Lecture 7: (chap. 2, cont.)

We now turn to the proof of the strong Markov property.

Proof:

We will prove the theorem p.38 in a number of steps.

The first step is to show that:

(2.63) $\theta_T : (\{T<\infty\}, \mathcal{F} \cap \{T<\infty\}) \longrightarrow (C, \mathcal{F})$ is measurable.

Due to (1.5) we only need to show that for $s \geq 0$,

(2.64) $X_s \circ \theta_T : (\{T<\infty\}, \mathcal{F} \cap \{T<\infty\}) \longrightarrow (\mathbb{R}^d; \mathcal{B}(\mathbb{R}^d))$ is measurable,

and in the spirit of the proof in (2.36), we write for $w \in \{T<\infty\}$

$$X_s \circ \theta_T(w) = X_{s+T(w)}(w) = \lim_{n\to\infty} \sum_{k\geq 1} \underbrace{X_{s+\frac{k}{n}}(w)}_{\text{meas. } \mathcal{F}\cap\{T<\infty\}} \, 1\underbrace{\left\{\frac{(k-1)}{n} \leq T < \frac{k}{n}\right\}}_{\in \mathcal{F}\cap\{T<\infty\}}$$

whence (2.64) and therefore (2.63).

The second step is then to show that:

(2.65) $X_T : (\{T<\infty\}, \mathcal{F}_T^+ \cap \{T<\infty\}) \longrightarrow (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ is measurable.

To this effect we note that $X_t(\omega)$ is a progressively measurable process due to (2.60) and the claim (2.65) now follows from (2.61).

Note that $y \in \mathbb{R}^d \to E_y[Y]$ is measurable for any $Y \in b\mathcal{F}$, as shown in (2.9), (or in other words $y \in \mathbb{R}^d, A \in \mathcal{F} \to W_y(A) \in [0,1]$, is a stochastic kernel). Combining this observation with (2.63) and (2.65), the statement in (2.46) now makes sense.

The third step is to show that:

(2.66) when T take an at most denumerable set of values in $\mathbb{R}_+ \cup \{\infty\}$, then (2.46) is true.

This step will follow from the direct application of the simple Markov property. We write a_n, $0 \leq n < N$ $(\leq \infty)$ for the set of values of T in $[0,\infty)$.

Then for $A \in \mathcal{F}_T^+ \cap \{T<\infty\}$, $0=t_0<\dots<t_k$, $f \in b\mathcal{B}((\mathbb{R}^d)^{k+1})$ we find:

(2.67) $E_x[f(X_{t_0},\dots,X_{t_k})\circ\theta_T\, 1_A] = E_x[f(X_{T+t_0},\dots,X_{T+t_k})1_A] =$

$\sum_n E_x[f(X_{a_n+t_0},\dots,X_{a_n+t_k})\underbrace{1_{A\cap\{T=a_n\}}}_{\in \mathcal{F}_{a_n}^+}] \overset{(2.7)}{=}$

and with the simple Markov property:

$\sum_n E_x[E_{X_{a_n}}[f(X_{t_0},\dots,X_{t_k})]\,1_{A\cap\{T=a_n\}}] =$

$E_x[E_{X_T}[f(X_{t_0},\dots,X_{t_k})]\,1_A]$,

where the summation runs over the set of n such that $a_n \in [0,\infty)$ in lines two and four of (2.67). We can now use Dynkin's lemma and approximation to deduce that for $Y \in b\mathcal{F}$, one has

$E_x[Y\circ\theta_T\, 1_A] = E_x[E_{X_T}[Y]\,1_A]$,

and obtain (2.66).

The last step of the proof will be:

(2.68) the claim (2.46) holds for T a general $(\mathcal{F}_t^+)$-stopping time.

For this purpose, we use the discrete skeleton approximation of T:

(2.69) $T_n = \sum_{k\geq 0} \frac{k+1}{2^n}\, 1\{\frac{k}{2^n} \leq T < \frac{k+1}{2^n}\} + \infty\, 1\{T=\infty\}$.

