Lecture 12: (Chap. 4, cont.)

We now proceed with the construction of a good version of the stochastic integral $\int_0^t H_s\, dX_s$, for $H \in \Lambda_2$, cf. (4.44). We recall our standing assumptions (4.19), (4.20), (4.21).

Theorem:

For $H_s(\omega) \in \Lambda_2$, there is a process $(I_t)_{0 \le t \le \infty}$, essentially unique (i.e. two such processes except on a $\mathcal{G}$-negligible set agree for all $t \ge 0$), continuous, $(\mathcal{G}_t)$-adapted, such that:

(4.69) for each $0 \le t \le \infty$, $I_t = \int_0^t H_s\, dX_s$, P-a.s.,

(4.70) $(I_t)_{0 \le t \le \infty}$ is a continuous square integrable $(\mathcal{G}_t)$-martingale, (and of course $E[I_t^2] \overset{(4.46)}{=} E[\int_0^t H_s^2(\omega)\, ds]$, for $0 \le t \le \infty$).

Proof:

When $H \in \Lambda_1$, our definition of $\int_0^t H_s dX_s$ satisfies the above properties, see (4.37), (4.28), (4.29). When $H \in \Lambda_2$, we pick $H^n \in \Lambda_1$, $n \ge 0$, with $\lim_n \|H - H^n\|_{L^2(dP ds)} = 0$. As a result for $0 \le s \le t$, $A \in \mathcal{G}_s$,

$$\begin{array}{ccc} E[\int_0^t H_u^n\, dX_u\, 1_A] & = & E[\int_0^s H_u^n\, dX_u\, 1_A] \\ \downarrow n \to \infty & & \downarrow n \to \infty \\ E[\int_0^t H_u\, dX_u\, 1_A] & = & E[\int_0^s H_u\, dX_u\, 1_A] \end{array}$$

and with the discussion below (4.65) we thus find that

(4.71) $E[(H.X)_t \mid \mathcal{G}_s] = (H.X)_s$, P-a.s.,

so the martingale property comes for free. We thus only need to find $I(t,\omega)$ a continuous $(\mathcal{G}_t)$-adapted process for which (4.69) holds. We choose $n_k \to \infty$ such that

(4.72) $\sum_k k^4 \, \|H^{n_k} - H^{n_{k+1}}\|^2_{L^2(\mathcal{B}, dP ds)} < \infty$.

Then for each $k \ge 0$, $((H^{n_k} - H^{n_{k+1}}).X)_t^2$ is a continuous submartingale and by Doob's inequality (4.67), for $\lambda > 0$:

(4.73) $\lambda^2 P[\sup_{u \ge 0} |(H^{n_k}.X)_u - (H^{n_{k+1}}.X)_u| \ge \lambda] \le E[(\int_0^\infty (H_s^{n_k} - H_s^{n_{k+1}})\, dX_s)^2]$
$= \|H^{n_k} - H^{n_{k+1}}\|^2_{L^2(\mathcal{B}, dP ds)}$.

Choosing $\lambda = k^{-2}$, we obtain

$$P\Big[\sup_{u\geq 0} |(H^{n_k}.X)_u - (H^{n_{k+1}}.X)_u| \geq k^{-2}\Big] \leq k^4 \, \|H^{n_k} - H^{n_{k+1}}\|^2_{L^2(\mathcal{P}, dPds)}.$$

Applying Borel Cantelli's lemma, we can find $N \in \mathcal{G}$ with $P(N) = 0$, such that for $\omega \notin N$, we have $k_0(\omega) < \infty$, such that

$$(4.74) \quad \sup_{u\geq 0} |(H^{n_k}.X)_u(\omega) - (H^{n_{k+1}}.X)_u(\omega)| \leq \frac{1}{k^2}, \text{ for } k \geq k_0(\omega).$$

As a result for $\omega \notin N$, $(H^{n_k}.X)_u(\omega)$ converges uniformly on $[0,\infty]$.

