Lecture 8:

CHAPTER 3: SOME PROPERTIES OF THE BROWNIAN SAMPLE PATH

We will now discuss some typical properties of the Brownian sample paths. From this discussion the "roughness" of the typical sample path will be apparent. We begin with the quadratic variation and the variation of the sample path.

Theorem: ($d=1$, on the canonical space $(C, \mathcal{F}, W_0)$)

For $t>0$, W_0-a.s. and in $L^2(W_0)$,

(3.1) $\lim\limits_{n\to\infty} \sum\limits_{k\geq 0;\, \frac{k+1}{2^n}\leq t} |X_{\frac{k+1}{2^n}} - X_{\frac{k}{2^n}}|^2 = t$,

(3.2) W_0-a.s., the map $t\geq 0 \to X_t(w)\in \mathbb{R}$, has infinite variation on any $[a,b]$, $0\leq a<b$.

Proof:

– (3.1): we set

(3.3) $\Delta_{k,n} \overset{\text{def}}{=} |X_{\frac{k+1}{2^n}} - X_{\frac{k}{2^n}}|^2$, for $k,n\geq 0$.

For fixed n, by (1.1), the $\Delta_{k,n}$, $k\geq 0$, are i.i.d. under W_0, with mean 2^{-n}. Moreover we find that

(3.4) $E_0\Big[\Big(\sum\limits_{\frac{k+1}{2^n}\leq t} \Delta_{k,n} - t\Big)^2\Big] = E_0\Big[\Big\{\sum\limits_{\frac{k+1}{2^n}\leq t} (\Delta_{k,n} - 2^{-n}) - \underbrace{\Big(t - \frac{[t2^n]}{2^n}\Big)}_{= a_n}\Big\}^2\Big] =$

$$a_n^2 + 2a_n \underbrace{E_0\Big[\sum_{\frac{k+1}{2^n}\leq t} (\Delta_{k,n} - 2^{-n})\Big]}_{=0} + E_0\Big[\Big(\sum_{\frac{k+1}{2^n}\leq t} (\Delta_{k,n} - 2^{-n})\Big)^2\Big]$$

Since we are in presence of the variance of a sum of i.i.d. variables, we find:

(3.5) $E_0\Big[\Big(\sum\limits_{\frac{k+1}{2^n}\leq t} (\Delta_{k,n} - 2^{-n})\Big)^2\Big] = [2^n t]\, E_0\big[(\Delta_{0,n} - 2^{-n})^2\big]$

and since $\Delta_{0,n}$ is distributed as $2^{-n} X_1^2$ under W_0

$$= \frac{[2^n t]}{2^{2n}}\, E_0\big[(X_1^2 - 1)^2\big].$$

We have thus found that

(3.6) $E_0\Big[\Big(\sum\limits_{\frac{k+1}{2^n}\leq t} \Delta_{k,n} - t\Big)^2\Big] = a_n^2 + \frac{[2^n t]}{2^{2n}}\, E_0[(X_1^2 - 1)^2]$ is summable in n.

From this we deduce that $\Big(\sum\limits_{\frac{k+1}{2^n}\leq t} \Delta_{k,n} - t\Big)$ converges a.s. and in $L^2(W_0)$ to 0.

The claim (3.1) now follows.

(3.2):

The set of $w \in C$ for which there exists $0 \le a < b < \infty$, such that $t \to X_t(w)$ has finite variation on $[a,b]$ equals the event

(3.7) $\bigcup_{r<s \text{ in } \mathbb{Q}\cap[0,\infty)} \{w \in C;\ V_{r,s}(w) < \infty\}$,

where $V_{r,s}(w)$ denotes the random variable:

(3.8) $V_{r,s}(w) = \sup_{\substack{r=t_0<\dots<t_k=s \\ \text{rationals}}} \sum_{i=1}^{k} |X_{t_i}(w) - X_{t_{i-1}}(w)|$.

If (3.2) did not hold, then for some $0 \le r_0 < s_0 \in \mathbb{Q}\cap[0,\infty)$ one would have:

(3.9) $W_0[V_{r_0,s_0} < \infty] > 0$.

However on the event $\{V_{r_0,s_0} < \infty\}$,

(3.10) $\sum_{r_0 \le \frac{k}{2^n}, \frac{k+1}{2^n} \le s_0} |X_{\frac{k+1}{2^n}} - X_{\frac{k}{2^n}}|^2 \le \sup_{\substack{|u-v| \le 2^{-n} \\ u,v \le s_0}} |X_u - X_v| \sum_{r_0 \le \frac{k}{2^n}, \frac{k+1}{2^n} \le s_0} |X_{\frac{k+1}{2^n}} - X_{\frac{k}{2^n}}|$

$\le \sup_{\substack{|u-v| \le 2^{-n} \\ u,v \le s_0}} |X_u - X_v|\, V_{r_0,s_0} \xrightarrow[n\to\infty]{} 0$, thanks to the continuity of the trajectory $t \to X_t$.

