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})\to\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$. In particular, if $c:\mathcal{X}^3\to\mathbb{R}$ is a bounded continuous G-invariant alternating cocycle representing $\kappa\in\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$, then the cocycle

$$(z_1,z_2,z_3)\mapsto c\big(F_\rho(z_1),F_\rho(z_2),F_\rho(z_3)\big)$$

represents $\rho^*(\kappa)\in\mathrm{H}^2_{\mathrm{b}}(\Gamma,\mathbb{R})$ [9]. We then deduce from Lemma 3.8 that

$$\int_{\Gamma\backslash L}c\big(F_\rho(gz_1),F_\rho(gz_2),F_\rho(gz_3)\big)d\mu(g)=\frac{1}{2\pi}\mathrm{t_b}(\rho,\kappa)A(z_1,z_2,z_3)\,.$$

If now $\rho_n\to\rho$, then according to Lemma 3.13 $F_{\rho_n}\to F_\rho$ uniformly on compact sets and hence

$$c\big(F_{\rho_n}(gz_1),F_{\rho_n}(gz_2),F_{\rho_n}(gz_3)\big)\to c\big(F_\rho(gz_1),F_\rho(gz_2),F_\rho(gz_3)\big)$$

pointwise. Since

$$\big|c\big(F_{\rho_n}(gz_1),F_{\rho_n}(gz_2),F_{\rho_n}(gz_3)\big)\big|\le\|c\|_\infty\,,$$

the dominated convergence theorem implies that $\mathrm{t_b}(\rho_n,\kappa)\to\mathrm{t_b}(\rho,\kappa)$. □

4. Structure of maximal representations: the Zariski dense case

In this section we will investigate the structure of maximal homomorphisms $\rho : \Gamma \to G$, where, as before, $\Gamma < L$ is a lattice in a finite connected covering L of $\mathrm{PU}(1,1)$, $G = \mathrm{Iso}(\mathcal{X})^\circ$ is the connected component of the group of isometries of an irreducible Hermitian symmetric space $\mathcal{X}$, and we assume now that the image of ρ is Zariski dense. More precisely, if $\mathbf{G}$ is the connected adjoint $\mathbb{R}$-group associated to the complexification of the Lie algebra of G, we will prove the following:

THEOREM 4.1. *If $\rho : \Gamma \to \mathbf{G}(\mathbb{R})^\circ$ is a maximal representation with Zariski dense image*, *then*:

(1) *the Hermitian symmetric space $\mathcal{X}$ is of tube type*;

(2) *the image of ρ is discrete*;

(3) *the representation ρ is injective*, *modulo possibly the center $\mathcal{Z}(\Gamma)$ of Γ*.

4.1. *The formula.* Here we will use heavily the results of [14]. In particular, let $\mathcal{D}$ be the bounded domain realization of $\mathcal{X}$ and $\check{S}$ its Shilov boundary. Recall that $\check{S} = G/Q$, where Q is a specific maximal parabolic subgroup of G, and denote by $\check{S}^{(2)}$ the set of pairs of transverse points in $\check{S}$.

The lattice Γ acts on the boundary of the Poincaré disk $\partial\mathbb{D}$ and, as is well known, the space $(\partial\mathbb{D}, \lambda)$, where λ is the round measure on $\partial\mathbb{D}$ is a Poisson boundary for Γ; moreover, the Γ-action on $\partial\mathbb{D} \times \partial\mathbb{D}$ is ergodic. With this we can apply [10, Prop. 7.2] and [14, Th. 4.7] to conclude:

THEOREM 4.2. *Assume that $\rho : \Gamma \to \mathbf{G}(\mathbb{R})^\circ = G$ is a homomorphism with Zariski dense image. Then there exists a ρ-equivariant measurable map $\varphi : \partial\mathbb{D} \to \check{S}$ such that $\big(\varphi(x_1), \varphi(x_2)\big) \in \check{S}^{(2)}$ for almost all $(x_1, x_2) \in (\partial\mathbb{D})^2$.*

Using the boundary map φ, we now give an explicit cocycle on $(\partial\mathbb{D})^3$ representing the pullback $\rho^*(\kappa_G^{\mathrm{b}}) \in \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$. To this purpose, if $\beta_{\check{S}} : \check{S}^3 \to \mathbb{R}$ is the generalized Maslov cocycle (see §2.1.3), we have:

COROLLARY 4.3 ([14, Prop. 4.6]). *Under the canonical isomorphism*

$$\mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R}) \cong \mathcal{Z}L^\infty_{\mathrm{alt}}\big((\partial\mathbb{D})^3, \mathbb{R}\big)^\Gamma$$

the class $\rho_{\mathrm{b}}^*(\kappa_G^{\mathrm{b}})$ corresponds to the cocycle

$$\begin{aligned} (\partial\mathbb{D})^3 &\longrightarrow \mathbb{R} \\ (x, y, z) &\mapsto \beta_{\check{S}}\big(\varphi(x), \varphi(y), \varphi(z)\big). \end{aligned}$$

We then conclude:

COROLLARY 4.4. *Let $\rho : \Gamma \to G = \mathbf{G}(\mathbb{R})^\circ$ be a homomorphism with Zariski dense image and $\varphi : \partial\mathbb{D} \to \check{S}$ a measurable ρ-equivariant boundary map. Then if μ is the L-invariant probability measure on $\Gamma\backslash L$, we have that*

$$\int_{\Gamma\backslash L} \beta_{\check{S}}\big(\varphi(gx), \varphi(gy), \varphi(gz)\big) d\mu(g) = \mathrm{t_b}(\rho)\beta_{\partial\mathbb{D}}(x, y, z) \tag{4.1}$$

for almost every $(x,y,z)\in(\partial\mathbb{D})^3$. *In particular, if* ρ *is maximal,*

$$\beta_{\check{S}}\big(\varphi(x),\varphi(y),\varphi(z)\big)=\mathrm{r}_{\mathscr{X}}\,\beta_{\partial\mathbb{D}}(x,y,z) \tag{4.2}$$

for almost every $(x,y,z)\in\partial\mathbb{D}$ *and thus*

$$\rho^*(\kappa_G^{\mathrm{b}})=\mathrm{r}_{\mathscr{X}}\,\kappa_L^{\mathrm{b}}\,. \tag{4.3}$$

Proof. We have that

$$\mathrm{T_b}\big(\rho^*(\kappa_G^{\mathrm{b}})\big)=\mathrm{t_b}(\rho)\kappa_L^{\mathrm{b}}\,,$$

so that the formula follows from Corollary 4.3 and the functoriality of the transfer operator in [46, III.8].

Assume now that ρ is maximal, that is, $\mathrm{t_b}(\rho)=\mathrm{r}_{\mathscr{X}}$. Fix $(x_0,y_0,z_0)\in(\partial\mathbb{D})^3$ such that $\beta_{\partial\mathbb{D}}(x_0,y_0,z_0)=\frac{1}{2}$ and (4.1) holds. Since for every $a,b,c\in\check{S}$

$$|\beta_{\check{S}}(a,b,c)|\leq\frac{\mathrm{r}_{\mathscr{X}}}{2}\,,$$

([21] — see also [14, Th. 4.2]) and since μ is a probability measure, we deduce from (4.1) that for almost every $g\in L$

$$\beta_{\check{S}}\big(\varphi(gx_0),\varphi(gy_0),\varphi(gz_0)\big)=\mathrm{r}_{\mathscr{X}}\beta_{\partial\mathbb{D}}(x_0,y_0,z_0)\,. \tag{4.4}$$

Similarly, if $\beta_{\partial\mathbb{D}}(x_0,y_0,z_0)=-\frac{1}{2}$, by the same argument we deduce that (4.4) holds for almost every $g\in L$. Thus the function on $(\partial\mathbb{D})^3$

$$(x,y,z)\mapsto\beta_{\check{S}}\big(\varphi(x),\varphi(y),\varphi(z)\big)$$

is essentially L-invariant and (4.4) implies then that this function coincides almost everywhere with $\mathrm{r}_{\mathscr{X}}\beta_{\partial\mathbb{D}}$. □

4.2. *$\mathscr{X}$ is of tube type.* In this section we prove the first assertion of Theorem 4.1. For this we will use our characterization of tube type domains obtained in [14]. In particular we defined in [14, §2.4] the *Hermitian triple product*, a G-invariant map

$$\langle\langle\cdot,\cdot,\cdot\rangle\rangle:\check{S}^{(3)}\to\mathbb{R}^\times\backslash\mathbb{C}^\times$$

on the set $\check{S}^{(3)}$ of triples of points in $\check{S}$ which are pairwise transverse, and which is related to the generalized Maslov cocycle by

$$\langle\langle x,y,z\rangle\rangle\equiv e^{i\pi p_{\mathscr{X}}\beta_{\check{S}}(x,y,z)}\mod\mathbb{R}^\times$$

for all $(x,y,z)\in\check{S}^{(3)}$, where $p_{\mathscr{X}}$ is an integer defined in terms of the root system associated to G. Let $\check{S}=G/Q$ and $\mathbf{Q}$ be the $\mathbb{R}$-parabolic subgroup of $\mathbf{G}$ with $\mathbf{Q}(\mathbb{R})=Q$.

Then if $A^\times$ is the $\mathbb{R}$-algebraic group $\mathbb{C}^\times\times\mathbb{C}^\times$ (with real structure $(\lambda,\mu)\mapsto(\overline{\mu},\overline{\lambda})$) and $\mathbb{C}^\times\mathbf{1}=\{(\lambda,\lambda)\in A:\lambda\in\mathbb{C}^\times\}$, we construct a rational $\mathbf{G}$-invariant map defined over $\mathbb{R}$

$$\langle\langle\cdot,\cdot,\cdot\rangle\rangle_{\mathbb{C}}:(\mathbf{G}/\mathbf{Q})^3\to\mathbb{C}^\times\mathbf{1}\backslash A^\times\,,$$

which we call the *complex Hermitian triple product*, and which is related to the Hermitian triple product by the commutative diagram

$$\begin{tikzcd} (\mathbf{G}/\mathbf{Q})^3 \arrow[r, "\langle\langle\cdot,\cdot,\cdot\rangle\rangle_{\mathbb{C}}"] & \mathbb{C}^\times\mathbf{1}\backslash A^\times \\ \check{S}^{(3)} \arrow[u, "\iota^3"] \arrow[r, "\langle\langle\cdot,\cdot,\cdot\rangle\rangle"] & \mathbb{R}^\times\backslash\mathbb{C}^\times, \arrow[u, "\Delta"'] \end{tikzcd}$$

where ι is given by the G-map $\iota : \check{S} \to \mathbf{G}/\mathbf{Q}$ sending $\check{S}$ to $(\mathbf{G}/\mathbf{Q})(\mathbb{R})$, and $\Delta([\lambda]) = [(\lambda, \overline{\lambda})]$ (see [14, Cor. 2.11]); the statement includes the fact that the domain of definition of $\langle\langle\cdot,\cdot,\cdot\rangle\rangle_{\mathbb{C}}$ contains $\check{S}^{(3)}$.

Given now $(a, b) \in \check{S}^{(2)}$, let, as in [14, §5.1],

$$\mathcal{O}_{a,b} \subset \mathbf{G}/\mathbf{Q}$$

be the Zariski open subset on which the map

$$\begin{aligned} p_{a,b} : \mathcal{O}_{a,b} &\to \mathbb{C}^\times\mathbf{1}\backslash A^\times \\ x &\mapsto \langle\langle a, b, x\rangle\rangle_{\mathbb{C}} \end{aligned}$$

is defined. We have then (see [14, Lemma 5.1]) that if for some $m \in \mathbb{Z} \setminus \{0\}$ the map

$$\begin{aligned} \mathcal{O}_{a,b} &\to \mathbb{C}^\times\mathbf{1}\backslash A^\times \\ x &\mapsto p_{a,b}(x)^m \end{aligned}$$

is constant, then $\mathscr{X}$ is of tube type. Now we apply (4.2) in Corollary 4.4 to get that if $\rho : \Gamma \to G = \mathbf{G}(\mathbb{R})^\circ$ is maximal, then

$$\beta_{\check{S}}\big(\varphi(x), \varphi(y), \varphi(z)\big) = \pm\frac{\mathrm{r}_{\mathscr{X}}}{2}$$

for almost every (x, y, z), which implies that

$$\Big\langle\Big\langle\varphi(x), \varphi(y), \varphi(z)\Big\rangle\Big\rangle^2 \equiv 1 \mod \mathbb{R}^\times . \tag{4.5}$$

In particular, fix x and y such that $\big(\varphi(x), \varphi(y)\big) \in \check{S}^{(2)}$ and such that (4.5) holds for almost all $z \in \partial\mathbb{D}$: letting $E \subset \partial\mathbb{D}$ be this set of full measure, we may assume that E is Γ-invariant and $\varphi(E) \subset \mathcal{O}_{a,b}$ where $a = \varphi(x)$ and $b = \varphi(y)$. But $\varphi(E)$ being $\rho(\Gamma)$-invariant is Zariski dense in $\mathbf{G}/\mathbf{Q}$ and hence Zariski dense in the open set $\mathcal{O}_{a,b}$; since the map $x \mapsto p_{a,b}(x)^2$ is constant on $\varphi(E)$ it is so on $\mathcal{O}_{a,b}$, which implies that $\mathscr{X}$ is of tube type.

4.3. *The image of ρ is discrete.* Under the hypothesis of Theorem 4.1, we know now that $\mathscr{X}$ is of tube type. Then the generalized Maslov cocycle $\beta_{\check{S}}$ takes on $\check{S}^{(3)}$ exactly $\mathrm{r}_{\mathscr{X}} + 1$ values, namely

$$\Big\{-\frac{\mathrm{r}_{\mathscr{X}}}{2}, -\frac{\mathrm{r}_{\mathscr{X}}}{2} + 1, \ldots, \frac{\mathrm{r}_{\mathscr{X}}}{2} - 1, \frac{\mathrm{r}_{\mathscr{X}}}{2}\Big\} \tag{4.6}$$

so that

$$\check{S}^{(3)} = \cup_{i=0}^{r_{\mathscr{X}}} \mathbb{O}_{-r_{\mathscr{X}}+2i} ,$$

where $\mathbb{O}_{-r_{\mathscr{X}}+2i}$ is the preimage via $\beta_{\check{S}}$ of $-\frac{r_{\mathscr{X}}}{2}+i$, which incidentally is open since $\beta_{\check{S}}$ is continuous on $\check{S}^{(3)}$ (see [14, Cor. 3.7]). With the above notation, it follows from (4.2) in Corollary 4.4 that

$$\big(\varphi(x), \varphi(y), \varphi(z)\big) \in \mathbb{O}_{-r_{\mathscr{X}}} \cup \mathbb{O}_{r_{\mathscr{X}}} \tag{4.7}$$

for almost all $(x, y, z) \in (\partial\mathbb{D})^3$. Let us now denote by $\operatorname{Ess\,Im}\varphi \subset \check{S}$ the essential image of φ, that is the support of the pushforward $\varphi_*(\lambda)$ of the round measure λ on $\partial\mathbb{D}$. Then $\operatorname{Ess\,Im}\varphi$ is closed and $\rho(\Gamma)$-invariant. It then follows from (4.7) that

$$(\operatorname{Ess\,Im}\varphi)^3 \subset \overline{\mathbb{O}_{-r_{\mathscr{X}}}} \cup \overline{\mathbb{O}_{r_{\mathscr{X}}}} ,$$

where the closure on the right-hand side is taken in $\check{S}^3$. There are now two cases. Either $r_{\mathscr{X}} = 1$, $\mathscr{X} = \mathbb{D}$, $G = \mathrm{PU}(1,1)$, which is the case treated in [11]; or $r_{\mathscr{X}} \geq 2$, and then $\overline{\mathbb{O}_{-r_{\mathscr{X}}}} \cup \overline{\mathbb{O}_{r_{\mathscr{X}}}}$ is not the whole of $\check{S}^{(3)}$, since its complement contains at least $\mathbb{O}_{-r_{\mathscr{X}}+2}$; thus $(\operatorname{Ess\,Im}\varphi)^3 \neq \check{S}^3$ and, since $(\operatorname{Ess\,Im}\varphi)^3$ is $\rho(\Gamma)^3$-invariant closed and $\check{S}$ is G-homogeneous, this implies that $\rho(\Gamma)$ is not dense in G. Since a Zariski dense subgroup of G is either discrete or dense, we have that $\rho(\Gamma)$ is discrete.