We thus define:

$$(4.75) \quad \begin{aligned} I_u(\omega) &= \lim_k (H^{n_k}.X)_u(\omega), \text{ for } \omega \notin N, \\ &= 0 \qquad\qquad\qquad\quad, \text{ for } \omega \in N, \end{aligned}$$

so that $u \in [0,\infty] \to I_u(\omega)$ is continuous for all $\omega \in \Omega$, and $I_u(\cdot)$ is $\mathcal{G}_u$-measurable (we use here the fact that $\mathcal{G}_u$ contains all negligible sets of $\mathcal{G}$, see (4.5)).

Observe that $(H^{n_k}.X)_u \xrightarrow{L^2(P)} (H.X)_u$, for $0 \leq u \leq \infty$, and P-a.s., $(H^{n_k}.X)_u \to I_u$. As a result $I_u \overset{P\text{-a.s.}}{=} (H.X)_u$, and (4.69) holds. The theorem is proved. □

From now on $(H.X)_t$ will denote the essentially unique regular version I_t. We will use the following inequality:

Proposition:

If $(X_t)_{t\geq 0}$ is a continuous non-negative submartingale on a filtered probability space, then for $0 \leq t < \infty$, $p \in (1,\infty)$

$$(4.76) \quad E\Big[\sup_{s\leq t} X_s^p\Big]^{1/p} \leq \frac{p}{p-1} E[X_t^p]^{1/p}.$$

Proof:

We apply the discrete time inequality to $X_{\frac{k}{2^n}t}$, $0 \leq k \leq 2^n$, and let $n \to \infty$, (see for instance Durrett's book p. 216 for the discrete time inequality). □

As an immediate application we have:

$$(4.77) \quad E\Big[\sup_{t\geq 0} (H.X)_t^2\Big]^{1/2} \leq 2 \, \|H\|_{L^2(\mathcal{P}, dPds)}, \text{ for } H \in \Lambda_2.$$

We now proceed with the next

Proposition:

For $H, K \in \Lambda_2$, the essentially uniquely defined process

(4.78) $N_t \overset{def}{=} (H.X)_t (K.X)_t - \int_0^t H_s(\omega) K_s(\omega) ds$, $0 \le t \le \infty$,

is a continuous $(\mathcal{G}_t)$-martingale and

(4.79) $\sup_{t \ge 0} |N_t| \in L^1(\Omega, \mathcal{G}, P)$.

Proof:

Note that due to the Cauchy-Schwarz inequality $E[\int_0^\infty |H_s(\omega)||K_s(\omega)| ds] \le \|H\|_{L^2(dPds)} \|K\|_{L^2(dPds)} < \infty$, so that (4.78) is well defined for all $0 \le t \le \infty$, and ω outside the negligible set N where $\int_0^\infty |H_s(\omega)||K_s(\omega)| ds = \infty$. It also defines a process with continuous trajectories outside N, and setting for instance $\int_0^t H_s(\omega) K_s(\omega) ds \equiv 0$, for $\omega \in N$, the property (4.79) is an immediate consequence of (4.77), and the above inequality. We thus only need to check that N_t is a $(\mathcal{G}_t)$-martingale.

- 1^{st} case H, K are basic: (similar to (4.30))

we only need to treat the case of $H = C 1_{(a,b]}$, $K = D 1_{(c,d]}$, with either $(a,b] = (c,d]$ or "$(a,b] < (c,d]$".

If $(a,b] = (c,d]$, then for $t \ge 0$,

(4.80) $N_t = CD[(X_{t\wedge b} - X_{t \wedge a})^2 - (t\wedge b - t \wedge a)]$ $\quad$ ($(\mathcal{G}_t)$-measurable)

is a martingale because when for instance $a \le s < t \le b$:

$$
\begin{aligned}
(4.81) \quad E[N_t | \mathcal{G}_s] &= E[(X_t - X_a)^2 - (t-a) | \mathcal{G}_s] CD \\
&= E[X_t^2 - 2X_t X_a + X_a^2 - (t-a) | \mathcal{G}_s] CD \\
&\overset{(4.20),(4.21)}{=} (X_s^2 - s - 2X_s X_a + X_a^2 + a) CD \\
&= [(X_s - X_a)^2 - (s-a)] CD = N_s,
\end{aligned}
$$

and the other cases are easier to check.

