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Isometric embeddings in bounded cohomology

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This paper is devoted to the construction of norm-preserving maps between bounded cohomology groups. For a graph of groups with amenable edge groups, we construct an isometric embedding of the direct sum of the bounded cohomology of the vertex groups in

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the bounded cohomology of the fundamental group of the graph of groups. With a similar technique we prove that if (X, Y) is a pair of CW-complexes and the fundamental group of each connected component of Y is amenable, the isomorphism between the relative bounded cohomology of (X, Y) and the bounded cohomology of X in degree at least 2 is isometric. As an application we provide easy and self-contained proofs of Gromov's Equivalence Theorem and of the additivity of the simplicial volume with respect to gluings along π_1-injective boundary components with amenable fundamental group.

Keywords: Relative bounded cohomology; isometries in bounded cohomology; simplicial volume; graph of groups; additivity of the simplicial volume; Dehn filling; ℓ^1-homology; Gromov Equivalence Theorem.

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1. Introduction

Bounded cohomology of groups and spaces was introduced by Gromov in the mid-70s [24] and can be dramatically different from their usual cohomology. For example, in the context of bounded cohomology, the lack of a suitable Mayer–Vietoris sequence prevents the use of the usual "cut and paste" techniques exploited in the computation of singular cohomology. Another peculiarity of bounded cohomology is that, in positive degree, the bounded cohomology of any amenable group (or of any space with amenable fundamental group) vanishes.

Using the Mayer–Vietoris sequence it is easy to show that, in positive degree, the cohomology of a free product of groups is isomorphic to the direct sum of the cohomologies of the factors. The main result of this paper provides an analogous result in the context of bounded cohomology. Since amenable groups are somewhat invisible to bounded cohomology, it is natural to extend the object of our study from free products to amalgamated products (or HNN extensions) along amenable subgroups. In order to treat both these cases at the same time, we will exploit notions and results coming from the Bass–Serre theory of graphs of groups (we refer the reader to Sec. 4 for a brief account on this topic).

For every group Γ we denote by $\mathrm{H}_\mathrm{b}^\bullet(\Gamma)$ the bounded cohomology of Γ with trivial real coefficients, endowed with the ℓ^∞-seminorm. If $\mathcal{G}$ is a graph of groups based on the graph G, we denote by $V(G)$ the set of vertices of G, and by Γ_v, $v \in V(G)$, the vertex groups of $\mathcal{G}$. Moreover, if G is finite, then for every element $(\varphi_1, \ldots, \varphi_k) \in \bigoplus_{v\in V(G)} \mathrm{H}_\mathrm{b}^n(\Gamma_v)$ we set

$$\|(\varphi_1, \ldots, \varphi_k)\|_\infty = \max\{\|\varphi_1\|_\infty, \ldots, \|\varphi_k\|_\infty\}.$$

We denote by Γ the fundamental group of $\mathcal{G}$, by $i_v : \Gamma_v \hookrightarrow \Gamma$ the inclusion of Γ_v into Γ, and by $\mathrm{H}(i_v^n) : \mathrm{H}_\mathrm{b}^n(\Gamma) \to \mathrm{H}_\mathrm{b}^n(\Gamma_v)$ the map induced by i_v on bounded cohomology.

The main result of our paper is the following:

Theorem 1.1. *Let Γ be the fundamental group of a graph of groups $\mathcal{G}$ based on the finite graph G. Suppose that every vertex group of $\mathcal{G}$ is countable, and that every edge group of $\mathcal{G}$ is amenable. Then for every $n \in \mathbb{N}\backslash\{0\}$ there exists an isometric* *embedding*

$$\Theta^n : \bigoplus_{v\in V(G)} \mathrm{H}^n_{\mathrm{b}}(\Gamma_v) \to \mathrm{H}^n_{\mathrm{b}}(\Gamma)$$

which provides a right inverse to the map

$$\bigoplus_{v\in V(G)} \mathrm{H}(i^n_v) : \mathrm{H}^n_{\mathrm{b}}(\Gamma) \to \bigoplus_{v\in V(G)} \mathrm{H}^n_{\mathrm{b}}(\Gamma_v).$$

The isometric embedding Θ^n is in general far from being an isomorphism: for example, the real vector spaces $\mathrm{H}^2_{\mathrm{b}}(\mathbb{Z} * \mathbb{Z})$ and $\mathrm{H}^3_{\mathrm{b}}(\mathbb{Z} * \mathbb{Z})$ are infinite-dimensional (the case of degree 2 is dealt with in [9, 16] — see also [42] for a beautiful and slick proof, and $\mathrm{H}^3_{\mathrm{b}}(\mathbb{Z} * \mathbb{Z})$ is computed in [37, 48]), while $\mathrm{H}^n_{\mathrm{b}}(\mathbb{Z}) \oplus \mathrm{H}^n_{\mathrm{b}}(\mathbb{Z}) = 0$ for every $n \geq 1$, since $\mathbb{Z}$ is amenable.

Moreover, the hypothesis that edge groups are amenable is necessary, as the following example shows. Let $\Gamma < \mathrm{PSL}(2, \mathbb{Q}_p) \times \mathrm{PSL}(2, \mathbb{Q}_q)$ be an irreducible torsionfree co-compact lattice, so that Γ projects densely on each of the factors. From this it follows that since $\mathrm{PSL}(2, \mathbb{Q}_q)$ is simple, Γ acts faithfully on the Bruhat–Tits tree $\mathcal{T}_{p+1}$ associated to $\mathrm{PSL}(2, \mathbb{Q}_p)$. Furthermore, the action of Γ inherits also the property that $\mathrm{PSL}(2, \mathbb{Q}_p)$ acts without inversion on $\mathcal{T}_{p+1}$ and with an edge as fundamental domain. Thus Γ is the amalgamated product $\mathbb{F}_a *_{\mathbb{F}_c} \mathbb{F}_b$ of two non-Abelian free groups over a common finite index subgroup. It follows from [15, Theorem 1.1] that $\mathrm{H}^2_{\mathrm{b}}(\Gamma)$ is finite dimensional, while $\mathrm{H}^2_{\mathrm{b}}(\mathbb{F}_a)$ is infinite dimensional.

Our construction of the map Θ in Theorem 1.1 relies on the analysis of the action of Γ on its Bass–Serre tree, which allows us to define a projection from combinatorial simplices in Γ to simplices with values in the vertex groups. Our construction is inspired by [24, p. 54] and exploits the approach to bounded cohomology developed by Ivanov [25], Burger and Monod [16, 38].

Surprisingly enough, the proof of Theorem 1.1 runs into additional difficulties in the case of degree 2. In that case, even to define the map Θ, it is necessary to use the fact that bounded cohomology can be computed via the complex of pluriharmonic functions [15], and that such a realization has no coboundaries in degree 2 due to the double ergodicity of the action of a group on an appropriate Poisson boundary [28, 16].

A simple example of a situation to which Theorem 1.1 applies is the one in which G consists only of one edge e with vertices v and w. In this case, we can realize $\Gamma_v *_{\Gamma_e} \Gamma_w$ as the fundamental group of a space X that can be decomposed as $X = X_v \cup X_w$, where $X_v \cap X_w$ has amenable fundamental group. A fundamental result by Gromov implies that the bounded cohomology of a CW-complex[a] is isometrically isomorphic to the bounded cohomology of its fundamental group [24, p. 49]. Using this, Theorem 1.1 specializes to the statement that there is an isometric embedding

$$\mathrm{H}^n_{\mathrm{b}}(X_v) \oplus \mathrm{H}^n_{\mathrm{b}}(X_w) \hookrightarrow \mathrm{H}^n_{\mathrm{b}}(X)$$

[a]See [13] for a more general version for all path connected spaces.

that is a right inverse to the restriction map. This forces classes in the image of the map to have some compatibility condition on $X_v \cap X_w$ and leads naturally to considering the bounded cohomology of X_v and X_w relative to $X_v \cap X_w$.

To this purpose, let (X, Y) be a pair of countable CW-complexes, and denote by $j^n : \mathrm{C}_\mathrm{b}^n(X, Y) \to \mathrm{C}_\mathrm{b}^n(X)$ the inclusion of relative bounded cochains into bounded cochains.

Theorem 1.2. *Let $X \supseteq Y$ be a pair of countable CW-complexes. Assume that each connected component of Y has amenable fundamental group. Then the map*

$$\mathrm{H}(j^n) : \mathrm{H}_\mathrm{b}^n(X, Y) \longrightarrow \mathrm{H}_\mathrm{b}^n(X)$$

is an isometric isomorphism for every $n \geq 2$.

The amenability of $\pi_1(Y)$ insures immediately, using the long exact sequence in relative bounded cohomology, the isomorphism of $\mathrm{H}_\mathrm{b}^n(X, Y)$ and $\mathrm{H}_\mathrm{b}^n(X)$, but the fact that this isomorphism is isometric is, to our knowledge, not contained in Gromov's paper and requires a proof. This result was obtained independently by Kim and Kuessner [29], using the rather technical theory of multicomplexes. Our proof of Theorem 1.2 uses instead in a crucial way the construction of an amenable $\pi_1(X)$-space thought of as a discrete approximation of the pair $(\widetilde{X}, p^{-1}(Y))$, where $p : \widetilde{X} \to X$ is a universal covering. The same technique is at the basis of the proof of Theorem 1.1.

Applications. In the second part of the paper we show how Theorems 1.1 and 1.2 can be used to provide simple, self-contained proofs of two theorems in bounded cohomology due to Gromov and some new consequences. The proofs of Gromov's results available in the literature rely on the theory of multicomplexes [24, 32].

The first of our applications is Gromov's additivity theorem for the simplicial volume, from which we deduce the behavior of the simplicial volume under generalized Dehn fillings, thus generalizing a result of Fujiwara and Manning. We then establish Gromov's Equivalence Theorem, which states that various seminorms on the relative homology of a pair (X, Y) actually coincide, provided that the fundamental group of every component of Y is amenable. Moreover, we give an ℓ^1-homology version of Theorem 1.2 due to Thurston.