On the other hand due to (3.1) and the continuity of the trajectory, we find that W_0-a.s.,

(3.11) $\sum_{r_0 \le \frac{k}{2^n}, \frac{k+1}{2^n} \le s_0} |X_{\frac{k+1}{2^n}} - X_{\frac{k}{2^n}}|^2 \xrightarrow[n\to\infty]{} s_0 - r_0 > 0$,

thus contradicting (3.10).

This proves (3.2). $\square$

Remark:

Using Dini's second theorem (i.e. a sequence of non-decreasing functions on a compact interval $I \subseteq \mathbb{R}$, converging to a continuous function, converges uniformly on I to this function), we deduce from (3.1) that

(3.12) W_0-a.s., for any $N \ge 1$, $\lim_{n\to\infty} \sup_{0 \le t \le N} \left| \sum_{0 \le \frac{k+1}{2^n} \le t} |X_{\frac{k+1}{2^n}} - X_{\frac{k}{2^n}}|^2 - t \right| = 0$.

Exercise:

Consider for $0 \le r < s$ and $w \in C$, the function theoretic quadratic variation of w on $[r,s]$:

$$V_{2,r,s}(w) = \sup_{\substack{r \le t_0 < \dots < t_k \le s \\ \text{rationals}}} \sum_{i=1}^{k} |X_{t_i}(w) - X_{t_{i-1}}(w)|^2.$$

Show that (in spite of (3.1)):

W_0-a.s., $V_{2,r,s} = \infty$ for all $0 \le r < s < \infty$ in $\mathbb{Q} \cap [0,\infty)$.

(Hint: Take advantage of (2.19) to construct partitions of $[r,s]$ for which $|X_{t_i} - X_{t_{i-1}}| \ge K \sqrt{t_i - t_{i-1}}$ occurs often. See also Durrett Probability: Theory and Examples, exercise 2.4, p. 345). □

Our next objective is the law of the iterated logarithm

Theorem: (A. Khinchin, 1933).

(3.13) i) W_0-a.s., $\varlimsup_{t \to 0} X_t / (\sqrt{2t \log\log \frac{1}{t}}) = 1$, "small time behavior"

ii) W_0-a.s., $\varliminf_{t \to 0} X_t / (\sqrt{2t \log\log \frac{1}{t}}) = -1$,

and

(3.14) i) W_0-a.s., $\varlimsup_{t \to \infty} X_t / (\sqrt{2t \log\log t}) = 1$, "large time behavior"

ii) W_0-a.s., $\varliminf_{t \to \infty} X_t / (\sqrt{2t \log\log t}) = -1$.

Proof:

Under W_0, $(-X_t)_{t \ge 0}$ is also a Brownian motion, so that we only need to prove (3.13) i) and (3.14) i).

Moreover we know from (1.20), (1.21), that

(3.15) $\beta_s = s X_{1/s}$, $s > 0$
$ = 0$, $s = 0$

is a Brownian motion. Thus if we can prove (3.13) i), it follows that

W_0-a.s. $1 = \varlimsup_{s \to 0} s X_{1/s} / \sqrt{2s \log\log \frac{1}{s}} = \varlimsup_{s \to 0} X_{1/s} / \sqrt{\frac{2}{s} \log\log \frac{1}{s}}$.

Setting $t = \frac{1}{s}$, we then find (3.14) i).

As a result we only need to prove (3.13) i).

First step: "the upper bound".

We set $\varphi(t) \overset{\text{def}}{=} \sqrt{2t \log\log \frac{1}{t}}$, our goal is to prove that

(3.16) W_0-a.s., $\overline{\lim\limits_{t\to 0}}\ \dfrac{X_t}{\varphi(t)} \le 1$.

Indeed we choose $\delta > 0$ and $q \in (0,1)$, (δ will be small and q close to 1), so that:

(3.17) $(1+\delta)^2 q > 1$, and define

(3.18) $t_n = q^n$, $n \ge 0$, (note that $t_n \downarrow 0$),

(3.19) $A_n = \{w \in C;$ for some $t \in [t_{n+1}, t_n]$, $X_t > (1+\delta)\varphi(t)\}$, $n \ge 0$.