If "$(a,b] < (c,d]$":

(4.82) $N_t = (X_{t\wedge b} - X_{t\wedge a})(X_{t \wedge d} - X_{t\wedge c}) CD$, $t \ge 0$,

is a martingale because when for instance $c \le s < t \le d$:

$$
\begin{aligned}
(4.83) \quad E[N_t | \mathcal{G}_s] &= CD (X_b - X_a) \, E[X_t - X_c | \mathcal{G}_s] \overset{(4.20)}{=} \\
& CD (X_b - X_a)(X_s - X_c) = N_s,
\end{aligned}
$$

and the other cases are simpler to check.

- 2nd case $H, K \in \Lambda_1$:

immediate from the previous case by bilinearity.

- general case: $H, K \in \Lambda_2$:

We choose H^n, K^n, $n \geq 0$ in Λ_1 respectively converging to H and K in $L^2(\mathbb{P}, dP\,ds)$. Due to (4.77) we see that

(4.84) $$E[\sup_{t\geq 0} |(H^n.X)_t - (H.X)_t|^2] \leq 4\,\|H^n - H\|_{L^2(\mathbb{P}, dP ds)} \to 0.$$

and a similar inequality for K. Note also that

$$\left|\int_0^t H_s K_s\,ds - \int_0^t H_s^n K_s^n\,ds\right| \leq \int_0^\infty |H_s - H_s^n|\,|K_s|\,ds + \int_0^\infty |H_s^n|\,|K - K_s^n|\,ds,$$

so integrating and using Cauchy-Schwarz's inequality we find that:

(4.85) $$E\left[\sup_{t\geq 0} \left|\int_0^t H_s K_s\,ds - \int_0^t H_s^n K_s^n\,ds\right|\right] \leq \|H - H^n\|_{L^2(dP ds)} \|K\|_{L^2(dP ds)} + \|H^n\|_{L^2(dP ds)} \|K - K^n\|_{L^2(dP ds)} \to 0.$$

As a result we find that

(4.86) $$\sup_{t\geq 0} |N_t^{(n)} - N_t| \to 0 \quad \text{in } L^1(\Omega, \mathcal{G}, P),$$

if $N_t^{(n)}$ denotes the martingale attached to H^n, K^n via (4.78). This is more than enough to conclude that N_t, $t \geq 0$, satisfies the martingale property and thus conclude the proof of the Proposition. $\square$

<u>Remark:</u>

Note that the above proposition shows that for $H \in \Lambda_2$,

$(H.X)_t^2 - \int_0^t H_s^2(\omega)\,ds$ is a continuous $(\mathcal{G}_t)$-martingale,

and the non-decreasing adapted process:

$t \geq 0 \longrightarrow \int_0^t H_s^2(\omega)\,ds$,

fulfills the properties (4.22)-(4.25) relative to $M_t = (H.X)_t$. We have thus constructed by "bare hands"

(4.87) $$\langle (H.X)\rangle_t = \int_0^t H_s^2(\omega)\,ds, \; t \geq 0,$$

(as mentioned below (4.25), the process satisfying (4.22)-(4.25) is essentially unique). $\square$

The good version of the stochastic integral that we have produced is in essence based on an isometry. We will now reconstruct some trajectorial property of the integral.

When T is $(\mathcal{G}_t)$-stopping time, the process

(4.88) $(s,\omega) \to 1_{[0,T]}(s,\omega) \overset{def}{=} 1\{s \le T(\omega)\}$

is progressively measurable (it is adapted, left-continuous in s, and a simple variant of (2.60) yields the claim). For such a T we have two "natural ways" to define "$\int_0^T H_s dX_s$", when $H \in \Lambda_2$:

- we can for instance use:

the continuous version $(H.X)_t(\omega)$ and replace t by $T(\omega)$.

