Lecture 5: (chap. 2, cont.)

We continue our discussion of the simple Markov property, and will in the spirit of (1.15), (in the case of Gaussian processes), provide a Markovian characterization of Brownian motion. For this purpose, we introduce the Brownian transition semigroup:

(2.21) $R_t f(x) = E_x[f(X_t)]$ , $x \in \mathbb{R}^d, t \geq 0, f \in bB(\mathbb{R}^d)$

$= (2\pi t)^{-d/2} \int_{\mathbb{R}^d} f(y) \exp\{-\frac{|y-x|^2}{2t}\} dy$ , when $t > 0$.

$R_t, t \geq 0$, satisfies the semigroup property:

(2.22) $R_{t+s} = R_t \circ R_s$ , $t, s \geq 0$.

Indeed one has with the help of the simple Markov property:

$R_{t+s} f(x) = E_x[f(X_{t+s})] = E_x[f(X_s) \circ \theta_t] \overset{(2.7)}{=} E_x[E_{X_t}[f(X_s)]]$

$\overset{(2.21)}{=} E_x[R_s f(X_t)] \overset{(2.21)}{=} R_t(R_s f)(x)$.

One then has the following Markovian characterization of Brownian motion, (compare with (1.15) for Gaussian processes):

Proposition:

Let $B_t, t \geq 0$, be an $\mathbb{R}^d$-valued process defined on $(\Omega, \mathcal{A}, P)$, with $P$-a.s. continuous trajectories, and $\mathcal{G}_s = \sigma(B_u, u \leq s)$. Then

(2.23) $B_t, t \geq 0$, is a Brownian motion $\Longleftrightarrow$ $B_0 = 0$, $P$-a.s., and $E[f(B_{t+s}) | \mathcal{G}_s] = R_t f(B_s)$ , $P$-a.s., for $f \in bB(\mathbb{R}^d)$ and $t, s \geq 0$.

Proof:

$\Longrightarrow$:

Using (1.10) and (2.21), if $B_t, t \geq 0$, is a Brownian motion, for $0 = t_0 < \dots < t_n = s$, $t > 0$, and $f_0, \dots, f_n, f \in bB(\mathbb{R}^d)$:

$E[f_0(B_{t_0}) \dots f_n(B_{t_n}) f(B_{t+s})] \overset{(1.10)}{=} E[f_0(B_{t_0}) \dots f_n(B_{t_n = s}) R_t f(B_s)]$

and using Dynkin's lemma, for any $G \in \mathcal{G}_s$:

$E[1_G f(B_{t+s})] = E[1_G R_t f(B_s)]$,

from which we deduce that $P$-a.s. $E[f(B_{t+s}) | \mathcal{G}_s] = R_t f(B_s)$.

The fact that $B_0 = 0$, $P$-a.s. is automatic.

— $\Leftarrow$:

By induction we see that for $t_0 = 0 < \dots < t_n$, $f_0, \dots, f_n \in bB(\mathbb{R}^d)$:

$$E[f_0(B_{t_0}) \dots f_n(B_{t_n})] = \int_{(\mathbb{R}^d)^n} f_0(0)\, f_1(x_1) \dots f_n(x_n) \prod_{i=1}^{n} [2\pi(t_i - t_{i-1})]^{-d/2} e^{-\frac{1}{2}\sum_{i=1}^{n} \frac{|x_i - x_{i-1}|^2}{t_i - t_{i-1}}} dx_1 \dots dx_n$$

As a result $B_t$, $t \geq 0$, has same finite dimensional distributions as $X_t$, $t \geq 0$, under $W$ (= Wiener measure), cf. (1.7), and it thus satisfies (1.1). Our claim follows. □

## Strong Markov property

In order to discuss the strong Markov property we need to introduce the notion of stopping times.