Additivity of the simplicial volume. The simplicial volume is a homotopy invariant of manifolds introduced by Gromov in his seminal paper [24]. If M is a connected, compact and oriented manifold with (possibly empty) boundary, then the simplicial volume of M is equal to the ℓ^1-seminorm of the fundamental class of M (see Sec. 6 for the precise definition). It is usually denoted by $\|M\|$ if M is closed, and by $\|M, \partial M\|$ if $\partial M \neq \emptyset$. The simplicial volume may also be defined in the context of open manifolds [24], but in this paper we will restrict our attention to compact ones. More precisely, unless otherwise stated, every manifold will be assumed to be connected, compact and oriented.

The explicit computation of nonvanishing simplicial volume is only known for complete finite-volume hyperbolic manifolds (see [24, 50] for the closed case and e.g. [20, 21, 23, 11] for the cusped case) and for manifolds locally isometric to the product of two hyperbolic planes [10] (see also [34,12] for the non-compact case with amenable cusp groups). Gromov's Additivity Theorem can be used to establish more computations of the simplicial volume by taking connected sums or gluings along π_1-injective boundary components with amenable fundamental group. For example the simplicial volume of a closed 3-manifold M equals the sum of the simplicial volumes of its hyperbolic pieces [46].

Furthermore, without aiming at being exhaustive, here we just mention that Gromov Additivity Theorem has also been exploited in studying the possible degrees of maps between manifolds [43,51,17,8,18,19], in establishing results about the behavior of manifolds under collapse [7, 5], and in various other areas of low-dimensional topology [1, 39, 6, 4, 26, 49, 35, 31, 3].

Theorem 1.3. (Gromov Additivity Theorem) *Let $M_1, \ldots, M_k$ be n-dimensional manifolds, $n \geq 2$, suppose that the fundamental group of every boundary component of every M_j is amenable, and let M be the manifold obtained by gluing $M_1, \ldots, M_k$ along (some of) their boundary components. Then*

$$\|M, \partial M\| \leq \|M_1, \partial M_1\| + \cdots + \|M_k, \partial M_k\|.$$

In addition, if the gluings defining M are compatible, then

$$\|M, \partial M\| = \|M_1, \partial M_1\| + \cdots + \|M_k, \partial M_k\|.$$

Here a gluing $f : S_1 \to S_2$ of two boundary components $S_i \subseteq \partial M_{j_i}$ is called *compatible* if $f_*(K_1) = K_2$ where K_i is the kernel of the map $\pi_1(S_i) \to \pi_1(M_{j_i})$ induced by the inclusion.

An immediate consequence of this theorem is the fact that the simplicial volume is additive with respect to connected sums: given two n-dimensional manifolds M_1, M_2, if $n \geq 3$ and the fundamental group of every boundary component of M_i is amenable, then

$$\|M_1 \# M_2, \partial(M_1 \# M_2)\| = \|M_1, \partial M_1\| + \|M_2, \partial M_2\|,$$

where $M_1 \# M_2$ is constructed by removing an open ball from the interior of M_i and gluing the obtained manifolds along the boundary spheres.

According to the preprint [32], Theorem 1.3 holds even if the amenability of the fundamental group is required only for those boundary components of the M_j that are indeed glued in M (and not for the ones still appearing in ∂M). Unfortunately, our argument does not apply to this more general case. In fact, if N is a compact n-manifold with boundary, then the bounded cohomology modules $H_b^n(N, \partial N)$ and $H_b^n(N)$ are not isomorphic in general. In order to circumvent this difficulty, one should define the bounded cohomology of a group relative to a family of subgroups, and prove that the relative bounded cohomology of a pair of spaces is isometrically isomorphic to the corresponding relative bounded group cohomology. However, this approach seems to run into several technical difficulties (see e.g. [22, p. 95, Theorem 1.8 and Remark 4.9] for a discussion of this issue).

Generalized Dehn fillings. A consequence of the first part of Theorem 1.3 is an easy proof of a result of Fujiwara and Manning [23] about generalized Dehn fillings. Let $n \geq 3$ and let M be a compact orientable n-manifold such that $\partial M = N_1 \cup \cdots \cup N_m$, where N_i is an $(n-1)$-torus for every i. For each $i \in \{1, \ldots, m\}$ we put on N_i a flat structure, and we choose a totally geodesic k_i-dimensional torus $T_i \subseteq N_i$, where $1 \leq k_i \leq n-2$. Each N_i is foliated by parallel copies of T_i with leaf space L_i homeomorphic to an $(n-1-k_i)$-dimensional torus. The *generalized Dehn filling* $R = M(T_1, \ldots, T_m)$ is defined as the quotient of M obtained by collapsing N_i on L_i for every $i \in \{1, \ldots, m\}$. Observe that unless $k_i = 1$ for every i, the quotient R is not a manifold. However, as observed in [23, p. 2240], R is always a pseudomanifold in the sense of [52, Definition 2.1], so it admits a fundamental class, whence a well-defined simplicial volume [52, Proposition 2.2]. Fujiwara and Manning proved that, if the interior of M admits a complete finite-volume hyperbolic structure, then the inequality $\|R\| \leq \|M, \partial M\|$ holds. Their argument easily extends to the case in which the fundamental group of M is residually finite and the inclusion of each boundary torus in M induces an injective map on fundamental groups. Our proof of Theorem 1.3 works verbatim when each M_i is just a pseudomanifold, so we obtain the following generalization of Fujiwara and Manning's result:

Corollary 1.4. *Let M be a compact orientable n-manifold with boundary given by a union of tori, and let R be a generalized Dehn filling of M. Then*

$$\|R\| \leq \|M, \partial M\|.$$

Equivalence of Gromov norms. In [24] Gromov introduced a one-parameter family of seminorms on $\mathrm{H}_n(X, Y)$. More precisely, let $\theta \in [0, \infty)$ and consider the norm $\|\cdot\|_1(\theta)$ on $C_n(X)$ defined by $\|c\|_1(\theta) = \|c\|_1 + \theta \|\partial_n c\|_1$. Every such norm is equivalent to the usual norm $\|\cdot\|_1 = \|\cdot\|_1(0)$ for every $\theta \in [0, \infty)$ and induces a quotient seminorm on relative homology, still denoted by $\|\cdot\|_1(\theta)$. Since $\|\cdot\|_1(\theta)$ is increasing as a function of θ, by passing to the limit one can also define a seminorm $\|\cdot\|_1(\infty)$ that, however, may be nonequivalent to $\|\cdot\|_1$ (in fact, $\|\cdot\|_1(\infty)$ may even have values in $[0, +\infty]$). The following result is stated by Gromov in [24].

Theorem 1.5. (Equivalence Theorem, [24, p. 57]) *Let $X \supseteq Y$ be a pair of countable CW-complexes, and let $n \geq 2$. If the fundamental groups of all connected components of Y are amenable, then the seminorms $\|\cdot\|_1(\theta)$ on $\mathrm{H}_n(X, Y)$ coincide for every $\theta \in [0, \infty]$.*

In order to prove Theorem 1.5, we establish two isometric isomorphisms of independent interest (see Lemma 5.1 and Proposition 5.3), using the homological construction of a mapping cone complex and considering a one-parameter family of seminorms in bounded cohomology introduced by Park [41].

As noticed by Gromov, Theorem 1.5 admits the following equivalent formulation, which is inspired by Thurston [50, Sec. 6.5] and plays an important role in several results about the (relative) simplicial volumes of gluings and fillings:

Corollary 1.6. *Let* $X \supseteq Y$ *be a pair of countable CW-complexes, and suppose that the fundamental groups of all the components of* Y *are amenable. Let* $\alpha \in \mathrm{H}_n(X,Y)$, $n \geq 2$. *Then, for every* $\epsilon > 0$, *there exists an element* $c \in \mathrm{C}_n(X)$ *with* $\partial_n c \in \mathrm{C}_{n-1}(Y)$ *such that* $[c] = \alpha \in \mathrm{H}_n(X,Y)$, $\|c\|_1 < \|\alpha\|_1 + \epsilon$ *and* $\|\partial_n c\|_1 < \epsilon$.

Proof. Let $\theta = (\|\alpha\|_1 + \epsilon)/\epsilon$. By Theorem 1.5 we know that $\|\cdot\|_1(\theta)$ induces the norm $\|\cdot\|_1$ in homology, so we can find a representative $c \in \mathrm{C}_n(X)$ of α with $\|c\|_1(\theta) = \|c\|_1 + \theta\|\partial_n c\|_1 \leq \|\alpha\|_1 + \epsilon$. This implies that $\|c\|_1 \leq \|\alpha\|_1 + \epsilon$ and $\|\partial_n c\|_1 \leq (\|\alpha\|_1 + \epsilon)/\theta = \epsilon$. □

2. Resolutions in Bounded Cohomology

This section is devoted to recalling some results on bounded cohomology to be used in the proof of Theorems 1.1 and 1.2. Let X be a space, where here and in the sequel by a space we will always mean a countable CW-complex. We denote by $\mathrm{C}_\mathrm{b}^\bullet(X)$ the complex of bounded real valued singular cochains on X and, if $Y \subset X$ is a subspace, by $\mathrm{C}_\mathrm{b}^\bullet(X,Y)$ the subcomplex of those bounded cochains that vanish on simplices with image contained in Y. All these spaces are endowed with the ℓ^∞-norm and the corresponding cohomology groups are equipped with the corresponding quotient seminorm.

For our purposes, it is important to observe that the universal covering map $p : \widetilde{X} \to X$ induces an isometric identification of the complex $\mathrm{C}_\mathrm{b}^\bullet(X)$ with the complex $\mathrm{C}_\mathrm{b}^\bullet(\widetilde{X})^\Gamma$ of $\Gamma := \pi_1(X)$-invariant bounded cochains on $\widetilde{X}$. Similarly, if $Y' := p^{-1}(Y)$, we obtain an isometric identification of the complex $\mathrm{C}_\mathrm{b}^\bullet(X,Y)$ with the complex $\mathrm{C}_\mathrm{b}^\bullet(\widetilde{X},Y')^\Gamma$ of Γ-invariants of $\mathrm{C}_\mathrm{b}^\bullet(\widetilde{X},Y')$.