4.4. *The representation ρ is injective.* Assume that

$$\ker(\rho) \not\subset \mathscr{Z}(\Gamma) .$$

Then it is easy to see that there is $\gamma \in \ker\rho$ of infinite order, so that we may choose a nonempty open interval $I \subset \partial\mathbb{D}$ such that $I, \gamma I, \gamma^2 I$ are pairwise disjoint and positively oriented; that is, $\beta_{\mathbb{D}}(x, y, z) = \frac{1}{2}$ for all $(x, y, z) \in I \times \gamma I \times \gamma^2 I$. Now choose three open nonvoid intervals I_1, I_2, and I_3 in I which are pairwise disjoint and positively oriented. Then it follows from (4.2) in Corollary 4.4 that

$$E_1 := \Big\{(x, y, z) \in I_3 \times \gamma I_2 \times \gamma^2 I_1 : \beta_{\check{S}}\big(\varphi(x), \varphi(y), \varphi(z)\big) = \frac{r_{\mathscr{X}}}{2}\Big\}$$

is of full measure in $I_3 \times \gamma I_2 \times \gamma^2 I_1$, while

$$E_2 := \Big\{(x, y, z) \in I_3 \times I_2 \times I_1 : \beta_{\check{S}}\big(\varphi(x), \varphi(y), \varphi(z)\big) = -\frac{r_{\mathscr{X}}}{2}\Big\}$$

is of full measure in $I_3 \times I_2 \times I_1$. Thus

$$E_1' := \big\{(x, y, z) \in E_1 : (x, \gamma^{-1} y, \gamma^{-2} z) \in E_2\big\}$$

is of full measure; using the almost everywhere equivariance of φ and the assumption that $\rho(\gamma) = \mathrm{Id}$, we conclude that for almost every $(x, y, z) \in E_1'$

$$\frac{r_{\mathscr{X}}}{2} = \beta_{\check{S}}\big(\varphi(x), \varphi(y), \varphi(z)\big) = \beta_{\check{S}}\big(\varphi(x), \varphi(\gamma^{-1} y), \varphi(\gamma^{-2} z)\big) = -\frac{r_{\mathscr{X}}}{2} ,$$

which is a contradiction. This shows that $\ker\rho \subset \mathscr{Z}(\Gamma)$ and completes the proof of Theorem 4.1.

5. Regularity properties of the boundary map: the Zariski dense case

In this section we generalize a technique from [12] to study the boundary map $\varphi : \partial\mathbb{D} \to \check{S}$ associated to a maximal representation $\rho : \Gamma \to G = \mathbf{G}(\mathbb{R})^\circ$ with Zariski dense image and establish the existence of strictly equivariant maps with additional regularity properties. This relies in an essential way on the fact established in Theorem 4.1 asserting that, in the situation described above, the symmetric space $\mathscr{X}$ associated to G is irreducible and of tube type.

THEOREM 5.1. *Let Γ be a lattice in a finite connected covering of* $\mathrm{PU}(1,1)$ *and let $\rho : \Gamma \to G$ a maximal representation with Zariski dense image. Then there are two Borel maps $\varphi_\pm : \partial\mathbb{D} \to \check{S}$ with the following properties*:

(1) *φ_+ and φ_- are strictly ρ-equivariant*;

(2) *φ_- is left continuous and φ_+ is right continuous*;

(3) *for every $x \neq y$, $\varphi_\epsilon(x)$ is transverse to $\varphi_\delta(y)$ for all $\epsilon, \delta \in \{+,-\}$*;

(4) *for all $x, y, z \in \partial\mathbb{D}$,*

$$\beta_{\check{S}}\big(\varphi_\epsilon(x), \varphi_\delta(y), \varphi_\eta(z)\big) = \mathrm{r}_{\mathscr{X}}\, \beta_{\partial\mathbb{D}}(x, y, z)\,,$$

for all $\epsilon, \delta, \eta \in \{+,-\}$.

Moreover φ_+ and φ_- are the unique maps satisfying (1) *and* (2).

5.1. *General properties of boundary maps.* Let $\Gamma < L$ be a lattice in a connected finite covering L of $\mathrm{PU}(1,1)$ as above, $\mathbf{G}$ a connected semisimple group, $\mathbf{P}$ a parabolic subgroup, both defined over $\mathbb{R}$, $G = \mathbf{G}(\mathbb{R})$, $P = \mathbf{P}(\mathbb{R})$, $\rho : \Gamma \to G$ a homomorphism and λ the round measure on $\partial\mathbb{D}$. If $\varphi : \partial\mathbb{D} \to G/P$ is a ρ-equivariant measurable map and ρ has Zariski dense image, one immediately sees that the image of φ cannot be contained in a proper algebraic subset. The following proposition is a strengthening of this statement, showing that from the point of view of the round measure, the essential image of φ meets any proper algebraic subset in a set of measure zero. Namely:

PROPOSITION 5.2. *If $\mathbf{V} \subset \mathbf{G}/\mathbf{P}$ is any proper Zariski closed subset defined over $\mathbb{R}$, then*

$$\lambda\big(\varphi^{-1}(\mathbf{V}(\mathbb{R}))\big) = 0\,.$$

This will follow from the following two lemmas.

LEMMA 5.3 ([36]). *If $A \subset \partial\mathbb{D}$ is a set of positive measure, then there exists a sequence $\{\gamma_n\}_{n=1}^\infty$ in Γ such that* $\lim_{n\to\infty} \lambda(\gamma_n A) = 1$.

LEMMA 5.4. *Let $\mathbf{V} \subset \mathbf{G}/\mathbf{P}$ be a proper Zariski closed subset defined over $\mathbb{R}$ and $\{g_n\}_{n=1}^\infty$ a sequence in G. Then there exist a proper Zariski closed subset $\mathbf{W} \subset \mathbf{G}/\mathbf{P}$ defined over $\mathbb{R}$ and a subsequence $\{g_{n_k}\}_{k=1}^\infty$ such that for every $\varepsilon > 0$ there exists $K > 0$ such that for all $k \geq K$*

$$g_{n_k}\mathbf{V}(\mathbb{R}) \subset \mathscr{N}_\varepsilon\big(\mathbf{W}(\mathbb{R})\big)\,,$$

where $\mathscr{N}_\varepsilon$ denotes the ε-neighborhood for some fixed distance on G/P.

Proof. Passing to a subsequence $\{g_{n_k}\}$ we may assume that, if $V := \mathbf{V}(\mathbb{R})$, $g_{n_k}V$ converges to a compact set $F \subset G/P$ in the Gromov-Hausdorff topology. It suffices then to show that F is not Zariski dense in $\mathbf{G}/\mathbf{P}$. To this end, let $I(\mathbf{V}) \subset \mathbb{C}[\mathbf{G}/\mathbf{P}]$ be the defining ideal and pick $d \geq 1$ such that the homogeneous component $I(\mathbf{V})_d(\mathbb{R}) \subset \mathbb{C}[\mathbf{G}/\mathbf{P}]_d(\mathbb{R})$ is nonzero. Let $\ell := \dim I(\mathbf{V})_d(\mathbb{R})$; then we may assume that the sequence $g_{n_k}I(\mathbf{V})_d(\mathbb{R})$ converges to a point E in the Grassmannian $\mathrm{Gr}_\ell\big(\mathbb{C}[\mathbf{G}/\mathbf{P}]_d(\mathbb{R})\big)$. In particular, given $p \in E$, $p \neq 0$, there exists $p_n \in g_{n_k}I(\mathbf{V})_d(\mathbb{R})$ with $p_n \to p$. It is then not difficult to show that p vanishes on F and one can take $\mathbf{W} \subset \mathbf{G}/\mathbf{P}$ to be the Zariski closure of F. □

Proof of Proposition 5.2. Let $V := \mathbf{V}(\mathbb{R})$ and $A := \{x \in \partial\mathbb{D} : \varphi(x) \in V\}$. Assume that $\lambda(A) > 0$ and pick a sequence $\{\gamma_n\}_{n=1}^\infty$ in Γ as in Lemma 5.3 such that $\lambda(\gamma_n A) \to 1$. Passing to a subsequence, let $\mathbf{W} \subset \mathbf{G}/\mathbf{P}$ be the proper Zariski closed subset given by Lemma 5.4. For every $m \geq 1$, let $N(m)$ be such that for all $n \geq N(m)$,

$$\rho(\gamma_n)V \subset \mathcal{N}_{1/m}(W)\,,$$

where $W := \mathbf{W}(\mathbb{R})$. Setting $E_N := \cup_{n=N}^\infty \gamma_n A$, we thus have

$$\varphi\big(E_{N(m)}\big) \subset \mathcal{N}_{1/m}(W)\,. \tag{5.1}$$

But now $E_{N(m)} \subset \partial\mathbb{D}$ is a set of full measure and so is $E := \cap_{m\geq 1} E_{N(m)}$, which, by (5.1), implies now that $\varphi(E) \subset W$. This implies that

$$\operatorname{Ess\,Im}(\varphi) \subset W \subset \mathbf{W} \subset \mathbf{G}/\mathbf{P}$$

and contradicts the Zariski density of $\rho(\Gamma)$ since $\operatorname{Ess\,Im}(\varphi)$ is $\rho(\Gamma)$-invariant. □

5.2. *Exploiting maximality.* Let now $\rho : \Gamma \to G$ be a maximal representation with Zariski dense image and $\varphi : \partial\mathbb{D} \to \check{S}$ be the ρ-equivariant measurable map given by Theorem 4.2. Having introduced the essential image $\operatorname{Ess\,Im}(\varphi) \subset \check{S}$, we will now study the essential graph $\operatorname{Ess\,Gr}(\varphi) \subset \partial\mathbb{D} \times \check{S}$ of φ defined as the support of the pushforward of the round measure λ on $\partial\mathbb{D}$ under the map

$$\begin{aligned} \partial\mathbb{D} &\to \partial\mathbb{D} \times \check{S} \\ x &\mapsto \big(x, \varphi(x)\big)\,. \end{aligned}$$

For this we will use (4.2) in Corollary 4.4 in an essential way. The following properties of the generalized Maslov cocycle $\beta_{\check{S}}$ follow from [16], [19], and [21]:

LEMMA 5.5. (1) $\beta_{\check{S}} : \check{S}^3 \to \{-\frac{\mathrm{r}_{\mathcal{X}}}{2}\} + \mathbb{Z}$ *is a* G*-invariant cocycle*;

(2) $|\beta_{\check{S}}(x,y,z)| \leq \frac{\mathrm{r}_{\mathcal{X}}}{2}$;

(3) *if* $\beta_{\check{S}}(x,y,z) = \frac{\mathrm{r}_{\mathcal{X}}}{2}$, *then* x, y, *and* z *are pairwise transverse*;

(4) $\check{S}^{(3)} = \sqcup_{i=0}^{\mathrm{r}_{\mathcal{X}}} \mathcal{O}_{-\mathrm{r}_{\mathcal{X}}+2i}$, *where* $\mathcal{O}_{-\mathrm{r}_{\mathcal{X}}+2i}$ *is open in* $\check{S}^3$ *and* $\beta_{\check{S}}$ *takes on the value* $-\frac{\mathrm{r}_{\mathcal{X}}}{2} + i$ *on* $\mathcal{O}_{-\mathrm{r}_{\mathcal{X}}+2i}$;

(5) *if* $x, \{x_n\}, \{x'_n\} \in \check{S}$, $\lim_{n\to\infty} x_n = x$, *and* $\beta_{\check{S}}(x, x'_n, x_n) = \frac{\mathrm{r}_{\mathcal{X}}}{2}$, *then* $\lim_{n\to\infty} x'_n = x$.

Finally, for $x \in \check{S} = G/Q \subset \mathbf{G}/\mathbf{Q}$, let $\mathbf{V}_x \subset \mathbf{G}/\mathbf{Q}$ be the proper Zariski closed $\mathbb{R}$-subset of all points $y \in \mathbf{G}/\mathbf{Q}$ which are not transverse to x, so that $V_x := \mathbf{V}_x(\mathbb{R})$ is the set of points in $\check{S}$ which are not transverse to x.

