Lecture 6: (chap. 2, cont.)

We continue with the discussion of the strong Markov property. We consider $(C, \mathcal{F})$. We recall that $(\mathcal{F}_t^+)_{t\ge0}$ is a right-continuous filtration, cf. p. 35 and 34. We further observe that:

(2.45) for T an $(\mathcal{F}_t^+)_{t\ge0}$-stopping time, $\mathcal{F}_T^+ \overset{(2.25)}{=} \{A \in \mathcal{F};\ A \cap \{T \le t\} \in \mathcal{F}_t^+, \forall t \ge 0\}$
$= \{A \in \mathcal{F};\ A \cap \{T < t\} \in \mathcal{F}_t^+, \forall t \ge 0\}$.

Indeed "$\subseteq$" is immediate and for "$\supseteq$" when $A \cap \{T < t\} \in \mathcal{F}_t^+$, for all $t \ge 0$, then $A \cap \{T \le t\} = \bigcap_{n\ge1} A \cap \{T < t + \frac{1}{n}\} \in \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}^+ = \mathcal{F}_t^+$, for $t \ge 0$.

Theorem: (strong Markov property)

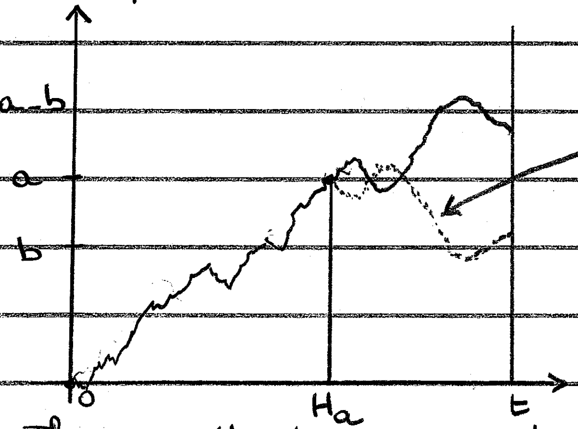
T an $(\mathcal{F}_t^+)_{t\ge0}$-stopping time, $Y \in b\mathcal{F}$, $x \in \mathbb{R}^d$, then

(2.46) $E_x[Y \circ \theta_T \mid \mathcal{F}_T^+] = E_{X_T}[Y]$, on $\{T < \infty\}$, W_x-a.s.,

(in other words, $\theta_T : (\{T<\infty\}, \mathcal{F} \cap \{T < \infty\}) \to (C, \mathcal{F})$ is measurable, and the random variable $E_{X_T}[Y]$ defined on $\{T < \infty\}$ is $\mathcal{F}_T^+ \cap \{T<\infty\}$-measurable and for any $A \in \mathcal{F}_T^+ \cap \{T < \infty\}$, $E_x[\underbrace{Y \circ \theta_T 1_A}_{\text{well-defined since } A \subseteq \{T<\infty\}}] = E_x[E_{X_T}[Y] 1_A]$.

Rather than discussing the proof right away we first provide an application.

The reflection principle:



$a \ge 0$, $b \le a$

reflection of the path after time H_a

$H_a = \inf\{s \ge 0, X_s = a\}$, "entrance time in a".

Theorem: ($d = 1$, $a \ge 0$, $b \le a$)

(2.47) $W_0(X_t \le b, \sup_{s \le t} X_s \ge a) = W_0(X_t \ge 2a - b)$, for $t > 0$, and

(2.48) $W_0(\sup_{s\le t} X_s \ge a) = 2 W_0(X_t \ge a)$

(in particular $\sup_{s \le t} X_s$ under W_0 has same law as $|X_t|$).

Proof:

– (2.47):

(2.49) $W_0(X_t \le b, \sup_{s\le t} X_s \ge a) = W_0(H_a \le t, X_t \le b) =$

$W_0(\{w \in C;\ H_a(w) \le t,\ X_{(t-H_a(w))_+}(\theta_{H_a(w)}(w)) \le b\})$.

We will use the following

Lemma: (T an $(\mathcal{F}_t^+)$-stopping time)

If $h(w_1, w_2)$ is $b\,\mathcal{F} \otimes \mathcal{F}_T^+$, then for any $x \in \mathbb{R}^d$, W_x-a.s on $\{T<\infty\}$,

(2.50) $E_x[h(\theta_{T(w)}(w), w) \mid \mathcal{F}_T^+] = \int_C h(w_1, w)\,dW_{X_T(w)}(w_1)$

variable of integration

Proof:

For $h = 1_{A_1}(w_1)\,1_{A_2}(w_2)$, $A_1 \in \mathcal{F}$, $A_2 \in \mathcal{F}_T^+$, (2.46) implies that for any $B \in \mathcal{F}_T^+ \cap \{T<\infty\}$, one has:

(2.51) $E_x[h(\theta_{T(w)}(w), w)\,1_B] = E_x[1_B \int_C h(w_1, w)\,dW_{X_T}(w_1)]$.

