Lecture 4:

CHAPTER 2: BROWNIAN MOTION AND MARKOV PROPERTY

We are going to successively discuss the "simple Markov property" and the "strong Markov property", and this chapter will revolve around the fact that Brownian motion is a canonical example of continuous Markov process.

Heuristically the simple Markov property states that if one "knows" the trajectory of a Brownian motion $X_.$ until time s, then the trajectory after time s: $X_{.+s}$, given this information behaves like a Brownian motion starting from the random initial position X_s. In particular only the knowledge of X_s matters in this prediction of the future after time s, given the past up to time s. The strong Markov property will extend this to stopping times in place of the fixed time s.

Notations: (as in Chapter 1)

$C = C(\mathbb{R}_+, \mathbb{R}^d)$,

$X_t : C \to \mathbb{R}^d$, $t \geq 0$, ("the canonical coordinates"),

$\mathcal{F} = \sigma(X_u, u \geq 0)$,

W = Wiener measure on $(\Omega, \mathcal{F})$.

For $x \in \mathbb{R}^d$, "Brownian motion starting from x" is described by the probability:

(2.1) W_x = the image of W under the map $w(\cdot) \in C \to w(\cdot) + x \in C$,

(we write E_x for the corresponding expectation).

In particular for $h(x_0, \dots, x_n) \in b\mathcal{B}((\mathbb{R}^d)^{n+1})$, $0 = t_0 < t_1 < \dots < t_n$,

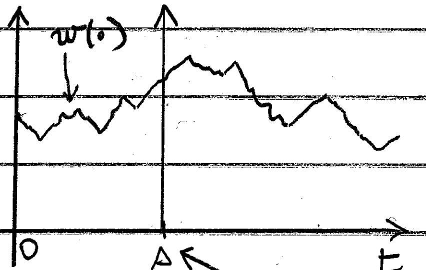
(2.2) $$E_x[h(X_{t_0}, X_{t_1}, \dots, X_{t_n})] = E^W[h(x + X_0, x + X_{t_1}, \dots, x + X_{t_n})]$$

$$\overset{(1.7)}{=} \int_{(\mathbb{R}^d)^n} h(x, x_1, \dots, x_n) \prod_{i=1}^n [2\pi(t_i - t_{i-1})]^{-d/2} \exp\Big\{-\sum_{i=1}^n \frac{|x_i - x_{i-1}|^2}{2(t_i - t_{i-1})}\Big\} dx_1 \dots dx_n$$

with $x_0 = x$

On C we have the time-shift operators:

(2.3) for $s \geq 0$, $\theta_s : (C, \mathcal{F}) \longrightarrow (C, \mathcal{F})$, $\theta_s(w)(\cdot) = w(s+\cdot)$
measurable by (1.5)



new origin of time.

Note that:

(2.4) $f(X_{t_0}, \dots, X_{t_n}) \circ \theta_s(w) = f(w(s+t_0), w(s+t_1), \dots, w(s+t_n))$
concerns the trajectory after time s.

The information contained in the past of the trajectory up to time s is described by:

(2.5) $\mathcal{F}_s = \sigma(X_u, u \leq s)$, and

(2.6) $\mathcal{F}_s^+ = \bigcap_{\varepsilon > 0} \mathcal{F}_{s+\varepsilon} \ (\supset \mathcal{F}_s)$,

so that "$\mathcal{F}_s^+$ peeks infinitesimally into the future after time s".

For instance the event "the trajectory immediately leaves its starting point":

$A = \bigcap_{n \geq 1} \left(\bigcup_{r \in \mathbb{Q} \cap [0, \frac{1}{n}]} \{X_r \neq X_0\} \right)$ is in $\mathcal{F}_0^+$ but not in $\mathcal{F}_0$.

Theorem: (simple Markov property)

$Y \in b\mathcal{F}$, $s \geq 0$, $x \in \mathbb{R}^d$, then

(2.7) $E_x[Y \circ \theta_s \mid \mathcal{F}_s^+] = E_{X_s}[Y]$, W_x-a.s.

(2.8) Under W_x, $(X_{s+u} - X_s)_{u \geq 0}$ is a Brownian motion independent of $\mathcal{F}_s^+$.

Proof:

(2.7):

Note that

(2.9) $y \in \mathbb{R}^d \to E_y[Y]$ is $b\mathcal{B}(\mathbb{R}^d)$-measurable for $Y \in b\mathcal{F}$.

Indeed this is true when $Y = 1_{A_0} \circ X_{t_0} \dots 1_{A_n} \circ X_{t_n}$, with

$t_0 < t_1 < \dots < t_n$ and $A_0, \dots, A_n \in \mathcal{B}(\mathbb{R}^d)$, thanks to (2.2).
We can then use Dynkin's lemma to conclude that this is also true when $Y = 1_F$, with $F \in \mathcal{F}$, and then approximate a general $Y \in b\mathcal{F}$ by step-functions of the form $\sum_{i=1}^m \lambda_i 1_{F_i}$ to obtain (2.9).