The main ingredient in the proof of Theorem 1.2 is the result of Ivanov that the complex of Γ-invariants of

$$0 \longrightarrow \mathrm{C}_\mathrm{b}^0(\widetilde{X}) \longrightarrow \mathrm{C}_\mathrm{b}^1(\widetilde{X}) \longrightarrow \cdots$$

computes the bounded cohomology of Γ (as $\mathrm{C}_\mathrm{b}^\bullet(\widetilde{X})^\Gamma$ coincides with $\mathrm{C}_\mathrm{b}^\bullet(X)$, this implies in particular that the bounded cohomology of X is isometrically isomorphic to the bounded cohomology of Γ). In fact, we will use the more precise statement that the obvious augmentation of the complex above is a strong resolution of $\mathbb{R}$ by relatively injective Banach Γ-modules (see [25, proof of Theorem 4.1]). We refer the reader respectively to [25, Sec. 3.2] and [25, Sec. 3.3] for the definitions of relatively injective module and of strong resolution.

By [25, Theorem 3.6], it follows from the fact that $\mathrm{C}^n_{\mathrm{b}}(\widetilde{X})$ is a strong resolution by Γ-modules that there exists a Γ-morphism of complexes

$$g^n : \mathrm{C}^n_{\mathrm{b}}(\widetilde{X}) \longrightarrow \ell^\infty(\Gamma^{n+1}) \qquad (\lozenge)$$

extending the identity of $\mathbb{R}$, and such that g^n is norm nonincreasing, i.e. $\|g^n\| \leq 1$, for $n \geq 0$. This map induces the isometric isomorphism $\mathrm{H}^\bullet_{\mathrm{b}}(X) \to \mathrm{H}^\bullet_{\mathrm{b}}(\Gamma)$ (see [25, Theorem 4.1]), and will be referred to as Ivanov's map.

The second result we need lies at the basis of the fact that the bounded cohomology of Γ can be computed isometrically from the complex of bounded functions on any amenable Γ-space. The notion of amenable space was introduced by Zimmer [53] in the context of actions of topological groups on standard measure spaces (see e.g. [38, Sec. 5.3] for several equivalent definitions). In our case of interest, i.e. when Γ is a discrete countable group acting on a countable set S (which may be thought as endowed with the discrete topology), the amenability of S as a Γ-space amounts to the amenability of the stabilizers in Γ of elements of S [2, Theorem 5.1]. Recall that, if Γ acts on a set S, then a map $f : S^{n+1} \to \mathbb{R}$ is *alternating* if

$$f(s_{\sigma(0)}, \ldots, s_{\sigma(n)}) = \varepsilon(\sigma) \cdot f(s_0, \ldots, s_n)$$

for every $(s_0, \ldots, s_n) \in S^{n+1}$ and every permutation σ of $\{0, \ldots, n\}$, where $\varepsilon(\sigma) = \pm 1$ is the sign of σ. We denote by $\ell^\infty_{\mathrm{alt}}(S^{\bullet+1})$ the complex of alternating bounded functions on $S^{\bullet+1}$.

Proposition 2.1. *Let S be an amenable Γ-set, where Γ is a discrete countable group. Then*:

(1) *There exists a Γ-morphism of complexes*

$$\mu^\bullet : \ell^\infty(\Gamma^{\bullet+1}) \longrightarrow \ell^\infty_{\mathrm{alt}}(S^{\bullet+1})$$

extending $\mathrm{Id}_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ that is norm nonincreasing in every degree.

(2) *The cohomology of the complex*

$$0 \longrightarrow \ell^\infty_{\mathrm{alt}}(S)^\Gamma \longrightarrow \ell^\infty_{\mathrm{alt}}(S^2)^\Gamma \longrightarrow \ell^\infty_{\mathrm{alt}}(S^3)^\Gamma \longrightarrow \cdots$$

is canonically isometrically isomorphic to $\mathrm{H}^\bullet_{\mathrm{b}}(\Gamma)$.

Proof. Point (1) is proved in [38, Lemma 7.5.6] (applied to the case $T = \Gamma$), point (2) in [38, Theorem 7.5.3].

Perhaps it is worth mentioning that, in the particular case at hand, the map $\mu^\bullet$ admits the following easy description. Since alternation gives a contracting Γ-morphism of complexes, it suffices to construct $\mu^n : \ell^\infty(\Gamma^{n+1}) \to \ell^\infty(S^{n+1})$. Let us fix the obvious componentwise action of Γ^{n+1} on S^{n+1}. Since S is an amenable Γ-space, for every $s \in S$ we may fix a mean μ_s on the stabilizer Γ_s of s. Let f be a bounded function on Γ^{n+1}, and let us consider an orbit $\Gamma^{n+1} \cdot \overline{s}_0 \subseteq S^{n+1}$, where $\overline{s}_0 = (s_0, \ldots, s_n)$ is an element of S^{n+1}. For every $\overline{s} \in \Gamma^{n+1} \cdot \overline{s}_0$, the set of elements of Γ^{n+1} taking $\overline{s}_0$ to $\overline{s}$ is a left coset $g_{\overline{s}}\Gamma_0$ of the stabilizer Γ_0 of $\overline{s}_0$ in Γ^{n+1}. Being the finite product of amenable groups, Γ_0 is amenable, and the product $\mu = \mu_{s_0} \otimes \cdots \otimes \mu_{s_n}$ is a mean on Γ_0. We define $\mu^n(f)(\overline{s})$ as the average of f on $g_{\overline{s}}\Gamma_0$ with respect to μ. We have thus defined $\mu^n(f)$ on every orbit, whence on the whole of S^{n+1}, and this concludes the construction of μ^n. $\square$

We point out that the computation of bounded cohomology via alternating cochains on amenable spaces is natural in the following sense:

Lemma 2.2. *Let $i : \Gamma_1 \to \Gamma$ be an inclusion of countable groups, let S_1 be a discrete amenable Γ_1-space, and S a discrete amenable Γ-space. If $\varphi : S_1 \to S$ is equivariant with respect to i, then the following diagram commutes*:

$$\begin{array}{ccc} \mathcal{Z}\ell^\infty_{\mathrm{alt}}(S^{\bullet+1})^\Gamma & \longrightarrow & \mathrm{H}^\bullet_{\mathrm{b}}(\Gamma) \\ \Big\downarrow{\scriptstyle \varphi^*} & & \Big\downarrow{\scriptstyle i^*} \\ \mathcal{Z}\ell^\infty_{\mathrm{alt}}(S_1^{\bullet+1})^{\Gamma_1} & \longrightarrow & \mathrm{H}^\bullet_{\mathrm{b}}(\Gamma_1). \end{array}$$

The third and last ingredient we need is a result from [15] where it is shown that the bounded cohomology of Γ is realized by yet another complex, namely the resolution via μ-pluriharmonic functions.

Let μ be a symmetric probability measure on Γ and denote by $\ell^\infty_{\mu,\mathrm{alt}}(\Gamma^{n+1})$ the subcomplex of $\ell^\infty_{\mathrm{alt}}(\Gamma^{n+1})$ consisting of μ-pluriharmonic functions on Γ^{n+1}, i.e. of elements $f \in \ell^\infty_{\mathrm{alt}}(\Gamma^{n+1})$ such that

$$f(g_0, \dots, g_n) = \int_{\Gamma^{n+1}} f(g_0\gamma_0, \dots, g_n\gamma_n) d\mu(\gamma_0) \dots d\mu(\gamma_n)$$

for every $(g_0, \dots, g_n) \in \Gamma^{n+1}$. By [15, Lemma 3.13], the inclusion $\ell^\infty_{\mu,\mathrm{alt}}(\Gamma^\bullet) \hookrightarrow \ell^\infty_{\mathrm{alt}}(\Gamma^\bullet)$ induces isometric isomorphisms in cohomology.

Moreover, if (B, ν) is the Poisson boundary of (Γ, μ), it is proven in [15, Proposition 3.11] that the Poisson transform

$$\mathcal{P} : L^\infty_{\mathrm{alt}}(B^{n+1}, \nu^{\otimes n+1}) \to \ell^\infty_{\mu,\mathrm{alt}}(\Gamma^{n+1})$$

$$\mathcal{P}(f)(g_0, \dots, g_n) = \int_{B^{n+1}} f(g_0\xi_0, \dots, g_n\xi_n) d\nu(\xi_0) \dots d\nu(\xi_n)$$

is a Γ-equivariant isometric isomorphism.

The main theorem of [28] (see also [16, Theorem 0.2] and [14, Proposition 4.2] for the case of finitely generated groups) implies that, if the support of μ generates Γ, then the action of Γ on B is doubly ergodic, in particular

$$\ell^\infty_{\mu,\mathrm{alt}}(\Gamma^2)^\Gamma = L^\infty_{\mathrm{alt}}(B^2, \nu^{\otimes 2})^\Gamma = 0,$$

and the projection of $\mathcal{Z}(\ell^\infty_{\mathrm{alt}}(\Gamma^3)^\Gamma)$ onto $\mathrm{H}^2_{\mathrm{b}}(\Gamma)$ restricts to an isometric isomorphism between the space of Γ-invariant μ-pluriharmonic alternating cocycles $\mathcal{Z}\ell^\infty_{\mu,\mathrm{alt}}(\Gamma^3)^\Gamma$ and the second bounded cohomology module of Γ. This implies

Proposition 2.3. *Let Γ be a countable group and μ a symmetric probability measure whose support generates Γ. Then there is an isometric linear section*

$$\sigma : \mathrm{H}^2_{\mathrm{b}}(\Gamma) \to \mathcal{Z}\ell^\infty(\Gamma^3)^\Gamma.$$

of the projection defining bounded cohomology.

