Lecture 3: (Chap. 1, cont.)

Complement:

We will now discuss an other construction of Brownian motion B_t, $0 \le t \le 1$, on time interval $[0,1]$, which gives a proof of a result of Paley and Wiener in (1934). This approach will bring into play some important methods on how to control the modulus of continuity of a stochastic process.

In place of the orthonormal basis of $L^2(\mathbb{R}_+, dt)$ given by the Haar functions, cf. (1.25), (1.26), we consider

(1.41) $\varphi_k(t), 0 \le t \le 1, k \ge 0$, an orthonormal basis of $L^2([0,1], dt)$,

as well as the sequence

(1.42) $\Phi_k(t) = \int_0^t \varphi_k(u)\,du, \quad 0 \le t \le 1, \ k \ge 0.$

Example: (Paley and Wiener)

The concrete choice of Paley and Wiener corresponds to:

$\varphi_0 \equiv 1$, $\varphi_k(t) = \sqrt{2}\cos(k\pi t)$, $k \ge 1$, $0 \le t \le 1$, so that

(1.43) $\Phi_0(t) = t$, and $\Phi_k(t) = \frac{\sqrt{2}}{k\pi}\sin(k\pi t), 0 \le t \le 1, k \ge 1.$

$\square$

In the spirit of (1.30), we consider on some probability space $(\Omega, \mathcal{A}, P)$, (for instance $\Omega = (0,1)$, $\mathcal{A} = \mathcal{B}((0,1))$, P = Lebesgue measure on $(0,1)$), a sequence η_k, $k \ge 0$, of iid $N(0,1)$-distributed variables.

Similarly to (1.30), we define for $n \ge 0$, $0 \le t \le 1$,

(1.44) $X_n(t) = \sum_{0 \le k \le n} \Phi_k(t)\,\eta_k.$

(Thus in the situation corresponding to the choice (1.43) of Paley and Wiener:

(1.45) $X_n(t) = t\eta_0 + \sum_{1 \le k \le n} \frac{\sqrt{2}}{k\pi}\sin(k\pi t)\,\eta_k$, for $0 \le t \le 1$, $n \ge 0$).

We will see that P a.s., $X_n(t,\omega)$ converges uniformly on $[0,1]$ to $X(t,\omega)$ distributed as the time restriction to $[0,1]$ of a Brownian motion. Here again the most delicate point has to do with the fact that the convergence is P-a.s. uniform on $[0,1]$. However unlike for the proof of (1.32), we cannot make use of the special properties of the orthonormal basis, (see for instance (1.34)). This will bring into play interesting considerations. We begin with a lemma concerning functions.

Lemma: $(T>0)$

For $r, \gamma > 0$, and $f \in C([0,T], \mathbb{R}^d)$, define

(1.46) $$I = \int_{[0,T]^2} \left(\frac{|f(t)-f(s)|}{|t-s|^\gamma} \right)^r ds\, dt,$$

Then for $0 \le s \le t \le T$, one has

(1.47) $$|f(t)-f(s)| \le 8 \int_0^{t-s} \left(\frac{4I}{u^2} \right)^{1/r} \gamma\, u^{\gamma-1}\, du$$

and when $\frac{2}{r} < \gamma$, this means that

(1.47)' $$|f(t)-f(s)| \le 8 \, \frac{\gamma}{\gamma - \frac{2}{r}} \, (4I)^{1/r} \, |t-s|^{\gamma - \frac{2}{r}}, \text{ for } 0 \le s \le t \le T.$$

Proof:

We only need to prove (1.47) when

(1.48) $$t = 1 = T, \quad s = 0, \quad \text{and} \quad \frac{2}{r} < \gamma.$$

The restriction $\frac{2}{r} < \gamma$ is clear, and note that given f and $0 \le s < t \le T$, as in the lemma, we can define

$\bar{f}(\tau) = f(s + (t-s)\tau)$, $0 \le \tau \le 1$,

so that if we know that (1.47) holds under (1.48), we find that:

(1.49) $$|f(t)-f(s)| = |\bar{f}(1) - \bar{f}(0)| \le 8 \, \frac{\gamma}{\gamma - \frac{2}{r}} \, (4\bar{I})^{1/r}, \text{ with}$$

$$\bar{I} = \int_{[0,1]^2} \left(\frac{|\bar{f}(u) - \bar{f}(v)|}{|u-v|^\gamma} \right)^r du\, dv \overset{\text{change of variable}}{\le} (t-s)^{-2+r\gamma}\, I$$

and inserting in (1.49) we find (1.47)'.

