BROWNIAN MOTION AND STOCHASTIC CALCULUS

SPRING TERM 2010
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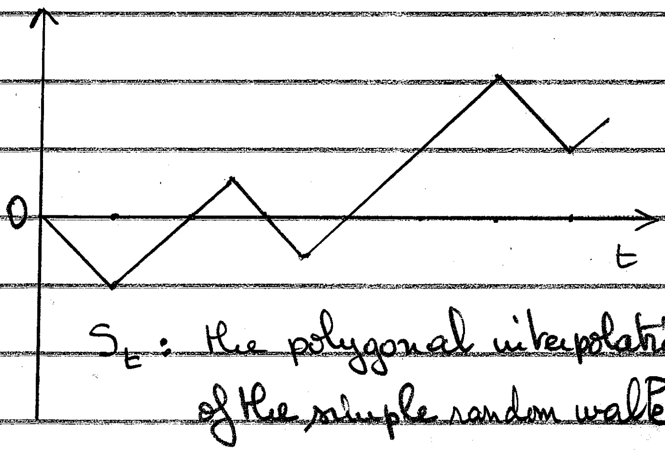
Lecture 1:

CHAPTER 0: INTRODUCTION

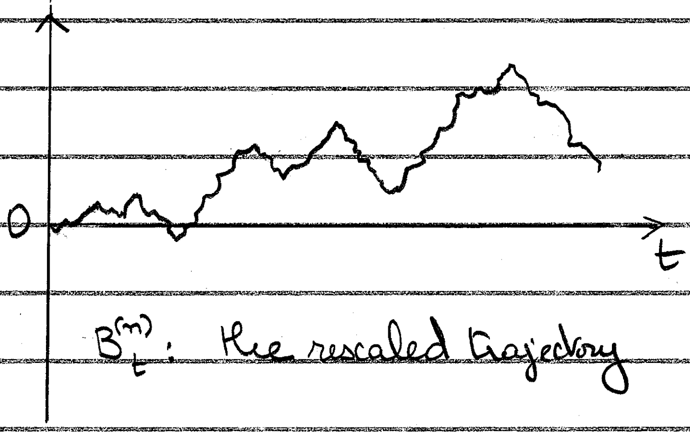
The object of this course is to present Brownian motion, develop the infinitesimal calculus attached to Brownian motion and discuss various applications to diffusion processes.

The name "Brownian motion" comes, who in 1827, director at the time of the British botanical museum, observed the disordered motion of "pollen grains suspended in water performing a continual swarming motion". Bachelier in his thesis in 1900 used Brownian as a model of the stock market, and Einstein considered it in 1905 when discussing the motion of small particles in suspension in a fluid, under the influence of shocks due to thermal agitation of molecules in the fluid. The mathematical theory of Brownian motion was then put on a firm basis with Wiener in 1923.

There are several ways to mathematically construct Brownian motion. One can for instance construct Brownian motion as the limit of rescaled polygonal interpolations of a simple random walk, choosing time units of order n^2 and space units of order n:



S_t: the polygonal interpolation of the simple random walk



$B_t^{(n)}$: the rescaled trajectory

$X_1, \dots, X_n, \dots$, are iid with $P[X_i = 1] = P[X_i = -1] = \frac{1}{2}$,

$S_n = X_1 + \dots + X_n$, $n \geq 1$, $S_0 = 0$,

S_t, $t \geq 0$, is the polygonal interpolation of S_n, $n \geq 0$, and

(0.1) $B_t^{(n)} = \frac{1}{n} S_{tn^2}$, $t \geq 0$, is the rescaled (in time and space) trajectory.

From the central limit theorem, one knows that $B_1^{(n)}$ converges in law to a $N(0,1)$-distribution, that is:
$P[B_1^{(n)} \leq a] \xrightarrow[n \to \infty]{} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} \exp\{-\frac{x^2}{2}\} dx$, for $a \in \mathbb{R}$.

In fact much more is true, and the law of $B_\cdot^{(n)}$ viewed as a random continuous trajectory converges in a suitable sense to the law of Brownian motion (this is a special case of the so-called "invariance principle" of Donsker).

