Lecture 13: (Chap 4, cont.)

Our next item of discussion is the "localization of stochastic integrals". We are going to relax the integrability condition $H \in \Lambda_2$, (i.e. $H \in L^2(\Omega \times \mathbb{R}, \mathcal{P}, dP\,ds)$) in the definition of stochastic integrals. As mentioned previously, cf. (4.64), $\int_0^1 e^{\alpha X_s^2} ds$ presently has no meaning when $\alpha \geq \frac{1}{4}$, (for the sake of definiteness we consider the canonical space $(C, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, W_0)$ and the canonical process $X_t, t \geq 0$). We are going to remedy this feature and $\int_0^1 e^{\alpha X_s^2} dX_s$ will become well-defined for any $\alpha \in \mathbb{R}$, as a result of the construction below (as well as many other stochastic integrals!).

We introduce

(4.96) $\Lambda_3 = \{ K$: $\mathcal{P}$-measurable functions on $\Omega \times [0,\infty)$, such that P-a.s., for all $t \geq 0$, $\int_0^t K_s^2(\omega)\,ds < \infty \}$.

Remark:

1) Note that when $K_s(\omega)$ is $(\mathcal{G}_s)$-adapted, for each $s \geq 0$, and continuous in s, for each ω, then automatically $K \in \Lambda_3$. In particular $\exp\{\alpha X_s^2\}$, or $\exp\{\exp\{X_s^2\}\}$ belong to Λ_3!
2) In the case where we consider a continuous square integrable martingale M. in place of X., the relevant condition will be $\int_0^t K_s^2(\omega)\, d\langle M\rangle_s(\omega) < \infty$, for all $t \geq 0$, outside a P-negligible set of ω. $\square$

Lemma:

When $H \in \Lambda_3$, there exist a non-decreasing sequence of $(\mathcal{G}_t)$-stopping times S_n, $n \geq 0$, which is P-a.s. tending to $+\infty$, such that for each $n \geq 0$:

(4.97) $H 1_{[0, S_n]} \in \Lambda_2$.

Proof:

Note that $(\omega, t) \to \int_0^t H_s^2(\omega)\,ds \in [0,\infty]$ is non-decreasing, $(\mathcal{G}_t)$-adapted.

As a result

(4.98) $S_n \overset{\text{def}}{=} \inf\{t \geq 0;\ \int_0^t H_s^2(\omega)ds \geq n\} \leq \infty$,

is a $(\mathcal{G}_t)$-stopping time, (cf. (2.26), (2.27), in fact the proof is simpler here because $\{S_n > t\} = \{\int_0^t H_s^2(\omega)ds < n\} \in \mathcal{G}_t$, for each $t \geq 0$).

In addition we have:

$E[\int_0^\infty (H^2 1_{[0,S_n]})_s(\omega)\, ds] \leq n < \infty$,

and (4.97) holds. Moreover since $H \in \Lambda_3$, it follows that $S_n(\omega) \uparrow \infty$ for P-a.e. ω. This proves our claim. □

We are now ready to extend the definition of the stochastic integral to integrands in Λ_3.

Definition and Theorem:

Let $H \in \Lambda_3$ and S_n, $n \geq 0$, be any sequence of $(\mathcal{G}_t)$-stopping times, non-decreasing in n, P-a.s. tending to $+\infty$, and such that (4.97) holds. Then the event

(4.99) $N = \bigcup_{n \geq 0} \{\omega \in \Omega;\ \exists t \in \mathbb{Q}_+,\ (H1_{[0,S_n]}.X)_{t \wedge S_n} \neq (H1_{[0,S_{n+1}]}.X)_{t\wedge S_n}\}$
$\cup \{\omega \in \Omega;\ \lim_n S_n(\omega) < \infty\}$ is P-negligible and

(4.100) $(H.X)_t(\omega) \overset{\text{def}}{=} (H1_{[0,S_n]}.X)_t(\omega)$, for $\omega \notin N$, and $t \leq S_n(\omega)$,
$\overset{\text{def}}{=} 0$, if $\omega \in N$,

is well-defined, continuous, adapted. Two such processes arising from two possible choices of sequences of S_n, $n \geq 0$, and versions of $(H1_{[0,S_n]}.X)_t(\omega)$, agree for all $t \geq 0$, except maybe on a negligible set, (i.e. (4.100) defines $(H.X)$ in an essentially unique fashion).