Note that φ is non-decreasing on $[0,T]$, T small and positive, because:

$$\psi(t) \overset{\text{def}}{=} \frac{\varphi^2}{2}(t) = t \log\log \frac{1}{t}, \text{ so that}$$

$$\psi'(t) = \log\log\frac{1}{t} + \frac{t}{\log\frac{1}{t}} \times -\frac{1}{t} = \log\log\frac{1}{t} - \frac{1}{\log\frac{1}{t}} > 0, \text{ for } t \text{ small.}$$

As a result we see that for large enough n

(3.20) $W_0(A_n) \le W_0\big(\sup\limits_{0\le s\le t_n} X_s > (1+\delta)\varphi(t_{n+1})\big)$, (recall $t_{n+1} < t_n$),

$$\overset{(2.48)}{=} 2 W_0\big(X_{t_n} > (1+\delta)\varphi(t_{n+1})\big)$$

$$\overset{(1.35)}{\le} \sqrt{\frac{2}{\pi}}\ \frac{1}{x_n} \exp\Big\{-\frac{x_n^2}{2}\Big\}, \text{ with } x_n \overset{\text{def}}{=} (1+\delta)\varphi(t_{n+1})/\sqrt{t_n}.$$

Note that $x_n = (1+\delta)\sqrt{2 q^{n+1-n} \log\log q^{-n-1}}$

$$= (1+\delta)\Big[2q \log\Big((n+1)\log\frac{1}{q}\Big)\Big]^{1/2}$$

$$= \big[2 \log\{(\alpha(n+1))^{\lambda}\}\big]^{1/2} \text{ with } \alpha = \log(1/q)$$

$$\lambda = q(1+\delta)^2 \overset{(3.17)}{>} 1$$

Coming back to the last line of (3.20), we find that

(3.21) $W_0(A_n) \le \sqrt{\dfrac{2}{\pi}}\ \dfrac{1}{\alpha^{\lambda}}\ \dfrac{1}{(n+1)^{\lambda}}$ for large n.

Since $\lambda > 1$, by (3.17), it follows that:

(3.22) $\sum\limits_n W_0(A_n) < \infty$,

and by the first lemma of Borel-Cantelli, we see that

(3.23) W_0-a.s., A_n occurs only finitely many times.

As a result we obtain W_0-a.s. $\overline{\lim\limits_{t\to 0}}\ X_t/\varphi(t) \le 1+\delta$. Letting $\delta \to 0$, (this is possible, cf. (3.17)), we obtain (3.16).

Second step: "the lower bound"

(3.24) W_0-a.s., $\varliminf_{t\to 0} \frac{X_t}{\varphi(t)} \geq 1$.

To this end we choose $q\in(0,1)$, $\varepsilon\in(0,\frac{1}{2})$, and define t_n, $n\geq 0$, as in (3.18). Here both ε and q will be chosen small, see (3.29) below.

We will use the lower bound (in the spirit of (1.35)):

(3.25) $P[\eta > x] \geq \frac{x}{x^2+1}\,\frac{1}{\sqrt{2\pi}}\,\exp\{-\frac{x^2}{2}\}$, for $x>0$ and η $N(0,1)$-distributed, (indeed

$x^{-1}e^{-\frac{x^2}{2}} = \int_x^{+\infty}(1+z^{-2})e^{-\frac{z^2}{2}}dz \leq (1+x^{-2})\int_x^{\infty}e^{-\frac{z^2}{2}}dz$, whence (3.25)).

As a result setting now $x_n = (1-\varepsilon)\,\varphi(t_n)\,(t_n-t_{n+1})^{-1/2}$, we find for large n that:

(3.26) $W_0[X_{t_n}-X_{t_{n+1}} > (1-\varepsilon)\varphi(t_n)] \overset{(1.1)}{=} W_0[X_1 > x_n] \overset{(3.25)}{\geq}$
$\sqrt{2\pi}^{-1}\, x_n(1+x_n^2)^{-1}\exp\{-\frac{x_n^2}{2}\} \geq \sqrt{2\pi}^{-1}(2x_n)^{-1}\exp\{-\frac{x_n^2}{2}\}$.

Moreover we have

(3.27) $x_n = \frac{1-\varepsilon}{\sqrt{1-q}}\sqrt{2\log(n\log\frac{1}{q})} = \sqrt{\beta\log(\gamma n)}$, with $\gamma = \log\frac{1}{q}$ and

(3.28) $\beta = 2\,\frac{(1-\varepsilon)^2}{1-q}$.

We assume that q is small enough so that

(3.29) $q < \varepsilon^2/4$, (and as a result $\beta < 2$).

Then the variables $X_{t_n}-X_{t_{n+1}}$, $n\geq 0$, are independent and

(3.30) $W_0(X_{t_n}-X_{t_{n+1}} > (1-\varepsilon)\varphi(t_n)] \geq \frac{c}{(\log n)^{1/2}}\, n^{-\beta/2}$, for large n.

The above expression is the general term of a divergent series. Hence the application of the second lemma of Borel Cantelli yields that

(3.31) W_0-a.s., for infinitely many n, $X_{t_n}-X_{t_{n+1}} \geq (1-\varepsilon)\varphi(t_n)$.