In the case of a discrete filtration $(\Omega, \mathcal{G}, (\mathcal{G}_n)_{n \geq 0})$, (i.e. the $\sigma$-algebras $\mathcal{G}_n$, $n \geq 0$, $\mathcal{G}$ satisfy $\mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \dots \subseteq \mathcal{G}_n \subseteq \dots \subseteq \mathcal{G}$), a stopping time is defined as a map $T: \Omega \to \mathbb{N} \cup \{\infty\}$, such that $\{T = n\} \in \mathcal{G}_n$, for each $n \geq 0$.

In other words "the decision to stop at a certain time $n$ is a function of the information known by time $n$".

In the case where time varies in $\mathbb{R}_+ = [0, \infty)$ in place of $\mathbb{N} (= \{0, 1, 2, \dots\})$, the "right way" to interpret the above sentence comes in the next:

Definition:

$(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0})$, where $(\mathcal{G}_t)_{t \geq 0}$ is assumed to be a filtration (i.e. the $\sigma$-algebras $\mathcal{G}_t$, $t \geq 0$, satisfy $\mathcal{G}_s \subseteq \mathcal{G}_t \subseteq \mathcal{G}$, for $0 \leq s \leq t$), then $T: \Omega \to [0, \infty]$ is a $(\mathcal{G}_t)$-stopping time if:

(2.24) $\{T \leq t\} \in \mathcal{G}_t$, for $t \geq 0$.

The "$\sigma$-algebra of the past of $T$" is defined as:

(2.25) $\mathcal{G}_T = \{A \in \mathcal{G};\ A \cap \{T \le t\} \in \mathcal{G}_t$, for each $t \ge 0\}$,
(this is indeed a $\sigma$-algebra!).

Examples:

1) Entrance time in a closed set:

Consider the canonical space $(C, \mathcal{F})$ and $\mathcal{F}_t$, $t \ge 0$, as in (2.5), as well as $A$ a closed subset of $\mathbb{R}^d$. The entrance time of $X_.$ in $A$ is

(2.26) $H_A = \inf\{s \ge 0;\ X_s \in A\}$, (by convention $H_A = \infty$, when $\{\dots\} = \phi$).

We will now see that

(2.27) $H_A$ is an $(\mathcal{F}_t)$-stopping time.

Indeed for $w \in C$, $\{s \ge 0,\ \overset{\text{closed}}{\underset{\nwarrow\ \text{continuous}}{X_s(w)}} \in A\}$ is a closed subset, which thus contains $H_A(w)$ when it is finite. Hence for $t \ge 0$:

$H_A(w) > t \iff \forall s \in [0,t],\ \text{dist}(X_s(w), A) > 0 \iff \inf\limits_{[0,t]} \text{dist}(X_s(w), A) > 0.$

Therefore we see that

$$\{H_A > t\} = \bigcup_{n \ge 1} \bigcap_{s \in \mathbb{Q} \cap [0,t]} \{\text{dist}(X_s(w), A) > \tfrac{1}{n}\} \in \mathcal{F}_t,$$

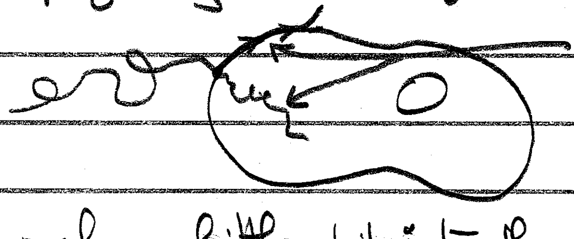
and (2.27) follows.

2) Entrance time in an open set

We now replace $A$ with $O$ open subset of $\mathbb{R}^d$, and of course set

(2.28) $H_O = \inf\{s \ge 0;\ X_s \in O\}$.

Observe that $\{H_O = t\}$ is not $\mathcal{F}_t$-measurable.



two possible trajectories, which agree up to time $t$, one of which has $H_O = t$ and the other not.

One needs to "peek a little bit into the future" to decide whether $H_O = t$ or not. This motivates the use of the filtration $\mathcal{F}_t^+ = \bigcap\limits_{\varepsilon > 0} \mathcal{F}_{t+\varepsilon}$, $t \ge 0$.