LEMMA 5.6. *Let* (x_1, f_1), (x_2, f_2), *and* (x_3, f_3) *be points in* $\operatorname{Ess}\operatorname{Gr}(\varphi)$ *so that* x_1, x_2, *and* x_3 *are pairwise distinct and* f_1, f_2, *and* f_3 *are pairwise transverse. Then*

$$\beta_{\check{S}}(f_1, f_2, f_3) = \mathrm{r}_{\mathcal{X}}\, \beta_{\partial\mathbb{D}}(x_1, x_2, x_3)\,.$$

Proof. Let I_i, $i = 1, 2, 3$ be pairwise disjoint open intervals containing x_i such that for all $y_i \in I_i$

$$\beta_{\partial\mathbb{D}}(y_1, y_2, y_3) = \beta_{\partial\mathbb{D}}(x_1, x_2, x_3)\,.$$

Let U_i, $i = 1, 2, 3$ be neighborhoods of f_i such that $U_1 \times U_2 \times U_3 \subset \check{S}^{(3)}$. Then

$$A_i = \{x \in I_i : \varphi(x) \in U_i\}$$

is of positive measure, and hence it follows from (4.2) in Corollary 4.4 that for almost every $(y_1, y_2, y_3) \in A_1 \times A_2 \times A_3$,

$$\beta_{\check{S}}\big(\varphi(y_1), \varphi(y_2), \varphi(y_3)\big) = \mathrm{r}_{\mathcal{X}}\, \beta_{\partial\mathbb{D}}(y_1, y_2, y_3) = \mathrm{r}_{\mathcal{X}}\, \beta_{\partial\mathbb{D}}(x_1, x_2, x_3)\,.$$

Thus setting $\epsilon = 2\beta_{\partial\mathbb{D}}(x_1, x_2, x_3) \in \{\pm 1\}$, we have for almost every $(y_1, y_2, y_3) \in A_1 \times A_2 \times A_3$, that

$$\big(\varphi(y_1), \varphi(y_2), \varphi(y_3)\big) \in (U_1 \times U_2 \times U_3) \cap \mathcal{O}_{\epsilon \mathrm{r}_{\mathcal{X}}}\,,$$

which implies, since the neighborhood U_i can be chosen arbitrarily small, that $(f_1, f_2, f_3) \in \overline{\mathcal{O}_{\epsilon \mathrm{r}_{\mathcal{X}}}}$. But

$$\overline{\mathcal{O}_{\epsilon \mathrm{r}_{\mathcal{X}}}} \cap \check{S}^{(3)} = \overline{\mathcal{O}_{\epsilon \mathrm{r}_{\mathcal{X}}}} \cap \Big(\cup_{i=0}^{\mathrm{r}_{\mathcal{X}}} \mathcal{O}_{-\mathrm{r}_{\mathcal{X}}+2i}\Big) = \mathcal{O}_{\epsilon \mathrm{r}_{\mathcal{X}}}$$

which, together with the assumption that $(f_1, f_2, f_3) \in \check{S}^{(3)}$, implies that $(f_1, f_2, f_3) \in \mathcal{O}_{\epsilon \mathrm{r}_{\mathcal{X}}}$ and hence proves the lemma. □

LEMMA 5.7. *Let* $(x_1, f_2), (x_2, f_2) \in \operatorname{Ess}\operatorname{Gr}(\varphi)$ *with* $x_1 \neq x_2$. *Then* f_1 *is transverse to* f_2.

Proof. For $x, y \in \partial\mathbb{D}$, let

$$((x, y)) := \left\{ z \in \partial\mathbb{D} :\ \beta_{\partial\mathbb{D}}(x, z, y) = \frac{1}{2} \right\}\,.$$

We will use the obvious fact that for almost every $x \in \partial\mathbb{D}$, $\big(x, \varphi(x)\big) \in \operatorname{Ess}\operatorname{Gr}(\varphi)$. Using Proposition 5.2, we can find $a \in ((x_1, x_2))$ such that $\big(a, \varphi(a)\big) \in \operatorname{Ess}\operatorname{Gr}(\varphi)$ and $\varphi(a) \notin V_{f_1} \cup V_{f_2}$, that is, $\varphi(a)$ is transverse to f_1 and f_2. Then by the same argument, we can find $b \in ((x_2, x_1))$ such that $\big(b, \varphi(b)\big) \in \operatorname{Ess}\operatorname{Gr}(\varphi)$, and

$\varphi(b)$ is transverse to f_1, f_2, and $\varphi(a)$. Applying the cocycle property of $\beta_{\check{S}}$ and Lemma 5.6, we obtain

$$\begin{aligned}0 &= \beta_{\check{S}}\big(\varphi(a), f_2, \varphi(b)\big) - \beta_{\check{S}}\big(f_1, f_2, \varphi(b)\big) \\ &\quad + \beta_{\check{S}}\big(f_1, \varphi(a), \varphi(b)\big) - \beta_{\check{S}}\big(f_1, \varphi(a), f_2\big) \\ &= \frac{\mathrm{r}_{\mathscr{X}}}{2} - \beta_{\check{S}}\big(f_1, f_2, \varphi(b)\big) + \frac{\mathrm{r}_{\mathscr{X}}}{2} - \beta_{\check{S}}\big(f_1, \varphi(a), f_2\big),\end{aligned}$$

which, together with Lemma 5.5(2), implies that $\beta_{\check{S}}\big(f_1, f_2, \varphi(b)\big) = \frac{\mathrm{r}_{\mathscr{X}}}{2}$; using Lemma 5.5(3) we conclude that f_1 and f_2 are transverse. □

For a subset $A \subset \partial\mathbb{D}$, let

$$F_A = \big\{f \in \check{S} : \text{ there exists } x \in A \text{ such that } (x, f) \in \operatorname{Ess}\operatorname{Gr}(\varphi)\big\},$$

and set

$$((x, y]] := ((x, y)) \cup \{y\}.$$

LEMMA 5.8. *Let $x \neq y$ in $\partial\mathbb{D}$. Then $\overline{F_{((x,y]]}} \cap F_x$ and $\overline{F_{[[y,x))}} \cap F_x$ consist each of one point.*

Proof. We start with two observations: first, if $A \cap B = \varnothing$, it follows from Lemma 5.7 that $F_A \cap F_B = \varnothing$; moreover, if A is closed, then F_A is also closed. We now prove that $\overline{F_{((x,y]]}} \cap F_x$ consists of one point; the other statement can be proved analogously.

Let $f, f' \in \overline{F_{((x,y]]}} \cap F_x$, and let $(x_n, f_n) \in \operatorname{Ess}\operatorname{Gr}(\varphi)$ be a sequence such that

$$x_n \in ((x, y]], \quad \lim x_n = x, \quad \text{and} \quad \lim f_n = f.$$

Observe now that if $z \in ((x, y]]$, writing $F_{((x,y]]} = F_{((x,z))} \cup F_{[[z,y]]}$ and taking into account that $F_{[[z,y]]}$ is closed and disjoint from F_x, we get that

$$\overline{F_{((x,y]]}} \cap F_x = \overline{F_{((x,z))}} \cap F_x,$$

and hence also that

$$\overline{F_{((x,y]]}} \cap F_x = \overline{F_{((x,z]]}} \cap F_x \tag{5.2}$$

for all $z \in ((x, y))$. Using (5.2), we may find another sequence $(y_n, f_n') \in \operatorname{Ess}\operatorname{Gr}(\varphi)$ such that

$$y_n \in ((x, x_n)), \quad \lim y_n = x, \quad \text{and} \quad \lim f_n' = f'.$$

Then it follows from Lemmas 5.6 and 5.7 that

$$\beta_{\check{S}}(f, f_n', f_n) = \mathrm{r}_{\mathscr{X}} \beta_{\partial\mathbb{D}}(x, y_n, x_n) = \frac{\mathrm{r}_{\mathscr{X}}}{2}.$$

Since $\lim f_n = f$, by Lemma 5.5(5), this however implies that $\lim f_n' = f$, and hence $f = f'$. □

Here is an interesting corollary about the structure of $\operatorname{Ess}\operatorname{Gr}(\varphi)$ which spells out precisely to which extent $\operatorname{Ess}\operatorname{Gr}(\varphi)$ is in general not the graph of a map.

COROLLARY 5.9. *For every $x \in \partial\mathbb{D}$, F_x consists of one or two points.*

Proof. Pick y_-, x, and y_+ positively oriented in $\partial\mathbb{D}$ and $f \in F_x$. For every neighborhood U of f, one of the sets

$$\{z \in [[y_-, x)) : \varphi(z) \in U\}$$
$$\{z \in ((x, y_+]] : \varphi(z) \in U\}$$

is of positive measure. This implies that

$$F_x \subset \overline{F_{[[y_-,x))}} \cup \overline{F_{((x,y_+]]}}$$

and thus

$$F_x = \left(\overline{F_{[[y_-,x))}} \cap F_x\right) \cup \left(\overline{F_{((x,y_+]]}} \cap F_x\right),$$

which, together with Lemma 5.8, proves the assertion. □

Proof of Theorem 5.1. We use Lemma 5.8 in order to define for every $x \in \partial\mathbb{D}$

$$\varphi_-(x) = \overline{F_{((x,y_+]]}} \cap F_x, \qquad \varphi_+(x) = \overline{F_{[[y_-,x))}} \cap F_x \tag{5.3}$$

where $y_+ \neq x$ and $y_- \neq x$ are arbitrary. Then φ_+ and φ_- are clearly respectively right and left continuous. The strict ρ-equivariance of φ_+ and φ_- follows from the invariance of $\operatorname{Ess}\operatorname{Gr}(\varphi) \subset \partial\mathbb{D} \times \check{S}$ under the diagonal Γ-action together with (5.3) and the fact that Γ acts in an orientation preserving way on $\partial\mathbb{D}$.

Properties (3) and (4) are immediate consequences of Lemmas 5.6 and 5.7.

Concerning the uniqueness of φ_+ for example, let ψ be a right continuous ρ-equivariant map. Since the $\rho(\Gamma)$-action on $\check{S}$ is proximal, we have $\psi(x) = \varphi_+(x)$ for almost every $x \in \partial\mathbb{D}$ [24]. Fix $x \in \partial\mathbb{D}$; then pick a sequence $y_n \in ((x, y))$ such that

$$\lim y_n = x \text{ and } \psi(y_n) = \varphi_+(y_n).$$

This implies, since ψ and φ_+ are both right continuous, that $\psi(x) = \varphi_+(x)$. □

6. Structure of maximal representations and boundary maps: the general case

In this section we present the proofs of Theorems 5 and 8 in the introduction; this relies on the results obtained in Sections 4 and 5 and on the relation between maximal representations and tight homomorphisms, which were introduced in [15].

Let G be a group of type (RH). We briefly recall the structure of $\mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$ in terms of simple components and the explicit form of the Gromov norm. Let $\mathscr{X}$ be the symmetric space associated to G; setting $G_{\mathscr{X}} := \operatorname{Iso}(\mathscr{X})^\circ$ we have a canonical projection $q : G \to G_{\mathscr{X}}$ which by hypothesis has compact kernel. Let $\mathscr{X} = \mathscr{X}_1 \times \cdots \times \mathscr{X}_n$ be the decomposition into irreducible factors and $p_i : G_{\mathscr{X}} \to G_{\mathscr{X}_i}$ the canonical projections. We then have the following isometric isomorphisms:

(1) $q^* : \mathrm{H}^2_{\mathrm{cb}}(G_{\mathscr{X}}, \mathbb{R}) \to \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$;

(2) $\prod \mathrm{H}^2_{\mathrm{cb}}(G_{\mathscr{X}_i}, \mathbb{R}) \to \mathrm{H}^2_{\mathrm{cb}}(G_{\mathscr{X}}, \mathbb{R})$, $(\alpha_i) \mapsto \sum_{i=1}^n p_i^*(\alpha_i)$.

Defining for ease of notation $\kappa^{\mathrm{b}}_{\mathscr{X}}$ to be the bounded Kähler class of $G_{\mathscr{X}}$, let

$$\kappa^{\mathrm{b}}_{\mathscr{X},i} := p_i^*(\kappa^{\mathrm{b}}_{\mathscr{X}_i}) \qquad \text{and} \qquad \kappa^{\mathrm{b}}_{G,i} := q^*(\kappa^{\mathrm{b}}_{\mathscr{X},i})\,.$$

Then

$$\{\kappa^{\mathrm{b}}_{G,i} : 1 \le i \le n\}$$

is a basis of $\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ and the norm of an element

$$\kappa = \sum_{i=1}^{n} \lambda_i \kappa^{\mathrm{b}}_{G,i}$$

equals (see [15, (2.15)])

$$\|\kappa\| = \sum_{i=1}^{n} |\lambda_i| \frac{\mathrm{r}_{\mathscr{X}_i}}{2}\,. \tag{6.1}$$

In the rest of this section, L will always denote a finite connected covering of $\mathrm{PU}(1,1)$ and $\Gamma < L$ a lattice. The following lemma is a routine verification using the Definition 3.5 of maximality and the Milnor-Wood type bounds in Corollary 3.4.

LEMMA 6.1. *Let $\rho : \Gamma \to G$ be a homomorphism.*

(1) *If $\Gamma_0 < \Gamma$ is a subgroup of finite index, then ρ is maximal if and only if $\rho|_{\Gamma_0}$ is maximal*;

(2) *ρ is maximal if and only if $q \circ \rho : \Gamma \to G_{\mathscr{X}}$ is maximal*;

(3) *ρ is maximal if and only if $p_i \circ q \circ \rho : \Gamma \to G_{\mathscr{X}_i}$ is maximal for every $i = 1, \dots, n$.*

Recall now from [15, Def. 2.11] that if H is a locally compact group, a continuous homomorphism $\rho : H \to G$ is *tight* if

$$\|\rho^*(\kappa^{\mathrm{b}}_G)\| = \|\kappa^{\mathrm{b}}_G\|\,.$$

LEMMA 6.2. *If $\rho : \Gamma \to G$ is maximal, then it is tight.*

Proof. Using that

$$\mathrm{T}_{\mathrm{b}}\big(\rho^*(\kappa^{\mathrm{b}}_G)\big) = \mathrm{t}_{\mathrm{b}}(\rho)\kappa^{\mathrm{b}}_L$$

and that, because of maximality,

$$\mathrm{t}_{\mathrm{b}}(\rho) = \mathrm{r}_{\mathscr{X}} = 2\|\kappa^{\mathrm{b}}_G\|\,,$$

we get that

$$\mathrm{T}_{\mathrm{b}}\big(\rho^*(\kappa^{\mathrm{b}}_G)\big) = 2\|\kappa^{\mathrm{b}}_G\|\kappa^{\mathrm{b}}_L$$

which, together with $\big|\mathrm{T}_{\mathrm{b}}\big(\rho^*(\kappa^{\mathrm{b}}_G)\big)\big| \le \|\rho^*(\kappa^{\mathrm{b}}_G)\|$ and $\|\kappa^{\mathrm{b}}_L\| = 1/2$, implies that

$$\|\kappa^{\mathrm{b}}_G\| \le \|\rho^*(\kappa^{\mathrm{b}}_G)\|\,.$$

Since the reverse inequality holds always true, we have proved the lemma. □

Let H be of type (RH), $\mathcal{Y}$ the associated symmetric space, $\mathcal{Y} = \mathcal{Y}_1 \times, \dots, \times \mathcal{Y}_m$ the decomposition into irreducible factors, and $\{\kappa^{\mathrm{b}}_{H,i} : 1 \le i \le m\}$ the basis of $\mathrm{H}^2_{\mathrm{cb}}(H, \mathbb{R})$ obtained as above. Given a continuous homomorphisms $\sigma : H \to G$ and writing

$$\sigma^*(\kappa^{\mathrm{b}}_G) = \sum_{i=1}^m \lambda_i \kappa^{\mathrm{b}}_{H,i} \,,$$

it follows from (6.1) and (2.1) that σ is tight if and only if

$$\sum_{i=1}^m |\lambda_i| \mathrm{r}_{\mathcal{Y}_i} = \mathrm{r}_{\mathcal{X}} \,.$$

Finally we recall that a continuous homomorphism $\sigma : H \to G$ is *positive* if $\lambda_i \ge 0$ for $1 \le i \le m$.