We thus assume (1.48). We define for $0 \le u \le 1$,

(1.50) $$J(u) = \int_0^1 \left(\frac{|f(u)-f(v)|}{|u-v|^\gamma} \right)^2 dv.$$

Since $\int_0^1 J(u)\,du = I$, there is a $t_0 \in (0,1)$ s.t. $J(t_0) \le I$.
We will show that

(1.51) $$\begin{cases} \text{There are } t_n,\ n \ge 0, \text{ in } (0,1) \text{ such that with } d_n^\gamma = \frac{1}{2} t_n^\gamma, \\ t_{n+1} \in (0, d_n), \\ J(t_{n+1}) \le \dfrac{2I}{d_n}, \text{ and } \left(\dfrac{|f(t_{n+1}) - f(t_n)|^2}{|t_{n+1}-t_n|^\gamma} \right) \le 2\, \dfrac{J(t_n)}{d_n}. \end{cases}$$

Indeed given t_n, define d_n as indicated, and note that
$\left|\left\{ u \in (0,d_n);\ J(u) > \frac{2I}{d_n} \right\}\right| < \frac{d_n}{2}$, since $\int_0^1 J(u)\,du = I$
Lebesgue measure
and
$$\left|\left\{ u \in (0,d_n);\ \left(\frac{|f(u)-f(t_n)|}{|u-t_n|^\gamma} \right)^2 > 2\, \frac{J(t_n)}{d_n} \right\}\right| < \frac{d_n}{2},$$

since $\int \left(\frac{|f(u)-f(t_n)|}{|u-t_n|^\gamma} \right)^2 du = J(t_n)$.

Hence we must be able to find t_{n+1} in $(0, d_n)$, for which the last line of (1.51) holds. This proves (1.51). We now have

(1.52) $$1 > t_0 > d_0 > t_1 > d_1 > \dots > t_n > d_n > \dots,$$

and since $d_{n+1}^\gamma \le \frac{1}{2} d_n^\gamma$, $d_n \to 0$, as $n \to \infty$.
Note also that:

(1.53) $$|t_n - t_{n+1}|^\gamma \le t_n^\gamma = 2 d_n^\gamma = 4\left(d_n^\gamma - \tfrac{1}{2} d_n^\gamma\right) \le 4\left(d_n^\gamma - d_{n+1}^\gamma\right).$$

From the last line of (1.51), with the convention $d_{-1} = 1$, we find for $n \ge 0$:

(1.54) $$|f(t_n) - f(t_{n+1})| \overset{(1.51)}{\le} \left(\frac{2J(t_n)}{d_n} \right)^{1/2} |t_n - t_{n+1}|^\gamma \overset{(1.51)}{\le} \left(\frac{4I}{d_n d_{n-1}} \right)^{1/2} |t_n - t_{n+1}|^\gamma$$

$$\overset{(1.53)}{\le} 4 \left(\frac{4I}{d_n d_{n-1}} \right)^{1/2} \left(d_n^\gamma - d_{n+1}^\gamma\right) \le 4 \int_{d_{n+1}}^{d_n} \left(\frac{4I}{u^2} \right)^{1/2} \gamma u^{\gamma-1}\,du.$$

Summing over n, we find:

(1.55) $$|f(t_0) - f(0)| \le 4 \int_0^{t_0} \left(\frac{4I}{u^2} \right)^{1/2} \gamma u^{\gamma-1}\,du.$$

If we introduce the function $f'(\cdot) = f(1-\cdot)$, for which the corresponding I' of course equals I, we can pick $t_0' = 1 - t_0$, and obtain with (1.55) that

$$|f'(1-t_0) - f'(0)| = |f(t_0) - f(1)| \leq 4 \int_0^{t_0'} \left(\frac{4I}{u^2}\right)^{1/2} \gamma u^{\gamma-1} du,$$

and combining with (1.55) deduce that

$$(1.56) \quad |f(1) - f(0)| \leq 8 \int_0^1 \left(\frac{4I}{u^2}\right)^{1/2} \gamma u^{\gamma-1} du,$$

thus concluding the proof of the lemma. □

Remark:

The above lemma is a special case of a more general result of Garsia, Rodemich, and Rumsey, (see D. Stroock: "Probability Theory, an Analytic View", p. 170). The interest of the lemma is that it enables to control the modulus of continuity of f in terms of an integral of $f(\cdot)$ given by I. This will be handy when proving Kolmogorov criterion below. The quantity I is also related to certain Besov norms, (see for instance the book of Adams: "Sobolev spaces", p. 214). □

As an application of the lemma we have

Theorem: (Kolmogorov's criterion)

If $X_n(t,\omega)$, $0 \leq t \leq T$, $n \geq 1$, are d-dimensional stochastic processes on $(\Omega, \mathcal{Q}, P)$, with continuous trajectories such that for $r > 0$, $q > 0$,

$$(1.57) \quad E\left[\sup_{n \geq 1} |X_n(t) - X_n(s)|^r\right] \leq C\, |t-s|^{1+q}, \text{ for } 0 \leq s \leq t \leq T,$$

then for each $\beta \in (0, \frac{q}{r})$ there is a $K(r, q, \beta, T) > 0$, such that:

$$(1.58) \quad P\left[\sup_{\substack{n \geq 1,\\ 0 \leq s < t \leq T}} \frac{|X_n(t) - X_n(s)|}{(t-s)^\beta} \geq R\right] \leq \frac{KC}{R^r}, \text{ for all } R > 0.$$

(Note that the processes $X_n(\cdot)$ may very well all coincide with $X_1(\cdot)$).

Proof:

We set

our choice of β

(1.59) $\gamma \overset{\text{def}}{=} \frac{2}{r} + \beta < \frac{2+q}{r}$,

and observe that (1.57) and Fubini's theorem imply that

(1.60) $$E\Big[\underbrace{\int_{[0,T]^2} \sup_{n\geq 1} \frac{|X_n(t)-X_n(s)|^r}{|t-s|^{r\gamma}}\,dt\,ds}_{\overset{\text{def}}{=} J(\omega)}\Big] \leq C \int_{[0,T]^2} C\,|t-s|^{\overbrace{1+q-r\gamma}^{>-1 \text{ by } (1.59)}}\,ds\,dt \overset{\text{def}}{=} C\,K_1(r,q,\beta,T) < \infty.$$

From the previous lemma we know that for $0 \leq s \leq t \leq T$

(1.61) $$\sup_{n\geq 1} |X_n(t) - X_n(s)| \overset{(1.47)'}{\leq} 8\,\frac{\gamma}{\beta}\,(4J(\omega))^{1/r}\,(t-s)^{\beta}.$$

It thus follows that for $R > 0$

$$P\Big[\sup_{n\geq 1} \frac{|X_n(t)-X_n(s)|}{|t-s|^{\beta}} \geq R\Big] \overset{(1.61)}{\leq} P\Big[8^2\Big(\frac{\gamma}{\beta}\Big)^2 4\,J(\omega) \geq R^2\Big] \underset{\text{inequality}}{\overset{\text{Markov}}{\leq}}$$

$$4.8^2\Big(\frac{\gamma}{\beta}\Big)^2 \frac{E[J(\omega)]}{R^2} \overset{(1.60)}{\leq} 4\,8^2\Big(\frac{\gamma}{\beta}\Big)^2 \frac{CK_1}{R^2},$$ and (1.58) follows.

□

Kolmogorov's criterion gives a powerful way to estimate the modulus of continuity of stochastic processes.

The next result in the special case of (1.43), (1.45), recovers a famous result of Paley and Wiener (1934).

Theorem: (under (1.41), (1.42))

P-a.s., $X_n(t) = \sum_{0\leq k\leq n} \Phi_k(t)\,\xi_k$, $0 \leq t \leq 1$,

(1.62) converges uniformly on $[0,1]$ to $X(t,\omega)$, $0 \leq t \leq 1$, which has the same law on $C([0,1],\mathbb{R})$ as B_t, $0\leq t \leq 1$, if B_t, $t \geq 0$, is a Brownian motion.

Proof:

We introduce the filtration

(1.63) $\mathcal{F}_n = \sigma(\xi_0, \xi_1, \dots, \xi_n)$, $n \geq 0$,

and note that for each $t \in [0,1]$ and $p \geq 1$:

(1.64) $X_n(t)$ is an $\mathcal{F}_n$-martingale, with finite L^p-norm.