An important advantage of continuous models versus discrete models is the presence of the wide apparatus of "infinitesimal calculus". However in the case of a typical realization of Brownian motion, the <u>trajectory</u> $t \geq 0 \to B_t(\omega) \in \mathbb{R}$, is <u>continuous</u>, <u>but very rough</u> (in particular nowhere differentiable of infinite variation on any proper interval).

The basic formula of calculus:

(0.2) $\frac{d}{dt} f(b(t)) = f'(b(t))\, b'(t)$, for f and b, C^1-functions,

can still be given a meaning when b is continuous of finite variation, and f is C^1, namely:

(0.3) $f(b(t)) = f(b(0)) + \int_0^t f'(b(s))\, db(s)$, for $t \geq 0$,

where $db(s)$ stands for the Stieltjes measure on $[0,\infty)$, such that $\int_{[0,a]} db(s) = b(a) - b(0)$, for $0 \leq a < \infty$.

However this extension is of little help in the case of Brownian motion since $t \to B_t$ is of infinite variation.

Nonetheless we will develop an infinitesimal calculus based on a formula (Ito's formula), which brings into play an "extra term":

(0.4) $f(B_t) = f(B_0) + \int_0^t f'(B_s)\,dB_s + \frac{1}{2}\int_0^t f''(B_s)\,ds$, for $f \in C^2(\mathbb{R})$, $t \geq 0$,

or in differential notation:

$$df(B_t) = f'(B_t)\,dB_t + \frac{1}{2} f''(B_t)\,dt.$$

Of course part of the work has to do with defining what is meant by "$\int_0^t f'(B_s)\,dB_s$", since as explained above, this expression has no meaning as a Stieltjes integral. This task will correspond to the construction of stochastic integrals.

Once this infinitesimal calculus is at our disposal, we will be able to solve certain differential equations with random perturbations, the so-called "stochastic differential equations", (SDE):

(0.5) $dX_t = b(X_t)\,dt + \underbrace{\sigma(X_t)\,dB_t}_{\text{random perturbation.}}$

There turns out to be a deep connection between such solutions of stochastic differential equations and certain partial differential equations (PDE).

For instance when $B_t = (B_t^1, \dots, B_t^d)$, where the $B^i_.$ are independant real-valued Brownian motions, and $D \subseteq \mathbb{R}^d$ is a smooth bounded domain, e.g. a ball, one can consider the:

Dirichlet problem: given $f \in C(\partial D)$, find u such that

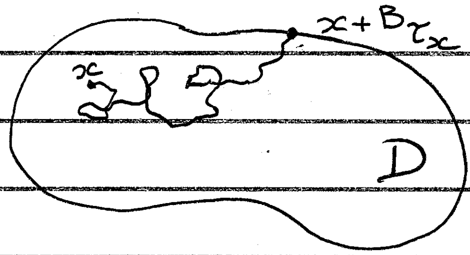
(0.6) $\begin{cases} \frac{1}{2}\Delta u = 0 \text{ in } D, \\ u|_{\partial D} = f, \end{cases}$

or the

Poisson equation: for $g \in C^\alpha(\bar{D})$, find u such that

(0.7) $\begin{cases} \frac{1}{2}\Delta u = g \text{ in } D, \\ u|_{\partial D} = 0 \end{cases}$

The two problems have solutions, which can be expressed in terms of Brownian motion:



Setting for $x \in D$,

(0.8) $\tau_x = \inf\{s \geq 0;\ x + B_s \in \partial D\}$,

one has

(0.9) $u_{\text{Dirichlet}}(x) = E[f(x + B_{\tau_x})]$

and

(0.10) $u_{\text{Poisson}}(x) = -E\left[\int_0^{\tau_x} g(x+B_s)\,ds\right]$.

With stochastic differential equations one is able to handle more general equations with $\frac{1}{2}\Delta$ replaced by:

(0.11) $L = \frac{1}{2}\sum_{i,j=1}^{d} (\sigma(x)\sigma^t(x))_{i,j}\,\partial^2_{i,j} + \sum_{i=1}^{d} b(x)_i\,\partial_i$,

and during this course we will describe a number of applications of these ideas and concepts.

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CHAPTER 1: BROWNIAN MOTION: DEFINITION AND CONSTRUCTION

We will see that Brownian motion plays a prominent role as a canonical example of three different notions:
- a continuous Gaussian process,
- a continuous Markov process,
- a continuous martingale.

In this chapter we will mainly rely on the first of these three notions.