LEMMA 6.3. *Let $\rho : \Gamma \to H$ and $\sigma : H \to G$ be homomorphisms, where σ is continuous and H and G are of type* (RH).

(1) *If $\sigma \circ \rho$ is maximal then σ is tight*;

(2) *if $\sigma \circ \rho$ is maximal and σ is positive then ρ is maximal*;

(3) *if ρ is maximal and σ is tight and positive*, *then $\sigma \circ \rho$ is maximal.*

Proof. With the notation introduced above, let $\sigma^*(\kappa^{\mathrm{b}}_G) = \sum_{i=1}^m \lambda_i \kappa^{\mathrm{b}}_{H,i}$. Thus

(6.3.a) $\mathrm{t}_{\mathrm{b}}(\sigma \circ \rho) = \sum_{i=1}^m \lambda_i \mathrm{t}_{\mathrm{b}}(\rho, \kappa^{\mathrm{b}}_{H,i})$;

(6.3.b) $|\mathrm{t}_{\mathrm{b}}(\rho, \kappa^{\mathrm{b}}_{H,i})| \le \mathrm{r}_{\mathcal{Y}_i}$;

(6.3.c) $\sum_{i=1}^m |\lambda_i| \mathrm{r}_{\mathcal{Y}_i} \le \mathrm{r}_{\mathcal{X}}$.

Thus if $\sigma \circ \rho$ is maximal, the equality (6.3.a) combined with (6.3.b) and (6.3.c) implies that we have equality in (6.3.c) and hence σ is tight.

If σ is positive, that is, $\lambda_i \ge 0$ for $1 \le i \le m$, and $\sigma \circ \rho$ is maximal, we get from (6.3.a), (6.3.b) and (6.3.c) that

$$\mathrm{t}_{\mathrm{b}}(\rho, \kappa^{\mathrm{b}}_{H,i}) = \mathrm{r}_{\mathcal{Y}_i}$$

which, together with Lemma 6.1(3), implies that ρ is maximal.

Finally, if ρ is maximal we get from Lemma 6.1(3) that $\mathrm{t}_{\mathrm{b}}(\rho, \kappa^{\mathrm{b}}_{H,i}) = \mathrm{r}_{\mathcal{Y}_i}$ for $1 \le i \le m$, and if σ is tight and positive, then

$$\mathrm{t}_{\mathrm{b}}(\sigma \circ \rho) = \sum_{i=1}^m \lambda_i \mathrm{t}_{\mathrm{b}}(\rho, \kappa^{\mathrm{b}}_{H,i}) = \sum_{i=1}^m \lambda_i \mathrm{r}_{\mathcal{Y}_i} = \mathrm{r}_{\mathcal{X}}$$

and hence $\sigma \circ \rho$ is maximal. $\square$

Proofs of Theorems 5 *and* 8. In order to prove these results we place ourselves, as we may, in the slightly more general context in which Γ is a lattice in a finite covering of $\mathrm{PU}(1,1)$. Let now $\mathbf{G}$ and $G = \mathbf{G}(\mathbb{R})^\circ$ be as in the statement of Theorem 5, that is, $\mathbf{G}$ is a connected semisimple algebraic group defined over

$\mathbb{R}$ such that $G = \mathbf{G}(\mathbb{R})^\circ$ is of Hermitian type, and let $\rho : \Gamma \to G$ be a maximal representation. Set $\mathbf{H} := \overline{\rho(\Gamma)}^Z$. Since ρ is maximal, it is in particular a tight homomorphism (Lemma 6.2) and hence [15, Th. 4] applies. In particular, $\mathbf{H}$ is reductive, $H := \mathbf{H}(\mathbb{R})^\circ$ has compact centralizer in G and is of type (RH); furthermore if $\mathcal{Y}$ denotes the symmetric space associated to H then there is a unique H-invariant complex structure on $\mathcal{Y}$ such that the inclusion $i : H \to G$ is tight and positive.

Setting $\Gamma_0 = \rho^{-1}(\Gamma \cap H)$ and $\rho_0 := \rho|_{\Gamma_0} : \Gamma_0 \to H$, we have from Lemma 6.1(1) that $i \circ \rho_0 : \Gamma_0 \to H$ is maximal and, since i is tight and positive, from Lemma 6.3(2) that $\rho_0 : \Gamma_0 \to H$ is maximal as well. Composing ρ_0 with $p_i \circ q : H \to H_{\mathcal{Y}} \to H_{\mathcal{Y}_i}$, where $\mathcal{Y}_i$, $1 \leq i \leq m$ are the irreducible factors of $\mathcal{Y}$, the resulting homomorphisms $\rho_{0,i} : \Gamma_0 \to H_{\mathcal{Y}_i}$ are maximal with Zariski dense image. Theorem 4.1 then implies that $\mathcal{Y}_i$ is of tube type and $\rho_{0,i}$ is injective, modulo the center of Γ_0, and with discrete image. This implies that $\rho : \Gamma \to G$ is injective (modulo the center) and with discrete image. Since $\mathcal{Y}$ is of tube type and $i : H \to G$ is tight, there is a unique maximal subdomain $\mathcal{T} \subset \mathcal{X}$ of tube type with $i(\mathcal{Y}) \subset \mathcal{T}$ (see [15, Th. 10(1)]); moreover, it is H-invariant and hence (by uniqueness) $\mathbf{H}(\mathbb{R})$-invariant and thus $\rho(\Gamma)$-invariant. This completes the proof of Theorem 5.

Applying Theorem 5.1 to every irreducible factor of $\mathcal{Y}$, we get, say, a left continuous strictly ρ_0-equivariant map $\varphi : \partial\mathbb{D} \to \check{S}_{\mathcal{Y}} = \check{S}_{\mathcal{Y}_1} \times \cdots \times \check{S}_{\mathcal{Y}_m}$. Since $i : H \to G$ is tight, we also have a canonical i-equivariant map $\check{\imath} : \check{S}_{\mathcal{Y}} \to \check{S}_{\mathcal{X}}$; applying judiciously the uniqueness property in [15, Th. 4.1], we deduce that $\check{\imath} \circ \varphi : \partial\mathbb{D} \to \check{S}_{\mathcal{X}}$ is ρ-equivariant. Finally, writing $i^*(\kappa^{\mathrm{b}}_{\mathcal{X}}) = \sum_{i=1}^m \lambda_i \kappa^{\mathrm{b}}_{\mathcal{Y},i}$, for $\lambda_i \geq 0$, we have (see [15, Lemma 5.9])

$$\beta_{\check{S}_{\mathcal{X}}}\big(\check{\imath}(x), \check{\imath}(y), \check{\imath}(z)\big) = \sum_{j=1}^m \lambda_j \beta_{\check{S}_{\mathcal{Y}_j}}(x_j, y_j, z_j),$$

where $x = (x_1, \cdots, x_m),\ y = (y_1, \cdots, y_m),\ z = (z_1, \cdots, z_m) \in \check{S}_{\mathcal{Y}}$. This, together with (4.2) in Corollary 4.4, implies

$$\beta_{\check{S}_{\mathcal{X}}}\big(\check{\imath}\varphi(a), \check{\imath}\varphi(b), \check{\imath}\varphi(c)\big) = \left(\sum_{i=1}^m \lambda_i \mathrm{r}_{\mathcal{Y}_i}\right) \beta_{\partial\mathbb{D}}(a, b, c)$$

and concludes the proof since i is tight and positive. □

7. Rotation numbers and applications to groups of Hermitian type

In this section we introduce and study rotation numbers on locally compact groups and compute them for groups of Hermitian type. The results are of independent interest. Here they are used in an essential way in the computation of the Toledo invariant in the case of surfaces with boundary.

7.1. *Basic definitions and properties.* Let G be a locally compact group and $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G, \mathbb{Z})$ a bounded integer valued Borel class. Let $B < G$ be a closed subgroup, and consider the first few terms of the long exact sequence

$$0 \longrightarrow \mathrm{Hom_c}(B, \mathbb{R}/\mathbb{Z}) \xrightarrow{\ \delta\ } \widehat{\mathrm{H}}^2_{\mathrm{cb}}(B, \mathbb{Z}) \longrightarrow \mathrm{H}^2_{\mathrm{cb}}(B, \mathbb{R}) \longrightarrow \cdots \tag{7.1}$$

coming from the coefficient sequence (2.4); denote by $\kappa_{\mathbb{R}}$ the image in $\mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$ of an element $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G, \mathbb{Z})$. Then if $\kappa_{\mathbb{R}}|_B = 0$, we let $f_B : B \to \mathbb{R}/\mathbb{Z}$ denote the unique continuous homomorphism with $\delta(f_B) = \kappa|_B$. In particular this applies to $B = \overline{\langle g \rangle}$ for any $g \in G$ and we define

Definition 7.1. The rotation number of $g \in G$ with respect to $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G, \mathbb{Z})$ is

$$\mathrm{Rot}_\kappa(g) := f_{\overline{\langle g \rangle}}(g)\,.$$

Using that the exact sequence in (7.1) is natural with respect to group homomorphisms, one easily verifies the following properties:

LEMMA 7.2. (1) $\mathrm{Rot}_\kappa : G \to \mathbb{R}/\mathbb{Z}$ *is invariant under conjugation*;

(2) *if* $\kappa_{\mathbb{R}}|_B = 0$, *then* $\mathrm{Rot}_\kappa|_B$ *is a continuous homomorphism and* $\delta(\mathrm{Rot}_\kappa|_B) = \kappa|_B$;

(3) *if* $\sigma : G_1 \to G_2$ *is a continuous homomorphism and* $\kappa_1 = \sigma^*(\kappa_2)$, *then*

$$\mathrm{Rot}_{\kappa_1}(g_1) = \mathrm{Rot}_{\kappa_2}\big(\sigma(g_1)\big)$$

for all $g_1 \in G_1$.

In the study of rotation numbers Rot_κ, quasimorphisms play an important role. We quickly review the basic definitions. If $A = \mathbb{Z}$ or $\mathbb{R}$, a function $f : G \to A$ is a quasimorphism if the function $df : G \to A$

$$df(x, y) = f(xy) - f(x) - f(y)$$

is bounded. When $A = \mathbb{R}$, a quasimorphism is homogeneous if

$$f(g^n) = nf(g)$$

for $n \in \mathbb{Z}$ and $g \in G$. Any quasimorphism $f : G \to A$ can be made homogeneous by setting

$$Hf(x) := \lim_{n\to\infty} \frac{f(x^n)}{n} \in \mathbb{R},$$

and it is a standard fact that $f - Hf$ is bounded.

LEMMA 7.3. *Assume that* κ *vanishes when considered as an ordinary class in* $\widehat{\mathrm{H}}^2_{\mathrm{c}}(G, \mathbb{Z})$. *Let* $f : G \to \mathbb{Z}$ *be a Borel map such that* df *is bounded and represents* κ *seen as a bounded class. Then* f *is a quasimorphism and*

$$\mathrm{Rot}_\kappa(g) \equiv Hf(g) \mod \mathbb{Z}$$

where Hf *is the homogenization of* f.

Proof. By Lemma 7.2(3), it suffices to show the assertion for $G = \mathbb{Z}$. Let $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the canonical projection; then $p \circ Hf : \mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a homomorphism since Hf is homogeneous, and we claim that $\kappa = \delta(p \circ Hf)$. Indeed let

$\sigma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be the Borel section of p with values in $[0, 1)$; then we have

$$\delta(p \circ HF) = -\mathrm{d}(\sigma \circ p \circ Hf) = df - d(f - Hf + \sigma \circ p \circ Hf)$$

where we have used in the last equality that $dHf = 0$. The claim then follows from the fact that f and $Hf - \sigma \circ p \circ Hf$ take integral values and both $f - Hf$ and $\sigma \circ p \circ Hf$ are bounded. From the claim we get that

$$\mathrm{Rot}_\kappa(g) = p \circ Hf(g)$$

for all $g \in \mathbb{Z}$, which is the assertion that needed to be proved. □

Next we have:

LEMMA 7.4. *Let $f : G \to \mathbb{R}$ be a homogeneous Borel quasimorphism. Then f is continuous.*

Proof. We show first that f is locally bounded. Let $C := \sup_{x,y} |df(x, y)|$ and $E_N = \{x \in G : |f(x)| \le N\}$. Then $E_N = E_N^{-1}$ and for N large enough is of positive Haar measure. For such N we deduce that $E_N \cdot E_N$ is a neighborhood of $e \in G$; since f is a quasimorphism we have that $E_N \cdot E_N \subset E_{N+2C}$, which implies that f is bounded in a neighborhood of e and hence, by the quasimorphism property, locally bounded. Fix now $\varphi : G \to [0, \infty)$ continuous with compact support and of total integral one. Since f is locally bounded, we have that for every $n \in \mathbb{N}$

$$F_n(x) = \frac{1}{n} \int_G \big(f(x^n y) - f(y)\big)\varphi(y)dy$$

is defined and continuous. Since f is homogeneous we have

$$\begin{aligned} \big|f(x) - F_n(x)\big| &= \left|\frac{1}{n} f(x^n) - F_n(x)\right| \\ &= \left|\frac{1}{n} \int_G \big(f(x^n) + f(y) - f(x^n y)\big)\varphi(y)dy\right| \le \frac{C}{n}, \end{aligned}$$

which implies that f is the uniform limit of a sequence of continuous functions and therefore continuous. □

Now we come to the main goal of this subsection, which is the continuity of Rot_κ; this is shown by exhibiting a direct relationship with a certain quasimorphism on a central extension of G. More precisely let, as before, $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G, \mathbb{Z})$; then κ can be seen as a class in $\mathrm{H}^2_{\mathrm{c}}(G, \mathbb{Z})$ and hence by [44] gives rise to a topological central extension

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ i\ } G_\kappa \xrightarrow{\ p\ } G \longrightarrow e;$$

that is, G_κ is a locally compact group, i and p are continuous, $i(\mathbb{Z})$ is a closed central subgroup of G_κ, and $G_\kappa / i(\mathbb{Z})$ is topologically isomorphic to G.

PROPOSITION 7.5. *There exists a continuous homogeneous quasimorphism $f : G_\kappa \to \mathbb{R}$ such that*

(1) $f\big(i(n)g\big) = n + f(g)$, *for* $n \in \mathbb{Z}$, $g \in G_\kappa$;

(2) $\mathrm{Rot}_\kappa\big(p(g)\big) \equiv f(g) \mod \mathbb{Z}$.