Then using Dynkin's lemma and approximation, (2.51) holds for any $h \in b\,\mathcal{F} \otimes \mathcal{F}_T^+$, and $\int_C h(w_1, w)\,dW_{X_T}(w_1)$ (defined on $\{T<\infty\}$) is $\mathcal{F}_T^+ \cap \{T<\infty\}$ measurable. Our claim follows.

□

We now apply the above lemma with $T = H_a$, and

$h(w_1, w_2) = 1\{X_{(t-H_a(w_2))_+}(w_1) \le b\}$,

which is $\mathcal{F} \otimes \mathcal{F}_{H_a}^+$ - measurable, because

$(w,t) \to X_t(w) = \lim_n \sum_{k\ge 0} X_{\frac{k}{2^n}}(w)\,1_{[\frac{k}{2^n}, \frac{k+1}{2^n})}(t)$ is $\mathcal{F} \times \mathcal{B}(\mathbb{R}_+)$-measurable

and one can realize $h(w_1, w_2)$ in the following steps

$(w_1, w_2) \in C \times C \to (w_1, (t-H_a(w_2))_+) \in C \times \mathbb{R}_+ \to X_{(t-H_a(w_2))_+}(w_1) \in \mathbb{R}^d$

$\to h(w_1, w_2) \in \mathbb{R}_+$,

and each step is induced by a measurable transformation relative to the natural σ-algebra.

We can thus apply (2.50) to the last line of (2.49) and find:

$$W_0(H_a \le t, X_t \le b) = E_0[H_a \le t, \tilde{W}_{a = X_{H_a}}(\tilde{X}_{(t-H_a)_+} \le b)]$$

$$\overset{\text{symmetry}}{=} E_0[H_a \le t, \tilde{W}_{X_{H_a}}(\tilde{X}_{(t-H_a)_+} \ge 2a-b)]$$ and going backward
$$\overset{(2.50)}{=} E_0[H_a \le t, X_t \ge 2a-b] = W_0[X_t \ge 2a-b].$$

We have thus shown that:

(2.52) $W_0(H_a \le t, X_t \le b) = W_0[X_t \ge 2a-b]$,

and together with (2.49) this proves (2.47).

– (2.48):

(2.53) $W_0[H_a \le t] = W_0[H_a \le t, X_t \ge a] + W_0[H_a \le t, X_t < a]$
$$= W_0[X_t \ge a] + W_0[H_a \le t, X_t < a]$$
$$\underset{\text{with } b=a}{\overset{(2.52)}{=}} W_0[X_t \ge a] + W_0[X_t \ge a] = 2W[X_t \ge a],$$

and this proves (2.48). □

Corollary:

For $a \in \mathbb{R}$, H_a is W_0-a.s. finite and has distribution:

(2.54) $W_0(H_a \in ds) = \frac{1}{\sqrt{2\pi}} \frac{|a|}{s^{3/2}} \exp\{-\frac{a^2}{2s}\} 1_{(0,\infty)}(s)\, ds.$

The joint law of X_t and $\sup_{s \le t} X_s$, for $t > 0$, is given by:

(2.55) $W_0(X_t \in db, \sup_{s \le t} X_s \in da) = \frac{2(2a-b)}{\sqrt{2\pi t^3}} \exp\{-\frac{(2a-b)^2}{2t}\} 1\{a > 0, b < a\}\, da\, db.$

Proof:

– (2.54):

$a > 0$, then $W_0(H_a \le s) \overset{(2.48)}{=} W_0(|X_s| \ge a) \overset{\text{scaling}}{=} W_0(|X_1| \ge a/\sqrt{s}) \underset{s \to \infty}{\longrightarrow} 1$,

an hence $W_0(H_a < \infty) = 1$. Moreover we have:

(2.56) $W_0[H_a \le s] = W_0[\sup_{u \le s} X_u \ge a] \overset{(2.48)}{=} 2W_0[X_1 \ge \frac{a}{\sqrt{s}}] = \frac{2}{\sqrt{2\pi}} \int_{\frac{a}{\sqrt{s}}}^{\infty} e^{-u^2/2}\, du.$

Differentiating in s we find (2.54).