Definition.

$(\Omega, \mathcal{A}, P)$ a probability space, a d-dimensional Brownian motion on $(\Omega, \mathcal{A}, P)$ is an $\mathbb{R}^d$-valued stochastic process, (i.e. for each $t \geq 0$, $B_t(\cdot)$ is an $\mathbb{R}^d$-valued random variable defined on $(\Omega, \mathcal{A}, P)$), such that:

i) $B_0 = 0$, P-a.s.,

ii) for any $0 = t_0 < t_1 < \dots < t_n$, $B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \dots, B_{t_n} - B_{t_{n-1}}$ are independent

(1.1) random variables ("independent increments"),

iii) for $t > 0$, $s \geq 0$, $A \in \mathcal{B}(\mathbb{R}^d)$, $P[B_{t+s} - B_s \in A] = \int_A (2\pi t)^{-\frac{d}{2}} \exp\{-\frac{|x|^2}{2t}\} dx$,
("$B_{t+s} - B_s$ is $N(0, tI)$-distributed"),

iv) P-a.s., $t \geq 0 \longrightarrow B_t(\omega) \in \mathbb{R}^d$ is continuous.

In the above definition $(\Omega, \mathcal{A}, P)$ is "arbitrary". As we will see there is a way to construct a "canonical Brownian motion", once we know that at least one Brownian motion in the sense of the above definition exists.

We take as model (i.e. "canonical") space

(1.2) $C = C(\mathbb{R}_+, \mathbb{R}^d) = \{\text{continuous functions } \mathbb{R}_+ \to \mathbb{R}^d\}$.

On C we have the canonical coordinates:

(1.3) $X_u : C \to \mathbb{R}^d$, $u \geq 0$, such that $X_u(w) = w(u)$, for $w \in C$,

and the σ-algebra generated by these coordinates:

(1.4) $\mathcal{F} = \sigma(X_u, u \geq 0)$, (i.e. the smallest σ-algebra on C for which all X_u, $u \geq 0$, are measurable).

<u>Lemma:</u> (Ψ: a map $\Omega \to C$)

(1.5) $(\Omega, \mathcal{A}) \xrightarrow{\Psi} (C, \mathcal{F})$ is measurable if and only if $(\Omega, \mathcal{A}) \xrightarrow{X_u \circ \Psi} (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is measurable for each $u \geq 0$.

<u>Proof:</u>

$\Rightarrow$: immediate (composition of two measurable maps is measurable).

$\Leftarrow$: the collection $\mathcal{S}$ of $B \subseteq C$ such that $\Psi^{-1}(B) \in \mathcal{A}$ is a σ-algebra, and it contains $X_u^{-1}(D)$ for $D \in \mathcal{B}(\mathbb{R}^d)$, and $u \geq 0$. Hence $\mathcal{S}$ contains $\mathcal{F}$, the smallest σ-algebra for which all X_u, $u \geq 0$ are measurable. As a result for all $F \in \mathcal{F}$, $\Psi^{-1}(F) \in \mathcal{A}$, and Ψ is measurable. $\square$

We will later see that on a suitable $(\Omega, \mathcal{A}, P)$ we can construct a Brownian motion. For such a Brownian motion we can pick by (1.1) iv), a negligible $N \in \mathcal{A}$, (i.e. $P(N) = 0$), and define

$(\Omega \setminus N, \underline{\mathcal{A} \cap (\Omega \setminus N)}) \xrightarrow{B_\cdot} (C, \mathcal{F})$

the notation means the collection of sets $A \cap (\Omega \setminus N)$, with $A \in \mathcal{A}$.

The above map is measurable, indeed:

for $u \geq 0$, $X_u \circ B_\cdot(\omega) = B_u(\omega)$ is measurable $(\Omega \setminus N, \mathcal{A} \cap (\Omega \setminus N)) \to (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and we can apply (1.5).

We can consider the image under $B_\cdot$ of the probability measure P restricted to $\Omega \setminus N$, (i.e. $P: \mathcal{A} \cap \Omega \setminus N \to [0,1]$). We denote by W this image probability.

<u>Proposition:</u>

(1.6) The law W on $(C, \mathcal{F})$ is uniquely determined, (it is the so-called <u>d-dimensional Wiener measure</u>).

