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Bounded cohomology and totally real subspaces in complex hyperbolic geometry

MARC BURGER and ALESSANDRA IOZZI

Department Mathematik, ETH Zentrum, CH-8092 Zürich, Switzerland
(e-mail: burger@math.ethz.ch, iozzi@math.ethz.ch)

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To Dan

Abstract. We characterize representations of finitely generated discrete groups into (the connected component of) the isometry group of a complex hyperbolic space via the pullback of the bounded Kähler class.

1. *Introduction*

In this paper we continue our investigation of the properties of actions of discrete groups on Hermitian symmetric spaces via an invariant called the *bounded Kähler class* of the action.

Among the (irreducible) symmetric spaces, the Hermitian ones are those which admit an invariant complex structure, whose existence gives immediately an invariant differential two-form ω, which is hence closed [**12**], called the *Kähler form*. The easiest such example is the symmetric space associated to $\mathrm{PU}(1, 1)$, whose bounded domain realization is the Poincaré disk with the usual invariant volume form, or more generally the symmetric space associated to $\mathrm{PU}(p, q)$, whose bounded domain realization consists of complex $q \times p$ matrices Z such that $ZZ^* - \mathrm{Id}$ is strictly negative definite.

If X is Hermitian symmetric and $G := \mathrm{Iso}(X)^\circ$, the second bounded cohomology $\mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$ is one-dimensional, with generator κ^{b}_G admitting the representative

$$(g_0, g_1, g_2) \longmapsto \int_{\Delta(g_0, g_1, g_2)} \omega,$$

where $\Delta(g_0, g_1, g_2)$ is any C^1-simplex in X with geodesic sides and with vertices g_0x, g_1x, g_2x, x being a base point [**13**]. We shall refer to $\kappa^{\mathrm{b}}_G \in \mathrm{H}^2_{\mathrm{cb}}(G, \mathbb{R})$ as the *bounded Kähler class*. If $\rho : \Gamma \to G$ is a representation, the object of our interest is then the bounded Kähler class of the action $\rho^*(\kappa^{\mathrm{b}}_G) \in \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$.

A Hermitian symmetric space that admits also a holomorphic realization corresponding to the upper half plane in the case of $\mathrm{PU}(1, 1)$ is called of *tube type* and, among the above examples, the symmetric space associated to $\mathrm{PU}(p, q)$ is of tube type if and only if $p = q$. Symmetric spaces of tube type and not of tube type behave very differently in many aspects and these differences are reflected in our investigation via the bounded Kähler class [**3–9**].

For example we proved in [**3, 7**] that if X is a Hermitian symmetric space that is not of tube type, then the bounded Kähler class of a representation with Zariski dense image does not vanish and determines the representation up to conjugacy. If on the other hand X is of tube type, this characterization fails: for example any two hyperbolizations of a compact surface have the same invariant, [**14**].

In all cases, however, tube type or not tube type, the problem of determining properties of the representation with vanishing bounded Kähler class remains wide open. In this paper we establish relations between its vanishing and properties of the representation ρ in the case in which the Hermitian symmetric space is of rank one, that is a complex hyperbolic space.

The content of the next theorem is to pin down exactly that the class $\rho^*(\kappa_G^{\mathrm{b}})$ does vanish only when the image of ρ is 'small' in the following sense. Recall that a *totally real subspace* of $\mathcal{H}_{\mathbb{C}}^n$ is a subset isometric to $\mathcal{H}_{\mathbb{R}}^k$, for some $k \leq n$, with curvature -1 (if $\mathcal{H}_{\mathbb{C}}^n$ is normalized as to have sectional curvature between -4 and -1), see §2.

THEOREM 1.1. *Let Γ be a finitely generated group, $\rho : \Gamma \to \mathrm{PU}(n, 1)$ a homomorphism and $\kappa_n^{\mathrm{b}} \in \mathrm{H}_{\mathrm{cb}}^2(\mathrm{PU}(n, 1), \mathbb{R})$ the bounded Kähler class. Then the following are equivalent.*
(1) *$\rho^*(\kappa_n^{\mathrm{b}}) \in \mathrm{H}_{\mathrm{b}}^2(\Gamma, \mathbb{R})$ vanishes.*
(2) *Either $\rho(\Gamma)$ fixes a point in the boundary $\partial\mathcal{H}_{\mathbb{C}}^n$ of complex hyperbolic space or it leaves a totally real subspace invariant.*

The above result, together with the following complementary structure theorem, gives the complete description of a representation of a general discrete group into $\mathrm{PU}(n, 1)$ in terms of its bounded Kähler class.

THEOREM 1.2. *Let $\rho : \Gamma \to \mathrm{PU}(n, 1)$ be a representation of a finitely generated discrete group and assume that $\rho^*(\kappa_n^{\mathrm{b}}) \neq 0$. Let $\mathbf{L} := \overline{\rho(\Gamma)}^Z$ be the Zariski closure of the image of ρ and set $L := \mathbf{L}(\mathbb{R})^\circ$ to be the connected component of the real points of $\mathbf{L}$.*

Then L is an almost direct product $L = K \cdot M$, where K is compact and M is locally isomorphic to $\mathrm{SU}(m, 1)$, for $1 \leq m \leq n$. Moreover the symmetric space associated to L is a copy of $\mathcal{H}_{\mathbb{C}}^m$ isometrically and holomorphically embedded in $\mathcal{H}_{\mathbb{C}}^n$.

The proof relies upon the above mentioned characterization in [**3**] of representations with Zariski dense image into $\mathrm{PU}(p, q)$, $p \neq q$, with methods borrowed from [**9**, Proof of Theorem 5].

To give a geometric interpretation of this result it is convenient to have the following definition.

Definition 1.3. We say that a representation $\rho : \Gamma \to \mathrm{PU}(n, 1)$ is *elementary in the complex sense* if one of the following holds:
- $\rho(\Gamma)$ fixes a point in $\overline{\mathcal{H}_{C}^n}$;
- it leaves invariant a totally real subspace $\mathcal{H}_{\mathbb{R}}^k \subset \mathcal{H}_{\mathbb{C}}^n$;
- it leaves invariant a complex geodesic.