3. Relative Bounded Cohomology: Proof of Theorem 1.2

Let (X, Y) be a pair of countable CW-spaces. Assume that X is connected and the fundamental group of every component of Y is amenable. Let $p : \widetilde{X} \to X$ be the universal covering map, set $\Gamma := \pi_1(X)$ and let $Y = \bigsqcup_{i \in I} C_i$ be the decomposition of Y into the union of its connected components. If $\check{C}_i$ is a choice of a connected component of $p^{-1}(C_i)$ and Γ_i denotes the stabilizer of $\check{C}_i$ in Γ, then

$$p^{-1}(C_i) = \bigsqcup_{\gamma \in \Gamma/\Gamma_i} \gamma \check{C}_i.$$

The group Γ acts by left translations on the set

$$S := \Gamma \sqcup \bigsqcup_{i \in I} \Gamma/\Gamma_i.$$

Being a quotient of $\pi_1(C_i)$, the group Γ_i is amenable, so S is an amenable Γ-space. We define a Γ-equivariant measurable retraction $r : \widetilde{X} \to S$ as follows: let $\mathcal{F} \subset \widetilde{X} \smallsetminus Y'$ be a fundamental domain for the Γ-action on $\widetilde{X} \smallsetminus Y'$, where $Y' = p^{-1}(Y)$. Define the map r as follows:

$$r(\gamma x) := \begin{cases} \gamma \in \Gamma & \text{if } x \in \mathcal{F}, \\ \gamma\Gamma_i \in \Gamma/\Gamma_i & \text{if } x \in \check{C}_i. \end{cases}$$

For every $n \geq 0$ define

$$r^n : \ell^\infty_{\mathrm{alt}}(S^{n+1}) \longrightarrow \mathrm{C}^n_{\mathrm{b}}(\widetilde{X})$$

by

$$r^n(c)(\sigma) = c(r(\sigma_0), \ldots, r(\sigma_n)),$$

where $c \in \ell^\infty_{\mathrm{alt}}(S^{n+1})$ and $\sigma_0, \ldots, \sigma_n \in \widetilde{X}$ are the vertices of a singular simplex $\sigma : \Delta^n \to \widetilde{X}$. Clearly $(r^n)_{n \geq 0}$ is a Γ-morphism of complexes extending the identity of $\mathbb{R}$ and $\|r^n\| \leq 1$ for all $n \geq 0$.

Observe that if $n \geq 1$ and $\sigma(\Delta^n) \subset Y'$, then there are $i \in I$ and $\gamma \in \Gamma$ such that $\sigma(\Delta^n) \subset \gamma \check{C}_i$. Thus

$$r(\sigma_0) = \cdots = r(\sigma_n) = \gamma\Gamma_i$$

and

$$r^n(c)(\sigma) = c(\gamma\Gamma_i, \ldots, \gamma\Gamma_i) = 0,$$

since c is alternating. This implies that the image of r^n is in $\mathrm{C}_\mathrm{b}^n(\widetilde{X}, Y')$. Thus we can write $r^n = j^n \circ r_1^n$, where $j^n : \mathrm{C}_\mathrm{b}^n(\widetilde{X}, Y') \hookrightarrow \mathrm{C}_\mathrm{b}^n(\widetilde{X})$ is the inclusion and $r_1^n : \ell^\infty_{\mathrm{alt}}(S^{n+1}) \to \mathrm{C}_\mathrm{b}^n(\widetilde{X}, Y')$ is a norm nonincreasing Γ-morphism that induces a norm nonincreasing map in cohomology

$$\mathrm{H}(r_1^n) : \mathrm{H}^n(\ell^\infty_{\mathrm{alt}}(S^{\bullet+1})^\Gamma) \longrightarrow \mathrm{H}_\mathrm{b}^n(X, Y),$$

for $n \geq 1$.

Using the map g^n defined in ($\Diamond$) and the map μ^n provided by Proposition 2.1, we have the following diagram

$$\begin{tikzcd}
\mathrm{C}_\mathrm{b}^n(\widetilde{X}) \arrow[r, "g^n"] \arrow[drrr, dashed, "\text{extends } \mathrm{Id}_\mathbb{R}"'] & \ell^\infty(\Gamma^{n+1}) \arrow[r, "\mu^n"] & \ell^\infty_{\mathrm{alt}}(S^{n+1}) \arrow[r, "r_1^n", "\text{for } n\geq 1"'] \arrow[dr, "r^n"] & \mathrm{C}_\mathrm{b}^n(\widetilde{X}, Y') \arrow[d, "j^n"] \\
 & & & \mathrm{C}_\mathrm{b}^n(\widetilde{X}),
\end{tikzcd}$$

where the dashed map is the composition $r^n \circ \mu^n \circ g^n$ which is a Γ-morphism of strong resolutions by relatively injective modules extending the identity, and hence induces the identity on $\mathrm{H}_\mathrm{b}^n(X) = \mathrm{H}^n(\mathrm{C}_\mathrm{b}^\bullet(\widetilde{X})^\Gamma)$.

We proceed now to show that, for $n \geq 2$, the map

$$\mathrm{H}(j^n) : \mathrm{H}_\mathrm{b}^n(X, Y) \longrightarrow \mathrm{H}_\mathrm{b}^n(X)$$

induced by j^n is an isometric isomorphism in cohomology. In view of the long exact sequence for pairs in bounded cohomology and the fact that $\mathrm{H}_\mathrm{b}^\bullet(Y) = 0$ in positive degree, we already know that $\mathrm{H}(j^n)$ is an isomorphism. Let us set $\psi^n = r_1^n \circ \mu^n \circ g^n$. From the above we have

$$\mathrm{H}(j^n) \circ \mathrm{H}(\psi^n) = \mathrm{Id}_{\mathrm{H}_\mathrm{b}^n(X)}.$$

The conclusion follows from the fact that the maps $\mathrm{H}(j^n)$ and $\mathrm{H}(\psi^n)$ are norm nonincreasing.

4. Graphs of Groups: Proof of Theorem 1.1

In order to fix the notation, we recall some definitions concerning graphs of groups, closely following [44]. A graph G is a pair $(V(G), E(G))$ together with a map $E(G) \to V(G)^2$, $e \mapsto (o(e), t(e))$ and a fixed point free involution $e \mapsto \bar{e}$ of $E(G)$ satisfying $o(e) = t(\bar{e})$. The set $\bar{E}(G)$ of geometric edges of G is defined by setting

$\bar{E}(G) = \{\{e, \bar{e}\} \mid e \in E(G)\}$. The geometric realization $|G|$ of a graph G is the 1-dimensional CW-complex with one vertex for every element in $V(G)$ and one edge for every geometric edge. Its first baricentric subdivision G' has as vertices the set $V(G') = V(G) \sqcup \bar{E}(G)$.

Let $\mathcal{G}$ be a graph of groups based on the finite graph G. Recall that to every vertex $v \in V(G)$ is associated a group Γ_v and to every edge $e \in E(G)$ is associated a group Γ_e together with an injective homomorphism $h_e : \Gamma_e \to \Gamma_{t(e)}$. Moreover, it is required that $\Gamma_e = \Gamma_{\bar{e}}$. Let $\Gamma = \pi_1(\mathcal{G})$ denote the fundamental group of $\mathcal{G}$. By the universal property of the fundamental group of a graph of groups [44, Corollary 1, p. 45], for every $v \in V(G)$, $e \in E(G)$, there exist inclusions $\Gamma_v \to \Gamma$ and $\Gamma_e \to \Gamma$. Henceforth we will regard each Γ_v and each Γ_e just as a subgroup of Γ. Observe that, since $\Gamma_e = \Gamma_{\bar{e}}$, it makes sense to speak about the subgroup Γ_e also for $e \in \bar{E}(G)$.

A fundamental result in Bass–Serre theory [44, Theorem 12, p. 52] implies that Γ acts simplicially on a tree $T = (V(T), E(T))$, where

$$V(T) = \bigsqcup_{v \in V(G)} \Gamma/\Gamma_v, \quad E(T) = \bigsqcup_{e \in \bar{E}(G)} \Gamma/\Gamma_e.$$

The action of Γ on $V(T)$ and $E(T)$ is by left multiplication. The tree T is known as the *Bass–Serre tree* of $\mathcal{G}$ (or of Γ, when the presentation of Γ as the fundamental group of a graph of group is understood). There is an obvious projection $V(T) \to V(G)$ which sends the whole of Γ/Γ_v to v. This projection admits a preferred section that takes any vertex $v \in V(G)$ to the coset $1 \cdot \Gamma_v \in \Gamma/\Gamma_v$. This allows us to canonically identify $V(G)$ with a subset of $V(T)$.

Now we consider the space

$$S_{\mathcal{G}} = (\Gamma \times V(G)) \sqcup \bigsqcup_{e \in \bar{E}(G)} \Gamma/\Gamma_e.$$

We may define an action of Γ on $S_{\mathcal{G}}$ by setting $g_0 \cdot (g, v) = (g_0 g, v)$ for every $(g, v) \in \Gamma \times V(G)$ and $g_0 \cdot (g\Gamma_e) = (g_0 g)\Gamma_e$ for every $g\Gamma_e \in \Gamma/\Gamma_e$, $e \in \bar{E}(G)$.

There exists a Γ-equivariant projection $p : S_{\mathcal{G}} \to V(T')$ defined as follows: $p(g, v) = g\Gamma_v$ for $(g, v) \in \Gamma \times V(G)$, and p is the identity on each Γ/Γ_e, $e \in \bar{E}(G)$.

Let us now suppose that our graph of groups $\mathcal{G}$ satisfies the hypothesis of Theorem 1.1, i.e. every Γ_v is countable and every Γ_e is amenable. Under this assumption, both Γ and $S_{\mathcal{G}}$ are countable, and Γ acts on $S_{\mathcal{G}}$ with amenable stabilizers. As a consequence of Proposition 2.1, the bounded cohomology of Γ can be isometrically computed from the complex $\ell^\infty_{\mathrm{alt}}(S_{\mathcal{G}}^{\bullet+1})$.

For every vertex $v \in V(G)$, let S_v be the set

$$S_v = \Gamma_v \sqcup \bigsqcup_{t(e)=v} \Gamma_v/\Gamma_e,$$

where we identify Γ_e with a subgroup of Γ_v via the map h_e. We have an obvious action of Γ_v on S_v by left multiplication. Since every Γ_e is amenable, this action turns S_v into an amenable Γ_v-space.

The inclusion $\varphi_v : S_v \to S_{\mathcal{G}}$ defined by $\varphi_v(g) = (g, v)$ and $\varphi_v(g\Gamma_e) = g\Gamma_e$, induces a chain map

$$\varphi_v^\bullet : \ell^\infty_{\mathrm{alt}}(S_{\mathcal{G}}^{\bullet+1}) \to \ell^\infty_{\mathrm{alt}}(S_v^{\bullet+1}).$$

By construction, $\varphi_v^\bullet$ is equivariant with respect to the inclusion $\Gamma_v \to \Gamma$, so Lemma 2.2 implies that $\varphi_v^\bullet$ induces the restriction map in bounded cohomology.