The key observation is that:

(2.70) T_n is an $(\mathcal{F}_t^+)$-stopping time and $T_n \downarrow T$ as $n\to\infty$.

Indeed the fact that $T_n \geq T$ and $T_n \downarrow T$ as $n\to\infty$ is obvious from (2.69). In addition for $k,n\geq 0$,

$\{T_n \leq \frac{k+1}{2^n}\} \overset{(2.69)}{=} \{T < \frac{k+1}{2^n}\} \in \mathcal{F}^+_{\frac{k+1}{2^n}}$,

and for $t \in [\frac{k}{2^n}, \frac{k+1}{2^n})$ we have

$\{T_n \leq t\} = \{T_n \leq \frac{k}{2^n}\} \in \mathcal{F}^+_{\frac{k}{2^n}} \subseteq \mathcal{F}_t^+$,

so that T_n is an $(\mathcal{F}_t^+)$-stopping time.

Since $T \leq T_n$, it follows, cf. exercise 2 a) p.37, that

(2.71) $\mathcal{F}_T^+ \subseteq \mathcal{F}_{T_n}^+$, for $n\geq 0$.

Consider $A \in \mathcal{F}^+_T \cap \{T < \infty\}$. Since $\{T<\infty\} = \{T_n < \infty\}$, we also have $A \in \mathcal{F}^+_{T_n} \cap \{T_n < \infty\}$, and applying (2.66) we see that for $Y \in b\mathcal{F}$:

$$E_x[Y \circ \theta_{T_n} 1_A] = E_x[E_{X_{T_n}}[Y] 1_A]$$

Specializing to the case where $0 = t_0 < \ldots < t_k$ and $Y = f(X_{t_0}, \ldots, X_{t_k})$, with f bounded continuous on $(\mathbb{R}^d)^{k+1}$, we obtain that for $n \geq 0$:

$$(2.72)\quad E_x[f(X_{T_n + t_0}, \ldots, X_{T_n + t_k}) 1_A] = E_x[E_{X_{T_n}}[f(X_{t_0}, \ldots, X_{t_k})] 1_A].$$

We also know from (2.12) that:

$y \in \mathbb{R}^d \longrightarrow E_y[f(X_{t_0}, \ldots, X_{t_k})]$ is a bounded continuous function.

Therefore letting n tend to infinity in (2.72) we find that

$$(2.73)\quad E_x[f(X_{T+t_0}, \ldots, X_{T+t_k}) 1_A] = E_x[E_{X_T}[f(X_{t_0}, \ldots, X_{t_k})] 1_A].$$

By the same argument as below (2.12), we then find that (2.73) holds for $f(x_0, \ldots, x_k) = \prod_{i=0}^{k} 1_{K_i}(x_i)$, with K_i, $i = 0, \ldots, k$, closed subsets of $\mathbb{R}^d$, and then with Dynkin's lemma and approximation we obtain that

$$E_x[Y \circ \theta_T 1_A] = E_x[E_{X_T}[Y] 1_A],$$

for $Y \in b\mathcal{F}$ and $A \in \mathcal{F}^+_T \cap \{T < \infty\}$. This, recall (2.65), (2.9), now completes the proof of (2.46).

$\square$

Complement:

What can go wrong when going from the simple to the strong Markov property.

A typical example is given by the following process:

state space is $\mathbb{R}_+$.

The process waits an exponential time of parameter 1 in 0 and moves after with unit speed to the right. If it starts in $x > 0$, it simply moves to the right with unit speed.

We denote by P_x, $x \geq 0$, the law on $(C(\mathbb{R}_+, \mathbb{R}_+), \mathcal{F})$ of the process starting at $x \geq 0$.