COROLLARY 7.6. *Let* $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G,\mathbb{Z})$. *Then* $\mathrm{Rot}_\kappa : G \to \mathbb{R}/\mathbb{Z}$ *is continuous.*

Proof of Proposition 7.5. Let $c : G^2 \to \mathbb{Z}$ be a bounded Borel cocycle which represents $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G,\mathbb{Z})$ and which we assume to be normalized. Then G_κ is a Borel group isomorphic to the Borel space $G \times \mathbb{Z}$ with multiplication given by

$$(g_1,n_1)(g_2,n_2) = \big(g_1g_2, n_1+n_2+c(g_1,g_2)\big)\,.$$

Define $f_1 : G_\kappa \to \mathbb{Z}$ by $f_1(g,m) := m$. Then f_1 is a Borel function and df_1 is a bounded Borel cocycle representing $p^*(\kappa) \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G_\kappa,\mathbb{Z})$. Let $f : G_\kappa \to \mathbb{R}$ be the homogenization of f_1. Then Lemma 7.3 implies that

$$\mathrm{Rot}_\kappa\big(p(g)\big) = \mathrm{Rot}_{p^*(\kappa)}(g) \equiv f(g) \mod \mathbb{Z}$$

and Lemma 7.4 that f is continuous. Finally f satisfies (1) because f_1 does and $i(\mathbb{Z})$ is central. $\square$

7.2. *Rotation numbers on groups of Hermitian type.* We begin first by determining $\widehat{\mathrm{H}}^2_{\mathrm{cb}}(G,\mathbb{Z})$ for G of Hermitian type. The main points are summarized in the following

PROPOSITION 7.7. *Let G be a group of Hermitian type and $K < G$ a maximal compact subgroup.*

(1) *The comparison map* $\widehat{\mathrm{H}}^2_{\mathrm{cb}}(G,\mathbb{Z}) \to \widehat{\mathrm{H}}^2_{\mathrm{c}}(G,\mathbb{Z})$ *is an isomorphism*;

(2) *the map*

$$\begin{aligned} \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G,\mathbb{Z}) &\to \mathrm{Hom}_{\mathrm{c}}(K,\mathbb{R}/\mathbb{Z}) \\ \kappa &\mapsto \quad \mathrm{Rot}_{\kappa|_K} \end{aligned} \tag{7.2}$$

is an isomorphism;

(3) *the change of coefficient map*

$$\widehat{\mathrm{H}}^2_{\mathrm{cb}}(G,\mathbb{Z}) \to \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$$

is injective and its image is a lattice.

Proof. (1) follows from the commutativity of the diagram

$$\begin{tikzcd}
0 \arrow[r] & \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G,\mathbb{Z}) \arrow[r] \arrow[d] & \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G,\mathbb{R}) \arrow[r] \arrow[d] & \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R}/\mathbb{Z}) \arrow[d, equal] \\
0 \arrow[r] & \widehat{\mathrm{H}}^2_{\mathrm{c}}(G,\mathbb{Z}) \arrow[r] & \widehat{\mathrm{H}}^2_{\mathrm{c}}(G,\mathbb{R}) \arrow[r] & \mathrm{H}^2_{\mathrm{c}}(G,\mathbb{R}/\mathbb{Z})
\end{tikzcd}$$

and the fact that with real coefficients the comparison map is an isomorphism.

(2) follows from (1) and the fact that in ordinary Borel cohomology the restriction

$$\widehat{\mathrm{H}}_{\mathrm{c}}^{2}(G,\mathbb{Z}) \to \widehat{\mathrm{H}}_{\mathrm{c}}^{2}(K,\mathbb{Z})$$

is an isomorphism [52].

(3) follows from (1) and the corresponding statement in ordinary cohomology. □

In view of the preceding proposition, we can refer for every $u \in \mathrm{Hom}_{\mathrm{c}}(K,\mathbb{R}/\mathbb{Z})$ to the class κ associated to u, and conversely.

We turn now to the explicit computation of the rotation number function. Let $G = KAN$ be an Iwasawa decomposition; recall the refined Jordan decomposition, namely that every $g \in G$ is a product $g = g_e g_h g_n$, where g_e is contained in a compact subgroup and g_h and g_n are conjugated to an element respectively in A and N; moreover the elements g_e, g_h, and g_n commute pairwise. Then we have:

PROPOSITION 7.8. *Let $\kappa \in \widehat{\mathrm{H}}_{\mathrm{cb}}^{2}(G,\mathbb{Z})$ and $u \in \mathrm{Hom}_{\mathrm{c}}(K,\mathbb{R}/\mathbb{Z})$ be the corresponding homomorphism.*

(1) $\mathrm{Rot}_\kappa|_{AN}$ *is the trivial homomorphism*;

(2) *for $g \in G$, let g_e be the elliptic component in the refined Jordan decomposition of g and $k \in C(g_e) \cap K$, where $C(g_e)$ denotes the G-conjugacy class of g_e. Then* $\mathrm{Rot}_\kappa(g) = u(k)$.

Proof. (1) Since $B = AN$ is amenable, $\kappa_{\mathbb{R}}|_B = 0$ and thus Lemma 7.2(2) implies that $\mathrm{Rot}_\kappa|_{AN}$ is a continuous homomorphism and hence differentiable. Since $[B,B] = N$, then $\mathrm{Rot}_\kappa(N) = 0$. The restriction $\mathrm{Rot}_\kappa|_A$ is invariant under the Weyl group $\mathcal{N}_K(A)/\mathcal{Z}_K(A)$ and hence its differential

$$D_e\mathrm{Rot}_\kappa|_A : \mathfrak{a} \to \mathbb{R}$$

is a linear form invariant under the Weyl group; it must therefore vanish since G is semisimple and thus $\mathrm{Rot}_\kappa|_A$ is trivial as well.

(2) Let $g = g_e g_h g_n$ be the refined Jordan decomposition of g. Since the subgroup C generated by g_e, g_h, and g_n is Abelian, $\mathrm{Rot}_\kappa|_C$ is a homomorphism; hence

$$\mathrm{Rot}_\kappa(g) = \mathrm{Rot}_\kappa(g_e) + \mathrm{Rot}_\kappa(g_h) + \mathrm{Rot}_\kappa(g_n)\,.$$

Taking into account (1) and the fact that g_h and g_n are conjugate respectively to elements in A and N, we get that

$$\mathrm{Rot}_\kappa(g) = \mathrm{Rot}_\kappa(g_e) = \mathrm{Rot}_\kappa(k) = u(k)$$

which concludes the proof. □

For the next result, if $u \in \mathrm{Hom}_{\mathrm{c}}(K,\mathbb{R}/\mathbb{Z})$, let us denote by $u_* : \pi_1(G) \to \mathbb{Z}$ the homomorphism induced by u on the level of fundamental groups, where we have identified $\pi_1(K)$ with $\pi_1(G)$.

THEOREM 7.9. *Let $u : K \to \mathbb{R}/\mathbb{Z}$ be a continuous homomorphism and $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G, \mathbb{Z})$ the associated class.*

(1) Rot_κ *is continuous*;

(2) *the unique continuous lift $\widetilde{\mathrm{Rot}_\kappa} : \widetilde{G} \to \mathbb{R}$ such that $\widetilde{\mathrm{Rot}_\kappa}(e) = 0$ is a continuous homogeneous quasimorphism and satisfies*
$$\widetilde{\mathrm{Rot}_\kappa}(zg) = u_*(z) + \widetilde{\mathrm{Rot}_\kappa}(g)$$
for all $z \in \pi_1(G)$ and for all $g \in \widetilde{G}$.

As a consequence we obtain a description of the space $\mathcal{Q}(\widetilde{G})_\mathbb{Z}$ of continuous homogeneous quasimorphism $f : \widetilde{G} \to \mathbb{R}$ such that $f\big(\pi_1(G)\big) \subset \mathbb{Z}$.

COROLLARY 7.10. *The maps*
$$\begin{array}{ccccc} \mathrm{Hom_c}(K, \mathbb{R}/\mathbb{Z}) & \to & \mathcal{Q}(\widetilde{G})_\mathbb{Z} & \to & \mathrm{Hom}\big(\pi_1(G), \mathbb{Z}\big) \\ u & \mapsto & \widetilde{\mathrm{Rot}_\kappa} & \mapsto & \widetilde{\mathrm{Rot}_\kappa}|_{\pi_1(G)} = u_* \end{array}$$
are group isomorphisms.

Proof of Theorem 7.9. The first assertion is a special case of Corollary 7.6. Let then $p : G_\kappa \to G$ be the Lie group central extension determined by κ and $\pi : \widetilde{G} \to (G_\kappa)^\circ$ the canonical projection. If $f : G_\kappa \to \mathbb{R}$ is the continuous homogeneous quasimorphism given by Proposition 7.5, then it follows from Proposition 7.5(2) that $f|_{G_\kappa} \circ \pi : \widetilde{G} \to \mathbb{R}$ is a continuous lift of Rot_κ to $\widetilde{G}$ which moreover vanishes at e. Hence $\widetilde{\mathrm{Rot}_\kappa} = f|_{(G_\kappa)^\circ} \circ \pi$, which implies the remaining assertion in the theorem. □

Proof of Corollary 7.10. Since the composition of the two arrows is the isomorphism
$$\begin{array}{ccc} \mathrm{Hom_c}(K, \mathbb{R}/\mathbb{Z}) & \to & \mathrm{Hom}\big(\pi_1(G), \mathbb{Z}\big) \\ u & \mapsto & u_* \end{array}$$
it suffices to show that the second morphism is injective. If $f_1, f_2 : \widetilde{G} \to \mathbb{R}$ are homogeneous continuous quasimorphisms which induce the same homomorphism $h : \pi_1(G) \to \mathbb{Z}$, then their difference $f_1 - f_2$ is $\pi_1(G)$-invariant and hence descends to a homogeneous quasimorphism $G \to \mathbb{R}$ which therefore vanishes, since G is connected semisimple with finite center. Thus $f_1 = f_2$, which completes the proof. □

Remark 7.11. For special classes $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G, \mathbb{Z})$ the rotation number Rot_κ coincides with previously known constructions:

(1) If $G = \mathrm{Homeo}_+(S^1)$ is the group of orientation preserving homeomorphisms of the circle (viewed as an abstract group) and $e^{\mathrm{b}} \in \mathrm{H}^2_{\mathrm{b}}(G, \mathbb{Z})$ is the bounded Euler class, Ghys [25] observed that $\mathrm{Rot}_{e^{\mathrm{b}}}(\varphi)$ is the classical rotation number of the homeomorphism φ.

(2) To obtain the symplectic rotation number defined by Barge and Ghys [1] for $G = \mathrm{Sp}(2n, \mathbb{R})$, we have to consider the class κ which corresponds to the homomorphism $u : K = \mathrm{U}(n) \to \mathbb{T}$ defined by $u(k) = (\det k)^2$.

(3) If $\mathcal{D}$ is an irreducible symmetric domain of tube type, $G = \mathrm{Aut}(\mathcal{D})^\circ$, and K is the stabilizer of $0 \in \mathcal{D}$, Clerc and Koufany construct a homomorphism $\chi : K \to \mathbb{T}$ using the Jordan algebra determinant. The rotation number function and the quasimorphism constructed in their paper [18, Th. 10.3 and Prop. 10.4], coincide then respectively with Rot_κ and $\widetilde{\mathrm{Rot}_\kappa}$, where κ is the class corresponding to χ.

We observe moreover that if $u : K \to \mathbb{T}$ is the complex Jacobian at 0, then for every $k \in K$ we have that

$$\chi(k)^{p_{\mathcal{X}}} = u(k)^2 ,$$

which incidentally shows that κ_G^{b} is in the image of $\widehat{\mathrm{H}}^2_{\mathrm{cb}}(G, \mathbb{Z})$.

8. Toledo numbers: formula and applications to representation varieties

8.1. *The formula.* Let Σ be a connected oriented surface and G a group of Hermitian type. In this subsection we establish a formula for the Toledo invariant $\mathrm{T}_\kappa(\Sigma, \rho)$ where κ is a bounded integral class, and we concentrate on the case in which $\partial\Sigma \neq \varnothing$; we mention at the end the formula in the case in which $\partial\Sigma = \varnothing$.

The boundary of Σ is the union $\partial\Sigma = \bigsqcup_{j=1}^n C_j$ of oriented circles, and we fix a presentation

$$\pi_1(\Sigma) = \left\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n : \prod_{i=1}^g [a_i, b_i] \prod_{j=1}^n c_j = e \right\rangle,$$

where g is the genus of Σ and c_j is freely homotopic to C_j with positive orientation. Combining now the long exact sequence in bounded cohomology associated to the pair of spaces $(\Sigma, \partial\Sigma)$ and the one associated to the usual coefficient sequence in (2.4), we obtain

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^2_{\mathrm{b}}(\Sigma, \partial\Sigma, \mathbb{R}) \arrow[r, "j_{\partial\Sigma}"] & \mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbb{R}) \arrow[r] & \mathrm{H}^2_{\mathrm{b}}(\partial\Sigma, \mathbb{R}) = 0 \\
 & & \mathrm{H}^2_{\mathrm{b}}(\Sigma, \mathbb{Z}) \arrow[u] \arrow[r] & \mathrm{H}^2_{\mathrm{b}}(\partial\Sigma, \mathbb{Z}) \arrow[u] \\
 & & & \mathrm{H}^1(\partial\Sigma, \mathbb{R}/\mathbb{Z}) \arrow[u, "\delta"] \\
 & & & 0 \arrow[u]
\end{tikzcd}$$

where we have used that $\mathrm{H}^i_{\mathrm{b}}(\partial\Sigma,\mathbb{R}) = 0$, $i \geq 1$. We then have the following congruence relation:

LEMMA 8.1. *Let $\alpha \in \mathrm{H}^2_{\mathrm{b}}(\Sigma,\mathbb{Z})$ and denote by $\alpha_{\mathbb{R}}$ its image in $\mathrm{H}^2_{\mathrm{b}}(\Sigma,\mathbb{R})$.*

$$\begin{aligned}\left\langle j^{-1}_{\partial\Sigma}(\alpha_{\mathbb{R}}), [\Sigma,\partial\Sigma]\right\rangle &\equiv -\left\langle \delta^{-1}(\alpha|_{\partial\Sigma}), [\partial\Sigma]\right\rangle \mod \mathbb{Z}\\ &= -\sum_{j=1}^{n}\left\langle \delta^{-1}\alpha|_{C_j}, [C_j]\right\rangle\,,\end{aligned} \tag{8.1}$$

where we view $j^{-1}_{\partial\Sigma}(\alpha_{\mathbb{R}})$ as ordinary relative singular cohomology class.

Proof. Let $c \in \mathscr{Z}^2_{\mathrm{b}}(\Sigma,\mathbb{Z})$ be a $\mathbb{Z}$-valued bounded cocycle representing $\alpha \in \mathrm{H}^2_{\mathrm{b}}(\Sigma,\mathbb{Z})$, and let $c|_{\partial\Sigma} \in \mathscr{Z}^2_{\mathrm{b}}(\partial\Sigma,\mathbb{Z})$ be its restriction to the boundary $\partial\Sigma$. Since the fundamental groups of the components of $\partial\Sigma$ are amenable, there exists a bounded $\mathbb{R}$-valued 1-cochain $c' \in F^1_{\mathrm{b}}(\partial\Sigma,\mathbb{R})$ such that $dc' = c|_{\partial\Sigma}$. Let $c'' \in F^1_{\mathrm{b}}(\partial\Sigma,\mathbb{R}/\mathbb{Z})$ be the corresponding $\mathbb{R}/\mathbb{Z}$-valued 1-cochain on $\partial\Sigma$: for any 1-simplex $t \in S_1(\partial\Sigma)$ we have that

$$\langle c', t\rangle \equiv \langle c'', t\rangle \mod \mathbb{Z} \tag{8.2}$$

and moreover, since $c|_{\partial\Sigma}$ is $\mathbb{Z}$-valued, c'' is a 1-cocycle which represents the class $\delta^{-1}(\alpha|_{\partial\Sigma}) \in \mathrm{H}^1(\partial\Sigma,\mathbb{R}/\mathbb{Z})$.

