Lecture 10: (Chap. 4, cont.)

We will now begin the discussion of stochastic integrals.

We assume that

(4.19) $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ is a filtered probability space satisfying the usual conditions (4.5), (4.6),

(4.20) X_t, $t\geq 0$, is a continuous square integrable $(\mathcal{G}_t)$-martingale,

(4.21) $X_t^2 - t$, $t\geq 0$, is a $(\mathcal{G}_t)$-martingale.

A "concrete example of this situation occurs for instance in (4.10), (4.17), (4.18), when considering $(C, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, W_0)$ and the canonical process X_t, $t\geq 0$.

Remark:

We will later see, cf. Chapter 5, that when M_t, $t\geq 0$, is a continuous square integrable martingale on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$, as in (4.19), (i.e. for each $t\geq 0$, $E[M_t^2]<\infty$), one can construct a process $\langle M\rangle_t$, $t\geq 0$, such that

(4.22) $t \to \langle M\rangle_t(\omega)$ is non-decreasing continuous, for each $\omega\in\Omega$,

(4.23) $\langle M\rangle_0 = 0$,

(4.24) $\langle M\rangle_t$, $t\geq 0$, is $(\mathcal{G}_t)$-adapted, integrable,

(4.25) $M_t^2 - \langle M\rangle_t$, $t\geq 0$, is a $(\mathcal{G}_t)$-martingale.

Moreover $\langle M\rangle_t$ is essentially unique, (i.e. two such processes agree for all $t\geq 0$, except maybe on a negligible set of $\omega\in\Omega$), and it is called the "quadratic variation process". The terminology stems from the fact that for $t\geq 0$,

$$\sum_{\frac{k+1}{2^n}\leq t} \left(M_{\frac{k+1}{2^n}} - M_{\frac{k}{2^n}}\right)^2 \xrightarrow[n\to\infty]{} \langle M\rangle_t \quad \text{in } P\text{-probability},$$

see Karatzas–Shreve, section 1.5. $\square$

We are going to define the integral $\int_0^t H_s\,dX_s$ for some suitable basic integrands, for which the definition is "natural", and then we will use an isometry property to extend the class of processes we integrate. Later we will further extend the class of integrands by a so-called "localization argument".

Our building blocks are the <u>basic processes</u>:

(4.26) $\quad H_s(\omega) = C(\omega)\,1\{a < s \le b\},\ s \ge 0,\ \omega \in \Omega$, with $C \in b\mathcal{G}_a$, $0 \le a \le b$.

For such an H as in (4.26), we define:

(4.27) $\quad \int_0^\infty H_s\,dX_s \overset{\text{def}}{=} C(\omega)(X_b(\omega) - X_a(\omega)) \in L^2(P)$.

The restriction $C(\omega) \in b\mathcal{G}_a$ is not a priori natural. It is motivated by the fact that if we define

(4.28) $\quad \underset{\nwarrow}{(H.X)_t} \overset{\text{def}}{=} \int_0^\infty H_s\,1_{[0,t]}(s)\,dX_s \overset{(4.27)}{=} C(\omega)(X_{b\wedge t} - X_{a\wedge t})$

also denoted $\int_0^t H_s\,dX_s$

we have the

<u>Proposition:</u>

(4.29) $M_t = (H.X)_t$ is a continuous square integrable $(\mathcal{G}_t)$-martingale.

<u>Proof:</u>

$$
\begin{aligned}
(H.X)_t &= C(\omega)(X_{b\wedge t} - X_{a\wedge t}) \\
&= 0, \quad \text{if } 0 \le t \le a, \\
&= C(X_t - X_a), \quad \text{if } a \le t \le b, \\
&= C(X_b - X_a), \quad \text{if } b \le t < \infty,
\end{aligned}
$$

clearly defines a continuous adapted process which is square integrable. Considering the case $a \le s < t \le b$, (the other cases are simpler), we see that

$$E[M_t - M_s \mid \mathcal{G}_s] = E[\underset{b\mathcal{G}_a \subseteq b\mathcal{G}_s}{C}(X_t - X_s) \mid \mathcal{G}_s] \overset{P\text{-a.s.}}{=} C\,\underbrace{E[X_t - X_s \mid \mathcal{G}_s]}_{=0}$$

Our claim follows.

$\square$

The next step is the following

Proposition:

If H, K are basic processes, then

(4.30) $E[(H.X)_t\,(K.X)_t] = E[\int_0^t H_s(\omega)K_s(\omega)\,ds]$, for $0 \le t \le \infty$.

Proof:

It suffices to consider the case $t = \infty$, because
$(H.X)_t = (H 1_{[0,t]} . X)_\infty$.