Then note that for $u_0 = 0 < \dots < u_n = s$, $t_0 = 0 < \dots < t_k$, with f, g bounded measurable one has

(2.10) $$E_x[f(X_{u_0}, \dots, X_{u_n})\, g(X_{s+t_0}, \dots, X_{s+t_k})] =$$

$$\int_{(\mathbb{R}^d)^{n+k}} f(x, x_1, \dots, x_n)\, g(x_n, \dots, x_{n+k}) \prod_{i=1}^n [2\pi(u_i - u_{i-1})]^{-d/2} \prod_{j=1}^k [2\pi(t_j - t_{j-1})]^{-d/2}$$
$$\exp\Big\{-\frac{1}{2}\sum_{i=1}^n \frac{|x_i - x_{i-1}|^2}{u_i - u_{i-1}} - \frac{1}{2}\sum_{j=1}^k \frac{|x_{n+j} - x_{n+j-1}|^2}{t_j - t_{j-1}}\Big\}\, dx_1 \dots dx_{n+k} \overset{(2.2)}{=}$$

$$\int_{(\mathbb{R}^d)^n} f(x, x_1, \dots, x_n)\, E_{x_n}[g(X_{t_0}, \dots, X_{t_k})] \prod_{i=1}^n [2\pi(u_i - u_{i-1})]^{-d/2} dx_1 \dots dx_n \overset{(2.2)}{=}$$

$$E_x[f(X_0(w), \dots, X_{u_n}(w))\, E_{X_s(w)}[g(X_{t_0}, \dots, X_{t_k})]]$$

↑ this is the function: $w \to E_{X_s(w)}[g(X_{t_0}, \dots, X_{t_k})]$.

Using Dynkin's lemma, we see that for $s \ge 0$, and $A \in \mathcal{F}_s$:

(2.11) $$E_x[1_A\, g(X_{s+t_0}, \dots, X_{s+t_k})] = E_x[1_A\, E_{X_s}[g(X_{t_0}, \dots, X_{t_k})]].$$

If we now pick g continuous and bounded, clearly

(2.12) $$y \in \mathbb{R}^d \to E_y[g(X_{t_0}, \dots, X_{t_k})] = E_0[g(X_{t_0} + y, \dots, X_{t_k} + y)] \in \mathbb{R}$$

is a continuous bounded function thanks to dominated convergence.
We can apply (2.11) with $s + \varepsilon$ in place of s, $A \in \mathcal{F}_s^+ \subseteq \mathcal{F}_{s+\varepsilon}$, and letting $\varepsilon \to 0$, see by dominated convergence that (2.11) holds for $s \ge 0$, $A \in \mathcal{F}_s^+$, and g bounded continuous.
Then by approximation, it holds for g of the form
$g(x_0, \dots, x_k) = \prod_{i=0}^k 1_{K_i}(x_i)$, with K_i, $i = 0, \dots, k$, closed in $\mathbb{R}^d$.
Using Dynkin's lemma once more we see that
for $s \ge 0$, $A \in \mathcal{F}_s^+$, $A' \in \mathcal{F}$, $Y = 1_{A'}$:

(2.13) $$E_x[1_A\, Y \circ \theta_s] = E_x[1_A\, E_{X_s}[Y]].$$

Then using a uniform approximation by step functions, we see that (2.13) holds for $Y \in b\mathcal{F}$. This proves (2.7).

— (2.8):

$(X_{s+u} - X_s)_{u \geq 0}$ has continuous trajectories, and for $f \in bB((\mathbb{R}^d)^{n+1})$, $0 = t_0 < t_1 < \dots < t_n$,

(2.14) $E_x[f(X_{s+t_0} - X_s, \dots, X_{s+t_n} - X_s) \mid \mathcal{F}_s^+] =$

$E_x[f(X_{t_0} - X_0, \dots, X_{t_n} - X_0) \circ \theta_s \mid \mathcal{F}_s^+] \overset{(2.7)}{=}$

$E_{X_s}[f(X_{t_0} - X_0, \dots, X_{t_n} - X_0)] \overset{(2.1)}{=} E_0[f(X_{t_0}, \dots, X_{t_n})]$.

It now readily follows that $(X_{s+u} - X_s)_{u \geq 0}$ fulfills (1.1), and is a Brownian motion on $(C, \mathcal{F}, W_x)$. Moreover it straightforward from (2.14) with Dynkin's lemma to see that for any $F(\cdot): C \to \mathbb{R}$, bounded measurable $F(X_{s+\cdot} - X_s)$ is independent of $\mathcal{F}_s^+$. This proves (2.8).

□

Corollary: (Blumenthal's 0-1 law)

(2.15) For any $x \in \mathbb{R}^d$, $W_x(A) \in \{0,1\}$, when $A \in \mathcal{F}_0^+$.

Proof:

$1_A \circ \theta_0 = 1_A$, since θ_0 is the identity map.

Therefore we find that

$E_x[1_A \mid \mathcal{F}_0^+] = 1_A \quad W_x\text{-a.s. (since } A \in \mathcal{F}_0^+),$

and

$E_x[1_A \mid \mathcal{F}_0^+] = E_x[1_A \circ \theta_0 \mid \mathcal{F}_0^+] \overset{(2.7)}{=} E_{X_0}[1_A] = W_x(A)$, W_x-a.s., since $W_x(X_0 = x) = 1$.