If $\rho : \Gamma \to \mathrm{PU}(n, 1)$ is a representation of a finitely generated group, we denote by $c(\mathcal{L}_{\rho(\Gamma)})$ the convex hull of the limit set $\mathcal{L}_{\rho(\Gamma)} \subset \partial\mathcal{H}^n_{\mathbb{C}}$ of $\rho(\Gamma)$.

COROLLARY 1.4. *Let $\rho_1, \rho_2 : \Gamma \to \mathrm{PU}(n, 1)$ be representations of a finitely generated discrete group and assume that ρ_1 and ρ_2 are not elementary in the complex sense. If $\rho_1^*(\kappa^{\mathrm{b}}) = \rho_2^*(\kappa^{\mathrm{b}})$, then, up to conjugation by an element of* $\mathrm{SU}(n, 1)$*, we have that*

$$c(\mathcal{L}_{\rho_1(\Gamma)}) = c(\mathcal{L}_{\rho_2(\Gamma)}) =: C$$

and the actions of Γ on C via ρ_1 and ρ_2 coincide.

We turn now to a geometric counterpart of Theorem 1.1. To this end, let M be a quotient of $\mathcal{H}^n_{\mathbb{C}}$, and ω_M the induced Kähler form. Given any C^1-simplex $\sigma : \Delta^2 \to M$, let σ^* be a C^1-simplex with geodesic sides, homotopic to σ via a homotopy fixing the vertices. Then

$$\kappa_M(\sigma) := \int_{\sigma^*} \omega_M$$

defines a bounded singular cohomology class $\kappa_M \in \mathrm{H}^2_{\mathrm{s,b}}(M)$. We shall see that for compact arithmetic quotients M, the presence of a compact submanifold $V \subset M$ such that the restriction $\kappa_M|_V \in \mathrm{H}^2_{\mathrm{s,b}}(V)$ vanishes forces the existence of a totally real compact submanifold $R \subset M$ (that is a submanifold which is the compact quotient in M of a totally real subspace of $\mathcal{H}^n_{\mathbb{C}}$). We shall see moreover that V can be homotoped into R; it hence follows that if V is not homotopic to a point or a circle, then $\dim R \geq 2$. More generally we have the following corollary.

COROLLARY 1.5. *Let $M = \Lambda\backslash\mathbb{H}^n_{\mathbb{C}}$ be a compact arithmetic manifold, let V be a compact manifold and $f : V \to M$ a continuous map. Then $f^*(\kappa_M) \in \mathrm{H}^2_{\mathrm{s,b}}(V)$ vanishes if and only if there exists a compact, totally real immersed submanifold $R \subset M$ such that f is homotopic to a map with image in R.*

2. *Complex hyperbolic space*

We recall here the main points of complex hyperbolic geometry that we shall need and refer to [**1**, Ch. II.10] and [**15**] for details.

Let

$$\langle z, w\rangle = \sum_{k=1}^{n} z_k \overline{w}_k - z_{n+1}\overline{w}_{n+1}$$

be the Hermitian form of signature $(n, 1)$ on $\mathbb{C}^{n+1}$; recall that the complex hyperbolic n-space $\mathcal{H}^n_{\mathbb{C}}$ is the set of points $[x] \in \mathbb{P}^n(\mathbb{C})$ in complex projective n-space with $\langle x, x\rangle < 0$, equipped with the distance

$$\cosh^2 d([x], [y]) = \frac{\langle x, y\rangle\langle y, x\rangle}{\langle x, x\rangle\langle y, y\rangle}.$$

This distance, which comes from a Kähler metric, turns $\mathcal{H}^n_{\mathbb{C}}$ into a CAT(-1) space with sectional curvature $-4 \leq K \leq -1$. The same construction over the field of the real numbers gives rise to the real hyperbolic n-space $\mathcal{H}^n_{\mathbb{R}}$, whose sectional curvature is constant and equal to -1.

A real vector subspace $V \subset \mathbb{C}^{n+1}$ is *totally real* if $\langle z, w\rangle \in \mathbb{R}$ for all $z, w \in V$. A *totally real subspace of $\mathcal{H}^n_{\mathbb{C}}$* of dimension k is then the image in $\mathcal{H}^n_{\mathbb{C}}$ of a totally real subspace of $\mathbb{C}^{n+1}$ of real dimension $k+1$, provided the latter contains a negative vector. The totally real subspaces of dimension k in $\mathcal{H}^n_{\mathbb{C}}$ are precisely those subsets of $\mathcal{H}^n_{\mathbb{C}}$ which are isometric to a real hyperbolic space $\mathcal{H}^k_{\mathbb{R}}$ with curvature -1.

Obviously any one-dimensional real subspace of $\mathbb{C}^{n+1}$ is totally real. Also, given any two vectors $v_1, v_2 \in \mathbb{C}^{n+1}$ it is easy to see that the subspace $\mathbb{R}v_1 \oplus \mathbb{R}\langle v_2, v_1\rangle v_2$ is totally real. However, given three vectors, it is not always the case that some complex multiple of them spans a totally real subspace. To detect whether this is the case the Hermitian triple product is a useful tool. Recall that if $z_1, z_2, z_3 \in \mathbb{C}^{n+1}$, their *Hermitian triple product* is defined as

$$\langle z_1, z_2, z_3\rangle = \langle z_1, z_2\rangle\langle z_2, z_3\rangle\langle z_3, z_1\rangle.$$

Observe that, by definition, if $\alpha, \beta, \gamma \in \mathbb{C}$, then

$$\langle \alpha z_1, \beta z_2, \gamma z_3\rangle = |\alpha|^2|\beta|^2|\gamma|^2\langle z_1, z_2, z_3\rangle$$

so that, in particular, $\langle z_1, z_2, z_3\rangle \in \mathbb{R}$ if and only if $\langle \alpha z_1, \beta z_2, \gamma z_3\rangle \in \mathbb{R}$, provided $\alpha, \beta, \gamma \in \mathbb{C}^*$.

