Lecture 2: (chap 1, cont.)

In this chapter the fact that Brownian motion is a continuous Gaussian process will play an important role.

Definition:

- An $\mathbb{R}^d$-valued process, $X_t$, $t \in T$, ($T$ is some arbitrary non-empty set), defined on $(\Omega, \mathcal{A}, P)$ is a centered Gaussian process, if for any $n \geq 1$, $t_1, \dots, t_n \in T$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}^d$, $\sum_{i=1}^n \lambda_i \cdot X_{t_i}$ is a real-valued centered Gaussian variable (possibly $\equiv 0$). scalar product when $d \geq 2$
- The $d \times d$-matrix valued function on $T^2$:

$$(1.12) \quad \Gamma(u,v) = E[X_u \, {}^t X_v] = \left(E[X_u^i X_v^j]\right)_{1 \leq i,j \leq d}, \quad u, v \in T$$

(note that $\Gamma(v,u) = {}^t\Gamma(u,v)$),

is the covariance function of the process,

(note that $E[X_t] = 0$, for each $t \in T$).

Lemma: ($(X_t)_{t \in T}$ a centered Gaussian process with covariance function $\Gamma$)

The function $\Gamma(u,v)$, $u, v \in T$, completely determines all

(1.13) finite distributions $X_{t_1}, \dots, X_{t_n}$ on $(\mathbb{R}^d)^n$, for any $n \geq 1$, and $t_1, \dots, t_n \in T$.

Proof:

For $\xi = (\lambda_1, \dots, \lambda_n) \in (\mathbb{R}^d)^n$, we set

$$(1.14) \quad \varphi(\xi) \stackrel{\text{def}}{=} E\Big[\exp\Big\{i \underbrace{\sum_{j=1}^n \lambda_j \cdot X_{t_j}}_{\text{real valued centered Gaussian variable}}\Big\}\Big] = \exp\Big\{-\frac{1}{2} E\Big[\Big(\sum_{j=1}^n \lambda_j \cdot X_{t_j}\Big)^2\Big]\Big\}$$

$$= \exp\Big\{-\frac{1}{2} \sum_{i,j=1}^n E\big[(\lambda_i \cdot X_{t_i})(\lambda_j \cdot X_{t_j})\big]\Big\}$$

$$= \exp\Big\{-\frac{1}{2} \sum_{i,j=1}^n \underbrace{{}^t\lambda_i}_{\text{raw vector}} \underbrace{E[X_{t_i} \, {}^t X_{t_j}]}_{d \times d \text{ matrix}} \underbrace{\lambda_j}_{\text{column vector}}\Big\} \stackrel{(1.12)}{=} \exp\Big\{-\frac{1}{2} \sum_{i,j=1}^n {}^t\lambda_i \, \Gamma(t_i, t_j) \lambda_j\Big\}.$$

But the characteristic function $\varphi(\cdot)$ completely determines the law of $(X_{t_1}, \dots, X_{t_n})$ on $(\mathbb{R}^d)^n$. $\square$

We will now provide a characterization of Brownian motion as a continuous centered Gaussian process.

Proposition:

Let $B_t, t \geq 0$, be an $\mathbb{R}^d$-valued process defined on $(\Omega, \mathcal{A}, P)$ with $P$-a.s. continuous trajectories.

(1.15) $B_t, t \geq 0$, is a Brownian motion $\Longleftrightarrow$ $B_t, t \geq 0$ is a centered Gaussian process with $\Gamma(s,t) = (s \wedge t)\, \underset{\text{identity matrix}}{I_{d \times d}}$

Proof:

$\Longrightarrow$: for $0 \leq t_1 < t_2 < \dots < t_n$, $\lambda_1, \dots, \lambda_n \in \mathbb{R}^d$, $P$-a.s.,

$$a \overset{\text{def}}{=} \sum_{i=1}^n \lambda_i \cdot B_{t_i} \overset{(1.1)(i)}{=} \sum_{i=1}^n \lambda_i \cdot \Big(\sum_{j=1}^i B_{t_j} - B_{t_{j-1}}\Big) = \sum_{j=1}^n \Big(\sum_{i=j}^n \lambda_i\Big) \cdot (B_{t_j} - B_{t_{j-1}}).$$

By (1.1), the $(B_{t_j} - B_{t_{j-1}})$, $1 \leq j \leq n$, are independent respectively $N(0, (t_j - t_{j-1}) I_{d\times d})$-distributed. Therefore $a$ is a real valued centered Gaussian variable (use characteristic function), we have shown that $B_t, t \geq 0$, is a centered Gaussian process. Moreover for $0 \leq s \leq t$, we have

(1.16) $$\Gamma(s,t) = E[B_s\, {}^t B_t] = E[B_s\, {}^t B_s] + \underbrace{E[B_s\, {}^t (B_t - B_s)]}_{=0 \ \leftarrow \text{ independent and centered}}$$
$$= s I_{d\times d} = (s \wedge t) I_{d \times d}.$$

Then for $t \leq s$, $\Gamma(s,t) = {}^t\Gamma(s,t) \overset{(1.16)}{=} (t \wedge s)\, {}^t I_{d\times d} = (s \wedge t) I_{d \times d}$.

$\Longleftarrow$: If $0 \leq t_1 < \dots < t_n$ are given and on some auxiliary probability space $Y_j$, $1 \leq j \leq n$, are independent $N(0, (t_j - t_{j-1}) I_{d\times d})$-distributed, we can define $X_j = \sum_{k=1}^j Y_k$.

The argument above shows that $X_j, 1 \leq j \leq n$, is a centered Gaussian process, and for $1 \leq i \leq j \leq n$:

(1.17) $$\Gamma(i,j) \overset{\text{def}}{=} E[X_i\, {}^t X_j] = E[X_i\, {}^t X_i] + \underbrace{E[X_i\, {}^t(X_j - X_i)]}_{\shortparallel\, 0 \ \text{independent and centered}}$$

$$= E\Big[\Big(\sum_1^i Y_k\Big)\, {}^t\Big(\sum_1^i Y_k\Big)\Big] \underset{\text{centered}}{\overset{\text{indep.}}{=}} \sum_{1 \leq k \leq i} E[Y_k\, {}^t Y_k] = t_i I_{d \times d}.$$

As below (1.16), we thus see that:

(1.18) $\Gamma(i,j) = (t_i \wedge t_j) I_{d\times d}$, for $1 \le i,j \le n$.

By (1.13) we thus see that $(X_1,\dots,X_n)$ has same law as $(B_{t_1},\dots,B_{t_n})$. Therefore $(B_{t_1}, B_{t_2}-B_{t_1},\dots,B_{t_n}-B_{t_{n-1}})$ has same law as $(X_1, X_2-X_1,\dots,X_n-X_{n-1}) = (Y_1,\dots,Y_n)$. Hence (1.1) ii), iii) follow. Moreover (1.1) iv) holds by assumption and $E[B_0\,{}^tB_0]=0$ implies by looking at the diagonal coefficients of this matrix that P-a.s., $B_0=0$. This proves that $B_t$, $t\ge 0$, is a Brownian motion. $\square$

The above characterization is very helpful.

Examples:

1) Invariance by scaling:

Consider $B_t$, $t\ge0$, an $\mathbb{R}^d$-valued Brownian motion, $\lambda>0$, then

(1.19) $\lambda B_{t/\lambda^2}$, $t\ge0$, is also an $\mathbb{R}^d$-valued Brownian motion.

