Lecture 14: (Chap. 5, cont.)

We now proceed with the construction of $\langle M \rangle$, when M is a continuous square integrable martingale. The result is a special case of the so-called Doob-Meyer decomposition, (see for instance Karatzas-Shreve, p. 24).

Theorem:

Let $M_t, t \geq 0$, be a continuous square integrable $(\mathcal{G}_t)$-martingale. Then there exists a continuous, non-decreasing, $(\mathcal{G}_t)$-adapted process $A_t, t \geq 0$, such that:

(5.7) $A_0 = 0$,

(5.8) A_t is integrable for each $t \geq 0$,

(5.9) $M_t^2 - A_t$ is a $(\mathcal{G}_t)$-martingale,

and $A_t, t \geq 0$, is essentially unique.

Proof:

The essential uniqueness follows from (5.2). We only need to prove the existence of $A_t, t \geq 0$. Without loss of generality we assume that $M_0 = 0$, (otherwise we replace M_t with $\tilde{M}_t = M_t - M_0$).

We are going to construct $A.$ as a suitable limit of a discrete quadratic variations of $M.$, along certain random grids with mesh tending to 0. For this purpose we define for each $n \geq 0$, (n controls the mesh of the discrete grid), a sequence $\tau_k^n, k \geq 0$, of stopping times as follows:

(5.10) $\tau_k^0 = k$, for $k \geq 0$,

and for $n \geq 1$, by induction:

(5.11) $\tau_0^n = 0$, and for $\ell \geq 0$, on the event $\{\tau_k^{n-1} \leq \tau_\ell^n < \tau_{k+1}^{n-1}\}$, $k \geq 0$,

$$\tau_{\ell+1}^n = \inf\{t \geq \tau_\ell^n ; |M_t - M_{\tau_\ell^n}| \geq \tfrac{1}{n}\} \wedge (\tau_\ell^n + \tfrac{1}{n}) \wedge \tau_{k+1}^{n-1}.$$

Using the continuity of $M.$, we see that for $\omega \in \Omega$,

$$
(5.12)\quad \begin{cases}
\tau^n_k(\omega) < \tau^n_{k+1}(\omega), \text{ for } n, k \geq 0, \\
\underbrace{\{\tau^n_0(\omega), \tau^n_1(\omega), \dots\}}_{\text{subset of } \mathbb{R}_+} \subseteq \underbrace{\{\tau^{n+1}_0(\omega), \tau^{n+1}_1(\omega), \dots\}}_{\text{subset of } \mathbb{R}_+}, \\
\tau^n_k(\omega) \to \infty, \text{ as } k \to \infty, \\
|M_{\tau^n_{k+1}(\omega)}(\omega) - M_{\tau^n_k(\omega)}(\omega)| \leq \frac{1}{n}.
\end{cases}
$$

We then choose $K_0 < K_1 < \dots < K_n < \dots$ in $\mathbb{N}$ so that

(5.13) $P(\tau^n_{K_n} \leq n) \leq \frac{1}{n}$, for $n \geq 0$,

and define:

(5.14) $I_n(t) = \sum_{k=0}^{K_n-1} M_{\tau^n_k}(M_{\tau^n_{k+1}\wedge t} - M_{\tau^n_k \wedge t})$

as well as

(5.15) $A_n(t) = \sum_{k=0}^{K_n-1} (M_{\tau^n_{k+1}\wedge t} - M_{\tau^n_k\wedge t})^2$.

Note that $I_n(0)=0$, $A_n(0)=0$, that $I_n(\cdot)$, $A_n(\cdot)$ are continuous and adapted (for instance in the case of (5.14), the generic term vanishes on $\{\tau^n_k > t\}$ and one has

$$\underbrace{1\{\tau^n_k \leq t\} M_{\tau^n_k}}_{\mathcal{G}_t\text{-measurable}} \Big(\underbrace{M_{\tau^n_{k+1}\wedge t}}_{\mathcal{G}_{\tau^n_{k+1}\wedge t} \subseteq \mathcal{G}_t\text{-meas.}} - \underbrace{M_{\tau^n_k \wedge t}}_{\mathcal{G}_{\tau^n_k\wedge t}\subseteq \mathcal{G}_t\text{-meas.}}\Big) \text{ is } \mathcal{G}_t\text{-measurable,}$$

and the case of (5.15) is easier). In fact one has

(5.16) $I_n(t)$, $t \geq 0$, is a continuous $(\mathcal{G}_t)$-martingale, bounded for each t.