For any set F, we use the notation $F^{(n)}$ to denote the set of n-tuples of distinct points in F.

LEMMA 2.1. *Let $F \subset \mathbb{C}^{n+1}$ be a subset such that $\langle x, y, z\rangle \neq 0$ for all $(x, y, z) \in F^{(3)}$. Then $\langle x, y, z\rangle \in \mathbb{R}$ for every $(x, y, z) \in F^{(3)}$ if and only if there exist a totally real subspace V of $\mathbb{C}^{n+1}$ and a function $\lambda : F \to \mathbb{C}^*$ such that the $\mathbb{R}$-linear span of $\{\lambda_z z : z \in F\}$ is contained in V.*

Proof. ($\Leftarrow$) If $\lambda_x x, \lambda_y y, \lambda_z z \in \mathbb{C}^{n+1}$ are contained into a totally real subspace then all the pairwise Hermitian products are real, and hence their triple scalar product is real as well.

($\Rightarrow$) To see the converse, let us first associate to every finite subset $S \subset F$ a totally real subspace V_S that is the real span of the set $\{\lambda_z z : z \in S\}$, where $\lambda : S \to \mathbb{C}^*$ is a function to be determined.

The construction goes as follows. Let $S = \{z_1, \ldots, z_{|S|}\}$ be any listing of the elements of S. For $|S| = 3$, it is easy to check that if one chooses $\lambda_1 = 1$, $\lambda_2 = \langle z_1, z_2\rangle$ and $\lambda_3 = \langle z_1, z_3\rangle$, then the condition $\langle z_1, z_2, z_3\rangle \in \mathbb{R}$ implies that the subspace V_3 spanned by $\{\lambda_1 z_1, \lambda_2 z_2, \lambda_3 z_3\}$ is totally real. Notice that $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}^*$ because $\langle \lambda_2 z_2, \lambda_3 z_3\rangle = \langle z_1, z_2, z_3\rangle$ and, by hypothesis, we know that $\langle z_1, z_2, z_3\rangle \neq 0$.

We proceed now by induction. Let us assume that if $\langle z_i, z_j, z_k\rangle \in \mathbb{R}$ for all $1 \leq i, j, k \leq \ell \leq |S| - 1$ with $i \neq j \neq k \neq i$, then $\{\lambda_1 z_1, \ldots, \lambda_\ell z_\ell\}$ span a totally real subspace V_ℓ, with $\lambda_1 = 1$ and $\lambda_j = \langle z_1, z_j\rangle \in \mathbb{C}^*$. Notice that by construction $V_3 \subseteq \cdots \subseteq V_{\ell-1} \subseteq V_\ell$.

Define now $\lambda_{\ell+1} = \langle z_1, z_{\ell+1}\rangle$. As before, $\lambda_{\ell+1} \in \mathbb{C}^*$. Moreover, by definition,

$$\langle \lambda_1 z_1, \lambda_{\ell+1} z_{\ell+1}\rangle = |\langle z_1, z_{\ell+1}\rangle|^2 \in \mathbb{R}$$

and, by inductive hypothesis, if $j > 1$

$$\begin{aligned}\langle \lambda_j z_j, \lambda_{\ell+1} z_{\ell+1}\rangle &= \langle z_1, z_j\rangle\overline{\langle z_1, z_{\ell+1}\rangle}\langle z_j, z_{\ell+1}\rangle \\ &= \langle z_1, z_j, z_{\ell+1}\rangle \in \mathbb{R},\end{aligned}$$

which shows that the real subspace $V_{\ell+1}$ generated by $\lambda_1 z_1, \ldots, \lambda_\ell z_\ell, \lambda_{\ell+1} z_{\ell+1}$ is totally real and $V_\ell \subseteq V_{\ell+1}$.

Let now S be a fixed finite subset of F such that $\dim_{\mathbb{R}} V_S$ is maximal. We need to show that every $f \in F$ can be multiplied by a non-zero complex number λ_f so that $\lambda_f f \in V_S$. In fact, if for every $\lambda \in \mathbb{C}^*$ $\lambda f \notin V_S$, then it is easy to see that $\dim_{\mathbb{R}} V_{S\cup\{f\}} > \dim_{\mathbb{R}} V_S$, contradicting maximality. □

3. *The Cartan invariant as a bounded cohomology class*

Let $\partial\mathcal{H}^n_{\mathbb{C}}$ be the sphere at infinity of the CAT(-1) space $\mathcal{H}^n_{\mathbb{C}}$, which can be identified with the image of the null cone $\mathcal{C}^0 = \{z \in \mathbb{C}^{n+1}\backslash\{0\} : \langle z, z\rangle = 0\}$ under the projection $p : \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{P}^n(\mathbb{C})$.

Since the diagonal action of $\mathrm{PU}(n, 1)$ is not transitive on the set $(\partial\mathcal{H}^n_{\mathbb{C}})^{(3)}$, one can associate to distinct triples of points in $\partial\mathcal{H}^n_{\mathbb{C}}$ an invariant which plays the role of the crossratio for quadruples of points in the boundary of real hyperbolic space. This is the 'invariant angulaire' or *Cartan invariant*, defined by

$$c_n(\xi_1, \xi_2, \xi_3) = \frac{2}{\pi}\,\mathrm{Arg}(-\langle z_1, z_2, z_3\rangle), \tag{3.1}$$

where $p(z_i) = \xi_i$, with $z_i \in \mathcal{C}^0$ for $i = 1, 2, 3$.

It follows from the fact that the Hermitian form has signature $(n, 1)$, that the Hermitian triple product has negative real part. If we take the convention in (3.1) that $\mathrm{Arg}(z) \in [-\pi/2, \pi/2]$ for $\mathrm{Re}\, z \geq 0$, it follows that c_n takes values in $[-1, 1]$.

Moreover, the Cartan invariant has the following important properties (see [**15**]).

(i) $c_n(\xi_1, \xi_2, \xi_3) = c_n(\eta_1, \eta_2, \eta_3)$ if and only (ξ_1, ξ_2, ξ_3) and (η_1, η_2, η_3) are in the same $\mathrm{PU}(n, 1)$-orbit in $(\partial\mathcal{H}^n_{\mathbb{C}})^{(3)}$.