On the other hand, we can extend c' to a 1-cochain $\widetilde{c'}$ on Σ by setting

$$\widetilde{c'}(\sigma) = \begin{cases} c'(\sigma) & \text{if } \sigma \in S_1(\partial\Sigma)\\ 0 & \text{otherwise}\,,\end{cases}$$

so that $c - d\widetilde{c'} \in \mathscr{Z}^2_{\mathrm{b}}(\Sigma,\partial\Sigma,\mathbb{R})$ is a cocycle which represents $j^{-1}_{\partial\Sigma}(\alpha_{\mathbb{R}})$.

Let now s be a two-chain which represents the relative fundamental class $[\Sigma,\partial\Sigma]$, so that ∂s represents the fundamental class $[\partial\Sigma]$. From the definition of $\widetilde{c'}$, from (8.2), and the fact that $\langle c, s\rangle \in \mathbb{Z}$, it follows that

$$\langle c - d\widetilde{c'}, s\rangle = \langle c, s\rangle - \langle d\widetilde{c'}, s\rangle \equiv -\langle c'', \partial s\rangle \mod \mathbb{Z}\,,$$

thus completing the proof. □

We apply the above general lemma to the situation at hand and show the following

LEMMA 8.2. *Let $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G,\mathbb{Z})$ and $\rho : \pi_1(\Sigma) \to G$ a homomorphism. Then*

$$\mathrm{T}_\kappa(\Sigma,\rho) \equiv -\sum_{j=1}^{n} \mathrm{Rot}_\kappa\big(\rho(c_j)\big) \mod \mathbb{Z}\,.$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{H}^2_{\mathrm{b}}\big(\pi_1(\Sigma),\mathbb{Z}\big) \arrow[r, "g_\Sigma"] \arrow[d] & \mathrm{H}^2_{\mathrm{b}}(\Sigma,\mathbb{Z}) \arrow[d] \\
\mathrm{H}^2_{\mathrm{b}}\big(\pi_1(C_j),\mathbb{Z}\big) \arrow[r] & \mathrm{H}^2_{\mathrm{b}}(C_j,\mathbb{Z}) \\
\mathrm{H}^1\big(\pi_1(C_j),\mathbb{R}/\mathbb{Z}\big) \arrow[r] \arrow[u, "\delta"] & \mathrm{H}^1(C_j,\mathbb{R}/\mathbb{Z})\,, \arrow[u]
\end{tikzcd}$$

where the horizontal arrows are isomorphisms, the vertical arrows between first and second row are restriction maps and those between third and second are connecting homomorphisms. The commutativity implies the first equality

$$\big\langle \delta^{-1}\big([g_\Sigma\rho^*(\kappa)]|_{C_j}\big),[C_j]\big\rangle = \delta^{-1}\big(\rho^*(\kappa)|_{\pi_1(C_j)}\big)(c_j) = \mathrm{Rot}_\kappa\big(\rho(c_j)\big)\,,$$

while the second is the definition of Rot_κ (see Definition 7.1). This, together with Lemma 8.1 applied to $\alpha = g_\Sigma\big(\rho^*(\kappa)\big)$, implies the result. □

Now we come to the formula for the Toledo invariant. Observe first that when $\partial\Sigma \neq \varnothing$, $\pi_1(\Sigma)$ is a free group and thus any homomorphism $\rho : \pi_1(\Sigma) \to G$ admits a lift $\widetilde{\rho} : \pi_1(\Sigma) \to \widetilde{G}$.

Proof of Theorem 12. Using the equivariance property of $\widetilde{\mathrm{Rot}_\kappa}$ in Theorem 7.9(2), one checks that

$$\mathrm{R}(\rho) := -\sum_{j=1}^{n} \widetilde{\mathrm{Rot}_\kappa}\big(\widetilde{\rho}(c_j)\big)$$

does not depend on the choice of the lift $\widetilde{\rho}$. Thus the map

$$\begin{aligned} \mathrm{Hom}\big(\pi_1(\Sigma),G\big) &\to \mathbb{R} \\ \rho \quad &\mapsto \mathrm{R}(\rho) \end{aligned}$$

is well defined and continuous since $\widetilde{\mathrm{Rot}_\kappa}$ is continuous and $\widetilde{G} \to G$ is a covering. This implies, with Proposition 3.10, that the map

$$\rho \mapsto \mathrm{T}_\kappa(\Sigma,\rho) - \mathrm{R}(\rho) \tag{8.3}$$

is continuous. On the other hand (from Lemma 8.2) we know that this map is $\mathbb{Z}$-valued and hence, since $\mathrm{Hom}\big(\pi_1(\Sigma),G\big)$ is connected, that (8.3) is constant: evaluation at the trivial homomorphism implies that this constant is zero, thus showing the theorem. □

Finally, we indicate briefly the formula when $\partial\Sigma = \varnothing$. Let

$$[\,\cdot\,,\,\cdot\,]^{\sim} : G \times G \to \widetilde{G}$$

denote the $\widetilde{G}$-valued commutator map. Recall that when $\partial\Sigma = \varnothing$ the consideration of ordinary cohomology suffices.

THEOREM 8.3. *Let $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{c}}(G, \mathbb{Z})$ and $u \in \mathrm{Hom}_{\mathrm{c}}(K, \mathbb{R}/\mathbb{Z})$ the associated homomorphism. If $\rho : \pi_1(\Sigma) \to G$ is a representation, we have*

$$\mathrm{T}_\kappa(\Sigma, \rho) = -u_* \left(\prod_{j=1}^{g} \left[\rho(a_i), \rho(b_i)\right]^{\sim} \right)$$

where, as usual, $u_* : \pi_1(K) = \pi_1(G) \to \mathbb{Z}$ is the homomorphism induced by u.

Remark 8.4. One may prove the formula in Theorem 8.3 by cutting Σ along the separating curve $[a_1, b_1]$ and combine the formula in Theorem 8.3 applied to each component together with the additivity property in Proposition 3.2(1).

Remark 8.5. The formula in Theorem 8.3 generalizes Milnor's classical formula for the Euler number of a representation into $\mathrm{GL}^+(2)$ [45].

8.2. *The bounded fundamental class and generalized w_1-classes.* The group $\mathrm{PU}(1, 1)$ acts effectively on the circle $\partial\mathbb{D}$ and we have seen that if ρ_1 and ρ_2 are maximal representations of $\pi_1(\Sigma)$ into $\mathrm{PU}(1, 1)$, then the resulting actions on $\partial\mathbb{D}$ are semiconjugate in the sense of Ghys [25]. If $e^{\mathrm{b}} \in \mathrm{H}^2_{\mathrm{b}}\big(\mathrm{PU}(1, 1), \mathbb{Z}\big)$ denotes the bounded Euler class, or more precisely the restriction to $\mathrm{PU}(1, 1)$ of the bounded Euler class of the group of orientation preserving homeomorphisms of $\partial\mathbb{D}$, then $\rho_1^*(e^{\mathrm{b}}) = \rho_2^*(e^{\mathrm{b}})$ [25]. Thus we obtain a canonical class

$$\kappa^{\mathrm{b}}_{\Sigma,\mathbb{Z}} \in \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(\Sigma), \mathbb{Z}\big)$$

associated to the oriented surface Σ and which plays the role of the classical fundamental class when $\partial\Sigma \neq \varnothing$. We propose to call it the *bounded fundamental class of* Σ; observe that even when $\partial\Sigma = \varnothing$, this class contains more information than the usual fundamental class since

$$\mathrm{H}^2_{\mathrm{b}}\big(\pi_1(\Sigma), \mathbb{Z}\big) \to \mathrm{H}^2\big(\pi_1(\Sigma), \mathbb{Z}\big)$$

is never injective. Thus we obtain that for a representation $\rho : \pi_1(\Sigma) \to \mathrm{PU}(1, 1)$ the following are equivalent:

(1) ρ is maximal;

(2) ρ comes from a complete hyperbolic structure on Σ°;

(3) $\rho^*(e^{\mathrm{b}}) = \kappa^{\mathrm{b}}_{\Sigma,\mathbb{Z}}$.

For general groups G of Hermitian type, an analogue of the equivalence of (1) and (3) holds for real coefficients; the extent to which it does not hold for integral coefficients will lead to nontrivial invariants for maximal representations.

Let now G be of Hermitian type and, as usual, let $\{\kappa^{\mathrm{b}}_{G,i} : 1 \le i \le n\}$ be the basis of $\mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$ determined by the decomposition of the associated symmetric space $\mathscr{X} = \mathscr{X}_1 \times \dots \mathscr{X}_n$ into irreducible factors. We define a linear form λ_G on $\mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$ by

$$\lambda_G(\kappa^{\mathrm{b}}_{G,i}) = \mathrm{r}_{\mathscr{X}_i} .$$

Let $\kappa^{\mathrm{b}}_\Sigma \in \mathrm{H}^2_{\mathrm{b}}\big(\pi_1(\Sigma), \mathbb{R}\big)$ denote the real class which is the image of $\kappa^{\mathrm{b}}_{\Sigma,\mathbb{Z}}$ by change of coefficients. Then it follows from our results obtained so far that:

COROLLARY 8.6. *For a homomorphism $\rho : \pi_1(\Sigma) \to G$ the following are equivalent*:

(1) ρ *is maximal*;

(2) $\rho^*(\kappa) = \lambda_G(\kappa)\kappa^{\mathrm{b}}_\Sigma$ *for all* $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$.

Let now $\mathscr{D}$ be the bounded symmetric domain associated to G, $\check{S}$ its Shilov boundary, and Q the stabilizer of some point in $\check{S}$; let e_G be the exponent of the finite group Q/Q°. We will furthermore denote by $\mathrm{Hom}_{\max}\big(\pi_1(\Sigma), G\big)$ the set of maximal representations of $\pi_1(\Sigma)$ into G.

If $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G, \mathbb{Z})$ and $\rho_0 : \pi_1(\Sigma) \to G$ is a maximal representation, then Theorem 13 states that for every maximal representation $\rho : \pi_1(\Sigma) \to G$ the map

$$\begin{aligned} \mathrm{R}^{\rho_0}_\kappa(\rho) : \pi_1(\Sigma) \to& \quad \mathbb{R}/\mathbb{Z} \\ \gamma \quad \mapsto& \ \mathrm{Rot}_\kappa\big(\rho(\gamma)\big) - \mathrm{Rot}_\kappa\big(\rho_0(\gamma)\big) \end{aligned}$$

is a homomorphism, which takes values in $e_G^{-1}\mathbb{Z}/\mathbb{Z}$ if $\mathscr{D}$ is of tube type.

Before we turn to the proof of Theorem 13 we give an example and state a few preliminary lemmas.

Example 8.7. Let V be a real symplectic vector space of dimension $2n$. Let $K = \mathrm{U}(V, J)$ be the maximal compact subgroup of $\mathrm{Sp}(V)$ given by the choice of a compatible complex structure J on V, $\det : K \to \mathbb{T}$ the complex determinant, and let $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}\big(\mathrm{Sp}(V), \mathbb{Z}\big)$ be the bounded integer class associated to the homomorphism $u : K \to \mathbb{R}/\mathbb{Z}$, where $e^{2\pi i u} = \det$.

When $n = 1$, the associated rotation number $\mathrm{Rot}_\kappa(\rho) : \pi_1(\Sigma) \to \mathbb{Z}/2\mathbb{Z}$, $\gamma \mapsto \mathrm{Rot}_\kappa\big(\rho(\gamma)\big)$ associates to an element $\gamma \in \pi_1(\Sigma)$ the sign of the eigenvalue of $\rho(\gamma) \in \mathrm{Sp}(V) \cong \mathrm{SL}(2, \mathbb{R})$. In particular, $\mathrm{Rot}_\kappa(\rho)$ itself is not a homomorphism.

On the other hand, when n is even, let $\Delta : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{Sp}(V)$ be the homomorphism corresponding to a diagonal disk $\mathbb{D} \to \mathscr{X}$, and choose a hyperbolization $h : \pi_1(\Sigma) \to \mathrm{SL}(2, \mathbb{R})$. Setting $\rho_0 = \Delta \circ h$, we have that $\mathrm{Rot}_\kappa(\rho_0) = 0$ and hence $\mathrm{Rot}_\kappa(\rho) = \mathrm{R}^{\rho_0}_\kappa(\rho)$ is a homomorphism. This homomorphism is related to the first Stiefel-Whitney class of the following bundle. Let $\mathscr{L}(V)$ the Grassmannian of Lagrangian subspaces in V and $\mathscr{L} \to \mathscr{L}(V)$ the tautological bundle. In [12] we have shown that if $S = \Gamma\backslash\mathbb{D}$ is a closed hyperbolic surface and $\rho : \Gamma \to \mathrm{Sp}(V)$ is a maximal representation, the equivariant map $\varphi : \partial\mathbb{D} \to \mathscr{L}(V)$ in Theorem 8 is continuous. Composing φ with the visual map $T_1\mathbb{D} \to \partial\mathbb{D}$ and pulling back the bundle $\mathscr{L}$, we get a vector bundle $\mathscr{L}_\rho \to T_1S$ with base the unit tangent bundle of S. Then the first Stiefel-Whitney class $w_1(\mathscr{L}_\rho) \in \mathrm{H}^1(T_1 S, \mathbb{Z}/2\mathbb{Z})$ is given by the composition of the projection $\pi_1(T_1 S) \to \Gamma$ and $\mathrm{Rot}_\kappa(\rho) : \Gamma \to \mathbb{Z}/2\mathbb{Z}$.

This example shows that in some cases $\mathrm{Rot}_\kappa(\rho)$ is a homomorphism, whereas in general only the difference $\mathrm{R}_\kappa^{\rho_0}(\rho)$ for a fixed maximal representation ρ_0 is a homomorphism, which generalizes the first Stiefel-Whitney class; in fact when $\partial\Sigma \neq \varnothing$, the map φ is in general not continuous as the case of $\mathrm{PU}(1,1)$ already shows, and there is no (continuous) bundle in sight.

