Lecture 11: (Chap 4, cont.)

We have thus reduced the proof of the important Proposition p.65, to the

Proof of the lemma p.65:

Since $E[A_t] < \infty$, for each $t \geq 0$, it follows that indeed
$\Lambda_1 \subset \Lambda_2 \overset{\text{def}}{=} L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, dP dA_s)$.

We further observe that
$\widetilde{A}_t = t + A_t$, $t \geq 0$,
satisfies the same assumptions as $A_t, t \geq 0$, and proving (4.47) for $\widetilde{A}_t, t \geq 0$, implies (4.47) for $A_t, t \geq 0$. We thus assume that for $\omega \in \Omega$:

(4.48) $t \in [0,\infty) \to A_t(\omega) \in [0,\infty)$ is an increasing bijection,
and for $0 \leq s \leq t$, $t - s \leq A_t(\omega) - A_s(\omega)$.

Define for $H \in \Lambda_2$,

(4.49) $H^n = 1_{[0,n]} \times \{(-n) \vee (H \wedge n)\} \in \Lambda_2$,

then we find that due to dominated convergence:

(4.50) $\|H - H^n\|_{L^2(dPdA)} \underset{n \to \infty}{\longrightarrow} 0$.

We then introduce the inverse function of A.

(4.51) $\tau_u = \inf\{t \geq 0, A_t > u\}$, for $u \geq 0$.

Note that for $f \geq 0$, $\mathcal{B}(\mathbb{R}_+)$-measurable and $\omega \in \Omega$,

(4.52) $\int_0^\infty f(t)\, dA_t = \int_0^\infty f(\tau_u)\, du$ "change of variable formula",

(Indeed this identity holds when $f = 1_{[a,b]}$ with $a \leq b$, since $\tau_u \in [a,b]$ is equivalent to $u \in [A_a, A_b]$. Then with Dynkin's lemma (4.52) holds for any $f = 1_C$, with $C \in \mathcal{B}([0,T])$, $T > 0$, arbitrary, and the general case follows by approximation).

We then define for $n \geq 1$, $\ell \geq 0$,

(4.53) $H_t^{n,\ell}(\omega) = 2^\ell \int_{\tau(A_t - 2^{-\ell})}^t H_s^n(\omega)\, dA_s(\omega)$, for $t \geq 0, \omega \in \Omega$,
where by convention $A_t = 0$, for $t \leq 0$, $\tau(u) = 0$, for $u \leq 0$.

Clearly $H^{n,l}$ is bounded in absolute value by $\|H^n\|_\infty$, is a continuous function of t, and vanishes when $t > n+2^{-l}$, (indeed $A_{n+2^{-l}} \overset{(4.48)}{\geq} A_n + 2^{-l}$, so that for $t > n+2^{-l}$, $A_t - 2^{-l} > A_n$, and hence $\tau(A_t - 2^{-l}) > n$, which in turns implies that the integral in (4.53) vanishes in view of (4.49)).

Moreover

(4.54) $H^{n,l}$ is $(\mathcal{G}_t)$-adapted.

Indeed $\tau(A_t - 2^{-l}) = \inf\{s \geq 0, A_s > A_t - 2^{-l}\}$ is $\mathcal{G}_t$-measurable, (simply consider the events $\{\tau(A_t - 2^{-l}) < u\}$, for $u \leq t$). Moreover for any F $\mathcal{G}_t \otimes \mathcal{B}([0,t])$-measurable, $\int_0^t F_s(\omega)\, dA_s(\omega)$ is $\mathcal{G}_t$-measurable, as follows from Dynkin's lemma, approximation, and consideration of functions of the form $F = 1_{D\times[a,b]}$, with $D \in \mathcal{G}_t$, $0 \leq a \leq b \leq t$. Coming back to (4.53), the claim (4.54) follows.

Now as a result of (4.52) we find that:

(4.55) $\displaystyle\int_0^\infty (H^n_t(\omega) - H^{n,l}_t(\omega))^2 dA_t = \int_0^\infty (H^n_{\tau_u}(\omega) - H^{n,l}_{\tau_u}(\omega))^2 du$

and for $u \geq 0$,

(4.56) $\displaystyle H^{n,l}_{\tau_u}(\omega) \overset{(4.53)}{=} 2^l \int_{\tau(A_{\tau_u} - 2^{-l})}^{\tau_u} H^n_s(\omega)\, dA_s = 2^l \int_{\tau(u-2^{-l})}^{\tau_u} H^n_s(\omega)\, dA_s$

$$= 2^l \int_0^\infty 1\{\tau(u-2^{-l}) \leq s \leq \tau_u\} H^n_s(\omega)\, dA_s$$

$$\overset{(4.52)}{=} 2^l \int_0^\infty 1\{\tau(u-2^{-l}) \leq \tau_u \leq \tau_v\} H^n_{\tau_v}(\omega)\, dv = 2^l \int_{(u-2^{-l})_+}^{u} H^n_{\tau_v}(\omega)\, dv.$$