(ii) c_n is an alternating function on $(\partial\mathcal{H}^n_{\mathbb{C}})^{(3)}$, that is, for all $\sigma \in S_3$, we have that $c_n(\xi_{\sigma(1)}, \xi_{\sigma(2)}, \xi_{\sigma(3)}) = \mathrm{sign}(\sigma) c_n(\xi_1, \xi_2, \xi_3)$.

(iii) c_n is continuous on $(\partial\mathcal{H}^n_{\mathbb{C}})^{(3)}$.

(iv) Extending c_n to the whole of $(\partial\mathcal{H}^n_{\mathbb{C}})^3$ by setting $c_n(\xi_1, \xi_2, \xi_3) = 0$ if the triple is not distinct, we have the cocycle relation $c_n(\xi_2, \xi_3, \xi_4) - c_n(\xi_1, \xi_3, \xi_4) + c_n(\xi_1, \xi_2, \xi_4) - c_n(\xi_1, \xi_2, \xi_3) = 0$ for any quadruple $(\xi_1, \xi_2, \xi_3, \xi_4) \in (\partial\mathcal{H}^n_{\mathbb{C}})^4$.

(v) Furthermore, $|c_n(\xi_1, \xi_2, \xi_3)| = 1$ if and only if (ξ_1, ξ_2, ξ_3) lie on a chain, that is on the boundary of a complex geodesic.

Actually we shall never use property (v) in this paper; however, we chose to point it out here to illustrate, together with next corollary, what kind of geometric information can be obtained from the maximality or minimality of the (absolute value of the) Cartan invariant.

COROLLARY 3.1. *Let $\mathcal{L} \subset \partial\mathcal{H}^n_{\mathbb{C}}$ be any subset. Then $c_n(\xi_1, \xi_2, \xi_3) = 0$ for all $(\xi_1, \xi_2, \xi_3) \in (\mathcal{L})^{(3)}$ if and only if $\mathcal{L}$ is contained in the boundary of a totally real subspace of $\mathcal{H}^n_{\mathbb{C}}$.*

Proof. By definition, there exists a subset $F \subset \mathcal{C}^0 \subset \mathbb{C}^{n+1}\backslash\{0\}$ such that $\mathcal{L} = p(F)$. The corollary is then a restatement of Lemma 2.1 after observing that, again since the Hermitian form has signature $(n, 1)$, we have $\langle z_1, z_2, z_3\rangle \neq 0$ for all $(z_1, z_2, z_3) \in F^{(3)}$. □

The extension of c_n defined in (iv) defines a bounded measurable alternating function on $(\partial\mathcal{H}^n_{\mathbb{C}})^3$, and we want to describe the other essential property that it enjoys, namely that it defines a bounded cohomology class in $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{PU}(n, 1), \mathbb{R})$ that coincides with $(1/\pi)\kappa^{\mathrm{b}}_n$. To this end, recall that if H is a locally compact group, the continuous bounded cohomology of H can be defined as the cohomology of the complex of H-invariants of

$$0 \longrightarrow C_{\mathrm{b}}(H, \mathbb{R}) \xrightarrow{d} C_{\mathrm{b}}(H^2, \mathbb{R}) \xrightarrow{d} \cdots$$

where $C_{\mathrm{b}}(H^j, \mathbb{R})$ is the space of real-valued continuous bounded functions on the cartesian product of j copies of H [**18**] and the coboundary operator is given by

$$df(h_0, \ldots, h_k) = \sum_{i=0}^{k} (-1)^i f(h_0, \ldots, \hat{h}_i, \ldots, h_k).$$

In order to see how c_n defines a cohomology class, we need, however, some more definitions that will also be used later in the proof of Proposition 4.2. If H is a locally compact group, a *continuous Banach H-module* is a Banach space on which H acts continuously by isometric automorphisms, and *H-morphisms* are linear continuous H-equivariant maps between continuous Banach H-modules. Then a continuous Banach H-module E is *relatively injective* if for every injective H-morphism $\imath : A \hookrightarrow B$ of continuous Banach H-modules A, B which admits a left inverse of norm bounded by 1 and every H-morphism $\alpha : A \to E$ there is a H-morphism $\beta : B \to E$ which extends α and such that $\|\beta\| \leq \|\alpha\|$.

$$\begin{tikzcd} A \arrow[r, hook, "\imath"] \arrow[dr, "\alpha"'] & B \arrow[d, "\beta"] \\ & E \end{tikzcd}$$

The following theorem is a characterization of amenable actions that provides an essential tool to compute continuous bounded cohomology. Before stating it, recall that X is a *standard Borel space* if it is a set endowed with a Borel σ-algebra that is Borel isomorphic to the interval $[0, 1]$. A *regular H-space* is a pair (X, ν), where X is a standard Borel space on which H acts measurably and ν is a quasi-invariant measure such that the corresponding H-action on $L^1(X, \nu)$ is continuous, [**18**, Definition 2.1.1].

THEOREM 3.2. [**10**] *Let H be a locally compact group, and (B, ν) a regular H-space. The H-action on B is amenable if and only if $L^\infty(B, \mathbb{R})$ is a relatively injective H-module. Moreover, the cohomology of the complex*

$$0 \longrightarrow L^\infty(B, \mathbb{R})^H \longrightarrow L^\infty(B^2, \mathbb{R})^H \longrightarrow \cdots$$

is canonically isomorphic to $\mathrm{H}^*_{\mathrm{cb}}(H, \mathbb{R})$.

We do not recall here the definition of amenable action for which we refer to [**21**, Ch. 4], and we limit ourselves to recall that the actions of $\mathrm{PU}(n, 1)$ on $\partial\mathcal{H}^n_{\mathbb{C}}$ and of the free group $\mathbb{F}_r$ in r generators on the Poisson boundary of the regular infinite tree $\mathcal{T}_r$ of valence r (see the proof of Proposition 4.2) are both amenable.