LEMMA 8.8. *Let $\rho : \pi_1(\Sigma) \to G$ be a maximal representation. Then for every $\gamma \in \pi_1(\Sigma)$, $\rho(\gamma)$ has at least one fixed point in $\check{S}$.*

Proof. This follows at once from Theorem 8, more specifically from the strict equivariance of the left continuous map $\varphi : \partial\mathbb{D} \to \check{S}$. □

LEMMA 8.9. *The restriction map $\mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R}) \to \mathrm{H}^2_{\mathrm{cb}}(Q, \mathbb{R})$ is the zero map.*

Proof. Let $\mathscr{X} = \mathscr{X}_1 \times \cdots \times \mathscr{X}_n$ be a decomposition of the symmetric space associated to G into irreducible factors. Then $\check{S} = \check{S}_1 \times \cdots \times \check{S}_n$, where $\check{S}_i$ is the Shilov boundary of $\mathscr{X}_i$. Let $p_i : \check{S} \to \check{S}_i$ be the projection onto the i-th factor and set $\beta_{\check{S},i} = p_i{}^* \beta_{\check{S}_i}$, where $\beta_{\check{S}_i}$ is the generalized Maslov cocycle of $\check{S}_i$.

Let $\kappa = \sum_{i=1}^n \lambda_i \kappa^{\mathrm{b}}_{G,i}$, where $\{\kappa^{\mathrm{b}}_{G,i} : 1 \le i \le n\}$ is the basis of $\mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$. Applying [9, Cor. 2.3], we have for any Q-invariant Borel set $Z \subset \check{S}$ a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^2\big(\mathscr{B}^\infty_{\mathrm{alt}}(\check{S}^\bullet)^Q\big) & \longrightarrow & \mathrm{H}^2_{\mathrm{cb}}(Q, \mathbb{R}) \\ \big\downarrow & \nearrow & \\ \mathrm{H}^2\big(\mathscr{B}^\infty_{\mathrm{alt}}(Z^\bullet)^Q\big) & & \end{array}$$

where the class $[\beta := \sum_{i=1}^n \lambda_i \beta_{\check{S},i}] \in \mathrm{H}^2\big(\mathscr{B}^\infty_{\mathrm{alt}}(\check{S}^\bullet)^Q\big)$ goes to $\kappa|_Q$ (see Section 2.1.3); taking now Z to be the Q-fixed point in $\check{S}$ and observing that $\beta|_{Z^3} = 0$, we get that $\kappa|_Q = 0$. □

LEMMA 8.10. $\mathrm{Rot}_\kappa|_Q : Q \to \mathbb{R}/\mathbb{Z}$ *is a homomorphism, and if $\mathscr{D}$ is of tube type,* Rot_κ *is trivial on Q°, and hence* $\mathrm{Rot}_\kappa(Q) \subset e_G^{-1}\mathbb{Z}/\mathbb{Z}$.

Proof. The first assertion follows from the fact that $\kappa_{\mathbb{R}}|_Q = 0$ (see Lemma 8.9) and from Lemma 7.2(2).

Let $Q = M A_Q N_Q$ be the Langlands decomposition of Q; then Rot_κ is trivial on $A_Q N_Q$ (see Proposition 7.8(1)). Now M° is reductive with compact center and we may assume that $\mathscr{Z}(M^\circ) \subset K \cap Q$. If then $\mathscr{D}$ is of tube type, the Lie algebra of $K \cap Q$ is contained in the Lie algebra of $[K, K]$ [37, Th. 4.11] and since $\mathrm{Rot}_\kappa|_K$ is a homomorphism, it is therefore trivial on $[K, K]$ and hence on $\mathscr{Z}(M^\circ)^\circ$. Since Rot_κ is also trivial on every connected almost simple factor of M°, we obtain finally that it is trivial on Q°. □

Proof of Theorem 13. Let $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G,\mathbb{Z})$ and $\rho, \rho_0 : \pi_1(\Sigma) \to G$ be maximal representations. Corollary 8.6 implies that the real class in $\mathrm{H}^2_{\mathrm{b}}(\pi_1(\Sigma),\mathbb{R})$ which corresponds to $\rho^*(\kappa) - \rho(\kappa) \in \mathrm{H}^2_{\mathrm{b}}(\pi_1(\Sigma),\mathbb{Z})$ vanishes and hence

$$\rho^*(\kappa) - \rho_0^*(\kappa) = \delta(h) \tag{8.4}$$

for a unique homomorphism $h : \pi_1(\Sigma) \to \mathbb{R}/\mathbb{Z}$. Restricting the equality (8.4) to cyclic subgroups, we get

$$\mathrm{R}_\kappa^{\rho_0}(\rho) = \mathrm{Rot}_\kappa\big(\rho(\gamma)\big) - \mathrm{Rot}_\kappa\big(\rho_0(\gamma)\big) = h(\gamma)$$

for all $\gamma \in \pi_1(\Sigma)$.

Now from Lemma 8.8 we know that every $\rho(\gamma)$ and $\rho_0(\gamma)$ is conjugate to an element of Q and thus if $\mathcal{D}$ is of tube type we have from Lemma 8.10 that $\mathrm{R}_\kappa^{\rho_0}(\rho) \in e_G^{-1}\mathbb{Z}/\mathbb{Z}$. The last assertion follows then from the fact that Rot_κ is continuous and $\mathrm{Hom}\big(\pi_1(\Sigma), e_G^{-1}\mathbb{Z}/\mathbb{Z}\big)$ is finite. □

8.3. *Applications to representation varieties.* Let G be a group of Hermitian type. If $\partial\Sigma = \varnothing$, it is well known that $\mathrm{Hom}_{\max}\big(\pi_1(\Sigma), G\big)$ is a union of components of $\mathrm{Hom}\big(\pi_1(\Sigma), G\big)$ and hence if G is real algebraic, the set of maximal representations is a real semialgebraic set. In the case in which $\partial\Sigma \neq \varnothing$, $\mathrm{Hom}\big(\pi_1(\Sigma), G\big)$ is connected; it is then necessary to study certain naturally defined subsets of the representation variety.

We assume $\partial\Sigma \neq \varnothing$ and use the presentation in (1.1). Let $\mathscr{C} = (\mathscr{C}_1, \dots, \mathscr{C}_n)$ be a set of conjugacy classes in G. Then

$$\mathrm{Hom}^{\mathscr{C}}\big(\pi_1(\Sigma), G\big) := \big\{\rho \in \mathrm{Hom}\big(\pi_1(\Sigma), G\big) :\ \rho(c_i) \in \mathscr{C}_i,\ 1 \le i \le n\big\}$$

is a real semialgebraic set and

COROLLARY 8.11. *For any $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ the map $\rho \mapsto \mathrm{T}_\kappa(\Sigma,\rho)$ is constant on connected components of* $\mathrm{Hom}^{\mathscr{C}}\big(\pi_1(\Sigma), G\big)$.

Proof. Since $\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$ is spanned by integral classes (see Proposition 7.7(3)), we may assume that κ is integral, in which case $\mathrm{T}_\kappa(\Sigma,\rho)$ is congruent mod $\mathbb{Z}$ to $-\sum_{i=1}^n \mathrm{Rot}_\kappa\big(\rho(\mathrm{c}_i)\big)$; the latter is then constant for $\rho \in \mathrm{Hom}^{\mathscr{C}}\big(\pi_1(\Sigma), G\big)$. Thus, since $\rho \mapsto \mathrm{T}_\kappa(\Sigma,\rho)$ is continuous, it is locally constant which proves the corollary. □

In view of Lemma 8.8 a particularly suitable space of representations in relation with the study of maximal representations is

$$\mathrm{Hom}^{\check{S}}\big(\pi_1(\Sigma), G\big) := \big\{\rho \in \mathrm{Hom}\big(\pi_1(\Sigma), G\big) :\ \rho(c_i) \text{ has at least one fixed point in } \check{S},\ 1 \le i \le n\big\}.$$

Indeed we have:

$$\mathrm{Hom}_{\max}\big(\pi_1(\Sigma), G\big) \subset \mathrm{Hom}^{\check{S}}\big(\pi_1(\Sigma), G\big).$$

COROLLARY 8.12. *Let $\kappa \in \widehat{\mathrm{H}}^2_{\mathrm{cb}}(G,\mathbb{Z})$ and assume that $\mathcal{D}$ is of tube type. Then we have that*

$$\mathrm{T}_\kappa(\Sigma,\rho) \in e_G^{-1}\mathbb{Z}$$

for every $\rho \in \mathrm{Hom}^{\check{S}}\big(\pi_1(\Sigma),G\big)$, and $\mathrm{Hom}_{\max}\big(\pi_1(\Sigma),G\big)$ is a union of connected components of $\mathrm{Hom}^{\check{S}}\big(\pi_1(\Sigma),G\big)$. In particular, if G is a real algebraic group, we conclude that the set of maximal representations of $\pi_1(\Sigma)$ into G is a real semialgebraic set.

Proof. If $\mathcal{D}$ is of tube type and $\rho(c_i)$ fixes a point in $\check{S}$, we have by Lemma 8.10 that $\mathrm{Rot}_\kappa\big(\rho(c_i)\big) \in e_G^{-1}\mathbb{Z}$ and hence, by Lemma 8.2, $\mathrm{T}_\kappa(\Sigma,\rho) \in e_G^{-1}\mathbb{Z}$. Since $\widehat{\mathrm{H}}^2_{\mathrm{cb}}(G,\mathbb{Z})$ spans $\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$, we get that for every $\kappa \in \mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$, the map $\rho \mapsto \mathrm{T}_\kappa(\Sigma,\rho)$ is locally constant on $\mathrm{Hom}^{\check{S}}\big(\pi_1(\Sigma),G\big)$ which implies the assertion. □

9. **Examples**

The aim of this section is to prove Theorem 7 in the introduction. In this case $G = \mathbf{G}(\mathbb{R})^\circ$, where $\mathbf{G}$ is a connected algebraic group defined over $\mathbb{R}$ and G is of Hermitian type. As usual, $t : \mathbb{D}^{\mathrm{r}} \to \mathcal{X}$ is a maximal polydisk (where $\mathrm{r} = \mathrm{r}_{\mathcal{X}}$), and $d : \mathbb{D} \to \mathcal{X}$ is the composition of the diagonal embedding $\mathbb{D} \to \mathbb{D}^{\mathrm{r}}$ with t. Accordingly, we have homomorphisms

$$\tau : \mathrm{SU}(1,1)^{\mathrm{r}} \to G \qquad \text{and} \qquad \Delta : \mathrm{SU}(1,1) \to G$$

which satisfy

$$\tau^*(\kappa_G^{\mathrm{b}}) = \kappa^{\mathrm{b}}_{\mathrm{SU}(1,1)^{\mathrm{r}}} \qquad \text{and} \qquad \Delta^*(\kappa_G^{\mathrm{b}}) = \mathrm{r}\,\kappa^{\mathrm{b}}_{\mathrm{SU}(1,1)}\,.$$

As a consequence, if $h, h_1, \dots, h_{\mathrm{r}} : \pi_1(\Sigma) \to \mathrm{SU}(1,1)$ are hyperbolizations, the composition of

$$\begin{aligned} \pi_1(\Sigma) &\to \mathrm{SU}(1,1)^{\mathrm{r}} \\ \gamma &\mapsto \big(h_1(\gamma),\dots,h_{\mathrm{r}}(\gamma)\big) \end{aligned}$$

with τ, h_r, and $\Delta \circ h$ define maximal representations. We will need the following

LEMMA 9.1. *If $\mathcal{X}$ is of tube type there exists $u \in \mathcal{Z}_G(\mathrm{Image}\,\Delta)$ such that G is generated by $\mathrm{Image}\,\tau \cup u(\mathrm{Image}\,\tau)u^{-1}$.*

Proof. For every $u \in \mathcal{Z}_G(\mathrm{Image}\,\Delta)$ let H_u denote the subgroup of G generated by $\mathrm{Image}\,\tau$ and $u(\mathrm{Image}\,\tau)u^{-1}$, and let $\mathfrak{h}_u$ denote its Lie algebra. Then H_u is of Hermitian type and of the same rank as G, and the embedding of the symmetric space $\mathcal{Y}_u$ associated to H_u into the symmetric space $\mathcal{X}$ associated to G is holomorphic. Moreover, if Z_0 is the generator of the center of the maximal compact subgroup of $\mathrm{Image}\,\Delta$ which gives the complex structure on the disk $d(\mathbb{D})$, then $Z_0 \in \mathfrak{h}_u \subset \mathfrak{g}$ gives the complex structure on $\mathcal{Y}_u$ and on $\mathcal{X}$. In particular, the embedding of Lie algebras $\mathfrak{h}_u \hookrightarrow \mathfrak{g}$ is an (H_2)-homomorphism, [48].

Fixing a base point in the image of the tight holomorphic disk $d(\mathbb{D})$ in $\mathcal{X}$, we may assume that the Cartan decompositions of $\mathfrak{h}_u$ and $\mathfrak{g}$ are compatible. Let $\mathfrak{k}_0 \subset \mathfrak{g}$

denote the Lie algebra of $\mathscr{Z}_G(\mathrm{Image}\,\Delta)$ and $\mathfrak{t} \subset \mathfrak{g}$ the Lie algebra of Image τ, and let $\mathfrak{t} = \mathfrak{l} \oplus \mathfrak{r}$ be the Cartan decomposition of $\mathfrak{t}$. With a case by case analysis using the Satake-Ihara classification of (H_2)-homomorphisms, [47], [34], one can determine elements $v \in \mathfrak{k}_0$ such that $\mathrm{Ad}(\exp v)\mathfrak{r} \cup \mathfrak{r}$ will not be contained in any noncompact Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ given by an (H_2)-homomorphism. $\square$

Now let $h : \pi_1(\Sigma) \to \mathrm{SU}(1,1)$ be a hyperbolization as in the statement of the theorem and choose a simple closed geodesic $C \subset \Sigma^\circ$ separating Σ into two components Σ_1, Σ_2. With the hypotheses at hand, we can find simple closed geodesics $C_i \subset \Sigma_i$ not intersecting C. Let $h_t^{(i)}$ denote the hyperbolizations of Σ° obtained by multiplying the length of C_i by a factor $(1+t)$, $t \geq 0$, while keeping $h_t^{(i)}$ constant on $\pi_1(C) \hookrightarrow \pi_1(\Sigma_i)$. Fix $0 < \epsilon_1 < \cdots < \epsilon_{\mathrm{r}}$; then the composition $\rho_t^{(i)}$ with

$$\begin{aligned} \pi_i(\Sigma_i) \to \quad & \mathrm{SU}(1,1)^{\mathrm{r}} \\ \gamma \mapsto \; & \big(h^{(i)}_{\epsilon_1 t}(\gamma), \ldots, h^{(i)}_{\epsilon_{\mathrm{r}} t}(\gamma)\big) \end{aligned}$$

is maximal and its Zariski closure coincides with Image τ. Choose now $u \in \mathscr{Z}_G(\mathrm{Image}\,\Delta)$ as in Lemma 9.1 and define the representation $\rho_t : \pi_1(\Sigma) \to G$ by

$$\rho_t(\gamma) := \begin{cases} \rho_t^{(1)} & \text{if } \gamma \in \pi_1(\Sigma_1) \\ u\rho_t^{(2)}u^{-1} & \text{if } \gamma \in \pi_1(\Sigma_2)\,. \end{cases}$$

Then ρ_t is maximal by the additivity property (see Proposition 3.2) and from Lemma 9.1 we deduce that ρ_t has Zariski dense image for $t > 0$.