Here we only need to check that for $s \leq t$:

(5.17) $E[M_{\tau^n_k}(M_{\tau^n_{k+1}\wedge t} - M_{\tau^n_k\wedge t}) \,|\, \mathcal{G}_s] = M_{\tau^n_k}(M_{\tau^n_{k+1}\wedge s} - M_{\tau^n_k \wedge s})$.

But from the stopping theorem and the observation above (5.16), we have

$E[\underbrace{1\{\tau^n_k \leq s\} M_{\tau^n_k}}_{\in \mathcal{G}_s} (M_{\tau^n_{k+1}\wedge t} - M_{\tau^n_k \wedge t}) \,|\, \mathcal{G}_s]$ = right-hand side of (5.17).

On the other hand $1\{s < \tau^n_k \leq t\} M_{\tau^n_k}$ is $\mathcal{G}_{\tau^n_k}$-measurable and for $A \in \mathcal{G}_s$, $1_A 1\{s < \tau^n_k\}$ is also $\mathcal{G}_{\tau^n_k}$-measurable, hence

$E[1_A 1\{s<\tau^n_k \leq t\} M_{\tau^n_k} (M_{\tau^n_{k+1}\wedge t} - M_{\tau^n_k\wedge t})] = E[1_A 1\{s < \tau^n_k \leq t\} M_{\tau^n_k}$
$(M_{\tau^n_k \vee (\tau^n_{k+1}\wedge t)} - M_{\tau^n_k})] = 0$, by the stopping theorem, cf. Karatzas-Shreve, p.19.

Hence $E[1\{s<\tau_k^n\le t\}\, M_{\tau_k^n}(M_{\tau_{k+1}^n\wedge t}-M_{\tau_k^n\wedge t})\,|\,\mathcal{G}_s]=0$, and (5.17) now easily follows since $1\{\tau_k^n>t\}(M_{\tau_{k+1}^n\wedge t}-M_{\tau_k^n\wedge t})=0$. This proves (5.16).

By direct inspection of (5.15) and (5.11) we see that

(5.18) for $t\ge s+\frac{1}{n}$, $\quad A_n(t)+\frac{1}{n^2}\ge A_n(s)$,

Moreover for $n\ge 1$, on $\{\tau^n_{K_n}>n\}$, $M_t=\sum_{0\le k<K_n} M_{\tau^n_{k+1}\wedge t}-M_{\tau^n_k\wedge t}$, for $0\le t\le n$, so that expanding the square and regrouping terms, we see that, cf. (5.14), (5.15):

(5.19) $M_t^2=2I_n(t)+A_n(t)$, $0\le t\le n$, $\quad$ on $\{\tau^n_{K_n}>n\}$.

The next step is to prove the P-a.s. uniform convergence on compact intervals of $I_{n_\ell}(\cdot)$ for a suitably chosen subsequence n_ℓ. To this end we will use the next

Lemma: $(T>0,\ \varepsilon>0)$

(5.20) $\lim\limits_{m\to\infty}\ \sup\limits_{n\ge m} P\big[\sup\limits_{0\le t\le T}|I_n(t)-I_m(t)|\ge\varepsilon\big]=0.$

Proof:

Choose $n\ge m\ge T$, and define $S=T\wedge\tau^m_{K_m}\wedge\tau^n_{K_n}$. By Doob's inequality, cf. (4.67), we find that

(5.21) $P\big[\sup\limits_{0\le t\le T}|I_n(t)-I_m(t)|\ge\varepsilon\big]\le P\big[\tau^m_{K_m}\le m \text{ or } \tau^n_{K_n}\le n\big]+$
$P\big[\sup\limits_{0\le t\le T}|I_n(t\wedge S)-I_m(t\wedge S)|\ge\varepsilon\big]\underset{(5.13)}{\overset{(4.67)}{\le}}\frac{2}{m}+\frac{1}{\varepsilon^2}E[(I_n(S)-I_m(S))^2]$,

where we have used (5.16) and the stopping theorem.