The following result establishes the existence of a partial retraction of the chain map $\varphi^\bullet = \bigoplus_{v\in V(G)} \varphi_v^\bullet$, and plays a fundamental role in the proof of Theorem 1.1.

Theorem 4.1. *There is a (partial) norm nonincreasing chain map*

$$\psi^n : \bigoplus_{v\in V(G)} \ell^\infty_{\mathrm{alt}}(S_v^{n+1})^{\Gamma_v} \to \ell^\infty_{\mathrm{alt}}(S_{\mathcal{G}}^{n+1})^{\Gamma}, \quad n \geq 2$$

such that the composition $\varphi^n \circ \psi^n$ is the identity of $\bigoplus_{v\in V(G)} \ell^\infty_{\mathrm{alt}}(S_v^{n+1})^{\Gamma_v}$ for every $n \geq 2$.

Proof. To define the map ψ^n we need the notion of a *barycenter* of an $(n+1)$-tuple $(y_0, \ldots, y_n)$ in $V(T')^{n+1}$. Given a vertex $v \in V(T')$, let $N(v) \subseteq V(T')$ be the set of vertices having combinatorial distance (in T') at most one from v. The vertex $\bar{y} \in V(T) \subseteq V(T')$ is a barycenter of $(y_0, \ldots, y_n) \in V(T')^{n+1}$ if for any y_i, y_j in $V(T')\backslash\{\bar{y}\}$, $i \neq j$, the points y_i and y_j belong to different connected components of $|T'|\backslash\{\bar{y}\}$. It follows readily from the definitions that there exists at most one barycenter for any n-tuple provided that $n \geq 3$.

Let $p : S_{\mathcal{G}} \to V(T')$ be the projection defined above. For $v \in V(G)$, let us identify S_v with $\varphi_v(S_v) \subseteq S_{\mathcal{G}}$, and recall that $V(G)$ is canonically identified with a subset of $V(T) \subseteq V(T')$. Under these identifications we have $S_v = p^{-1}(N(v))$ for every $v \in V(G)$, and we coherently set $S_w = p^{-1}(N(w)) \subseteq S_{\mathcal{G}}$ for every $w \in V(T)$.

Let us fix $w \in V(T)$. We define a retraction $r_w^0 : S_{\mathcal{G}} \to S_w$ as follows: if $x_0 \in S_w$, then $r_w^0(x_0) = x_0$; otherwise, if y_0 is the endpoint of the first edge of the combinatorial path $[w, p(x_0)]$ in T', then $r_w^0(x_0)$ is the unique preimage of y_0 via p. We extend r_w^0 to a chain map $r_w^\bullet : S_{\mathcal{G}}^{\bullet+1} \to S_w^{\bullet+1}$ by setting $r_w^n(x) = (r_w^0(x_0), \ldots, r_w^0(x_n))$ for $x = (x_0, \ldots, x_n)$. Notice that if w is not a barycenter of $(p(x_0), \ldots, p(x_n))$, then the $(n+1)$-tuple $r_w^n(x)$ has at least two coordinates that are equal, so any alternating cochain vanishes on $r_w^n(x)$.

We are now ready to define the (partial) chain map $\psi^\bullet$. Recall that every vertex $w \in V(T)$ is a coset in Γ/Γ_v for some $v \in V(G)$. For every $w \in V(T)$ we choose a representative $\sigma(w) \in \Gamma$ of w, and we observe that $\sigma(w)^{-1}w \in V(G) \subseteq V(T)$. Let $x \in S_{\mathcal{G}}^{n+1}$, $n \geq 2$. We have $\sigma(w)^{-1}r_w^n(x) \in S^{n+1}_{\sigma(w)^{-1}w}$, so for every $(\bigoplus_{v\in V(G)} f_v) \in \bigoplus_{v\in V(G)} \ell^\infty_{\mathrm{alt}}(S_v^{n+1})$ it makes sense to set

$$\psi^n\left(\bigoplus_{v\in V(G)} f_v\right)(x) = \sum_{w\in V(T)} f_{\sigma(w)^{-1}w}(\sigma(w)^{-1}r_w^n(x)).$$

Since the f_v are alternating there is at most one nonzero term in the sum, corresponding to the barycenter (if any) of $(p(x_0), \ldots, p(x_n))$. Moreover ψ^n, $n \geq 2$, is a (partial) chain map and it is easy to check that $\psi^n(\bigoplus_{v\in V(G)} f_v)$ is Γ-invariant provided that f_v is Γ_v-invariant for every $v \in V(G)$. $\square$

We are now ready to finish the proof of Theorem 1.1.

Proof of Theorem 1.1. Since the first bounded cohomology of any group vanishes in degree one, it is sufficient to consider the case $n \geq 2$. Being a norm nonincreasing chain map defined for every degree $n \geq 2$, ψ^n induces a norm nonincreasing map $\Theta^n = \mathrm{H}(\psi^n)$ in bounded cohomology for every $n \geq 3$. Moreover, being induced by a right inverse of φ^n, the map Θ^n is a right inverse of $\bigoplus_{v\in V(G)} \mathrm{H}(i_v^n)$ for every $n \geq 3$. This implies that Θ^n is an isometric embedding.

If $n = 2$, it is not clear why ψ^2 should send coboundaries of bounded 1-cochains to coboundaries of bounded 1-cochains. In fact, we will show in the last part of this section that this is not the case in general. This difficulty may be circumvented by exploiting the fact, proved in Sec. 2, that every element in every $\mathrm{H}^2_{\mathrm{b}}(\Gamma_v)$ admits a special norm-minimizing representative.

In fact let us define the map Θ^2 as the composition of the maps

$$\oplus \mathrm{H}^2_{\mathrm{b}}(\Gamma_v) \xrightarrow{\oplus\sigma_v} \oplus \mathcal{Z}\ell^\infty_{\mu,\mathrm{alt}}(\Gamma_v^3)^{\Gamma_v} \xrightarrow{\oplus\mu_v} \oplus \mathcal{Z}\ell^\infty_{\mathrm{alt}}(S_v^3)^{\Gamma_v} \xrightarrow{\psi^2} \mathcal{Z}\ell^\infty_{\mathrm{alt}}(S_{\mathcal{G}}^3)^{\Gamma} \longrightarrow \mathrm{H}^2_{\mathrm{b}}(\Gamma)$$

where $\sigma_v : \mathrm{H}^2_{\mathrm{b}}(\Gamma_v) \to \mathcal{Z}\ell^\infty_{\mu,\mathrm{alt}}(\Gamma_v^3)^{\Gamma_v}$ is the map described in Proposition 2.3, μ_v is the morphism constructed in Proposition 2.1, and ψ^2 is the map of Theorem 4.1.

All the maps involved are norm nonincreasing, hence the same holds for Θ^2. Moreover, Θ^2 induces a right inverse of the restriction since the following diagram is commutative

$$\begin{tikzcd}
 & \mathcal{Z}\ell^\infty_{\mathrm{alt}}(S_{\mathcal{G}}^3)^{\Gamma} \arrow[r] \arrow[d, "\varphi^2"] & \mathrm{H}^2_{\mathrm{b}}(\Gamma) \arrow[d] \\
\bigoplus \mathcal{Z}\ell^\infty_{\mu,\mathrm{alt}}(\Gamma_v^3)^{\Gamma_v} \arrow[ur, "\psi^2\circ\oplus\mu_v"] \arrow[r, "\oplus\mu_v"] & \bigoplus \mathcal{Z}\ell^\infty_{\mathrm{alt}}(S_v^3)^{\Gamma_v} \arrow[r] & \bigoplus \mathrm{H}^2_{\mathrm{b}}(\Gamma_v). \arrow[ll, bend left, "\oplus\sigma_v"]
\end{tikzcd}$$

This finishes the proof of the theorem. $\square$

Remark 4.2. Let us now briefly comment on the fact that the map ψ^2 does not send, in general, coboundaries of bounded 1-cochains to coboundaries of bounded 1-cochains. We will only be considering free products, that is the case in which the graph G is a tree and all edge groups are trivial. In [42, Proposition 4.2], Rolli constructed a linear map

$$\bigoplus_{v\in V(G)} \ell^\infty_{\mathrm{odd}}(\Gamma_v) \to \mathrm{H}^2_{\mathrm{b}}(\Gamma) \tag{4.1}$$

and showed that this map is injective. Here $\ell^\infty_{\mathrm{odd}}(\Gamma_v)$ is the set of bounded functions on Γ_v such that $f(g^{-1}) = -f(g)$.

We denote by $(\bar{\mathrm{C}}^\bullet(\Gamma), \bar{d}^\bullet)$ (respectively, $(\bar{\mathrm{C}}_{\mathrm{b}}^\bullet(\Gamma), \bar{d}^\bullet)$) the space of inhomogeneous (respectively, bounded inhomogeneous) cochains on Γ, and recall that $\bar{\mathrm{C}}^\bullet(\Gamma)$ (respectively, $\bar{\mathrm{C}}_{\mathrm{b}}^\bullet(\Gamma)$) is isometrically isomorphic to the corresponding module of homogeneous Γ-invariant cochains via the chain map $h^\bullet$ given by $h^n(f)(x_0, \ldots, x_n) = f(x_0^{-1}x_1, \ldots, x_{n-1}^{-1}x_n)$. We denote by $\bar{\mathrm{C}}_{\mathrm{alt}}^n(\Gamma)$ (respectively, $\bar{\mathrm{C}}_{\mathrm{b,alt}}^n(\Gamma)$) the subspace of $\bar{\mathrm{C}}^n(\Gamma)$ (respectively, $\bar{\mathrm{C}}_{\mathrm{b}}^n(\Gamma)$) corresponding via h^n to alternating cochains on Γ^{n+1}.