As a particularly important consequence of Theorem 3.2, we record the following fact. If H and B are as in Theorem 3.2 and if in addition H acts doubly ergodically on B (that is ergodically on $B \times B$ with respect to the product measure†), then there is an isometric

† In classical ergodic theory this is equivalent to the concept of mixing. The way this condition is used here is to infer that any H-invariant measurable map $B \times B \to \mathbb{R}$ is essentially constant. However a slightly different conclusion is needed when one considers bounded cohomology with coefficients, from which the need of a different terminology.

isomorphism

$$\mathrm{H}^2_{\mathrm{cb}}(H, \mathbb{R}) \cong \mathcal{Z}L^\infty_{\mathrm{alt}}(B^3, \mathbb{R})^H, \tag{3.2}$$

that is, $\mathrm{H}^2_{\mathrm{cb}}(H, \mathbb{R})$ is identified with the Banach space of bounded, alternating, H-invariant cocycles on B^3 with values in the trivial H-module $\mathbb{R}$, see [**10**].

Taking $H = \mathrm{PU}(n, 1)$ and $B = \partial\mathcal{H}^n_{\mathbb{C}}$ the boundary of hyperbolic n-space, we see that the Cartan cocycle, defined in (3.1) on $B^{(3)}$ and then extended to B^3, defines via (3.2) a bounded cohomology class in $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{PU}(n, 1), \mathbb{R})$, which equals $(1/\pi)\kappa^{\mathrm{b}}_n$ by [**3**, Lemma 6.2]; the explicit relation between the Kähler form and the Cartan invariant is given by the formula

$$c_n(\xi_1, \xi_2, \xi_3) = \int_{\Delta(\xi_1,\xi_2,\xi_3)} \omega,$$

where $\Delta(\xi_1, \xi_2, \xi_3)$ is an ideal simplex with geodesic sides and vertices ξ_1, ξ_2, ξ_3 at infinity.

4. *Functoriality and boundaries*

Given two locally compact groups H_1 and H_2, and a continuous homomorphism $\rho : H_1 \to H_2$, it is obvious on the appropriate resolutions by bounded continuous cochains that ρ induces a morphism

$$\rho^* : \mathrm{H}^*_{\mathrm{cb}}(H_2) \to \mathrm{H}^*_{\mathrm{cb}}(H_1).$$

While on the one hand it is much more convenient to compute these cohomology groups using resolutions by L^∞ functions on amenable H_i-regular spaces, on the other, doing so, it is much less clear how the map ρ^* looks in these resolutions. However, the following proposition makes an essential point, obtained using fully the homological algebra approach to bounded cohomology. We refer to [**2**] for a more detailed discussion.

PROPOSITION 4.1. [**2**] *Let $\rho : H_1 \to H_2$ be a continuous homomorphism of locally compact groups. Let (Y_1, ν_1) be a regular amenable H_1-space, Y_2 a compact metric separable H_2-space on which H_2 acts by homeomorphisms and let $\varphi : Y_1 \to \mathcal{M}(Y_2)$ be a ρ-equivariant measurable map.*

Then to any strict bounded Borel cocycle $c : Y_2^{n+1} \to \mathbb{R}$ one can canonically associate a bounded class $[c] \in \mathrm{H}^n_{\mathrm{cb}}(H_2, \mathbb{R})$ and $\rho^*([c])$ can be represented by the cocycle in $L^\infty(Y_1^{n+1})$ defined by

$$(y_1, \ldots, y_{n+1}) \to \varphi(y_1) \otimes \cdots \otimes \varphi(y_{n+1})(c).$$

Once we know that any finitely generated group Γ admits amenable, doubly ergodic standard Γ-spaces, we shall be able to put to use Proposition 4.1 and Theorem 3.2.

To this purpose, we recall the following proposition which is a particular case of a theorem in [**10**, §1], (see also [**17**] for the most general setting). Since in our setting the proof is very transparent, we present it here.

PROPOSITION 4.2. [**10**, Theorem 0.2] *Let Γ be a finitely generated group. Then there exists a Γ-space B with a quasi-invariant measure μ, such that the Γ-action on (B, μ) is both amenable and doubly ergodic.*

Proof. Let S be a finite generating set for Γ of cardinality r, and let $\mathbb{F}_r$ be the free group on r generators, so that we have a surjective homomorphism $\rho : \mathbb{F}_r \to \Gamma$ with kernel N. Let $\mathcal{T}_r$ be the regular infinite tree of valence r with automorphism group $\mathrm{Aut}(\mathcal{T}_r)$, so that $\mathbb{F}_r \subset \mathrm{Aut}(\mathcal{T}_r)$.

Let $\partial\mathcal{T}_r$ be the natural Poisson boundary of $\mathcal{T}_r$ (consisting of reduced words of infinite length) and let $\tilde{\mu}$ be the natural quasi-invariant probability measure defined by $\tilde{\mu}(E(x)) = (2r(2r-1)^{n-1})^{-1}$, where $|x| = n$ and $E(x) \subset \partial\mathcal{T}_r$ consists of the infinite reduced words starting with x (so that $\{E(x) : x \in \mathcal{T}_r\}$ is a basis for the topology of $\partial\mathcal{T}_r$). The space (B, μ) will then be realized as the point realization of the algebra of N-invariant L^∞ functions on $(\partial\mathcal{T}_r, \tilde{\mu})$ (see [**20**, Theorem 3.3] for details). Namely, let $\mathcal{B}^N$ and $\mathcal{B}$ be the measure algebras generated respectively by $L^\infty(\partial\mathcal{T}_r, \tilde{\mu})^N$ and $L^\infty(\partial\mathcal{T}_r, \tilde{\mu})$. Since $\mathcal{B}^N \subset \mathcal{B}$, corresponding to $\mathcal{B}^N$ there exists a factor (B, μ) of $(\partial\mathcal{T}_r, \tilde{\mu})$, namely a measure space (B, μ) with a probability measure μ and a measurable map $p : (\partial\mathcal{T}_r, \tilde{\mu}) \to (B, \mu)$ such that $\mu = p_*(\tilde{\mu})$ (where, if $A \subset B$ is a measurable set, $p_*(\tilde{\mu})$ is defined by $p_*(\tilde{\mu})(A) = \tilde{\mu}(p^{-1}(A))$). The space (B, μ) carries a Γ-action (since it carries an action of $\mathbb{F}_r$ which factors through the action of N), with respect to which the projection map is $\mathbb{F}_r$-equivariant.