As a result W_x-a.s. $1_A = W_x(A)$, and the claim (2.15) follows. □

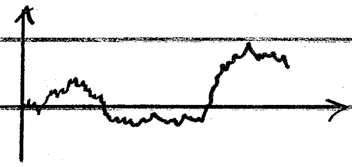
As we now see the σ-algebra $\mathcal{F}_0^+$ contains some interesting events and this explains the interest of the Blumenthal's 0-1 law.

Examples:

1) $d=1$, let $\tilde{H}_+ \overset{def}{=} \inf\{s>0;\, X_s>0\}$, $\tilde{H}_- \overset{def}{=} \inf\{s>0;\, X_s<0\}$ denote the respective hitting times of $(0,\infty)$ and $(-\infty,0)$.

Proposition:

(2.16) W_0-a.s. , $\tilde{H}_+ = \tilde{H}_- = 0$



Proof:

$\{\tilde{H}_+=0\} = \bigcap_{n\geq 1}\Big(\bigcup_{r\in[0,\frac{1}{n}]\cap\mathbb{Q}}\{X_r\in(0,\infty)\}\Big)\in \mathcal{F}_0^+$

and for $t>0$,

$W_0(\tilde{H}_+\leq t) \geq W_0(X_t>0) = \frac{1}{2}$

$\downarrow$ decreases for $t\to 0$

$W_0(\tilde{H}_+=0)\geq \frac{1}{2}$.

Thus by (2.15), we find that $W_0(\tilde{H}_+=0)=1$.

Of course in the same way $W_0(\tilde{H}_-=0)=1$. $\square$

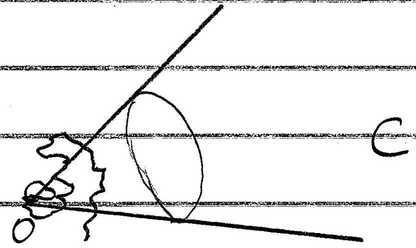
2) $d\geq 2$, C some open cone with tip 0 in $\mathbb{R}^d$ (i.e. C is open, and for $x\in\mathbb{R}^d$, $x\in C \Leftrightarrow \lambda x\in C$, for all $\lambda>0$).

Define the hitting time of C:

(2.17) $\tilde{H}_C = \inf\{s>0;\, X_s\in C\}$.

Proposition:

(2.18) W_0-a.s., $\tilde{H}_C = 0$.



Proof:

The argument is similar to the proof of (2.16).
We use the fact that $\{\tilde{H}_C = 0\} \in \mathcal{F}_0^+$ and

$$W_0(\tilde{H}_C \le t) \ge W_0(X_t \in C) \overset{\text{scaling}}{=} W_0(\sqrt{t}\, X_1 \in C) \underset{\text{with tip } 0}{\overset{C \text{ is a cone}}{=}} W_0(X_1 \in C) > 0.$$

One then concludes as for (2.16). □

3) $d = 1$, $t_n > 0$, $n \ge 1$, with $\lim t_n = 0$,

Proposition:

(2.19) W_0-a.s., $\overline{\lim\limits_n} \dfrac{X_{t_n}}{\sqrt{t_n}} = \infty$.

Proof:

For $c > 0$, note that $A \overset{\text{def}}{=} \limsup_n \{X_{t_n} > c\sqrt{t_n}\} \in \mathcal{F}_0^+$.
Indeed $A \overset{\text{def}}{=} \bigcap_{n \ge 1} A_n = \bigcap_{n \ge n_0} A_n$ with $A_n = \bigcup_{m \ge n} \{X_{t_m} > c\sqrt{t_m}\}$,
so $A \in \mathcal{F}_\varepsilon$, if $t_m \le \varepsilon$, for $m \ge n_0$, but $\varepsilon > 0$ can be chosen arbitrary.
By (2.15), $W_0(A) = 0$ or 1, moreover A_n decreases with n, so that:

(2.20) $W_0(A) = \lim_n W_0(A_n) \ge \varliminf_n W_0(X_{t_n} > c\sqrt{t_n}) \overset{\text{scaling}}{=} W_0(X_1 > c) > 0$.

As a result $W_0(A) = 1$, for arbitrary $c > 0$.
In particular, W_0-a.s., $\overline{\lim\limits_n} \dfrac{X_{t_n}}{\sqrt{t_n}} \ge c$, and choosing $c = k$, $k \ge 1$, we obtain (2.19). □

Exercise: Show that:
Under W_0, the asymptotic σ-field $\mathcal{Q} = \bigcap_{u < \infty} \sigma(X_v, v \ge u)$ is trivial, that is
$W_0(A) \in \{0, 1\}$, for any $A \in \mathcal{Q}$.
In fact more is true:
$A \in \mathcal{Q} \Rightarrow W_x(A) = 0$, for all $x \in \mathbb{R}^d$, or $W_x(A) = 1$, for all $x \in \mathbb{R}^d$.
(Hint: use (1.21), Blumenthal's 0-1 law and the Markov property). □