Indeed:

$B_t^\lambda \overset{\text{def}}{=} \lambda B_{t/\lambda^2}$, $t\ge0$, is also a continuous centered Gaussian process and for $s\ge0$, $t\ge0$:

$$E[B_s^\lambda\,{}^tB_t^\lambda] = \lambda^2 E[B_{s/\lambda^2}\,{}^tB_{t/\lambda^2}] = \lambda^2\Big(\frac{s}{\lambda^2}\wedge\frac{t}{\lambda^2}\Big) I_{d\times d} = (s\wedge t) I_{d\times d},$$

and (1.19) follows from (1.15).

2) Invariance by time inversion:

Consider $B_t$, $t\ge0$, an $\mathbb{R}^d$-valued Brownian motion and define

(1.20) $\beta_0 = 0$, and $\beta_s = s\,B_{1/s}$, for $s>0$.

Then one has:

(1.21) $\beta_s$, $s\ge0$, is an $\mathbb{R}^d$-valued Brownian motion.

Indeed

$\beta_s$, $s\ge0$, is a centered Gaussian process, and for $0<s,t$:

$$(1.22)\quad E[\beta_s\,{}^t\beta_t] = st\,E[B_{\frac1s}\,{}^tB_{\frac1t}] = st\Big(\frac1s\wedge\frac1t\Big) I_{d\times d} = \frac{st}{s\vee t} I_{d\times d} = (s\wedge t) I_{d\times d},$$

and this formula immediately extends to the case $0\le s,t$.

There only remains to see that P-a.s., $s \geq 0 \to \beta_s$ is continuous to conclude (1.21). To this end we note that by (1.22) and (1.13),

(1.23) the laws of $\beta_s$, $s > 0$, and $B_t$, $t > 0$, on $C((0,\infty), \mathbb{R}^d)$ are identical.
open in 0 !

We let $X_u$, $u > 0$, denote the canonical process on $C((0,\infty), \mathbb{R}^d)$, and $\mathcal{G} = \sigma(X_u, u > 0)$ the canonical $\sigma$-algebra. If $Q$ stands for the common law on $(C((0,\infty), \mathbb{R}^d), \mathcal{G})$ of $\beta_s$, $s > 0$, or $B_t$, $t > 0$, then

(1.24) $\{\lim_{u \to 0} X_u = 0\} \in \mathcal{G}$ (it is an "event").

Indeed one has:

$$\{\lim_{u \to 0} X_u = 0\} = \bigcap_{n \geq 1} \bigcup_{m \geq 1} \bigcap_{u \in \mathbb{Q} \cap (0, \frac{1}{m})} \{|X_u| \leq \tfrac{1}{n}\} \in \mathcal{G}.$$

As a result we find

$$\underbrace{Q(\lim_{u \to 0} X_u = 0)}_{\overset{(1.23)}{\|}} = P[\lim_{u \to 0} B_u = 0] \overset{(1.1)\, i), iv)}{=} 1$$

$P[\lim_{u \to 0} \beta_u = 0]$, and hence $\beta_s$, $s \geq 0$, fulfills (1.1) iv) as well. □

## Construction of Brownian motion:

We are now going to construct a Brownian motion on some $(\Omega, \mathcal{A}, P)$. It suffices to consider the case $d = 1$, since by taking $d$ independent copies of a real valued Brownian motion, we obtain a $d$-dimensional Brownian motion.

We follow a method of Paul Lévy (48), later simplified by Z. Ciesielski (61).

We recall the Haar functions on $\mathbb{R}_+$:

(1.25) $\varphi_\ell(t) = 1_{[\ell, \ell+1)}(t)$, $\ell \in \mathbb{N}$,

$$\varphi_{m,k}(t) = 2^{\frac{m}{2}} 1_{[\frac{k}{2^m}, \frac{k}{2^m} + \frac{1}{2^{m+1}})}(t) - 2^{\frac{m}{2}} 1_{[\frac{k}{2^m} + \frac{1}{2^{m+1}}, \frac{k+1}{2^m})}(t),$$
with $m, k \in \mathbb{N}$.