Now that the space (B, μ) has been constructed, we need to show the properties of its Γ-action. Observe first of all that the action of $\mathrm{Aut}(\mathcal{T}_r)$ on $\partial\mathcal{T}_r$ is doubly ergodic, and so is the action of $\mathbb{F}_r$ on $\partial\mathcal{T}_r$ [**10**, Propositions 1.5 and 1.6]. Since p is $\mathbb{F}_r$-equivariant, it follows that the action of Γ on (B, μ) is doubly ergodic as well.

To prove that the Γ-action on (B, μ) is amenable, we shall use the characterization of amenable actions given in Theorem 3.2, that is we shall prove that the Banach Γ-module $L^\infty(B, \mu)$ is relatively injective. Let A, B' be two continuous Banach Γ-modules with an injective Γ-morphism $\iota : A \hookrightarrow B'$, and let $\alpha : A \to L^\infty(B, \mu)$ a Γ-morphism. If j is the inclusion $j : L^\infty(B, \mu) = L^\infty(\partial\mathcal{T}_r, \tilde{\mu})^N \hookrightarrow L^\infty(\partial\mathcal{T}_r, \tilde{\mu})$ and if we think of A and B' as continuous Banach $\mathbb{F}_r$-modules (with a trivial N-action), then we have an $\mathbb{F}_r$-morphism $\alpha' = j \circ \alpha : A \to L^\infty(\partial\mathcal{T}_r, \tilde{\mu})$.

$$\begin{tikzcd}
A \arrow[r, hook, "\iota"] \arrow[dr, "\alpha"'] & B \arrow[d, "\beta"] \arrow[dr, dotted, "\beta'"] & \\
& L^\infty(B, \mu) \arrow[r, hook, "j"] & L^\infty(\partial\mathcal{T}_r, \tilde{\mu})
\end{tikzcd}$$

Since the $\mathbb{F}_r$-action is amenable on $(\partial\mathcal{T}_r, \tilde{\mu})$, by Theorem 3.2 there exists an $\mathbb{F}_r$-morphism $\beta' : B' \to L^\infty(\partial\mathcal{T}_r, \tilde{\mu})$ which extends α' and such that $\|\beta'\| \le \|\alpha'\|$. Since the N-action on A and B' was trivial, the image of β' lies in $L^\infty(\partial\mathcal{T}_r, \tilde{\mu})^N$, hence defining the desired extension $\beta : B' \to L^\infty(\partial\mathcal{T}_r, \tilde{\mu})^N = L^\infty(B, \mu)$ with $\|\beta\| \le \|\alpha\|$. □

5. *Proofs*

Proof of Theorem 1.1. ($\Leftarrow$) This implication follows immediately since the restriction of the Kähler form to a totally real subspace vanishes identically.

($\Rightarrow$) We may assume that $\rho(\Gamma)$ is not elementary. In fact, if this is not the case the conclusion is immediate since either $\rho(\Gamma)$ fixes a point in $\partial\mathcal{H}^n_{\mathbb{C}}$ or a point in $\mathcal{H}^n_{\mathbb{C}}$ or a geodesic.

Let (B, μ) be the amenable doubly ergodic Γ-space in Proposition 4.2 and let $\mathcal{L}_{\rho(\Gamma)} = \overline{\rho(\Gamma) \cdot x} \cap \partial\mathcal{H}^n_{\mathbb{C}}$ be the limit set of $\rho(\Gamma)$ (which is independent of $x \in \mathcal{H}^n_{\mathbb{C}}$). Then, since $\rho(\Gamma)$ is not elementary, there exists a Γ-equivariant measurable map $\varphi : B \to \mathcal{L}_{\rho(\Gamma)}$, [**11**, Corollary 3.2]. By Proposition 4.1 with $H_1 = \Gamma$, $H_2 = \mathrm{PU}(n, 1)$, $(Y_1, \mu) = (B, \nu)$, $Y_2 = \partial\mathcal{H}^n_{\mathbb{C}}$, and where we think of $\mathcal{L}_{\rho(\Gamma)}$ as embedded in $\mathcal{M}(\partial\mathcal{H}^n_{\mathbb{C}})$ as Dirac masses, the cocycle

$$
\begin{aligned}
c^\rho : B \times B \times B &\to [-1, 1] \\
(b_1, b_2, b_3) &\mapsto c_n(\varphi(b_1), \varphi(b_2), \varphi(b_3))
\end{aligned}
$$

is a representative of $\rho^*(\kappa_n^{\mathrm{b}}) \in \mathrm{H}^2_{\mathrm{b}}(\Gamma, \mathbb{R})$. Since Γ acts ergodically on $B \times B$, c^ρ is an alternating 2-cocycle and $\rho^*(\kappa_n^{\mathrm{b}}) = 0$, it follows, from (3.2) and from the properties of the Cartan invariant, that $c^\rho = 0$ almost everywhere, that is that $c_n(\varphi(b_1), \varphi(b_2), \varphi(b_3)) = 0$ for almost every $(b_1, b_2, b_3) \in B \times B \times B$ with respect to the product measure. The rest of the argument will consist of showing that, in fact, the following claim holds.

CLAIM 5.1. *c_n is identically zero on $(\mathcal{L}_{\rho(\Gamma)})^{(3)}$.*

Then Corollary 3.1, with $\mathcal{L} = \mathcal{L}_{\rho(\Gamma)}$, shows that $\mathcal{L}_{\rho(\Gamma)}$ is contained in the boundary of a totally real subspace of $\mathcal{H}^n_{\mathbb{C}}$. The intersection of all totally real subspaces of $\mathcal{H}^n_{\mathbb{C}}$ containing $\mathcal{L}_{\rho(\Gamma)}$ is then a totally real subspace left invariant by Γ.