Proof:

As a result of (4.95), the event N in (4.99) is P-negligible. Note that

(4.101) for $\omega \in N^c$, $(H1_{[0,S_n]}.X)_t(\omega) = (H1_{[0,S_{n'}]}.X)_t(\omega)$,
for $n, n' \geq 0$, and for $0 \leq t \leq S_n(\omega) \wedge S_{n'}(\omega)$.

Hence $(H.X)_t(\omega)$ in (4.100) is well-defined. Moreover for $t \geq 0$,
$(H.X)_t(\omega) = \lim_n (H1_{[0,S_n]}.X)_t(\omega)$, if $\omega \notin N$, $= 0$, if $\omega \in N$.

Since $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ satisfies the usual conditions, cf. (4.5), (4.6), $(H.X)_t$, $t\geq 0$, is $(\mathcal{G}_t)$-adapted. Further $t \to (H.X)_t(\omega)$ is continuous, for each $\omega \in \Omega$.

If S_n, S'_n, $n\geq 0$, are two sequences satisfying the assumptions of the theorem, the same holds for $T_n \overset{\text{def}}{=} S_n \wedge S'_n$. From (4.95) we thus find that

(4.102) P-a.s., for $0 \leq t \leq T_n(\omega)$, $(H 1_{[0,S_n]} . X)_t(\omega) = (H1_{[0,S'_n]} . X)_t(\omega)$
$$= (H1_{[0,T_n]}.X)_t(\omega).$$

The claim about the essential uniqueness in the claim (4.100) easily follows. □

Remark:

Of course $\Lambda_2 \subseteq \Lambda_3$, and for $H \in \Lambda_2$, we can choose $S_n \equiv \infty$, for $n\geq 0$, so that (4.97) holds. Noting that $(H1_{[0,\infty]}.X)_t$, $t\geq 0$, and $(H.X)_t$, $t\geq 0$, are indistinguishable, we see that the definition (4.100) is consistent when $H \in \Lambda_2 \subseteq \Lambda_3$. □

We now have given a meaning to expressions like $\int_0^t \exp\{\exp\{X_s^2\}\}dX_s$, and of course we should not expect that we still keep the martingale property for $H \in \Lambda_3$, (an indication of this feature appeared in (4.64)). The adequate notion comes in the next

Definition:

A process $(M_t)_{t\geq 0}$, such that there exists an increasing sequence of stopping times S_n, P-a.s. tending to ∞, such that for each n, $(M_{t\wedge S_n})_{t\geq 0}$ is a continuous square integrable martingale, is called a continuous local martingale.

Remark:

When M_0 is bounded, one can replace "continuous square integrable" with "continuous bounded". Indeed for such an $(M_t)_{t\geq 0}$ as above, with M_0 bounded, one defines the sequence of $(\mathcal{G}_t)$-stopping times:

(4.103) $T_m = \inf\{s \geq 0;\ |M_s| \geq m\} \leq \infty$,

so that $T_m \uparrow \infty$, as $m \to \infty$. Then for fixed $m \geq \|M_0\|_\infty$, $|M_{t\wedge T_m}| \leq m$, for $t \geq 0$, and when $0 \leq s < t$, $A \in \mathcal{G}_s$, we have:

(4.104) $E[M_{t\wedge T_m} 1_A] \overset{\text{dom conv.}}{=} \lim_n E[M_{t\wedge T_m \wedge S_n} 1_A]$.

By the stopping theorem $\tilde{M}_{t\wedge T_m}$ is a continuous martingale if $\tilde{M}_t$ is a continuous martingale, and applying this to $\tilde{M}_t \overset{\text{def}}{=} M_{t\wedge S_n}$, we find that the last term of (4.104) equals $\lim_n E[M_{s\wedge T_m \wedge S_n} 1_A] \overset{\text{dom. conv.}}{=} E[M_{s\wedge T_m} 1_A]$.

In other words $(M_{t\wedge T_m})_{t\geq 0}$ is a $(\mathcal{G}_t)$-martingale, which is bounded and continuous, and our claim follows. □

Exercises:

1) Deduce the continuous time stopping theorem we used above from the discrete time version, (see also Karatzas-Shreve, p.19).