Let $\alpha : \oplus\ell_{\mathrm{odd}}^\infty(\Gamma_v) \to \bar{\mathrm{C}}_{\mathrm{alt}}^1(\Gamma)$ be defined by $\alpha(\oplus f_v)(x) = \sum f_{v_i}(x_i)$, where $x_0 \ldots x_n$ is the reduced expression for x and $x_i \in \Gamma_{v_i}$. Even if the image of α is not contained in $\bar{\mathrm{C}}_{\mathrm{b,alt}}^1(\Gamma)$ in general, it is proved in [42] that the image of the composition $R = \bar{d}^1 \circ \alpha$ consists of bounded cocycles. Moreover, R admits the explicit expression

$$R(\oplus f_v)(x, y) = f_v(\gamma_2) - f_v(\gamma_1\gamma_2) + f_v(\gamma_1), \tag{4.2}$$

where $a\gamma_1 b$ and $b^{-1}\gamma_2 c$ are reduced expressions for x and y with γ_1 and γ_2 maximal subwords belonging to the same vertex group Γ_v and $\gamma_1 \neq \gamma_2^{-1}$.

Let us now consider the following diagram:

$$\begin{tikzcd}
\oplus\ell_{\mathrm{odd}}^\infty(\Gamma_v) \arrow[r, "R"] \arrow[d, "\mu\circ h^2\circ\bar{d}^1"] & \mathcal{Z}\bar{\mathrm{C}}_{\mathrm{b}}^2(\Gamma) \arrow[r] \arrow[d, "\mu\circ h^2"] & \mathrm{H}_{\mathrm{b}}^2(\Gamma) \arrow[d] \\
\oplus\mathcal{Z}\ell_{\mathrm{alt}}^\infty(S_v^3)^{\Gamma_v} \arrow[r, "\psi^2"] & \mathcal{Z}\ell_{\mathrm{alt}}^\infty(S_{\mathcal{G}}^3)^{\Gamma} \arrow[r] & \mathrm{H}_{\mathrm{b}}^2(\Gamma).
\end{tikzcd}$$

Rolli's map (4.1) is defined as the composition of the horizontal arrows on the top. We claim that the diagram is commutative. Since we are in the case of a free product, we have an obvious identification between $S_{\mathcal{G}}$ and $\Gamma \times V(G')$, and the map $\mu^\bullet : \mathrm{C}_{\mathrm{b}}^\bullet(\Gamma) \to \mathrm{C}_{\mathrm{b}}^\bullet(S_{\mathcal{G}})$ is induced by the projection $\Gamma \times V(G') \to \Gamma$. The commutativity of the square on the right is now a consequence of Lemma 2.2. To show that the left square commutes, let us consider a triple $((x_0, v_0), (x_1, v_1), (x_2, v_2)) \in S_{\mathcal{G}}^3$. Then one may verify that the barycenter of the triple $(p(x_0, v_0), p(x_1, v_1), p(x_2, v_2)) \in V(T')^3$ is the vertex $a\Gamma_v$ where $x_0^{-1}x_1 = a\gamma_1 b$, $x_1^{-1}x_2 = b^{-1}\gamma_2 c$, and $\gamma_1, \gamma_2 \in \Gamma_v$ satisfy $\gamma_1 \neq \gamma_2^{-1}$. Using this fact and equality (4.2) it is easy to verify that the square on the left is also commutative.

Summarizing, as a corollary of Rolli's result we have shown that the image of $\mu \circ h^2 \circ \overline{d}^1$ is a big subspace of coboundaries in $\oplus\ell_{\mathrm{alt}}^\infty(S_v^3)^{\Gamma_v}$ that are not taken by ψ^2 to coboundaries in $\ell_{\mathrm{alt}}^\infty(S_{\mathcal{G}}^3)^\Gamma$. In particular, the restriction of ψ^2 to bounded cocycles does not induce a well-defined map in bounded cohomology.

Remark 4.3. The assumption that G is finite did not play an important role in our proof of Theorem 1.1. Let us suppose that G is countable, and take an element $\varphi \in \mathrm{H}_{\mathrm{b}}^n(\Gamma)$. Then the restriction $\mathrm{H}(i_v^n)(\varphi) \in \mathrm{H}_{\mathrm{b}}^n(\Gamma_v)$ can be non-null for infinitely many $v \in V(G)$. However, we have $\|\mathrm{H}(i_v^n)(\varphi)\|_\infty \leq \|\varphi\|_\infty$ for every $v \in V(G)$, so there exists a well-defined map

$$\prod_{v\in V(G)} \mathrm{H}(i_v^n) : \mathrm{H}_\mathrm{b}^n(\Gamma) \to \left(\prod_{v\in V(G)} \mathrm{H}_\mathrm{b}^n(\Gamma_v)\right)^{\mathrm{ub}},$$

where $(\prod_{v\in V(G)} \mathrm{H}_\mathrm{b}^n(\Gamma_v))^{\mathrm{ub}}$ is the subspace of uniformly bounded elements of $\prod_{v\in V(G)} \mathrm{H}_\mathrm{b}^n(\Gamma_v)$. Our arguments easily extend to the case when G is countable to prove that, for every $n \geq 2$, there exists an isometric embedding

$$\Theta^n : \left(\prod_{v\in V(G)} \mathrm{H}_\mathrm{b}^n(\Gamma_v)\right)^{\mathrm{ub}} \to \mathrm{H}_\mathrm{b}^n(\Gamma)$$

which provides a right inverse to $\prod_{v\in V(G)} \mathrm{H}(i_v^n)$.

5. Mapping Cones and Gromov Equivalence Theorem

Let (X, Y) be a topological pair. As mentioned in the Introduction, Gromov considered in [24] the one-parameter family of norms on $\mathrm{C}_n(X)$ defined by $\|c\|_1(\theta) = \|c\|_1 + \theta\|\partial_n c\|_1$. All these norms are equivalent but distinct, and $\mathrm{C}_n(Y)$ is a closed subspace of $\mathrm{C}_n(X)$ with respect to any of these norms. Therefore, the norm $\|\cdot\|_1(\theta)$ descends to a quotient norm on $\mathrm{C}_n(X, Y)$, and to a quotient seminorm on $\mathrm{H}_n(X, Y)$. All these (semi)norms will be denoted by $\|\cdot\|_1(\theta)$. They admit a useful description that exploits a cone construction for relative singular homology analogous to Park's cone construction for relative ℓ^1-homology [41] (see also [33]).

Let us denote by $i_n : \mathrm{C}_n(Y) \to \mathrm{C}_n(X)$ the map induced by the inclusion $i : Y \to X$. The homology mapping cone complex of (X, Y) is the complex

$$(\mathrm{C}_n(Y \to X), \overline{d}_n)) = \left(\mathrm{C}_n(X) \oplus \mathrm{C}_{n-1}(Y), \begin{pmatrix} \partial_n & i_{n-1} \\ 0 & -\partial_{n-1} \end{pmatrix}\right),$$

where $\partial_\bullet$ denotes the usual differential both of $\mathrm{C}_\bullet(X)$ and of $\mathrm{C}_\bullet(Y)$. The homology of the mapping cone $(\mathrm{C}_\bullet(Y \to X), \overline{d}_\bullet)$ is denoted by $\mathrm{H}_\bullet(Y \to X)$. For every $n \in \mathbb{N}$, $\theta \in [0, \infty)$ one can endow $\mathrm{C}_n(Y \to X)$ with the norm

$$\|(u, v)\|_1(\theta) = \|u\|_1 + \theta\|v\|_1,$$

which induces in turn a seminorm, still denoted by $\|\cdot\|_1(\theta)$, on $\mathrm{H}_n(Y \to X)$.[b]

The chain map

$$\beta_n : \mathrm{C}_n(Y \to X) \to \mathrm{C}_n(X, Y), \quad \beta_n(u, v) = [u] \tag{5.1}$$

induces a map $\mathrm{H}(\beta_n)$ in homology.

Lemma 5.1. *The map*

$$\mathrm{H}(\beta_n) : (\mathrm{H}_n(Y \to X), \|\cdot\|_1(\theta)) \to (\mathrm{H}_n(X, Y), \|\cdot\|_1(\theta))$$

is an isometric isomorphism for every $\theta \in [0, +\infty)$.

[b] In [40], Park restricts her attention only to the case $\theta \geq 1$.

Proof. It is immediate to check that $\mathrm{H}(\beta_n)$ admits the inverse map

$$\mathrm{H}_n(X,Y) \to \mathrm{H}_n(Y \to X), \quad [u] \mapsto [(u, -\partial_n u)].$$

Therefore, $\mathrm{H}(\beta_n)$ is an isomorphism, and we are left to show that it is norm-preserving.

Let us set

$$\beta'_n : \mathrm{C}_n(Y \to X) \to \mathrm{C}_n(X), \quad \beta'_n(u,v) = u.$$

By construction, β_n is the composition of β'_n with the natural projection $\mathrm{C}_n(X) \to \mathrm{C}_n(X,Y)$. Observe that an element $(u,v) \in \mathrm{C}_n(Y \to X)$ is a cycle if and only if $\partial_n u = -i_{n-1}(v)$. As a consequence, although the map β'_n is not norm nonincreasing in general, it does preserve norms when restricted to $\mathcal{Z}\mathrm{C}_n(Y \to X)$. Moreover, every chain in $C_n(X)$ representing a relative cycle is contained in $\beta'_n(\mathcal{Z}\mathrm{C}_n(Y \to X))$, and this concludes the proof. $\square$

As is customary when dealing with seminorms in homology, in order to control the seminorm $\|\cdot\|_1(\theta)$ it is useful to study the topological dual of $(\mathrm{C}_n(Y \to X), \|\cdot\|_1(\theta))$, and exploit duality. If $(C_\bullet, d_\bullet)$ is a normed chain complex (i.e. a chain complex of normed real vector spaces), then for every $n \in \mathbb{N}$ one may consider the topological dual D^n of C_n, endowed with the dual norm. The differential $d_n : C_n \to C_{n-1}$ induces a differential $d^{n-1} : D^{n-1} \to D^n$, and we say that $(D^\bullet, d^\bullet)$ is the dual normed chain complex of $(C_\bullet, d_\bullet)$. The homology (respectively, cohomology) of the complex $(C_\bullet, d_\bullet)$ (respectively, $(D^\bullet, d^\bullet)$) is denoted by $\mathrm{H}_\bullet(C_\bullet)$ (respectively, $\mathrm{H}_\mathrm{b}^\bullet(D^\bullet)$). We denote the norms on C_n and D^n and the induced seminorms on $\mathrm{H}_n(C_\bullet)$ and $\mathrm{H}_\mathrm{b}^n(D^\bullet)$ respectively by $\|\cdot\|_C$ and $\|\cdot\|_D$. The duality pairing between D^n and C_n induces the *Kronecker product*

$$\langle \cdot, \cdot \rangle : \mathrm{H}_\mathrm{b}^n(D^\bullet) \times \mathrm{H}_n(C_\bullet) \to \mathbb{R}.$$

By the Universal Coefficient Theorem, taking (co)homology commutes with taking *algebraic* duals. However, this is no more true when replacing algebraic duals with topological duals, so $\mathrm{H}_\mathrm{b}^n(D^\bullet)$ is not isomorphic to the topological dual of $\mathrm{H}_n(C_\bullet)$ in general (see e.g. [33] for a thorough discussion of this issue). Nevertheless, the following well-known consequence of Hahn–Banach Theorem establishes an important relation between $\mathrm{H}_\mathrm{b}^n(D^\bullet)$ and $\mathrm{H}_n(C_\bullet)$. We provide a proof for the sake of completeness (and because in the available formulations of this result the maximum is replaced by a supremum).