To prove the claim, let $\lambda = \varphi_*\mu$ be the measure on $\mathcal{L}_{\rho(\Gamma)}$. Since φ is Γ-equivariant and μ is quasi-invariant, $\operatorname{supp} \lambda$ is a closed $\rho(\Gamma)$-invariant subset of $\mathcal{L}_{\rho(\Gamma)}$. Since $\rho(\Gamma)$ is not elementary, it acts minimally on $\mathcal{L}_{\rho(\Gamma)}$, which implies that $\operatorname{supp} \lambda = \mathcal{L}_{\rho(\Gamma)}$. Now let $\nu = \lambda \times \lambda \times \lambda$ be the product measure on $(\mathcal{L}_{\rho(\Gamma)})^3$. Then we have so far that $c_n(x_1, x_2, x_3) = 0$ for all triples of points $(x_1, x_2, x_3) \in \operatorname{supp}(\nu) = (\operatorname{supp} \lambda)^3 = (\mathcal{L}_{\rho(\Gamma)})^3$. Now let $(a, b, c) \in (\mathcal{L}_{\rho(\Gamma)})^{(3)}$, and let U_a, U_b, U_c be small neighborhoods in $\mathcal{L}_{\rho(\Gamma)}$ of a, b, c respectively which are pairwise disjoint. Since $\operatorname{supp} \lambda = \mathcal{L}_{\rho(\Gamma)}$, the measure λ of an open non-void set is positive. Hence $\operatorname{supp}(\nu) \cap (U_a \times U_b \times U_c) \neq \emptyset$ so that for $(a', b', c') \in \operatorname{supp}(\nu) \cap (U_a \times U_b \times U_c)$ we have $c_n(a', b', c') = 0$. Then, by continuity of c_n, we also have that $c_n(a, b, c) = 0$, hence completing the proof. $\square$

Proof of Theorem 1.2. Let $\mathbf{L} := \overline{\rho(\Gamma)}^Z$ and $L := \mathbf{L}(\mathbb{R})^\circ$; since $\rho^*(\kappa_n^{\mathrm{b}}) \neq 0$, we have in particular that $\kappa_n^{\mathrm{b}}|_L \neq 0$ and thus L is not amenable. Since $\mathrm{PU}(n, 1)$ has real rank one, this implies that L is reductive. Since L is not amenable, its semi-simple part has positive rank, and the rank must necessarily be one. Thus L is the almost direct product $L = K \cdot M$ of a compact connected subgroup K with a connected simple Lie group M of real rank one†. Since K is compact and $\kappa_n^{\mathrm{b}}|_L \neq 0$, we have that $\kappa_n^{\mathrm{b}}|_M \neq 0$. In particular $\mathrm{H}^2_{\mathrm{cb}}(M, \mathbb{R}) \cong \mathrm{H}^2_{\mathrm{c}}(M, \mathbb{R}) \neq 0$, which implies that M is of Hermitian type. Since M is of rank one, it is isomorphic to $\mathrm{SU}(m, 1)$, for $1 \leq m \leq n$. We can then choose a local isomorphism $\pi : \mathrm{SU}(m, 1) \to M$ such that $\pi^*(\kappa_n^{\mathrm{b}}) = \lambda\kappa_m^{\mathrm{b}}$, where $\lambda > 0$. If $m \geq 2$, then $\pi : \mathrm{SU}(m, 1) \to \mathrm{PU}(n, 1)$ corresponds necessarily to a totally geodesic embedding of $\mathcal{H}^m_{\mathbb{C}}$ which is holomorphic since $\lambda > 0$; if on the other hand $m = 1$, then, looking at the root space decomposition of $\mathrm{PU}(n, 1)$ coming from a maximal split torus in $\mathrm{SU}(1, 1)$ via π,

† Namely, $L = K \cdot M$, where $K \cap M$ is finite and K and M commute.

one concludes that either $\pi(\mathrm{SU}(1, 1))$ is totally real, and hence $\pi^*(\kappa_n^{\mathrm{b}}) = 0$, which is a contradiction, or π corresponds to a complex geodesic, and then $\lambda = 1$. $\square$

Proof of Corollary 1.4. Let $\mathbf{L}_i := \overline{\rho_i(\Gamma)^Z}$ and $L_i := \mathbf{L}_i(\mathbb{R})^\circ$. Then it follows from Theorem 1.1 that $\rho_1^*(\kappa_n^{\mathrm{b}}) = \rho_2^*(\kappa_n^{\mathrm{b}}) \neq 0$ and from Theorem 1.2 that $\mathbf{L}_i = K_i \cdot M_i$, with K_i compact and M_i locally isomorphic to $\mathrm{SU}(n_i, 1)$ with $2 \leq n_i \leq n$. From [**3**, Theorem 1.3] we deduce that $n_1 = n_2$ and that if $p_i : L_i \to \mathrm{Ad}(M_i)$ is the projection to the adjoint group of M_i then $p_1 \circ \rho_1 : \Gamma \to \mathrm{PU}(n_1, 1)$ and $p_2 \circ \rho_2 : \Gamma \to \mathrm{PU}(n_2, 1)$ are conjugated via a holomorphic isometry $\mathcal{H}_{\mathbb{C}}^{n_1} \to \mathcal{H}_{\mathbb{C}}^{n_2}$, which implies the corollary. $\square$

Proof of Corollary 1.5. ($\Rightarrow$) Let $\Lambda = \pi_1^*(V)$, $\Gamma = \pi_1^*(M)$, $\rho : \Lambda \to \Gamma$ the homomorphism induced by f, and $f^* : \mathrm{H}^*_{\mathrm{s,b}}(M) \to \mathrm{H}^*_{\mathrm{s,b}}(V)$ the map in singular bounded cohomology induced by f. According to Gromov [**16**], there is a natural isomorphism $\mathrm{H}^*_{\mathrm{b}}(X) \simeq \mathrm{H}^*_{\mathrm{b}}(\pi_1^*(X))$, for any manifold. In fact this is true much more generally, for example for any countable CW complex X. In particular, in our case we have that this isomorphism sends the class k_M to the class κ_n^{b}, so that the commutativity of the square

$$\begin{array}{ccc} \mathrm{H}^*_{\mathrm{s,b}}(M) & \xrightarrow{f^*_{\mathrm{b}}} & \mathrm{H}^*_{\mathrm{s,b}}(V) \\ \wr\downarrow & & \downarrow\wr \\ \mathrm{H}^*_{\mathrm{b}}(\Gamma) & \xrightarrow{\rho^*} & \mathrm{H}^*_{\mathrm{b}}(\Lambda) \end{array}$$

together with the hypothesis $f^*(k_M) = 0$ implies that $\rho^*(\kappa_n^{\mathrm{b}}) \in \mathrm{H}^2_{\mathrm{b}}(\Lambda)$ vanishes.