Observe that the essential uniqueness of the continuous version of $(H.X)_t(\omega)$ ensures that two different continuous versions of the stochastic integral give rise to resulting random variables which differ on an at most negligible set. In other words:

(4.89) $(H.X)_{T(\omega)}(\omega)$ is uniquely defined up to a negligible set.

- Alternatively we can use the definition

(4.90) $\int_0^\infty (1_{[0,T]} H)_s \, dX_s$,

once we note that $1_{[0,T]} H \in \Lambda_2$.

As we now explain both definitions coincide.

<u>Proposition</u>: (stopping theorem for stochastic integrals)

Let T be a $(\mathcal{G}_t)$-stopping time, and $H \in \Lambda_2$, then P-a.s.,

(4.91) $\int_0^{t\wedge T} H_s \, dX_s = \int_0^t (1_{[0,T]} H)_s \, dX_s$, for $0 \le t \le \infty$.

<u>Proof</u>:

We consider $(H.X)_t$, $t \ge 0$, and $(H 1_{[0,T]} . X)_{t \ge 0}$. For a given $u \ge 0$,

$(H 1_{[0,u]})_s(\omega) = (H 1_{[0,T]} 1_{[0,u]})_s$, for all $s \ge 0$, on $G \overset{def}{=} \{u \le T\}$.

It now follows from (4.63) that for $u \ge 0$, P-a.s. on $\{u \le T\}$, (4.92) $(H.X)_u = (H1_{[0,u]}.X)_\infty = (H1_{[0,T]}1_{[0,u]}.X)_\infty = (H1_{[0,T]}.X)_u$.

As a result we see that

P-a.s. for all $u \in \mathbb{Q} \cap [0,\infty)$, $u \leq T(\omega) \Rightarrow (H.X)_u(\omega) = (H1_{[0,T]}.X)_u(\omega)$,

and using continuity that

(4.93) P-a.s. for all $0 \leq t \leq T(\omega)$, $(H.X)_t(\omega) = (H1_{[0,T]}.X)_t(\omega)$.

Analogously for $u \geq 0$:

$H1_{[0,T]}1_{[0,u]} = H1_{[0,T]}$ on $\tilde{G} = \{T \leq u\}$,

so that P-a.s., for $u \geq 0$, on $\{T \leq u\}$

$(H1_{[0,T]}.X)_u = (H1_{[0,T]}.X)_\infty$.

From this we deduce that

P-a.s., for all $u \in \mathbb{Q} \cap [0,\infty)$, $T(\omega) \leq u \Rightarrow (H1_{[0,T]}.X)_u(\omega) = (H1_{[0,T]}.X)_\infty$

Using continuity as above we thus find that

(4.94) P-a.s. for all $t \geq T(\omega)$, $(H1_{[0,T]}.X)_t(\omega) = (H1_{[0,T]}.X)_\infty$.

Combining (4.93) and (4.94) we see that

P-a.s., for all $t \geq 0$, $(H.X)_{t \wedge T(\omega)}(\omega) = (H1_{[0,T]}.X)_{t\wedge T(\omega)}(\omega)$
$= (H1_{[0,T]}.X)_t(\omega)$,

and this proves (4.91). □

We then have the following

Corollary:

Given $H, K \in \Lambda_2$, T a $(\mathcal{G}_t)$-stopping time such that "$H = K$ on the random interval $[0,T]$" (i.e. $H1_{[0,T]} = K1_{[0,T]}$), then one has

(4.95) P-a.s. $\int_0^t H_s dX_s = \int_0^t K_s dX_s$, for $0 \leq t \leq T(\omega)$.

Proof:

P-a.s., for $t \geq 0$, $\int_0^{t\wedge T} H_s dX_s \overset{(4.91)}{=} \int_0^t (1_{[0,T]}H)_s dX_s = \int_0^t (1_{[0,T]}K)_s dX_s \overset{(4.91)}{=} \int_0^{t \wedge T} K_s dX_s$,

and the claim follows. □

The above corollary provides some "pathwise feeling" to the stochastic integral and also has important consequences.