Lemma 5.2. *Let $(C_\bullet, \|\cdot\|_C)$ be a normed chain complex with dual normed chain complex $(D^\bullet, \|\cdot\|_D)$. Then, for every $\alpha \in \mathrm{H}_n(C_\bullet)$ we have*

$$\|\alpha\|_C = \max\{\langle \beta, \alpha \rangle \mid \beta \in \mathrm{H}_\mathrm{b}^n(D^\bullet), \|\beta\|_D \leq 1\}.$$

Proof. The inequality $\geq$ is obvious. Let $a \in C_n$ be a representative of α. In order to conclude it is enough to find an element $b \in D^n$ such that $d^n b = 0$, $b(a) = \|\alpha\|_C$ and

$\|b\|_D \leq 1$. If $\|\alpha\|_C = 0$ we may take $b = 0$. Otherwise, let $V \subseteq C_n$ be the closure of $d_{n-1}C_{n-1}$ in C_n, and put on the quotient $W := C_n/V$ the induced seminorm $\|\cdot\|_W$. Since V is closed, such seminorm is in fact a norm. By construction, $\|\alpha\|_C = \|[a]\|_W$. Therefore, Hahn–Banach Theorem provides a functional $\overline{b} : W \to \mathbb{R}$ with operator norm one such that $\overline{b}([a]) = \|\alpha\|_C$. We obtain the desired element $b \in D^n$ by pre-composing $\overline{b}$ with the projection $C_n \to W$. □

Let us come back to the mapping cone for the homology of a pair (X, Y). For $\theta \in (0, \infty)$, the dual normed chain complex of $(\mathrm{C}_n(Y \to X), \|\cdot\|_1(\theta))$ is Park's mapping cone for relative bounded cohomology [41], that is the complex

$$(\mathrm{C}_\mathrm{b}^n(Y \to X), \overline{d}^n) = \left(\mathrm{C}_\mathrm{b}^n(X) \oplus \mathrm{C}_\mathrm{b}^{n-1}(Y), \begin{pmatrix} d^n & 0 \\ i^n & -d^{n-1} \end{pmatrix}\right)$$

endowed with the norm

$$\|(f, g)\|_\infty(\theta) = \max\{\|f\|_\infty, \theta^{-1}\|g\|_\infty\}.$$

We endow the cohomology $\mathrm{H}_\mathrm{b}^n(Y \to X)$ of the complex $(\mathrm{C}_\mathrm{b}^n(Y \to X), \overline{d}^n)$ with the quotient seminorm, which will still be denoted by $\|\cdot\|_\infty(\theta)$. We denote by $\beta^\bullet$ the chain map dual to the chain map $\beta_\bullet$ defined in (5.1), i.e. we set

$$\beta^n : \mathrm{C}_\mathrm{b}^n(X, Y) \to \mathrm{C}_\mathrm{b}^n(Y \to X), \quad \beta^n(f) = (f, 0)$$

for every $n \in \mathbb{N}$. Then β^n induces an isomorphism between $\mathrm{H}_\mathrm{b}^n(X, Y)$ and $\mathrm{H}_\mathrm{b}^n(Y \to X)$ (see [41], or the first part of the proof of Proposition 5.3). If we assume that the fundamental group of every component of Y is amenable, then we can improve this result as follows:

Proposition 5.3. *Suppose that the fundamental group of every component of Y is amenable. Then, for every $n \geq 2$, $\theta \in (0, \infty)$, the map*

$$\mathrm{H}(\beta^n) : (\mathrm{H}_\mathrm{b}^n(X, Y), \|\cdot\|_\infty) \to (\mathrm{H}_\mathrm{b}^n(Y \to X), \|\cdot\|_\infty(\theta))$$

is an isometric isomorphism.

Proof. Let us first prove that $\mathrm{H}(\beta^n)$ is an isomorphism (here we do not use any hypothesis on Y). To this aim, it is enough to show that the composition

$$\mathcal{Z}\mathrm{C}_\mathrm{b}^n(X, Y) \xrightarrow{\beta^n} \mathcal{Z}\mathrm{C}_\mathrm{b}^n(Y \to X) \longrightarrow \mathrm{H}_\mathrm{b}^n(Y \to X) \tag{5.2}$$

is surjective with kernel $d\mathrm{C}_\mathrm{b}^{n-1}(X, Y)$. For any $g \in \mathrm{C}_\mathrm{b}^\bullet(Y)$ we denote by $g' \in \mathrm{C}_\mathrm{b}^\bullet(X)$ the extension of g that vanishes on simplices with image not contained in Y. Let us take $(f, g) \in \mathcal{Z}\mathrm{C}_\mathrm{b}^n(Y \to X)$. From $\overline{d}^n(f, g) = 0$ we deduce that $f - d^{n-1}g' \in \mathcal{Z}\mathrm{C}_\mathrm{b}^n(X, Y)$. Moreover, $(f - d^{n-1}g', 0) - (f, g) = -\overline{d}^{n-1}(g', 0)$, so the map (5.2) above is surjective. Finally, if $f \in \mathcal{Z}\mathrm{C}_\mathrm{b}^n(X, Y)$ and $(f, 0) = \overline{d}^{n-1}(\alpha, \beta)$, then $\alpha - d^{n-2}\beta'$ belongs to $\mathrm{C}_\mathrm{b}^{n-1}(X, Y)$ and $d(\alpha - d^{n-2}\beta') = f$. This concludes the proof that $\mathrm{H}(\beta^n)$ is an isomorphism.

Let us now suppose that the fundamental group of each component of Y is amenable. We consider the chain map

$$\gamma^\bullet : \mathrm{C}_\mathrm{b}^\bullet(Y \to X) \to \mathrm{C}_\mathrm{b}^\bullet(X), \quad (f, g) \mapsto f.$$

For every $n \in \mathbb{N}$ the composition $\gamma^n \circ \beta^n$ coincides with the inclusion $j^n : \mathrm{C}_\mathrm{b}^n(X, Y) \to \mathrm{C}_\mathrm{b}^n(X)$. By Theorem 1.2, for every $n \geq 2$ the map $\mathrm{H}(j^n)$ is an isometric isomorphism. Moreover, both $\mathrm{H}(\gamma^n)$ and $\mathrm{H}(\beta^n)$ are norm nonincreasing, so we may conclude that the isomorphism $\mathrm{H}(\beta^n)$ is isometric for every $n \geq 2$. □

Putting together Proposition 5.3 and the main theorem of [33] we obtain the following result (which may be easily deduced also from Proposition 5.3 and Lemma 5.2).

Corollary 5.4. *Suppose that the fundamental group of every component of Y is amenable. Then, for every $n \geq 2$, $\theta \in (0, \infty)$, the map*

$$\mathrm{H}(\beta_n) : (\mathrm{H}_n(Y \to X), \|\cdot\|_1(\theta)) \to (\mathrm{H}_n(X, Y), \|\cdot\|_1)$$

is an isometric isomorphism.

We are now ready to conclude the proof of Gromov's Equivalence Theorem (Theorem 1.5 here). Under the assumption that the fundamental group of every component of Y is amenable, Lemma 5.1 and Corollary 5.4 imply that the identity between $(\mathrm{H}_n(X, Y), \|\cdot\|_1)$ and $(\mathrm{H}_n(X, Y), \|\cdot\|_1(\theta))$ is an isometry for every $\theta > 0$. The conclusion follows from the fact that, by definition, $\|\cdot\|_1(0) = \|\cdot\|_1$ and $\|\cdot\|_1(\infty) = \lim_{\theta\to\infty} \|\cdot\|_1(\theta)$.

6. Additivity of the Simplicial Volume

Let us recall that if M is a compact connected orientable n-manifold, the simplicial volume of M is defined as

$$\|M, \partial M\| = \|[M, \partial M]\|_1,$$

where $[M, \partial M] \in \mathrm{C}_n(M, \partial M)$ is the image of the integral fundamental class of M via the change of coefficients homomorphism induced by the inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$.

Let G be a finite graph. We associate to any vertex $v \in V(G)$ a compact oriented n-manifold $(M_v, \partial M_v)$ such that the fundamental group of every component of ∂M_v is amenable, and to any edge $e \in E(G)$ a closed oriented $(n-1)$-manifold S_e together with an orientation preserving homeomorphism $f_e : S_e \to \partial_e M_{t(e)}$, where $\partial_e M_{t(e)}$ is a connected component of $\partial M_{t(e)}$. We also require that $S_{\overline{e}}$ is equal to S_e with reversed orientation, and that the images of f_e and $f_{e'}$ are distinct whenever e, e' are distinct edges of G. We denote by M the quotient of the union $(\bigcup_{v\in V(G)} M_v) \cup (\bigcup_{e\in \bar{E}(G)} S_e)$ with respect to the identifications induced by the maps f_e, $e \in E(G)$. Of course, M is just the manifold obtained by gluing the M_v along the maps $f_e \circ f_{\overline{e}}^{-1}$, $e \in \bar{E}(G)$. We also assume that M is connected.