If we now define for $k,\ell\ge 0$,

(5.22) $\rho_k=\tau^m_k\wedge S$, $\quad\sigma_\ell=\tau^n_\ell\wedge S$,

it follows from the second line of (5.12) that

$\{\rho_0(\omega),\dots,\rho_k(\omega),\dots\}\subseteq\{\sigma_0(\omega),\sigma_1(\omega),\dots,\sigma_\ell(\omega),\dots\}$,

and from (5.14) that

(5.23) $I_n(S)-I_m(S)=\sum\limits_{\ell\ge 0} M_{\sigma_\ell}(M_{\sigma_{\ell+1}}-M_{\sigma_\ell})-\sum\limits_{k\ge 0} M_{\rho_k}(M_{\rho_{k+1}}-M_{\rho_k})=$

$\sum\limits_{k,\ell\ge 0} 1\{\rho_k\le\sigma_\ell<\rho_{k+1}\}\, M_{\sigma_\ell}(M_{\sigma_{\ell+1}}-M_{\sigma_\ell})-\sum\limits_{k,\ell\ge 0} 1\{\rho_k\le\sigma_\ell<\rho_{k+1}\}\, M_{\rho_k}(M_{\sigma_{\ell+1}}-M_{\sigma_\ell})=$

$$\sum_{k,\ell\geq 0} 1\{\rho_k \leq \sigma_\ell < \rho_{k+1}\}\,(M_{\sigma_\ell}-M_{\rho_k})(M_{\sigma_{\ell+1}}-M_{\sigma_\ell}) \overset{\text{def}}{=} \sum_{k,\ell\geq 0} a_{k,\ell}(\omega).$$

Note that

(5.24) the $a_{k,\ell}$, $k,\ell\geq 0$, are pairwise orthogonal in $L^2(P)$.

Indeed for $\ell<\ell'$, cf. exercise 2 p. 37,

$a_{k,\ell}=1\{\rho_k\leq\sigma_\ell<\rho_{k+1}\}\,(M_{\sigma_\ell}-M_{\rho_k})(M_{\sigma_{\ell+1}}-M_{\sigma_\ell})$ is $\mathcal{G}_{\sigma_{\ell+1}}\subseteq\mathcal{G}_{\sigma_{\ell'}}$-measurable, and:

$1\{\rho_{k'}\leq\sigma_{\ell'}<\rho_{k'+1}\}\,(M_{\sigma_{\ell'}}-M_{\rho_{k'}})$ is $\mathcal{G}_{\sigma_{\ell'}}$-measurable as well.

Since $E[M_{\sigma_{\ell'+1}}\,|\,\mathcal{G}_{\sigma_{\ell'}}]=M_{\sigma_{\ell'}}$, (see for instance Karatzas-Shreve, p. 19), it follows that $E[a_{k,\ell}\,a_{k',\ell'}]=0$, for $\ell<\ell'$. To obtain (5.24) one simply notes that for $\ell\geq 0$, $k<k'$, $a_{k,\ell}(\omega)\,a_{k',\ell}(\omega)=0$.

Coming back to (5.23) we conclude with (5.24) that

(5.25) $E[(I_n(S)-I_m(S))^2]=\sum_{k,\ell\geq 0}E[a_{k,\ell}^2]\overset{(5.11)}{\leq}$

$\frac{1}{m^2}E\Big[\sum_{k,\ell\geq 0}1\{\rho_k\leq\sigma_\ell<\rho_{k+1}\}\,(M_{\sigma_{\ell+1}}-M_{\sigma_\ell})^2\Big]=\frac{1}{m^2}E\Big[\sum_{\ell\geq 0}(M_{\sigma_{\ell+1}}-M_{\sigma_\ell})^2\Big]$

and since the above increments are pairwise orthogonal

$=\frac{1}{m^2}E[M_S^2]\leq\frac{1}{m^2}E[M_T^2]$,

since M_t^2, $t\geq 0$, is a continuous submartingale, and $S\leq T$.