Proposition:

(2.29) If $O$ is an open set of $\mathbb{R}^d$, $H_O$ is an $(\mathcal{F}_t^+)$-stopping time.

Proof:

$\{H_0 < s\} = \bigcup_{u\in\mathbb{Q}\cap[0,s)} \{X_u \in O\} \in \mathcal{F}_s$, for $s \geq 0$,

and hence

$\{H_0 \leq s\} = \bigcap_{\varepsilon>0} \{H_0 < s+\varepsilon\} \in \mathcal{F}_s^+$, for $s \geq 0$. □

Remark:

By the above argument we also see that:

when $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0})$ is such that the filtration $(\mathcal{G}_t)_{t\geq 0}$ is right continuous, (i.e. $\mathcal{G}_t = \bigcap_{\varepsilon>0} \mathcal{G}_{t+\varepsilon}$, for all $t \geq 0$), then

(2.30) $T$ is a $(\mathcal{G}_t)$-stopping time $\Longleftrightarrow \{T<t\} \in \mathcal{G}_t$, for all $t>0$.

(indeed "$\Rightarrow$" immediate and for "$\Leftarrow$": $\{T \leq s\} = \bigcap_{n\geq 1} \underbrace{\{T < s+\frac{1}{n}\}}_{\text{decreasing in } n}$ which belongs to $\bigcap_{\varepsilon>0} \mathcal{G}_{t+\varepsilon} = \mathcal{G}_t$). □

Here are now some simple useful properties:

Proposition: $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0})$

(2.31) $T$ stopping time $\Rightarrow$ $T$ is $\mathcal{G}_T$-measurable.

(2.32) $S, T$ stopping times, then $T \wedge S$ $(= \min(T,S))$ and $T \vee S$ $(= \max(T,S))$ are stopping times.

(2.33) In the case of $(C, \mathcal{F})$, if $T, S$ are $(\mathcal{F}_t^+)$-stopping times then

$$T + S\circ\theta_T = T(w) + S(\theta_{T(w)}(w)), \text{ when } T(w)<\infty,$$
$$= \infty, \text{ when } T(w)=\infty,$$

is also an $(\mathcal{F}_t^+)$-stopping time.

Proof:

(2.31):

It suffices to show that $\{T\leq u\} \in \mathcal{G}_T$, for $u \geq 0$, and indeed $\{T \leq u\} \cap \{T \leq t\} = \{T \leq u \wedge t\} \in \mathcal{G}_{u\wedge t} \subseteq \mathcal{G}_t$, for all $t \geq 0$, and by (2.25), $\{T \leq u\} \in \mathcal{F}_T$, for all $u \geq 0$.

(2.32):

$\{T\wedge S \leq t\} = \{T \leq t\} \cup \{S \leq t\} \in \mathcal{G}_t$, for $t \geq 0 \Rightarrow T \wedge S$ is a $(\mathcal{G}_t)$-stopping time.

$\{T \vee S < t\} = \{T<t\} \cap \{S<t\} \in \mathcal{G}_t$, for $t \geq 0 \Rightarrow T \vee S$ is a $(\mathcal{G}_t)$-stopping time.

(2.33):

$(\mathcal{F}_t^+)_{t \geq 0}$ is a right-continuous filtration (check it!), and by (2.30) we only need to show that for $t>0$:

(2.34) $\{T + S \circ \theta_T < t\} \in \mathcal{F}_t^+$.

To this effect note that

(2.35) $\{T + S \circ \theta_T < t\} = \bigcup_{\substack{u,v \in \mathbb{Q} \cap (0,\infty) \\ u+v<t}} \{T<u, S \circ \theta_T < v\}$.

We will use the following claim:

(2.36) Assume $\{T<u\} \neq \phi$, then $\theta_T : (\{T<u\}, \mathcal{F}_{v+u} \cap \{T<u\}) \rightarrow (C, \mathcal{F}_v)$ is measurable.