Fact:

(1.26) The $\varphi_\ell$, $\ell \geq 0$, $\varphi_{m,k}$, $m, k \geq 0$, form a complete orthonormal basis of $L^2(\mathbb{R}_+, dt)$.

Indeed:
- The functions $\varphi_\ell$, $\varphi_{m,k}$ have unit $L^2(\mathbb{R}_+, dt)$-norms
- They are pairwise orthogonal in $L^2(\mathbb{R}_+, dt)$.
- The $L^2$-closure of the span of the $\varphi_\ell$, $\varphi_{m,k}$ is $L^2(\mathbb{R}_+, dt)$, because one sees by induction on $m \geq 0$, that all $1_{[\frac{j}{2^m}, \frac{j+1}{2^m})}$, $j \geq 0$, belong to the space generated by $\varphi_\ell$, $\ell \geq 0$, and $\varphi_{m',k}$, $0 \leq m' < m$, $0 \leq k$, and the above claim follows.

Heuristic (non-rigorous) description of the construction of B.M.
The idea is to use the formal development of $\dot{B}$ (the derivative of Brownian motion!!!) in the above Haar basis. Formally we have:

$$\dot{B}(\cdot) \text{"="} \sum_{\ell \geq 0} \varphi_\ell(\cdot) \Big(\int_0^\infty \dot{B} \varphi_\ell \, dt\Big) + \sum_{m,k \geq 0} \varphi_{m,k}(\cdot) \Big(\int_0^\infty \dot{B} \varphi_{m,k} \, dt\Big).$$

We then write

$$(1.27) \quad B(t) \text{"="} \int_0^t \dot{B}(u)\,du \text{"="} \sum_{\ell \geq 0} \int_0^t \varphi_\ell(u)\,du \Big(\int_0^\infty \dot{B} \varphi_\ell \, dt\Big) + \sum_{m,k \geq 0} \int_0^t \varphi_{m,k}(u)\,du \Big(\int_0^\infty \dot{B} \varphi_{m,k} \, dt\Big).$$

We now define

$$(1.28) \quad \eta_\ell \text{"="} \int_0^\infty \dot{B} \varphi_\ell \, dt \text{"="} \int_\ell^{\ell+1} \dot{B}\, dt = B(\ell+1) - B(\ell),$$

$$(1.29) \quad \eta_{m,k} \text{"="} \int_0^\infty \dot{B} \varphi_{m,k} \, dt \text{"="} 2^{\frac{m}{2}} \Big(\int_{\frac{k}{2^m}}^{\frac{k}{2^m}+\frac{1}{2^{m+1}}} \dot{B}\, dt - \int_{\frac{k}{2^m}+\frac{1}{2^{m+1}}}^{\frac{k+1}{2^m}} \dot{B}\, dt\Big)$$

$$\text{"="} 2^{\frac{m}{2}} \Big(B\Big(\frac{k}{2^m}+\frac{1}{2^{m+1}}\Big) - B\Big(\frac{k}{2^m}\Big)\Big) - 2^{\frac{m}{2}} \Big(B\Big(\frac{k+1}{2^m}\Big) - B\Big(\frac{k}{2^m}+\frac{1}{2^{m+1}}\Big)\Big).$$

Now if a Brownian motion exists, then the right-hand sides of (1.28) and (1.29) make sense and the above $\eta_\ell$, $\eta_{m,k}$ are $N(0,1)$-variables, and form a centered gaussian family, (since they are linear combinations of $B(t)$, $t \geq 0$). Moreover the variables $\eta_\ell$, $\ell \geq 0$, $\eta_{m,k}$, $m, k \geq 0$, are pairwise orthogonal:

$E[\xi_\ell \xi_{\ell'}]=0$, $\ell \neq \ell'$, $E[\xi_\ell \xi_{m,k}]=0$, for $\ell \neq \ell' \geq 0$, $m,k \geq 0$,
$E[\xi_{m,k}\xi_{m',k'}]=0$, for $(m,k) \neq (m',k')$.
This orthogonality follows from (1.1) ii), iii), (for instance $E[\xi_\ell \xi_{\ell'}]=0$, for $\ell \neq \ell'$ is immediate to check, likewise $E[\xi_\ell \xi_{m,k}]=0$; if $[\frac{k}{2^m}, \frac{k+1}{2^m}) \cap [\ell,\ell+1)=\phi$ is immediate, and on the other hand when the intersection is not empty, then $[\frac{k}{2^m}, \frac{k+1}{2^m}) \subseteq [\ell,\ell+1)$, and one has

$$E[\xi_\ell \xi_{m,k}] = 2^{\frac{m}{2}} E\Big[\big(B(\ell+1)-B(\ell)\big)\Big\{\Big(B\big(\tfrac{k+1}{2^m}\big)-B\big(\tfrac{k}{2^m}+\tfrac{1}{2^{m+1}}\big)\Big)-\Big(B\big(\tfrac{k}{2^m}+\tfrac{1}{2^{m+1}}\big)-B\big(\tfrac{k}{2^m}\big)\Big)\Big\}\Big]$$

and writing

$$B(\ell+1)-B(\ell) = \underbrace{B(\ell+1)-B\big(\tfrac{k+1}{2^m}\big)} + \underbrace{B\big(\tfrac{k+1}{2^m}\big)-B\big(\tfrac{k}{2^m}+\tfrac{1}{2^{m+1}}\big)} + \underbrace{B\big(\tfrac{k}{2^m}+\tfrac{1}{2^{m+1}}\big)-B\big(\tfrac{k}{2^m}\big)} + \underbrace{B\big(\tfrac{k}{2^m}\big)-B(\ell)}$$

we can use (1.1) ii), (iii) to conclude that $E[\xi_\ell \xi_{m,k}]=0$ as well, the last equality $E[\xi_{m,k}\xi_{m',k'}]=0$ for $(m,k)\neq(m',k')$ is shown with analogous considerations).
Since the $\xi_\ell$, $\xi_{m,k}$ form a centered Gaussian family, are $N(0,1)$-distributed and are pairwise uncorrelated, they are in fact $N(0,1)$-distributed, (cf. (1.13)). Hence the formal formulas (1.27), (1.28), (1.29) tell us where we should "look for a Brownian motion".
We will now see how the above non-rigorous considerations can be transformed into a real proof.

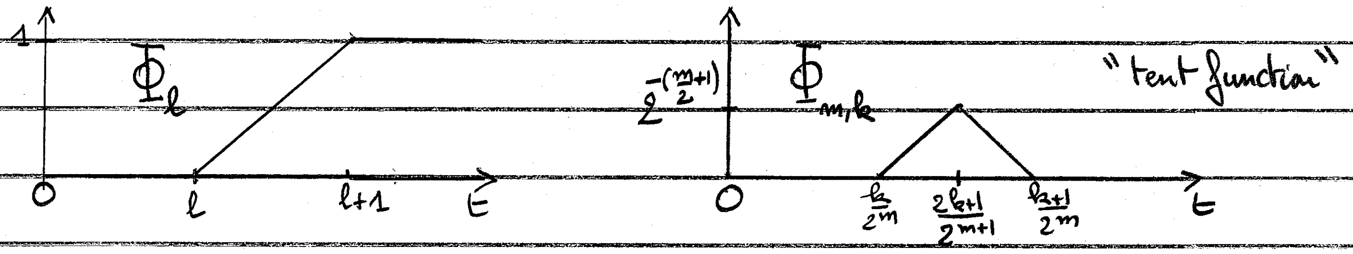
Mathematical construction:
We consider on some suitable probability space $(\Omega, \mathcal{Q}, P)$ a (countable) family of iid $N(0,1)$-distributed variables $\xi_\ell$, $\xi_{m,k}$, $\ell \geq 0$, $m,k \geq 0$, (for instance $\Omega=(0,1)$, $\mathcal{Q}=\mathcal{B}((0,1))$, $P$ = Lebesgue measure on $(0,1)$, will do the job).
We then define for $n \geq 1$ and $t \geq 0$:

(1.30) $X_n(t) = \sum_{0\le \ell<n} \Phi_\ell(t)\xi_\ell + \sum_{0\le m<n} \left(\sum_{0\le k<n2^m} \Phi_{m,k}(t)\xi_{m,k}\right)$,

where

(1.31) $\Phi_\ell(t) = \int_0^t \varphi_\ell(u)\,du$ and $\Phi_{m,k}(t) = \int_0^t \varphi_{m,k}(u)\,du$

(these are called Schauder functions)




Lemma:

(1.32) P-a.s., $X_n(\cdot,\omega)$ converges uniformly on compact intervals of $\mathbb{R}_+$ to a finite limit $X(\cdot,\omega)$.

Proof:

It suffices to prove that for any $n_0\ge 1$, P-a.s., $X_n(\cdot,\omega)$ converges uniformly on $[0,n_0]$ to a finite limit, (because we can then find $N\in\mathcal{Q}$, with $P[N]=0$, so that on $N^c$, $X_n(\cdot,\omega)$ converges uniformly on any $[0,n_0]$, and then define

$X(t,\omega) = \lim_n X_n(t,\omega)$, $\omega\in N^c$, $t\ge 0$,

$\quad = 0$, when $\omega\in N$, $t\ge 0$ ).

For $t\in[0,n_0]$, we have for $n\ge n_0$:

(1.33) $X_n(t) = \sum_{\ell=0}^{n_0-1} \Phi_\ell(t)\xi_\ell + \sum_{0\le m<n} \left(\sum_{0\le k<n_0 2^m} \Phi_{m,k}(t)\xi_{m,k}\right)$,

and for each $t\in[0,n_0]$, and each $m\ge 0$,

there is at most one $k\ge 0$, such that $\Phi_{m,k}(t)\ne 0$.

We then define:

(1.34) $a_m(\omega) = \sup_{t\in[0,n_0]} \left|\sum_{k<n_0 2^m} \Phi_{m,k}(t)\xi_{m,k}\right| \le 2^{-(\frac{m}{2}+1)} \sup_{k<n_0 2^m} |\xi_{m,k}|$.

We will control this supremum. To this end we note that for $\xi$ $N(0,1)$-distributed

(1.35) $P[|\xi|>a] \le \sqrt{\frac{2}{\pi}}\,\frac{1}{a}\exp\{-\frac{a^2}{2}\}$, for $a>0$,

(indeed $\int_0^\infty \exp\{-\frac{(a+u)^2}{2}\}du \le \int_0^\infty \exp\{-\frac{a^2}{2} - au\}du = \frac{1}{a}\exp\{-\frac{a^2}{2}\}$ ).

It follows that

$$\sum_{m\geq 1} P\Big[\sup_{0\leq k<n_0 2^m} |\eta_{m,k}| > \sqrt{2m}\Big] \leq \sum_{m\geq 1} \sqrt{\tfrac{2}{\pi}}\ \frac{n_0 2^m}{\sqrt{2m}}\ e^{-m} < \infty .$$

Thus Borel Cantelli's lemma implies that for P-a.e. $\omega$, there exists $m_0(\omega)$ such that for $m \geq m_0(\omega)$, $\sup_{k\leq n_0 2^m} |\eta_{m,k}(\omega)| \leq \sqrt{2m}$.
As a result:

(1.36) $\quad$ P-a.s., $\sum_{m\geq m_0(\omega)} a_m(\omega) \leq \sum_{m\geq m_0(\omega)} 2^{-(\frac{m}{2}+1)}\sqrt{2m} < \infty$.