From the upper bound (3.16) applied to $-X_\cdot$, we see that W_0-a.s., for large n, $X_{t_n} \geq -(1+\varepsilon)\varphi(X_{t_n})$, and therefore

(3.32) W_0-a.s., for infinitely many n

$X_{t_n} = X_{t_n}-X_{t_{n+1}} - X_{t_{n+1}} \geq (1-\varepsilon)\varphi(t_n) - (1+\varepsilon)\varphi(t_{n+1})$
$= \varphi(t_n)[1-\varepsilon-(1+\varepsilon)\,\varphi(t_{n+1})/\varphi(t_n)]$.

Note that due to (3.29):

(3.33) $\lim_n \varphi(t_{n+1})/\varphi(t_n) = \sqrt{q} \overset{(3.29)}{<} \frac{\varepsilon}{2}$,

and it follows from (3.32) that

(3.34) W_0-a.s., for infinitely many n, $X_{t_n} \geq \varphi(t_n)(1-2\varepsilon)$,

so that W_0-a.s., $\varlimsup_{t\to 0} X_t/\varphi(t) \geq 1-2\varepsilon$. Letting ε tend to zero along some sequence we deduce (3.24).

□

<u>Remark</u>: (further extensions and related results)

1) There is a "functional" extension of the law of the iterated logarithm due to V. Strassen (1964). Given $w \in C$, one considers the subset of $C([0,1];\mathbb{R})$, (endowed with the sup-norm):

$F_w = \{ f \in C([0,1];\mathbb{R}),$ for some $t \geq 10$, $f(u) = \dfrac{X_{ut}(w)}{\sqrt{2t\log\log t}}, 0 \leq u \leq 1\}$.

<u>Theorem</u>:

(3.35) W_0-a.s, F_w is relatively compact, and the set of limit points of $(X_{ut}/\sqrt{2t\log\log t})_{0\leq u\leq 1}$, as $t\to\infty$, coincides with:

(3.36) $K = \{ f \in C([0,1];\mathbb{R});\ f(u) = \int_0^u g(s)ds$ for some $g \in L^2([0,1],ds)$ with $\int_0^1 g^2(s)ds \leq 1\}$, (of course K is compact).

For the proof see the book of Deuschel-Stroock, "Large Deviations", p. 21.

Note that when f runs over K, $f(1)$ runs over $[-1,1]$, (indeed $|f(1)| \leq (\int_0^1 g^2(s)ds)^{1/2} \leq 1$, and $f(u) = au$, with $|a| \leq 1$ belongs to K).

From this one recovers that:

(3.37) W_0-a.s., the set of limit points of $\dfrac{X_t}{\sqrt{2t\log\log t}}$, as $t\to\infty$, equals $[-1,1]$,

which in essence is a restatement of (3.14).

<u>Exercise</u>:

Given $T>0$, what is the W_0-a.s. set of limit points as $t\to\infty$, of $(X_{ut}/\sqrt{2t\log\log t})_{0\leq u\leq T}$ in $C([0,T];\mathbb{R})$ endowed with the sup-norm?

□

2) An other related result is Lévy's modulus of continuity for Brownian motion:

Theorem: (P. Lévy, 1937)

(3.38) W_0-a.s., $\displaystyle\overline{\lim_{u\to 0}} \frac{1}{\sqrt{2u\log\frac{1}{u}}} \sup_{\substack{0\le s<t\le 1\\ t-s\le u}} |X_t - X_s| = 1$.

For the proof, which has a similar flavour as the proof of the law of the iterated logarithm, see for instance the book of Karatzas-Shreve, p. 114.

Note that in (3.38), $\sqrt{2t\log\log\frac{1}{t}}$ in (3.13) is replaced with the "bigger" function $\sqrt{2t\log\frac{1}{t}}$. This has to do with the fact that in (3.39) one also takes the supremum over the "starting point X_s", whereas for fixed s, W_0-a.s., $\displaystyle\overline{\lim_{u\to 0}} \frac{|X_{s+u} - X_s|}{\sqrt{2u\log\log\frac{1}{u}}} = 1$.

3) An other law of the iterated logarithm was proved by K.L. Chung (1948). It governs the small values of $\sup_{0\le s\le t} |X_s|$.

Theorem:

(3.39) W_0-a.s., $\displaystyle\underline{\lim_{t\to\infty}} \left(\frac{\log\log t}{t}\right)^{1/2} \sup_{0\le s\le t} |X_s| = \frac{\pi}{\sqrt{8}}$.

This shows that $\sup_{0\le s\le t} |X_s|$ cannot grow too slowly. On the other hand it follows from (3.35), (3.36) that it cannot grow too fast and

(3.40) W_0-a.s., $\displaystyle\overline{\lim_{t\to\infty}} \left(\frac{1}{2t\log\log t}\right)^{1/2} \sup_{0\le s\le t} |X_s| = 1$.

$\square$