Inserting this inequality in (5.21) yields that

(5.26) $P\big[\sup_{0\leq t\leq T}|I_n(t)-I_m(t)|\geq\varepsilon\big]\leq\frac{2}{m}+\frac{1}{m^2\varepsilon^2}E[M_T^2]$.

In particular (5.20) follows. $\square$

We can now extract $n_\ell\geq\ell^2$, such that

$\sup_{n\geq n_\ell}P\big[\sup_{0\leq t\leq\ell}|I_n(t)-I_{n_\ell}(t)|\geq\frac{1}{\ell^2}\big]\leq\frac{1}{2^\ell}$,

so that

(5.27) $P\big[\sup_{0\leq t\leq\ell}|I_{n_{\ell+1}}(t)-I_{n_\ell}(t)|\geq\frac{1}{\ell^2}\big]\leq\frac{1}{2^\ell}$,

With (5.13) and Borel Cantelli's lemma we have $N\in\mathcal{G}$ with $P(N)=0$, so that

(5.28) $\sup_{0\leq t\leq\ell}|I_{n_{\ell+1}}(t)-I_{n_\ell}(t)|\leq\frac{1}{\ell^2}$, $\tau^{n_\ell}_{K_{n_\ell}}>\ell^2$, for $\ell\geq\ell_0(\omega)$, when $\omega\notin N$.

Thus $I_{n_\ell}(\cdot,\omega)$ converges uniformly on compact intervals when $\omega \notin N$, and we define

(5.29) $I(t,\omega) = \lim_\ell I_{n_\ell}(t,\omega)$, for $\omega \notin N$,
$\qquad = \frac{1}{2} M_t^2(\omega)$, for $\omega \in N$.

Therefore $t \in \mathbb{R}_+ \to I(t,\omega)$ is continuous for $\omega \in \Omega$, and $I(t,\omega)$ is $(\mathcal{G}_t)$-measurable for each $t \geq 0$, (we use the fact that $\mathcal{G}_0$ contains all negligible sets of $\mathcal{G}$).

By (5.19) and the fact that $\tau^{n_\ell}_{k_{n_\ell}} \to \infty$, when $\omega \notin N$, we see that when $\omega \notin N$, $A_{n_\ell}(\cdot,\omega)$ converges uniformly on compact intervals of $\mathbb{R}_+$ to

(5.30) $\quad A_t(\omega) \stackrel{def}{=} M_t^2(\omega) - 2I(t,\omega), \qquad \omega \in \Omega.$

Thus $A_t(\omega)$ is $(\mathcal{G}_t)$-measurable, for all $t \geq 0$, continuous in t for all $\omega \in \Omega$. Note that $A_0 = 0$, and due to (5.18) when $\omega \notin N$, and to (5.29), (5.30) when $\omega \in N$, $t \to A_t(\omega)$ is non-decreasing in t for all $\omega \in \Omega$.

We will now prove that for $n_0 \geq 1$, $k_0 \geq 1$, t,

(5.31) $I_{n_\ell}(\tau^{n_0}_{k_0} \wedge t) \xrightarrow[\ell \to \infty]{} I(\tau^{n_0}_{k_0} \wedge t)$ in $L^1(P)$.

We already know the P-a.s. convergence, cf. (5.29). It thus suffices to prove that $I_{n_\ell}(\tau^{n_0}_{k_0} \wedge t)$, $\ell \geq 0$, are uniformly integrable. However writing for $m \geq n_0$,

$\nu_k = \tau^m_k \wedge t \wedge \tau^{n_0}_{k_0}$,

we see as in (5.23) that

(5.32) $I_m(\tau^{n_0}_{k_0} \wedge t) = \sum_{k \geq 0} M_{\nu_k}(M_{\nu_{k+1}} - M_{\nu_k})$.