It follows that P-a.s., $X_m(\cdot,\omega)$ converges uniformly on $[0,n_0]$ to a finite limit.

$\square$

We will now see that the above defined $X(t,\omega)$, $t\geq 0$, is a Brownian motion. First we observe that each $X_m(t)$, $t\geq 0$, is a centered Gaussian process, (the $X_m(t)$ are finite linear combinations of the iid $N(0,1)$-distributed $\eta_\ell$, $\ell\geq 0$, $\eta_{m,k}$, $m,k\geq 0$).
Note also that

(1.37) $\quad$ for $t\geq 0$, $X_m(t,\omega) \xrightarrow{L^2(\Omega,\mathcal{Q},P)} X(t,\omega)$.

Indeed the $\eta_\ell$, $\eta_{m,k}$ are orthogonal in $L^2(P)$ so that, cf (1.30), $X_m(t)$ and $X_{m+m}(t) - X_m(t)$ are orthogonal, and we only need to check that $\sup_m E[X_m^2(t)] < \infty$, that is:

(1.38) $\quad \sum_{\ell\geq 0} \Phi_\ell(t)^2 + \sum_{m,k\geq 0} \Phi_{m,k}(t)^2 < \infty$.

To check this last point we observe that the above sum equals:

(1.39) $\quad \sum_{\ell\geq 0}\Big(\int_0^t \varphi_\ell(u)\,du\Big)^2 + \sum_{m,k\geq 0}\Big(\int_0^t \varphi_{m,k}(u)\,du\Big)^2 \underset{\text{Parseval relation}}{\overset{(1.26)}{=}}$

$\|1_{[0,t]}\|^2_{L^2(\mathbb{R}_+,du)} = t < \infty$.

In a very similar vein we calculate $E[X_m(s)X_m(t)]$ for $0\leq s,t$, as follows:

(1.40) $E[X_n(s)X_n(t)] \overset{(1.30)}{=} \sum_{0\le l<n} \Phi_l(t)\Phi_l(s) + \sum_{\substack{0\le m<n\\ 0\le k<n2^m}} \Phi_{m,k}(t)\Phi_{m,k}(s)$

$$= \langle \pi_n(1_{[0,t]}), \pi_n(1_{[0,s]})\rangle_{L^2(\mathbb{R}_+,du)},$$

with $\pi_n$ the orthogonal projection in $L^2(\mathbb{R}_+,du)$ on the space spanned by $\varphi_l$, $0\le l<n$, $\varphi_{m,k}$, $0\le m<n$, $0\le k<n2^m$.

Combining (1.37) and (1.40), we find that

$$\begin{array}{ccc} E[X_n(s)X_n(t)] & = & \langle \pi_n(1_{[0,s]}), \pi_n(1_{[0,t]})\rangle_{L^2(\mathbb{R}_+,du)} \\ \big\downarrow n\to\infty & & \big\downarrow n\to\infty \\ E[X(s)X(t)] & = & \langle 1_{[0,s]}, 1_{[0,t]}\rangle_{L^2(\mathbb{R}_+,du)} = s\wedge t. \end{array}$$

Note that weak limits of centered Gaussian distributions are centered Gaussian, (take characteristic functions). It follows that linear combinations of $X(t)$, which are limit, in $L^2(P)$ by (1.37), and thus in distribution, of linear combinations of $X_n(t)$, are centered Gaussian variables.

So $X(t,\omega)$, $t\ge 0$, is a centered Gaussian process. From the above calculation $\Gamma(s,t)=s\wedge t$, and due to (1.32) $P$-a.s., $t\ge 0 \to X(t,\omega)$ is continuous.

We have thus proved that $X(t,\omega)$ is a Brownian motion on the probability space $(\Omega,\mathcal{Q},P)$ selected at the bottom of p. 14.