It also suffices to check (4.30) when $(a,b] = (c,d]$ or $(a,b]$ "<" $(c,d]$, (i.e. $a \le b \le c \le d$), and $H = C 1_{(a,b]}$, $K = D 1_{(c,d]}$.

Indeed one makes repeated use of identities such as

(4.31) for $0 \le \alpha \le \beta \le \gamma$, $H = C 1_{(\alpha,\gamma]}$, $H^1 = C 1_{(\alpha,\beta]}$ $H^2 = C 1_{(\beta,\gamma]}$,
$\int_0^\infty H_s\,dX_s = \int_0^\infty H_s^1\,dX_s + \int_0^\infty H_s^2\,dX_s$.

We will thus only need to check (4.30) in two cases:

– case $(a,b] = (c,d]$:

(4.32) $E[(H.X)_\infty (K.X)_\infty] = E[\overbrace{CD}^{\in b\mathcal{G}_a}(X_b - X_a)^2] =$
$E[CD\,E[(X_b - X_a)^2 | \mathcal{G}_a]]$.

Note that:

(4.33) $E[(X_b - X_a)^2 | \mathcal{G}_a] = E[X_b^2 - 2X_bX_a + X_a^2 | \mathcal{G}_a] =$
$E[X_b^2 | \mathcal{G}_a] - 2X_a E[X_b | \mathcal{G}_a] + X_a^2 \overset{(4.20)}{=} E[X_b^2 | \mathcal{G}_a] - X_a^2 =$
$E[X_b^2 - b | \mathcal{G}_a] + b - X_a^2 \overset{(4.21)}{=} X_a^2 - a + b - X_a^2 = b - a$.

Thus coming back to (4.32) we have shown that
$E[(H.X)_\infty (K.X)_\infty] = E[CD(b-a)] = E[\int_0^\infty H_s(\omega)K_s(\omega)\,ds]$,
i.e. (4.30) holds.

– case $(a,b]$ "<" $(c,d]$:

$E[\underbrace{(H.X)_\infty}_{\mathcal{G}_b\text{-meas}} (K.X)_\infty] = E[(H.X)_\infty E[(K.X)_\infty | \mathcal{G}_b]]$
$\overset{(4.29)}{=} E[(H.X)_\infty \underbrace{(K.X)_b}_{=0}] = 0$.

Analogously we have
$E[\int_0^\infty H(s,\omega)K(s,\omega)\,ds] = 0$, and (4.30) holds.

□

Remark:

If one replaces $(X_t)_{t\geq 0}$ satisfying (4.20), (4.21), with $(M_t)_{t\geq 0}$, a continuous square integrable martingale, and defines for basic processes $H_s(\omega) = C(\omega)1\{a<s\leq b\}$, with $C \in b\mathcal{G}_a$,

(4.34) $\int_0^\infty H_s\, dM_s = C(M_b - M_a)$, and $\int_0^t H_s dM_s = \int_0^\infty H_s 1_{[0,t]}(s) dM_s$, $0 \leq t \leq \infty$,

then (4.30) is replaced by

(4.35) $E[(H.M)_t (K.M)_t] = E[\int_0^t H_s(\omega) K_s(\omega)\, d\langle M\rangle_s(\omega)]$,

with $\langle M\rangle_.$ the quadratic variation of M, and $(H.M)_.$ defined as in (4.28) with M replacing X.

$\square$

We now define the class Λ_1 of simple processes:

(4.36) $\Lambda_1 = \{ H_s(\omega) = H^1_s(\omega) + \dots + H^n_s(\omega),\ H^i \text{ are basic}\}$.

Proposition and definition:

(4.37) For $H \in \Lambda_1$, $\int_0^\infty H_s\, dX_s \overset{def}{=} \sum_{i=1}^n \int_0^\infty H^i_s dX_s$, is well defined.

Proof:

The only point to check is that when $H^1, \dots, H^p$ are basic processes such that $H^1 + \dots + H^p = 0$, then $\sum_{i=1}^p \int_0^\infty H^i_s dX_s = 0$.

Making repeated use of (4.31) and

(4.38) $H = C1_{(\alpha,\beta]}$, $K = D1_{(\alpha,\beta]}$ basic processes, then $\int_0^\infty H_s dX_s + \int_0^\infty K_s dX_s = \int_0^\infty L_s dX_s$, with $L = (C+D)1_{(\alpha,\beta]}$ basic process,

we can assume that

$H^i = C_i 1_{I^i}$, $1 \leq i \leq p$, with $I^i \cap I^j = \phi$, when $i \neq j$. In this case $H^1 + \dots + H^p = 0$ implies $H^1 = H^2 = \dots = H^p = 0$, and hence $\sum_{i=1}^p \int_0^\infty H^i_s dX_s = 0$. Our claim is thus proved.