(1.7) For $0 = t_0 < t_1 < \dots < t_n$ and $h \in b\mathcal{B}((\mathbb{R}^d)^{n+1})$ (i.e. bounded measurable on $(\mathbb{R}^d)^{n+1}$),

$$E^W[h(X_{t_0}, X_{t_1}, \dots, X_{t_n})] = \int_{(\mathbb{R}^d)^n} h(0, x_1, \dots, x_n) \prod_{i=1}^n [2\pi(t_i - t_{i-1})]^{-d/2} \exp\left\{-\frac{|x_i - x_{i-1}|^2}{2(t_i - t_{i-1})}\right\} dx_1 \dots dx_n$$

(1.8) $X_t(w), t \geq 0$, is a Brownian motion on $(C, \mathcal{F}, W)$, (it is the canonical d-dimensional Brownian motion).

Proof:

– (1.6): for $h \in bB((\mathbb{R}^d)^{n+1})$ and $0 = t_0 < t_1 < \dots < t_n$, we have

(1.9) $a \overset{def}{=} E^W[h(X_{t_0}, X_{t_1}, \dots, X_{t_n})] = E^P[h(B_0, B_{t_1}, \dots, B_{t_n})]$

$= E^P[h(B_0, B_0 + B_{t_1} - B_{t_0}, \dots, B_0 + B_{t_1} - B_{t_0} + B_{t_2} - B_{t_1} + \dots + B_{t_n} - B_{t_{n-1}})]$.

By (1.1), the $B_{t_i} - B_{t_{i-1}}$, $1 \leq i \leq n$, are independent, respectively distributed $N(0, (t_i - t_{i-1})I)$-distributed, and $B_0 = 0$, P-a.s.. Hence we find that

(1.10) $a = \int_{(\mathbb{R}^d)^n} h(0, y_1, y_1 + y_2, \dots, y_1 + \dots + y_n) \prod_{i=1}^{n} [2\pi(t_i - t_{i-1})]^{-d/2} \exp\{-\frac{|y_i|^2}{2(t_i - t_{i-1})}\} dy_1 \dots dy_n$

Picking $h = 1_D$, where $D \in \mathcal{B}((\mathbb{R}^d)^{n+1})$, we see that (1.9), (1.10) determines: $W(\{(X_0, X_{t_1}, \dots, X_{t_n}) \in D\})$.

The class of sets of the form $\{(X_0, X_{t_1}, \dots, X_{t_n}) \in D\}$, $n \geq 1$, $0 = t_0 < t_1 < \dots < t_n$, and $D \in \mathcal{B}((\mathbb{R}^d)^{n+1})$, arbitrary, is a π-system, (i.e. stable under intersection), which generates $\mathcal{F}$. From Dynkin's lemma, W is completely determined on $\mathcal{F}$, and in particular does not depend on the specific $(\Omega, \mathcal{Q}, P, B)$ and N we used.

– (1.7): we perform the change of variable in (1.10)

(1.11) $x_1 = y_1, x_2 = y_1 + y_2, \dots, x_n = y_1 + y_2 + \dots + y_n$.

Note that $\mathrm{Jac}(y_1, \dots, y_n | x_1, \dots, x_n) = 1$, so that

$a \overset{(1.10)}{=} \int_{(\mathbb{R}^d)^n} h(0, x_1, \dots, x_n) \prod_{i=1}^{n} [2\pi(t_i - t_{i-1})]^{-d/2} \exp\{-\frac{|x_i - x_{i-1}|^2}{2(t_i - t_{i-1})}\} dx_1 \dots dx_n$,

and this proves (1.7).

– (1.8): We pick h in (1.9), (1.10) of the form

$h(x_0, x_1, \dots, x_n) = g(x_0, x_1 - x_0, x_2 - x_1, \dots, x_n - x_{n-1})$, so that

$h(0, y_1, y_1 + y_2, \dots, y_1 + \dots + y_n) = g(0, y_1, y_2, \dots, y_n)$.

It then follows that $X_t, t \geq 0$, fulfills (1.1) i), ii), iii). Since for all $w \in C$, $t \geq 0 \to X_t(w) \in \mathbb{R}^d$ is continuous, (1.1) iv) holds as well, and $X_t, t \geq 0$, is a Brownian motion on $(C, \mathcal{F}, W)$.

□