Note that for any $g \in L^2(\mathbb{R}_+, du)$

$$g_l(u) = 2^l \int_0^\infty 1\{u-2^{-l} \leq v \leq u\}\, g(v)\, dv \xrightarrow[l\to\infty]{L^2(du)} g(u),$$

(this follows directly from the continuity of translations in $L^2(\mathbb{R}, du)$).

Thus combining (4.55) and (4.56), it follows by dominated convergence that

(4.57) $\displaystyle \|H^n - H^{n,l}\|^2_{L^2(dP\,dA)} = E\Big[\int_0^\infty (H^n_t(\omega) - H^{n,l}_t(\omega))^2 dA_t(\omega)\Big] \xrightarrow[l\to\infty]{} 0,$

for any $n \geq 1$.

We can now define for $n \geq 1$, $\ell, m \geq 0$:

(4.58) $H_t^{n,\ell,m}(\omega) = \sum_{k \geq 0} H^{n,\ell}_{\frac{k}{2^m}}(\omega)\, 1\left(\frac{k}{2^m}, \frac{k+1}{2^m}\right](t)$, for $t \geq 0, \omega \in \Omega$.

Clearly $H^{n,\ell,m} \in \Lambda_1$ are uniformly bounded in m, and for $t > 0$, $\omega \in \Omega$, thanks to the continuity of $H^{n,\ell}_{\cdot}(\omega)$, $H_t^{n,\ell,m}(\omega) \xrightarrow[m \to \infty]{} H_t^{n,\ell}(\omega)$.

Since dA_t does not give positive mass to $\{0\}$, we find that:

(4.59) $\| H^{n,\ell} - H^{n,\ell,m} \|_{L^2(dPdA)} \xrightarrow[m \to \infty]{} 0$, for $n \geq 1, \ell \geq 0$.

Combining (4.50), (4.57), (4.59) we have proved (4.45). $\square$

This concludes the proof of the Proposition p.65. $\square$

<u>Remarks:</u>

1) Reconstructing some trajectorial character of the stochastic integral.

Note that when H and K in Λ_2, and $G \in \mathcal{G}$ are such that

(4.60) $H_s(\omega) = K_s(\omega)$ for all $s \geq 0$, and $\omega \in G$,

then we see from (4.49) that a similar identity holds for H^n and K^n, from (4.53) that the same holds for $H^{n,\ell}$ and $K^{n,\ell}$, and finally from (4.58) that the same holds for $H^{n,\ell,m}$ and $K^{n,\ell,m}$.

As a result we can find $H^{(i)}$ and $K^{(i)}$ in Λ_1, $i \geq 1$, with the property:

(4.61) $H^{(i)} \to H$ in $L^2(dPds)$, $K^{(i)} \to K$ in $L^2(dPds)$, and for all $i \geq 1$, $H_s^{(i)}(\omega) = K_s^{(i)}(\omega)$, for all $s \geq 0$ and $\omega \in G$.

On the other hand when $H, K \in \Lambda_1$ are such that $H_{\cdot}(\omega) = K_{\cdot}(\omega)$ for $\omega \in G$, one checks from (4.27), (4.28), (4.37), (4.39) that

(4.62) $(H.X)_t(\omega) = (K.X)_t(\omega)$, for $0 \leq t \leq \infty$, and $\omega \in G$.

Combining this observation with (4.61) and (4.46), we see that:

(4.63) when $H, K \in \Lambda_2$ satisfy (4.60), then $\int_0^\infty H_s dX_s = \int_0^\infty K_s dX_s$, P-a.s. on G.

This somehow reconstructs some trajectorial character to the stochastic integral.