$\square$

As a consequence of (4.29) and (4.37) we see that for $H, K \in \Lambda_1$,

(4.39) $\int_0^t H_s\, dX_s$ $\left(\overset{def}{=} \int_0^\infty H_s 1_{[0,t]}(s) dX_s\right)$ is a continuous square integrable martingale, and $E[\int_0^\infty H_s dX_s \int_0^\infty K_s dX_s] = E[\int_0^\infty H_s K_s\, ds]$.

Remark:

In the case of a general continuous square integrable $(\mathcal{G}_t)$-martingale $M_t, t \geq 0$, in place of $X_t, t \geq 0$, we can use the same construction as above. The role of ds is simply replaced by $d\langle M\rangle_s(\omega)$ so that for $H, K \in \Lambda_1$ we have

$$(4.40)\quad E\left[\int_0^\infty H_s dM_s \int_0^\infty K_s dM_s\right] = E\left[\int_0^\infty H_s(\omega) K_s(\omega)\, d\langle M\rangle_s(\omega)\right]. \qquad \square$$

As a result of (4.39), we see that

(4.41) $K \in \Lambda_1 \longrightarrow \int_0^\infty K_s dX_s \in L^2(\Omega, \mathcal{G}, P)$ is an isometry, if Λ_1 is viewed as a subspace of $L^2(\Omega \times \mathbb{R}_+, \mathcal{G} \otimes \mathcal{B}(\mathbb{R}_+), dP \otimes ds)$.

Note that Λ_1 is in general not dense in $L^2(\Omega \times \mathbb{R}_+, \mathcal{G} \otimes \mathcal{B}(\mathbb{R}_+), dP \otimes ds)$, since all K in Λ_1 are progressively measurable processes. We hence consider

(4.42) $\mathcal{P}$ = the σ-algebra of progressively measurable sets in $\Omega \times \mathbb{R}_+$, (i.e. of $A \in \mathcal{G} \otimes \mathcal{B}(\mathbb{R}_+)$ such that for all $t \geq 0$, $A \cap (\Omega \times [0,t]) \in \mathcal{G}_t \otimes \mathcal{B}([0,t])$).

Remark:

A process $Z(u,\omega)$ on $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ is progressively measurable in the sense of the definition below (2.61) exactly when

$$(4.43)\quad (\Omega \times \mathbb{R}_+, \mathcal{P}) \xrightarrow{Z} (\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)) \text{ is measurable.} \qquad \square$$

We then define:

(4.44) $\Lambda_2 = L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, dP\,ds)$,
the set of progressively measurable processes $H(s,\omega)$ for which $E\left[\int_0^\infty H^2(s,\omega)\, ds\right] < \infty$. The interest of this definition comes from the next:

Proposition:

(4.45) Λ_1 is a dense subset of Λ_2 for the $L^2(\Omega\times\mathbb{R}_+,\mathcal{P},dPds)$-distance,

(4.46) $H \to \int_0^\infty H_s\,dX_s$ extends uniquely into an isometry from Λ_2 into $L^2(\Omega,\mathcal{G},P)$.

Proof:

In view of (4.41) we see that (4.46) immediately follows from (4.45).

The proof of (4.45) will in fact rely on a lemma, which is more general than what is needed to prove (4.45), but applies as well to the subsequent discussion of stochastic integrals with respect to continuous square integrable martingales. The non-decreasing process $t \to A_t(\omega)$ in the next lemma plays the role of $t \to \langle M\rangle_t(\omega)$, cf. (4.22).

Lemma:

Suppose that A_t, $t\geq 0$, is a continuous $(\mathcal{G}_t)$-adapted process, non-decreasing in t, with $A_0=0$, and $E[A_t]<\infty$, for every $t\geq 0$, then

(4.47) Λ_1 is a dense subset of $L^2(\Omega\times\mathbb{R}_+,\mathcal{P},dPdA_s)$ for the $L^2(dPdA_s)$-distance.

(the σ-finite measure $d\mu = dPdA_s$ is defined via

$$\mu(B) = \int_\Omega\Big(\int_0^\infty 1_B(\omega,u)\,dA_u(\omega)\Big)dP(\omega),$$

for $B \in \mathcal{G}\otimes\mathcal{B}(\mathbb{R}_+)\supset\mathcal{P}$).

With the above lemma, (4.45) clearly follows.

There remains to prove the lemma.