Appendix A. **Index of Notation**

G°	connected component of the identity in G
$\check{S}$	Shilov boundary of a bounded symmetric domain
$\mathrm{r}_{\mathscr{X}}$	rank of the symmetric space $\mathscr{X}$
$G_{\mathscr{X}}$	connected component of the group of isometries of $\mathscr{X}$
$\Delta(x,y,z)$	smooth triangle with geodesic sides and vertices x, y, z
$\Delta : L \to G$	homomorphism associated to a diagonal disk
$\tau : L^{\mathrm{r}_{\mathscr{X}}} \to G$	homomorphism associated to a maximal polydisk
$\Omega^\bullet(\mathscr{X})^G$	complex of G-invariant differential forms on $\mathscr{X}$
$\mathrm{H}^\bullet_{\mathrm{c}}(G,\mathbb{R})$	continuous cohomology of G with $\mathbb{R}$ coefficients
$\mathrm{H}^\bullet_{\mathrm{cb}}(G,\mathbb{R})$	bounded continuous cohomology of G with $\mathbb{R}$- coefficients
$\widehat{\mathrm{H}}^\bullet_{\mathrm{cb}}(G,A)$	Borel cohomology of G with $A = \mathbb{R}$, $\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$ coefficients
$\widehat{\mathrm{H}}^\bullet_{\mathrm{c}}(G,A)$	bounded Borel cohomology of G with $A = \mathbb{R}$, $\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$ coefficients
$\mathrm{H}^\bullet(X,Y,A)$	relative singular cohomology with $A = \mathbb{R}$, $\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$ coefficients
$\mathrm{H}^\bullet_{\mathrm{b}}(X,Y,A)$	relative bounded singular cohomology with $A = \mathbb{R}$, $\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$ coefficients
$\mathrm{H}^\bullet(X,A)$	singular cohomology with $A = \mathbb{R}$, $\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$ coefficients
$\mathrm{H}^\bullet_{\mathrm{b}}(X,A)$	bounded singular cohomology with $A = \mathbb{R}$, $\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$ coefficients
$\mathrm{H}^\bullet\big(\pi_1(X),A\big)$	group cohomology with $A = \mathbb{R}$, $\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$ coefficients
$\mathrm{H}^\bullet_{\mathrm{b}}\big(\pi_1(X),A\big)$	bounded group cohomology with $A = \mathbb{R}$, $\mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$ coefficients
$\big(\mathscr{B}^\infty_{\mathrm{alt}}(\check{S}^\bullet)\big)$	complex of bounded alternating Borel cocycles on $\check{S}$

$\mathcal{Z}L^\infty_{\mathrm{alt}}\big((\partial\mathbb{D})^\bullet,\mathbb{R}\big)^\Gamma$	cocycles in the complex of Γ-invariant alternating bounded measurable functions on $\partial\mathbb{D}$
$S_m(Y)$	set of singular m-simplices in Y
$F_{\mathrm{b}}(Y,\mathbb{R})$	space of bounded m-cochains
κ_G	Kähler class in $\mathrm{H}^2_{\mathrm{c}}(G,\mathbb{R})$
κ_G^{b}	bounded Kähler class in $\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$
$\mathrm{T}_\kappa(\Sigma,\rho)$	Toledo number with respect to $\kappa\in\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$
$\mathrm{T}(\Sigma,\rho)$	Toledo number with respect to bounded Kähler class $\kappa_G^{\mathrm{b}}\in\mathrm{H}^2_{\mathrm{cb}}(G,\mathbb{R})$
Rot_κ	rotation number associated to $\kappa\in\widehat{\mathrm{H}}^2_{\mathrm{cb}}(G,\mathbb{Z})$
$\widetilde{\mathrm{Rot}}_\kappa$	homogeneous quasimorphism which lifts Rot_κ
$\mathcal{N}_K(A)$	normalizer of A in K
$\mathcal{Z}_K(A)$	centralizer of A in K
$\mathcal{Z}(\Gamma)$	center of Γ

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References

[1] J. Barge and É. Ghys, Cocycles d'Euler et de Maslov, *Math. Ann.* **294** (1992), 235–265. MR 95b:55021 Zbl 0894.55006

[2] P. Blanc, Sur la cohomologie continue des groupes localement compacts, *Ann. Sci. École Norm. Sup.* **12** (1979), 137–168. MR 80k:22009 Zbl 0429.57012

[3] A. Borel, Class functions, conjugacy classes and commutators in semisimple Lie groups, in *Algebraic Groups and Lie Groups*, *Austral. Math. Soc. Lect. Ser.* **9**, Cambridge Univ. Press, Cambridge, 1997, pp. 1–19. MR 99j:22020 Zbl 0872.22011

[4] S. B. Bradlow, O. García-Prada, and P. B. Gothen, Surface group representations and $\mathrm{U}(p,q)$-Higgs bundles, *J. Differential Geom.* **64** (2003), 111–170. MR 2004k:53142 Zbl 1070.53054

[5] ———, Maximal surface group representations in isometry groups of classical Hermitian symmetric spaces, *Geom. Dedicata* **122** (2006), 185–213. MR 2008e:14013 Zbl 1132.14029

[6] R. Brooks, Some remarks on bounded cohomology, in *Riemann Surfaces and Related Topics*: *Proceedings of the* 1978 *Stony Brook Conference* (State Univ. New York, Stony Brook, N.Y., 1978), *Ann. of Math. Stud.* **97**, Princeton Univ. Press, Princeton, N.J., 1981, pp. 53–63. MR 83a:57038 Zbl 0457.55002

[7] M. Burger and N. Monod, Continuous bounded cohomology and applications to rigidity theory, *Geom. Funct. Anal.* **12** (2002), 219–280. MR 2003d:53065a Zbl 1006.22010

[8] M. Burger and A. Iozzi, A useful formula in bounded cohomology, to appear in *Séminaires et Congrès* **18**. Available at http://www.math.ethz.ch/~iozzi/grenoble.ps

[9] M. BURGER and A. IOZZI, Boundary maps in bounded cohomology. Appendix to: "Continuous bounded cohomology and applications to rigidity theory" [Geom. Funct. Anal. **12** (2002), no. 2, 219–280; MR1911660 (2003d:53065a)] by M. Burger and N. Monod, *Geom. Funct. Anal.* **12** (2002), 281–292. MR 2003d:53065b

[10] ———, Bounded Kähler class rigidity of actions on Hermitian symmetric spaces, *Ann. Sci. École Norm. Sup.* **37** (2004), 77–103. MR 2005b:32048 Zbl 1061.32016

[11] ———, Bounded differential forms, generalized Milnor-Wood inequality and an application to deformation rigidity, *Geom. Dedicata* **125** (2007), 1–23. MR 2009c:53053 Zbl 1134.53020

[12] M. BURGER, A. IOZZI, F. LABOURIE, and A. WIENHARD, Maximal representations of surface groups: symplectic Anosov structures, *Pure Appl. Math. Q.* **1** (2005), 543–590. MR 2007d:53064 Zbl 1157.53025

[13] M. BURGER, A. IOZZI, and A. WIENHARD, Surface group representations with maximal Toledo invariant, *C. R. Math. Acad. Sci. Paris* **336** (2003), 387–390. MR 2004e:53076 Zbl 1035.32013

[14] ———, Hermitian symmetric spaces and Kähler rigidity, *Transform. Groups* **12** (2007), 5–32. MR 2009c:32044 Zbl 1138.32012

[15] ———, Tight homomorphisms and Hermitian symmetric spaces, *Geom. Funct. Anal.* **19** (2009), 678–721. MR 2563767 Zbl 05652309

[16] J.-L. CLERC, L'indice de Maslov généralisé, *J. Math. Pures Appl.* **83** (2004), 99–114. MR 2004k:53127 Zbl 1061.53056

[17] ———, An invariant for triples in the Shilov boundary of a bounded symmetric domain, *Comm. Anal. Geom.* **15** (2007), 147–173. MR 2008a:32021 Zbl 1123.32014

[18] J.-L. CLERC and K. KOUFANY, Primitive du cocycle de Maslov généralisé, *Math. Ann.* **337** (2007), 91–138. MR 2007j:53091 Zbl 1110.32009

[19] J.-L. CLERC and K.-H. NEEB, Orbits of triples in the Shilov boundary of a bounded symmetric domain, *Transform. Groups* **11** (2006), 387–426. MR 2007j:32016 Zbl 1112.32010

[20] J.-L. CLERC and B. ØRSTED, The Maslov index revisited, *Transform. Groups* **6** (2001), 303–320. MR 2002j:53103 Zbl 1078.53076

[21] ———, The Gromov norm of the Kaehler class and the Maslov index, *Asian J. Math.* **7** (2003), 269–295. MR 2005b:53127 Zbl 1079.53120

[22] V. FOCK and A. GONCHAROV, Moduli spaces of local systems and higher Teichmüller theory, *Publ. Math. Inst. Hautes Études Sci.* (2006), 1–211. MR 2009k:32011 Zbl 1099.14025

[23] ———, Moduli spaces of convex projective structures on surfaces, *Adv. Math.* **208** (2007), 249–273. MR 2008g:57015 Zbl 1111.32013

[24] H. FURSTENBERG, A Poisson formula for semi-simple Lie groups, *Ann. of Math.* **77** (1963), 335–386. MR 26 #3820 Zbl 0192.12704

[25] É. GHYS, Groupes d'homéomorphismes du cercle et cohomologie bornée, in *The Lefschetz Centennial Conference*, *Part* III (Mexico City, 1984), *Contemp. Math.* **58**, Amer. Math. Soc., Providence, RI, 1987, pp. 81–106. MR 88m:58024 Zbl 0617.58009

[26] W. M. GOLDMAN, Discontinuous groups and the Euler class, Ph.D. thesis, Univ. California Berkeley, 1980.

[27] ———, Topological components of spaces of representations, *Invent. Math.* **93** (1988), 557–607. MR 89m:57001 Zbl 0655.57019

[28] ———, The modular group action on real SL(2)-characters of a one-holed torus, *Geom. Topol.* **7** (2003), 443–486. MR 2004k:57001 Zbl 1037.57001

[29] P. B. GOTHEN, Components of spaces of representations and stable triples, *Topology* **40** (2001), 823–850. MR 2002k:14017 Zbl 1066.14012

[30] M. GROMOV, Volume and bounded cohomology, *Inst. Hautes Études Sci. Publ. Math.* (1982), 5–99 (1983). MR 84h:53053 Zbl 0516.53046

[31] O. GUICHARD, Composantes de Hitchin et représentations hyperconvexes de groupes de surface, *J. Differential Geom.* **80** (2008), 391–431. MR 2009h:57031 Zbl pre05504163

[32] L. HERNÁNDEZ, Maximal representations of surface groups in bounded symmetric domains, *Trans. Amer. Math. Soc.* **324** (1991), 405–420. MR 91f:32040 Zbl 0733.32024

[33] N. HITCHIN, Lie groups and Teichmüller space, *Topology* **31** (1992), 449–473. MR 93e:32023 Zbl 0769.32008

[34] S.-I. IHARA, Holomorphic imbeddings of symmetric domains, *J. Math. Soc. Japan* **19** (1967), 261–302. MR 35 #5656 Zbl 0159.11102

[35] N. V. IVANOV, Foundations of the theory of bounded cohomology, *J. Soviet Math.* **37** (1987), 1090–1115. Zbl 0612.55006

[36] V. A. KAIMANOVICH, SAT actions and ergodic properties of the horosphere foliation, in *Rigidity in Dynamics and Geometry* (Cambridge, 2000), Springer-Verlag, New York, 2002, pp. 261–282. MR 2003e:37006 Zbl 1054.37002

[37] A. KORÁNYI and J. A. WOLF, Realization of hermitian symmetric spaces as generalized half-planes, *Ann. of Math.* **81** (1965), 265–288. MR 30 #4980 Zbl 0137.27402

[38] V. KOZIARZ and J. MAUBON, Harmonic maps and representations of non-uniform lattices of $\mathrm{PU}(m, 1)$, *Ann. Inst. Fourier* (*Grenoble*) **58** (2008), 507–558. MR 2009f:22011 Zbl 1147.22009

[39] F. LABOURIE, Anosov flows, surface groups and curves in projective space, *Invent. Math.* **165** (2006), 51–114. MR 2007c:20101 Zbl 1103.32007

[40] ———, Cross ratios, Anosov representations and the energy functional on Teichmüller space, *Ann. Sci. École Norm. Sup.* **41** (2009), 439–471. Zbl 1160.37021

[41] C. LÖH-STROHM, The proportionality principle of simplicial volume, preprint, 2005. Available at http://www.arXiv.org/math.AT/0504106

[42] G. LUSZTIG, Total positivity in reductive groups, in *Lie Theory and Geometry*, *Progr. Math.* **123**, Birkhäuser, Boston, MA, 1994, pp. 531–568. MR 96m:20071 Zbl 0845.20034

[43] ———, Total positivity in partial flag manifolds, *Represent. Theory* **2** (1998), 70–78. MR 2000b:20060 Zbl 0895.14014

[44] G. W. MACKEY, Les ensembles boréliens et les extensions des groupes, *J. Math. Pures Appl.* **36** (1957), 171–178. MR 19,752a Zbl 0080.02303

[45] J. MILNOR, On the existence of a connection with curvature zero, *Comment. Math. Helv.* **32** (1958), 215–223. MR 20 #2020 Zbl 0196.25101

[46] N. MONOD, *Continuous Bounded Cohomology of Locally Compact Groups*, *Lecture Notes in Math.* **1758**, Springer-Verlag, New York, 2001. MR 2002h:46121 Zbl 0967.22006

[47] I. SATAKE, Holomorphic imbeddings of symmetric domains into a Siegel space, *Amer. J. Math.* **87** (1965), 425–461. MR 33 #4326 Zbl 0144.08202

[48] ———, *Algebraic Structures of Symmetric Domains*, *Kanô Memorial Lectures* **4**, Iwanami Shoten, Tokyo, 1980. MR 82i:32003 Zbl 0483.32017

[49] D. TOLEDO, Representations of surface groups in complex hyperbolic space, *J. Differential Geom.* **29** (1989), 125–133. MR 90a:57016 Zbl 0676.57012

[50] W. T. VAN EST, Group cohomology and Lie algebra cohomology in Lie groups, I, II, *Nederl. Akad. Wetensch. Proc. Series* A. **56**, *Indag. Math.* **15** (1953), 484–504.

[51] A. WIENHARD, The action of the mapping class group on maximal representations, *Geom. Dedicata* **120** (2006), 179–191. MR 2008g:20112 Zbl 1175.32007

[52] D. WIGNER, Algebraic cohomology of topological groups, *Trans. Amer. Math. Soc.* **178** (1973), 83–93. MR 49 #2898 Zbl 0264.22001

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