It then follows from Doob's inequality (with $p=4$) that for $0 \leq s \leq t \leq 1$:

$$(1.65)\quad E\Big[\sup_{n\geq 0}(X_n(t)-X_n(s))^4\Big] \leq \Big(\frac{4}{3}\Big)^4 \sup_{n\geq 0} E\big[(X_n(t)-X_n(s))^4\big]$$

Note that $X_n(t)-X_s(s)$ is a centered Gaussian variable with variance

$$(1.66)\quad E\big[(X_n(t)-X_n(s))^2\big] \underset{(1.42)}{\overset{(1.44)}{=}} \sum_{0\leq k\leq n}\Big(\int_s^t \varphi_k(u)\,du\Big)^2 \leq \sum_{k\geq 0}\Big(\int_s^t \varphi_k(u)\,du\Big)^2$$

$$\underset{\text{relation}}{\overset{\text{Parseval}}{=}} \|1_{[s,t]}\|^2_{L^2([0,1],dt)} = t-s.$$

Now for Z an $N(0,1)$ variable we have

$E[Z^4] = 3$,

(indeed: $\int_{\mathbb{R}} x^4 e^{-x^2/2}\frac{dx}{\sqrt{2\pi}} \underset{\text{by parts}}{\overset{\text{integration}}{=}} \Big[x^3\Big(-\frac{e^{-x^2/2}}{\sqrt{2\pi}}\Big)\Big] + \int_{\mathbb{R}} 3x^2 e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} = 3$),

and hence by scaling we find that for Z, $N(0,\sigma^2)$ distributed,

$$(1.67)\quad E[Z^4] = 3\sigma^4$$

Combining this identity with (1.65), (1.66), we see that

$$(1.68)\quad E\Big[\sup_{n\geq 0}(X_n(t)-X_n(s))^4\Big] \leq C\,(t-s)^2, \text{ for } 0\leq s\leq t\leq 1,$$

$$\Big(\text{with } C = 3\Big(\frac{4}{3}\Big)^4\Big).$$

We can then apply Kolmogorov's criterion, where $\alpha = 1$, $r=4$, in (1.57) and choosing $\beta = \frac{1}{8}$ $\big(< \frac{1}{4} = \frac{\alpha}{r}\big)$, we deduce from (1.58) that

$$(1.69)\quad P\text{-a.s., } \sup_{\substack{n\geq 1\\ 0\leq s<t\leq 1}} \frac{|X_n(t)-X_n(s)|}{(t-s)^{1/8}} < \infty.$$

Since $X_n(0)=0$, for each n, it follows from Ascoli-Arzela's theorem that

(1.70) P-a.s., $X_n(\cdot,\omega)$, $n\geq 0$, is a relatively compact sequence in $C([0,1],\mathbb{R})$ endowed with the topology of uniform convergence.

From (1.68) with $s=0$, (1.64) and the martingale convergence theorem we also see that

(1.71) P-a.s., for all $t \in \mathbb{Q}\cap(0,1)$, $X_n(r,\omega)$ has a finite limit.

Combining (1.70) and (1.71), we have thus shown that

(1.72) P-a.s., $X_n(t,\omega)$ converges uniformly on $[0,1]$.
(this plays the role of (1.32)).
We can thus define a stochastic process $X(t,\omega)$, $0\le t\le 1$, such that
(1.73) $t\in[0,1] \to X(t,\omega)$ is continuous for each $\omega\in\Omega$, and
P-a.s., $\lim_n \sup_{t\in[0,1]} |X_n(t,\omega)-X(t,\omega)|=0$.
For each $t\in[0,1]$ the $(\mathcal{F}_n)$ martingale $X_n(t)$ also converges in $L^2(P)$, due to (1.66), with $s=0$, see also below (1.37). Proceeding as in (1.40), (see also (1.66)), we find that for $0\le s,t\le 1$:

$$E[X_n(s)X_n(t)] \underset{n}{\longrightarrow} \langle 1_{[0,s]}, 1_{[0,t]}\rangle_{L^2([0,1],du)} = s\wedge t$$

$\downarrow n\to\infty$

(1.74) $E[X(s)X(t)] = s\wedge t$.

Moreover $X(t)$, $0\le t\le 1$, by the same arguments as on page 17 is also a centered gaussian process. Applying (1.13), we thus find that $X(t)$, $0\le t\le 1$, has the same law on $C([0,1],\mathbb{R})$ as B_t, $0\le t\le 1$, if B_t, $t\ge 0$, is a Brownian motion. $\square$