2) Show that a bounded continuous local martingale is a martingale. □

Continuous local martingales naturally arise in our context as shown by the next

Proposition:

(4.105) For $H \in \Lambda_3$, $(H.X)_t$, $t \geq 0$, is a continuous local martingale.

Proof:

Consider an increasing sequence of stopping times $S_n \uparrow \infty$, P-a.s., such that for each n, $H 1_{[0,S_n]} \in \Lambda_2$, then

(4.106) P-a.s., for $t \geq 0$, $(H.X)_{t\wedge S_n} \overset{(4.100)}{=} (H.1_{[0,S_n]}.X)_{t\wedge S_n} \overset{(4.91)}{=} \underbrace{(H.1_{[0,S_n]}.X)_t}_{\text{continuous square integrable martingale}}$,

Our claim follows. □

CHAPTER 5: STOCHASTIC INTEGRALS FOR CONTINUOUS LOCAL MARTINGALES

In this chapter we are going to define the stochastic integral $\int_0^t H_s dM_s$ when the integrator M is a continuous local martingale and H is progressively measurable and such that $\int_0^t H_s^2(\omega) d\langle M\rangle_s(\omega) < \infty$, where $\langle M\rangle$. is the so-called "quadratic variation of the local martingale M". As in the previous chapter the filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ satisfies the "usual conditions", see (4.5), (4.6). Our first task will be the construction of $\langle M\rangle$. We begin with the

Lemma:

Suppose $A_t, t\geq 0$, $B_t, t\geq 0$, are continuous $(\mathcal{G}_t)$-adapted, non-decreasing processes such that $A_0 = B_0 = 0$, and

(5.1) $A_t - B_t$ is a $(\mathcal{G}_t)$-local martingale,

then

(5.2) P-a.s., for all $t\geq 0$, $A_t(\omega) = B_t(\omega)$.

Proof:

Introduce the non-decreasing sequence of $(\mathcal{G}_t)$-stopping times

(5.3) $S_n = \inf\{s\geq 0,\ A_s \text{ or } B_s \geq n\}$,

and note that $S_n \uparrow \infty$ as $n\to\infty$. As in (4.103) we see that

(5.4) $A_{t\wedge S_n} - B_{t\wedge S_n}$, $t\geq 0$, is a bounded martingale, for each $n\geq 0$.

It thus suffices to prove the theorem in the case where $A_t, t\geq 0$, and $B_t, t\geq 0$, are uniformly bounded, and

(5.5) $M_t = A_t - B_t$, $t\geq 0$, is a bounded continuous martingale.

We now observe that for $t\geq 0$:

$$E[M_t^2] = E\Big[\Big(\sum_{k=0}^{2^m-1} M_{\frac{k+1}{2^m}t} - M_{\frac{k}{2^m}t}\Big)^2\Big]$$

and expanding the square, the cross terms disappear by the martingale property.

So we find

$$E[M_t^2] = \sum_{k=0}^{2^m-1} E\Big[\big(M_{\frac{k+1}{2^m}t} - M_{\frac{k}{2^m}t}\big)^2\Big] = E\Big[\sum_{0\le k<2^m} \big(M_{\frac{k+1}{2^m}t} - M_{\frac{k}{2^m}t}\big)^2\Big]$$

$$E\Big[\underbrace{\sup_{0\le k<2^m} \big|M_{\frac{k+1}{2^m}t} - M_{\frac{k}{2^m}t}\big|}_{\substack{\text{by continuity } \downarrow m\to\infty \\ 0}} \times \underbrace{\sum_{0\le k<2^m} \big|M_{\frac{k+1}{2^m}t} - M_{\frac{k}{2^m}t}\big|}_{\le A_\infty + B_\infty \le \text{Const} < \infty}\Big] \xrightarrow[m\to\infty]{\text{dom. conv.}} 0.$$

We have thus shown that for $t \ge 0$:

(5.6) $E[M_t^2] = 0$,

and hence P-a.s., for $t \in \mathbb{Q}_+$, $M_t = A_t - B_t = 0$. But by continuity we have P-a.s., for $t \ge 0$, $A_t(\omega) = B_t(\omega)$. This completes the proof of the lemma. $\square$