2) The class of processes we can integrate has severe limitations.
If we consider the canonical space $(C, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, W_0)$ with $X_t, t\geq 0$, the canonical process we can now consider
$\int_0^1 e^{\alpha X_s} dX_s \left(= \int_0^\infty 1_{[0,1]}(s)\, e^{\alpha X_s} dX_s\right)$, for $\alpha \in \mathbb{R}$
because $e^{\alpha X_s}$ is progressively measurable and
$E_0\left[\int_0^\infty 1_{[0,1]}(s)\, e^{2\alpha X_s} ds\right] = \int_0^1 E_0[e^{2\alpha X_s}] ds = \int_0^1 e^{2\alpha^2 s} ds < \infty$,
so that $1_{[0,1]}(s)\, e^{\alpha X_s}$ belongs to Λ_2, for all $\alpha \in \mathbb{R}$.
On the other hand if we consider for $\alpha \in \mathbb{R}$,

(4.64) $\int_0^1 e^{\alpha X_s^2} dX_s$,

then we observe that

$$E_0\left[\int_0^\infty 1_{[0,1]}(s)\, e^{2\alpha X_s^2} ds\right] = \int_0^1 \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi s}}\, e^{(2\alpha - \frac{1}{2s})x^2} dx\, ds$$

$$= \int_0^1 (1-4\alpha s)_+^{-1/2} ds < \infty \quad \text{when } \alpha < \frac{1}{4},$$
$$= \infty \quad \text{when } \alpha \geq \frac{1}{4}.$$

Thus at the present stage of the construction of stochastic integrals, $\int_0^1 e^{\frac{1}{10} X_s^2} dX_s$ is meaningful, but $\int_0^1 e^{X_s^2} dX_s$ is not!

We will later extend the definition of stochastic integrals so that $\int_0^1 e^{X_s^2} dX_s$, (or even $\int_0^1 e^{(e^{X_s})} dX_s$!) are well-defined.
However in the theory we develop

(4.65) $\int_0^1 X_1\, dX_s$ is not defined because $1_{[0,1]} X_1$ is not $\mathcal{P}$-measurable. $\square$

Observe that given $H \in \Lambda_2$ and $\|H^n - H\|_{L^2(dP\,ds)} \underset{n\to\infty}{\longrightarrow} 0$, with $H^n \in \Lambda_1$, for each n, we know that for each $t \geq 0$, $(H^n.X)_t \underset{n\to\infty}{\longrightarrow} (H.X)_t$ in $L^2(\Omega, \mathcal{G}, P)$ and in fact $(H.X)_t \in L^2(\Omega, \mathcal{G}_t, P)$. We are now going to <u>select a nice version of the process $(H.X)_t, t\geq 0$</u>, so that it defines a continuous square integrable $(\mathcal{G}_t)$-martingale. We recall Doob's inequality in the discrete setting:

Proposition:

Consider a filtered probability space $(\Omega, F, (F_m)_{m\geq 0}, P)$ and X_m, $m\geq 0$, an (F_m)-submartingale, (i.e. X_m is F_m measurable and integrable, and $E[X_{m+1} \mid F_m] \geq X_m$, for $m\geq 0$). Then for $\lambda > 0$, $n\geq 0$, $A = \{\omega\in\Omega;\ \sup_{0\leq m\leq n} X_m \geq \lambda\}$, one has

(4.66) $\lambda P[A] \leq E[X_n 1_A] \leq E[X_n^+]$,

(see Durrett's book Probability: Theory and Examples, p. 215).

In the continuous time set-up we obtain:

Proposition:

Consider a filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ and X_t, $t\geq 0$, a continuous submartingale. Then for $\lambda > 0$, $t\geq 0$, and $A = \{\sup_{0\leq u\leq t} X_u \geq \lambda\}$ one has

(4.67) $\lambda P[\sup_{0\leq u\leq t} X_u \geq \lambda] \leq E[X_t 1_A] \leq E[X_t^+]$.

Proof:

Letting $\lambda_n \uparrow \lambda$, it suffices to prove that for $\lambda > 0$:

(4.68) $\lambda P[\sup_{0\leq u\leq t} X_u > \lambda] \leq E[X_t 1\{\sup_{0\leq u\leq t} X_u > \lambda\}]$.

By the same argument with $\lambda_m \downarrow \lambda$, we deduce from (4.66) that for $\lambda > 0$, one has:

$$\lambda P[\sup_{0\leq m\leq 2^\ell} X_{\frac{m}{2^\ell}t} > \lambda] \leq E[X_t 1\{\sup_{0\leq m\leq 2^\ell} X_{\frac{m}{2^\ell}t} > \lambda\}].$$

Letting $\ell \uparrow \infty$, since $\{\sup_{0\leq m\leq 2^\ell} X_{\frac{m}{2^\ell}t} > \lambda\} \uparrow \{\sup_{0\leq u\leq t} X_u > \lambda\}$, as $\ell\uparrow\infty$, we obtain (4.68), and our claim is proved. $\square$

Doob's inequality will be a key tool for the construction of a good version of $\int_0^t H_s\, dX_s$, when $H \in \Lambda_2$.