Indeed for $0 \leq s \leq v$, $X_s \circ \theta_T$ is measurable as a map: $(\{T<u\}, \mathcal{F}_{v+u} \cap \{T<u\}) \longrightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, as follows from the equality valid for $w \in \{T<u\}$:

$$X_s \circ \theta_T (w) = X_{s+T(w)}(w) = \lim_{n \to \infty} \sum_{1 \leq k \leq n} \underbrace{X_{s+\frac{k}{n}u}(w)}_{\substack{\text{measurable} \\ \mathcal{F}_{s+u} \cap \{T<u\} \\ \overset{s \leq v}{\subseteq} \mathcal{F}_{v+u} \cap \{T<u\}}} \underbrace{1\{\tfrac{(k-1)}{n}u \leq T < \tfrac{k}{n}u\}}_{\substack{\in \mathcal{F}_u^+ \cap \{T<u\} \\ \subseteq \mathcal{F}_{v+u} \cap \{T<u\}}}$$

The claim (2.36) now follows from a similar argument as (1.5).

On the other hand $\{S<v\} = \bigcup_{\substack{r<v \\ r \in \mathbb{Q} \cap (0,\infty)}} \{S<r\} \in \mathcal{F}_v$, and we see that the event in the union in the right-hand side of (2.35) satisfies:

$$\{T<u, S \circ \theta_T < v\} = \{w \in \{T<u\}, \theta_T(w) \in \{S<v\}\} \overset{(2.36)}{\in} \mathcal{F}_{u+v} \cap \overbrace{\{T<u\}}^{\in \mathcal{F}_u^+ (\text{even } \mathcal{F}_u)} \subseteq \mathcal{F}_{u+v}.$$

Thus coming back to (2.35) we have shown that $\{T + S \circ \theta_T < t\} \in \mathcal{F}_t \subseteq \mathcal{F}_t^+$, and (2.34) is proved, whence (2.33). □

Complement:

Special characterization of $\mathcal{F}_T$, when $T$ is an $(\mathcal{F}_t)$-stopping time on the canonical space $C$. We have the identity:

(2.37) $\mathcal{F}_T = \sigma(X_{T \wedge s}, s \geq 0)$,

(in other words $\mathcal{F}_T$ describes the information of the trajectory $X$, stopped at time $T$).

Proof:

"$\supseteq$":

This is the easier direction, we need to show that

(2.38) $X_{T\wedge s}$ is $\mathcal{F}_T$-measurable for each $s\geq 0$.

For this purpose we write

(2.39) $X_{T\wedge s} = \lim\limits_{n\to\infty} \sum\limits_{k=0}^{\infty} X_{\frac{k}{2^n}\wedge s}\, 1\{\frac{k}{2^n}\leq T<\frac{k+1}{2^n}\}$.

Observe now that for $A\in\mathcal{F}_{\frac{k}{2^n}\wedge s}$, and $u\geq 0$,
$A\cap\{\frac{k}{2^n}\leq T<\frac{k+1}{2^n}\}\cap\{T\leq u\}$
is empty if $u<\frac{k}{2^n}$, and when $\frac{k}{2^n}\leq u<\frac{k+1}{2^n}$ coincides with
$A\cap\{T<\frac{k}{2^n}\}^c\cap\{T\leq u\}\in\mathcal{F}_u$, and when $u\geq\frac{k+1}{2^n}$ equals
$A\cap\{\frac{k}{2^n}\leq T<\frac{k+1}{2^n}\}\in\mathcal{F}_u$.
It thus follows that $A\cap\{\frac{k}{2^n}\leq T<\frac{k+1}{2^n}\}\in\mathcal{F}_T$. Then using an approximation of $X_{\frac{k}{2^n}\wedge s}$ by step functions we conclude that $X_{\frac{k}{2^n}\wedge s}\,1\{\frac{k}{2^n}\leq T<\frac{k+1}{2^n}\}$ is $\mathcal{F}_T$-measurable, and (2.38) follows from (2.39).

"$\subseteq$":

This step is more involved. We introduce the notation

(2.40) $w_t(\cdot) \overset{\text{déf}}{=} w(\cdot\wedge t)\,(\in C)$, for any $t\geq 0$, and $w\in C$.