The main point is now to show that there is a totally real subspace $T \subset \mathcal{H}_{\mathbb{C}}^n$ such that the following hold.

(1) $\rho(\Lambda) \subset \mathrm{Stab}_\Gamma(T)$.

(2) $\mathrm{Stab}_\Gamma(T)$ acts cocompactly on T.

Indeed, setting $R = \mathrm{pr}(T)$, where $\mathrm{pr} : \mathcal{H}_{\mathbb{C}}^n \to M$ is the canonical projection, we have that $R = T/\mathrm{Stab}_\Gamma(T)$ is a compact immersed submanifold. Let $p_T : \mathcal{H}_{\mathbb{C}}^n \to T$ be the orthogonal projection, and for every pair $x, y \in \mathcal{H}_{\mathbb{C}}^n$ of points, let $g_{x,y} : [0, 1] \to \mathcal{H}_{\mathbb{C}}^n$ be the constant speed geodesic connecting x to y. Define

$$\tilde{f}_t(x) = g_{\tilde{f}(x), p_T(\tilde{f}(x))}(t).$$

Clearly $\tilde{f}_t$ is Λ-equivariant and thus descends to a homotopy $t \to f_t$ between $f_0 = f$ and f_1 which has image in R.

Thus we turn to the construction of T. Because of our opening remarks, we know that $\rho^*(\kappa_n^{\mathrm{b}}) = 0$ and we are hence in the position of applying Theorem 1.1.

There are two cases. First, assume that $\rho(\Lambda)$ is elementary. Since Γ is torsion free and cocompact, either $\rho(\Lambda) = \{e\}$, in which case we take $T = \{pt\}$, or $\rho(\Lambda)$ is infinite cyclic, in which case we take as T the axis of a generator of $\rho(\Lambda)$. In both cases, T satisfies the properties (1) and (2) and we are done.

Assume now that $\rho(\Lambda) := \Delta$ is non-elementary. Let $\mathcal{L}_\Delta$ be its limit set and T the minimal totally real subspace of $\mathcal{H}_{\mathbb{C}}^n$ such that ∂T contains $\mathcal{L}_\Delta$. From Theorem 1.1 we know that T is Δ-invariant and what remains to show is that $\mathrm{Stab}_\Gamma(T)$ acts cocompactly on T. Here we bring in the hypothesis that Γ is arithmetic. Namely, let $\mathbf{G}$ be a connected, semi-simple adjoint group defined over $\mathbb{Q}$ such that $\mathbf{G}(\mathbb{R}) = \mathrm{PU}(n, 1) \times K$, where K is compact and $\Gamma' = \mathrm{pr}_1(\mathbf{G}(\mathbb{Z})) \leq \mathrm{PU}(1, 1)$ is commensurable with Γ. Define $\mathbf{H}$ to be the connected component of the Zariski closure of

$$\{\gamma \in \mathbf{G}(\mathbb{Z}) : \mathrm{pr}_1(\gamma) \text{ leaves } T \text{ invariant}\}.$$

Then $\mathbf{H}$ is a $\mathbb{Q}$-subgroup of $\mathbf{G}$; let $H = \mathrm{pr}_1(\mathbf{H}(\mathbb{R}))$, which is closed and with a finite number of connected components. We have $H \supset \Delta \cap \Gamma'$, where the latter is of finite index in Δ and non-elementary; hence

$$\mathcal{L}_H \supset \mathcal{L}_{\Delta\cap\Gamma'} = \mathcal{L}_\Delta.$$

Since H is non-elementary as well, $\mathcal{L}_H = \mathcal{L}_{H^\circ}$, so that finally

$$\partial T \supset \mathcal{L}_{H^\circ} \supset \mathcal{L}_\Delta.$$

Observe that if S is the image of H° in $\mathrm{Iso}(T)$ under the restriction map, then

$$\mathcal{L}_{H^\circ} = \mathcal{L}_S.$$

We claim now that S is reductive with compact center. Indeed, let $\mathcal{R}$ be the (connected) radical of S. Then the fixed point set of $\mathcal{R}$ in $\mathcal{L}_S$ is non-void, S-invariant, and hence equal to $\mathcal{L}_S$. Since $|\mathcal{L}_S| \geq 3$, $\mathcal{R}$ is compact and hence central. Now we use a theorem of Mostow [**19**] which guarantees the existence of a point $t \in T$ such that the orbit $S \cdot t \subset T$ is totally geodesic, and hence coincides with the symmetric space associated to S; but then $T' = S \cdot t$ is totally real, with $T' \supset \mathcal{L}_S$, which by minimality of T implies that $T' = T$ and hence S acts transitively on T. Since $\mathbf{H}(\mathbb{R})^\circ$ is a compact extension of S, we conclude firstly that $\mathbf{H}$ has no $\mathbb{Q}$-rational characters, and hence $\mathbf{H}(\mathbb{Z})$ is a (cocompact) lattice in $\mathbf{H}(\mathbb{R})$; secondly, that, T being the symmetric space associated to $\mathbf{H}(\mathbb{R})$, $\mathrm{pr}_1(\mathbf{H}(\mathbb{Z}))$ acts cocompactly on T. Thus $\mathrm{Stab}_{\Gamma'}(T)$, and hence $\mathrm{Stab}_\Gamma(T)$, act cocompactly on T. □

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