For $t \geq 0$, one defines the operator $R_t : b\mathcal{B}(\mathbb{R}_+) \to b\mathcal{B}(\mathbb{R}_+)$ in analogy with (2.21), via:

$$(2.74) \quad R_t f(x) \overset{\text{def}}{=} E_x[f(X_t)] = f(x+t), \text{ if } x > 0$$
$$= e^{-t} f(0) + \int_0^t e^{-u} f(t-u)\,du, \text{ if } x = 0.$$

Note that R_t, $t \geq 0$, has the semigroup property:

$$(2.75) \quad R_{t+s} = R_t \circ R_s \;, \text{ for } s,t \geq 0.$$

Indeed when $f \in b\mathcal{B}(\mathbb{R}_+)$ and $x > 0$

$R_t(R_s f)(x) = (R_s f)(x+t) = f(x+t+s) = R_{t+s} f(x)$,

and when $x = 0$,

$$R_t(R_s f)(0) = e^{-t} R_s f(0) + \int_0^t e^{-u} R_s f(t-u)\,du$$
$$= e^{-t-s} f(0) + e^{-t}\int_0^s e^{-v} f(s-v)\,dv + \int_0^t e^{-u} f(t+s-u)\,du$$
$$= e^{-(t+s)} f(0) + \int_0^s e^{-(t+v)} f(s-v)\,dv + \int_0^t e^{-u} f(t+s-u)\,du$$
$$\overset{\text{setting } t+v=u}{=} e^{-(t+s)} f(0) + \int_t^{t+s} f(t+s-u)\,du + \int_0^t e^{-u} f(t+s-u)\,du$$
$$= R_{t+s} f(0), \text{ whence (2.75).}$$

Moreover one has the regularity:

$$(2.76) \quad R_t f(x) \underset{t \to 0}{\longrightarrow} f(x), \text{ for } x \geq 0, \text{ when } f \text{ is continuous bounded,}$$

(by direct inspection of (2.74)).

One can further check that:

(2.77) $X_\cdot$ has the simple Markov property with respect to $(\mathcal{F}_t^+)$.

In essence as below (2.12) one uses the fact that for g continuous bounded, $x \geq 0$,

$$E_{X_{s+\varepsilon}}[g(X_{t_0}, \dots, X_{t_k})] \overset{\varepsilon \to 0}{\longrightarrow} E_{X_s}[g(X_{t_0}, \dots, X_{t_k})], \; P_x\text{-a.s.},$$

and this is done by looking separately at the event $\{X_s > 0\}$ and $\{X_s = 0\}$.

However the process is <u>not strong Markov</u>! For instance $H_{(0,\infty)}$ the entrance time in $(0,\infty)$ is an $(\mathcal{F}_t^+)$-stopping time, cf. (2.29), and P_0-a.s. $H_{(0,\infty)} > 0$, but $H_{(0,\infty)} \circ \theta_{H_{(0,\infty)}} = 0$, P_0-a.s., so that:

(2.78) $$0 = E_0[1\{H_{(0,\infty)} > 0\} \circ \theta_{H_{(0,\infty)}}] \neq E[E_{\underset{\substack{\| \\ 0 \; P_0\text{-a.s.}}}{X_{H(0,\infty)}}}[1\{H_{(0,\infty)} > 0\}]] = 1.$$

Roughly speaking the problem is that P_0 does not describe the motion of $X_{H_{(0,\infty)}+\cdot}$, i.e. of $X_\cdot$ after time $H_{(0,\infty)}$.

Note that even when f is smooth one can have for $t > 0$,

(2.79) $$\lim_{x \to 0_+} R_t f(x) = f(t) \neq R_t f(0): \quad R_t f \text{ is not continuous!}$$

So the crucial property (2.12) in the Brownian case, that was used below (2.72), is not satisfied in the present example. Indeed if $H^n_{(0,\infty)}$ is the discrete skeleton of $H_{(0,\infty)}$, cf (2.69), for bounded continuous g,

(2.80) $E_{X_{H^n_{(0,\infty)}}}[g(X_{t_0}, \dots, X_{t_k})]$ need not P_0-a.s. converge for $n \to \infty$, to

$$E_{X_{H(0,\infty)}}[g(X_{t_0}, \dots, X_{t_k})] \underset{P_0\text{-a.s.}}{=} E_0[g(X_{t_0}, \dots, X_{t_k})].$$

This should be contrasted with (2.77). □