— (2.55): Consider $0<a$, $b<a$, we have

$$(2.57)\quad W_0\big(X_t \le b, \sup_{s\le t} X_s \ge a\big) \overset{(2.47)}{=} W_0\big(X_t > 2a-b\big) = \frac{1}{\sqrt{2\pi t}} \int_{2a-b}^{\infty} e^{-x^2/2t}\, dx$$

setting $x = 2u-b$ ↓ ; derivating in b ↓

$$= \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-\frac{(2u-b)^2}{2t}}\, du = \int_{[a,\infty)\times(-\infty,b]} f(u,v)\, du\, dv, \text{ if}$$

$$f(u,v) = 2\,\frac{(2u-v)}{\sqrt{2\pi t^3}}\, \exp\Big\{-\frac{(2u-v)^2}{2t}\Big\}\, 1\{u>0, v<u\},$$

is the probability density that appears in the right-hand side of (2.55).
This probability density is concentrated on the open set:
$\Delta = \{(x,y)\in\mathbb{R}^2;\ x>0,\ y<x\}$.
Note that the same holds true for the joint law of $(\sup_{s\le t} X_s, X_t)$ under W_0. Indeed observe that

(2.58) $B_s \overset{\text{def}}{=} X_{t-s} - X_t$, $0\le s\le t$,

is a Brownian motion with time parameter $[0,t]$, (because it is a centered Gaussian process with continuous trajectories and covariance $E_0[B_s B_{s'}] = s\wedge s'$, $0\le s,s'\le t$). We know from (2.16) that the hitting time of $(0,\infty)$ by $B_.$ or by $X_.$ is a.s. equal to 0, (one can also see this from (2.47)). Therefore we have:

(2.59) W_0-a.s., $\sup_{s\le t} X_s > X_t$, and $\sup_{s\le t} X_s > 0$.

As a result the joint law of $(\sup_{s\le t} X_s, X_t)$ under W_0 is supported by Δ. Now the collection of subsets of Δ of the form $[a,\infty)\times(-\infty,b]$, with $a>0$, $b<a$, is a π-system, which generates $\mathcal{B}(\Delta)$ (the Borel subsets of Δ). By (2.57) and Dynkin's lemma we can conclude that (2.55) holds. □

Remark:

The collection of subsets $[a,\infty)\times(-\infty,b]$, of $\bar\Delta = \{(x,y)\in\mathbb{R}^2;\ x\ge 0,\ y\le x\}$, with $a\ge 0$, $b\le a$, is not rich enough to generate $\mathcal{B}(\bar\Delta)$, (their trace on $\{(x,y)\in\mathbb{R}^2,\ x=y\ge 0\}\subseteq\partial\Delta$ is at most a point). This is why we work with Δ and not $\bar\Delta$. □

As a preparation for the proof of the strong Markov property, we introduce the following

Definition:

Given $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0})$, an $\mathbb{R}^d$-valued process $Z(u,\omega)$, $u\geq 0$, $\omega\in\Omega$, is called progressively measurable if the restriction of Z to $[0,t]\times\Omega$ is $\mathcal{B}([0,t])\otimes\mathcal{G}_t$-measurable for each $t\geq 0$.

Example:

$Z(u,\omega)$ right-continuous in u, adapted, (i.e. $Z(u,\cdot)$ is $\mathcal{G}_u$-measurable for each $u\geq 0$), is progressively measurable because on $[0,t]\times\Omega$:

(2.60) $$Z(s,\omega) = \lim_{n\to\infty} \underbrace{\sum_{k=1}^{n} Z_{\frac{k}{n}t}\, 1\{\tfrac{k-1}{n}t \leq s < \tfrac{k}{n}t\} + Z_t(\omega)\, 1\{s=t\}}_{\mathcal{B}([0,t])\otimes\mathcal{G}_t\text{-measurable}}$$ □

The interest of this notion in our context comes from the next

Lemma:

Given $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0})$, $Z(t,\omega)$ progressively measurable, T a $(\mathcal{G}_t)$-stopping time one has

(2.61) Z_T (i.e. $\omega\in\{T<\infty\} \longrightarrow Z(T(\omega),\omega)\in\mathbb{R}^d$) is $\mathcal{G}_T\cap\{T<\infty\}$-measurable.

Proof:

It suffices to show that for any $t\geq 0$,

(2.62) $(\{T\leq t\}, \mathcal{G}_t\cap\{T\leq t\}) \xrightarrow{Z_T} (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is measurable.

But the above map is the composition of the measurable maps:

$$\underset{\mathcal{G}_t\cap\{T\leq t\}}{\omega\in\{T\leq t\}} \longrightarrow \underset{\mathcal{B}([0,t])\otimes\mathcal{G}_t}{(T(\omega),\omega)\in[0,t]\times\Omega \ni (u,\omega)} \longrightarrow \underset{\mathcal{B}(\mathbb{R}^d)}{Z(u,\omega)\in\mathbb{R}^d}$$

The first map is measurable because both maps

$(\{T\leq t\}, \mathcal{G}_t\cap\{T\leq t\}) \xrightarrow{T} ([0,t], \mathcal{B}([0,t]))$ and

$(\{T\leq t\}, \mathcal{G}_t\cap\{T\leq t\}) \xrightarrow{Id} (\Omega, \mathcal{G}_t)$ are measurable,

and the second map is measurable because Z is progressively measurable.

□