For every $e \in E(G)$ we identify S_e with the corresponding hypersurface in M, and we denote by $\mathcal{S}$ the union $\bigcup_{e\in\bar{E}(G)} S_e \subseteq M$. The inclusion $i_v : (M_v, \partial M_v) \to (M, \mathcal{S} \cup \partial M)$ is a map of pairs inducing a norm nonincreasing map in cohomology

$$i_v^n : \mathrm{H}_{\mathrm{b}}^n(M, \mathcal{S} \cup \partial M) \to \mathrm{H}_{\mathrm{b}}^n(M_v, \partial M_v).$$

We consider now the setting of Theorem 1.3, therefore any component of $\partial M \cup \mathcal{S}$ has amenable fundamental group. Moreover, since every compact manifold has the homotopy type of a finite CW-complex [30], we may compose the isomorphisms $\mathrm{H}_{\mathrm{b}}^n(M, \partial M) \cong \mathrm{H}_{\mathrm{b}}^n(M)$, $\mathrm{H}_{\mathrm{b}}^n(M) \cong \mathrm{H}_{\mathrm{b}}^n(M, \partial M \cup \mathcal{S})$ provided by Theorem 1.2, thus getting an isometric isomorphism

$$\zeta^n : \mathrm{H}_{\mathrm{b}}^n(M, \partial M) \to \mathrm{H}_{\mathrm{b}}^n(M, \partial M \cup \mathcal{S}).$$

This map is the inverse of the map induced by the inclusion of pairs $(M, \partial M) \to (M, \partial M \cup \mathcal{S})$. Finally, we define the norm nonincreasing map

$$\zeta_v^n = i_v^n \circ \zeta^n : \mathrm{H}_{\mathrm{b}}^n(M, \partial M) \to \mathrm{H}_{\mathrm{b}}^n(M_v, \partial M_v).$$

Lemma 6.1. *For every $\varphi \in \mathrm{H}_{\mathrm{b}}^n(M, \partial M)$ we have*

$$\langle \varphi, [M, \partial M] \rangle = \sum_{v \in V(G)} \langle \zeta_v^n(\varphi), [M_v, \partial M_v] \rangle.$$

Proof. Let $c_v \in \mathrm{C}_n(M_v)$ be a real chain representing the fundamental class of M_v. We identify any chain in M_v with the corresponding chain in M, and we set $c = \sum_{v\in V(G)} c_v \in \mathrm{C}_n(M)$. We now suitably modify c in order to obtain a relative fundamental cycle for M. It is readily seen that ∂c_v is the sum of real fundamental cycles of the boundary components of M_v. Therefore, since the gluing maps defining M are orientation-reversing, we may choose a chain $c' \in \bigoplus_{e\in\bar{E}(G)} \mathrm{C}_n(S_e)$ such that $\partial c - \partial c' \in \mathrm{C}_{n-1}(\partial M)$. We set $c'' = c - c'$. By construction c'' is a relative cycle in $\mathrm{C}_n(M, \partial M)$, and it is immediate to check that it is in fact a relative fundamental cycle for M. Let now $\psi \in \mathrm{C}_{\mathrm{b}}^n(M, \mathcal{S} \cup \partial M)$ be a representative of $\zeta^n(\varphi)$. By definition we have

$$\psi(c) = \sum \psi(c_v) = \sum \langle \zeta_v^n(\varphi), [M_v, \partial M_v] \rangle.$$

On the other hand, since ψ vanishes on chains supported on $\mathcal{S}$, we also have

$$\psi(c) = \psi(c'' + c') = \psi(c'') = \langle \varphi, [M, \partial M] \rangle,$$

and this concludes the proof. □

Let us now proceed with the proof of Theorem 1.3. In order to match the notation with the statement of Theorem 1.3, we henceforth denote by $\{1, \ldots, k\}$ the set of vertices of G. By Lemma 5.2 we may choose an element $\varphi \in \mathrm{H}_{\mathrm{b}}^n(M, \partial M)$

such that

$$\|M, \partial M\| = \langle \varphi, [M, \partial M] \rangle, \quad \|\varphi\|_\infty \leq 1.$$

Observe that $\|\zeta_v^n(\varphi)\|_\infty \leq \|\varphi\|_\infty \leq 1$ for every $v \in V(G)$, so by Lemma 6.1

$$\|M, \partial M\| = \langle \varphi, [M, \partial M] \rangle = \sum_{v=1}^{k} \langle \zeta_v^n(\varphi), [M_v, \partial M_v] \rangle \leq \sum_{v=1}^{k} \|M_v, \partial M_v\|. \quad (6.1)$$

This proves the first part of Theorem 1.3.

Remark 6.2. Inequality (6.1) may also be proved by using Matsumoto–Morita's boundary condition [36, Definition 2.1 and Theorem 2.8] and Corollary 1.6.

To conclude the proof of Theorem 1.3 we now consider the case when M is obtained via compatible gluings. Therefore, if K_e is the kernel of the map induced by f_e on fundamental groups, then $K_e = K_{\overline{e}}$ for every $e \in E(G)$ (recall that $S_e = S_{\overline{e}}$, so both K_e and $K_{\overline{e}}$ are subgroups of $\pi_1(S_e) = \pi_1(S_{\overline{e}})$). If we consider the graph of groups $\mathcal{G}$ with vertex groups $G_v = \pi_1(M_v)$ and edge groups $G_e = \pi_1(S_e)/K_e$, then van Kampen Theorem implies that $\pi_1(M)$ is the fundamental group of the graph of groups $\mathcal{G}$ (see [45] for full details).

Proposition 6.3. *For every* $(\varphi_1, \ldots, \varphi_k) \in \bigoplus_{v=1}^{k} \mathrm{H}_\mathrm{b}^n(M_v, \partial M_v)$, *there exists* $\varphi \in \mathrm{H}_\mathrm{b}^n(M, \partial M)$ *such that*

$$\|\varphi\|_\infty \leq \|(\varphi_1, \ldots, \varphi_k)\|_\infty, \quad \zeta_v^n(\varphi) = \varphi_v, \quad v = 1, \ldots, k.$$

Proof. The proposition follows at once from Theorem 4.1 and the commutativity of the following diagram:

$$\begin{tikzcd}
\mathrm{H}_\mathrm{b}^n(M, \partial M) \arrow[r] \arrow[d, "\oplus\zeta_v^n"'] & \mathrm{H}_\mathrm{b}^n(M) \arrow[r] \arrow[d] & \mathrm{H}_\mathrm{b}^n(\pi_1(M)) \arrow[d, "\oplus i_v^n"] \\
\oplus\mathrm{H}_\mathrm{b}^n(M_v, \partial M_v) \arrow[r] & \oplus\mathrm{H}_\mathrm{b}^n(M_v) \arrow[r] & \oplus\mathrm{H}_\mathrm{b}^n(\pi_1(M_v)),
\end{tikzcd}$$

where the horizontal arrows are, respectively, the isometric isomorphisms constructed in Theorem 1.2 and Ivanov's maps, and the vertical arrows are given by restrictions. $\square$

By Lemma 5.2, for every $v = 1, \ldots, k$, we may choose an element $\varphi_v \in \mathrm{H}_\mathrm{b}^n(M_v, \partial M_v)$ such that

$$\|M_v, \partial M_v\| = \langle \varphi_v, [M_v, \partial M_v] \rangle, \quad \|\varphi_v\|_\infty \leq 1,$$

and Proposition 6.3 implies that there exists $\varphi \in \mathrm{H}_{\mathrm{b}}^n(M, \partial M)$ such that

$$\|\varphi\|_\infty \leq 1, \quad \zeta_v^n(\varphi) = \varphi_v, \quad v = 1, \ldots, k.$$

Using Lemma 6.1 we get

$$\sum_{v \in V(G)} \|M_v, \partial M_v\| = \sum_{v \in V(G)} \langle \varphi_v, [M_v, \partial M_v] \rangle = \langle \varphi, [M, \partial M] \rangle \leq \|M, \partial M\|,$$

which finishes the proof of Theorem 1.3.

Remark 6.4. The following examples show that the hypotheses of Theorem 1.3 should not be too far from being the weakest possible.

Let M be a hyperbolic 3-manifold with connected geodesic boundary. It is well-known that the genus of ∂M is bigger than one, and that ∂M is π_1-injective in M. We fix a pseudo-Anosov homeomorphism $f : \partial M \to \partial M$, and for every $m \in \mathbb{N}$ we denote by $D_m M$ the twisted double obtained by gluing two copies of M along the homeomorphism $f^m : \partial M \to \partial M$ (so $D_0 M$ is the usual double of M). It is shown in [27] that

$$\|D_0 M\| < 2 \cdot \|M, \partial M\|.$$

On the other hand, by [47] we have $\lim_{m\to\infty} \operatorname{Vol} D_m M = \infty$. But $\operatorname{Vol} N = v_3 \cdot \|N\|$ for every closed hyperbolic 3-manifold N, where v_3 is a universal constant [24, 50], so $\lim_{m\to\infty} \|D_m M\| = \infty$, and the inequality

$$\|D_m M\| > 2 \cdot \|M, \partial M\|$$

holds for infinitely many $m \in \mathbb{N}$. This shows that, even in the case when each S_e is π_1-injective in $M_{t(e)}$, no inequality between $\|M, \partial M\|$ and $\sum_{v=1}^k \|M_v, \partial M_v\|$ holds in general if one drops the requirement that the fundamental group of every S_e is amenable.

On the other hand, let M_1 be (the natural compactification of) the once-punctured torus. The interior of M_1 admits a complete finite-volume hyperbolic structure, so $\|M_1, \partial M_1\| = \operatorname{Area}(M_1)/v_2$, where $v_2 = \pi$ denotes the maximal area of hyperbolic triangles. By Gauss–Bonnet Theorem, this implies that $\|M_1, \partial M_1\| = 2$. If M_2 is the 2-dimensional disk, then the manifold M obtained by gluing M_1 with M_2 along $\partial M_1 \cong \partial M_2 \cong S^1$ is a torus, so $\|M\| = 0$ and

$$\|M\| < \|M_1, \partial M_1\| + \|M_2, \partial M_2\|.$$

This shows that, even in the case when the fundamental group of every S_e is amenable, the equality $\|M, \partial M\| = \sum_{j=1}^k \|M_j, \partial M_j\|$ does not hold in general if one drops the requirement that the gluings are compatible.

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