We will use the following

Claim:

(2.41) $f(w) = f(w_{T(w)})$, for any $f\in b\mathcal{F}_T$ and $w\in C$.

To see that the claim holds note that it is obviously true when $T(w)=\infty$, since $w_{T(w)}=w$ in this case, and we only need to check that

(2.42) $f(w)\,1\{T(w)=t\} = f(w_t)\,1\{T(w)=t\}$, for all $t\geq 0$, and $f\in b\mathcal{F}_T$.

To see this last point we argue as follows:

using Dynkin's lemma we find that for $t \geq 0$,

(2.43) $Y(w) = Y(w_t)$, for any $Y \in b\mathcal{F}_t$, $w \in C$,

and since $T$ is an $(\mathcal{F}_t)$-stopping time, $\{T = t\} \in \mathcal{F}_t$, and hence with (2.43):

$$1\{T(w) = t\} = 1\{T(w_t) = t\}, \text{ for } t \geq 0,\ w \in C.$$

Similarly, when $f(\cdot)$ is $b\mathcal{F}_T$, using (2.25), (2.31), we see that: $f(w)\,1\{T(w) = t\} = \underbrace{f(w)\,1\{T(w) = t\}}_{\in b\mathcal{F}_T}\,1\{T(w) \leq t\} \in b\mathcal{F}_t$. As a result with (2.43) and the previous identity we find that for $w \in C$, $t \geq 0$,

$$f(w)\,1\{T(w) = t\} = f(w_t)\,1\{T(w_t) = t\} = f(w_t)\,1\{T(w) = t\},$$

and this proves (2.41) and completes the proof of (2.41).

Now from Dynkin's lemma we see that for any $f \in b\mathcal{F}_T$ there exists $F$ bounded measurable on $(\mathbb{R}^d)^{\mathbb{N}}$ and a sequence $(t_k)_{k \geq 0}$ in $[0,\infty)$ such that:

(2.44) $f(w) = F(w(t_0), w(t_1), \dots, w(t_k), \dots)$

Applying the claim (2.41) we thus find that for $f \in b\mathcal{F}_T$

$$\begin{aligned} f(w) = f(w_{T(w)}) &\overset{(2.44)}{=} F(w_{T(w)}(t_0), w_{T(w)}(t_1), \dots, w_{T(w)}(t_k), \dots) \\ &\underset{(2.40)}{=} F(w(T(w) \wedge t_0), w(T(w) \wedge t_1), \dots, w(T(w) \wedge t_k), \dots) \\ &= F(X_{T \wedge t_0}(w), X_{T \wedge t_1}(w), \dots, X_{T \wedge t_k}(w), \dots) \end{aligned}$$

and this proves that $\mathcal{F}_T \subseteq \sigma(X_{T \wedge s}, s \geq 0)$. $\square$

Exercises:

1) Show that $\mathcal{F}_T$ is generated by a countable collection of events, (hint: use (2.37)).

2) Given $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t \geq 0})$ and $S, T$ two $(\mathcal{G}_t)$-stopping times, show that:

   a) if $S \leq T$, $\mathcal{G}_S \subseteq \mathcal{G}_T$,

   b) $\{S < T\}$ and $\{S \leq T\}$ belong to $\mathcal{G}_S \cap \mathcal{G}_T$ (hint: write $\{S < T\} \cap \{T \leq t\} = \bigcup_{s \in \mathbb{Q} \cap [0,t]} \{S < s\} \cap \{T > s\} \cap \{T \leq t\}$, and $\{S < T\} \cap \{S \leq t\} = \{S \leq t\} \cap \{T > t\} \cup \{S < T\} \cap \{T \leq t\}$, and use that $\{S < T\} \in \mathcal{G}_T$),

   c) for $A \in \mathcal{G}_S$, $A \cap \{S < T\}$ and $A \cap \{S \leq T\}$ belong to $\mathcal{G}_{S \wedge T}$.