Since we have the bound

$|M_{\nu_k}| \leq \sup_{0 \leq u \leq \tau^{n_0}_{k_0}} |M_u| \stackrel{(5.11)}{\leq} \frac{k_0}{n_0}$,

a similar (but easier) calculation as in (5.25) yields that for $m \geq n_0$:

(5.33) $E[I_m(\tau^{n_0}_{k_0} \wedge t)^2] \leq \left(\frac{k_0}{n_0}\right)^2 \sum_{k \geq 0} E[(M_{\nu_{k+1}} - M_{\nu_k})^2] \leq \left(\frac{k_0}{n_0}\right)^2 E[M_t^2]$.

This proves the asserted uniform integrability and (5.31) follows.

From the stopping theorem, $I_{m_\ell}(\tau_{k_0}^{n_0} \wedge t)$, $t \geq 0$, are martingales, and from (5.31) we deduce that $I(\tau_{k_0}^{n_0} \wedge t)$, $t \geq 0$, is a (continuous) martingale. By (5.30) we now find that

$$E[A(\tau_{k_0}^{n_0} \wedge t)] = E[M^2(t \wedge \tau_{k_0}^{n_0})]$$

$$\text{monotone convergence} \downarrow k_0 \to \infty \qquad \text{dominated convergence} \downarrow k_0 \to \infty \qquad \Big(\text{recall } E[\sup_{s \leq t} |M_s|^2] \overset{(4.76)}{\leq} 4E[M_t^2]\Big)$$

$$E[A(t)] = E[M^2(t)]$$

This proves (5.8), and it also follows that

(5.34) $M^2(\tau_{k_0}^{n_0} \wedge t) - A(\tau_{k_0}^{n_0} \wedge t) \xrightarrow[k_0 \to \infty]{L^1(P)} M^2(t) - A(t)$, for $t \geq 0$.

The claim (5.9) follows and the theorem is proved. □

Notation:

When M is a continuous square integrable martingale, the essentially unique process A constructed in the above theorem is denoted by $\langle M \rangle$, it is the so-called "quadratic variation" of M, (in some sense (5.15) explains the terminology).

When Z_t, $t \geq 0$, is a stochastic process and T a random time, one writes:

(5.35) $Z_t^T \overset{\text{def.}}{=} Z_{t \wedge T}$, $t \geq 0$, the so-called stopped process.

Corollary:

Let M_t, $t \geq 0$, be a continuous local martingale. Then there exists an essentially unique continuous, non-decreasing, $(\mathcal{G}_t)$-adapted process $\langle M \rangle_t$, $t \geq 0$, such that

(5.36) $\langle M \rangle_0 = 0$,

(5.37) $M_t^2 - M_0^2 - \langle M \rangle_t$, $t \geq 0$, is a continuous local martingale.

Moreover when T is a $(\mathcal{G}_t)$-stopping time, one has

(5.38) P-a.s. for all $t \geq 0$, $\langle M^T \rangle_t = \langle M \rangle_{T \wedge t} (= \langle M \rangle_t^T)$.

Proof:

The uniqueness part of the statement follows from (5.2). As for the

existence part, choose $T_n \uparrow \infty$ so that $M^{T_n}_{\cdot}$ is a continuous square integrable martingale. Note that from the stopping theorem
$(M^{T_{n+1}}_{t\wedge T_n})^2 - \langle M^{T_{n+1}}\rangle_{t\wedge T_n} = M^2_{T_n\wedge t} - \langle M^{T_{n+1}}\rangle_{t\wedge T_n}$, $t\geq 0$,
is a $(\mathcal{G}_t)$-martingale. From (5.2) it follows that

(5.39) P-a.s., for all $t\geq 0$, $\langle M^{T_{n+1}}\rangle_{t\wedge T_n} = \langle M^{T_n}\rangle_t$.

As a result we can find N in $\mathcal{G}$ with $P(N)=0$, so that

(5.40) when $\omega\notin N$, for all $m\geq n\geq 0$, $0\leq t\leq T_n(\omega)$, $\langle M^{T_m}\rangle_t(\omega) = \langle M^{T_n}\rangle_t(\omega)$.

We thus define

(5.41) $\langle M\rangle_t(\omega) = \langle M^{T_n}\rangle_t(\omega)$, for any $n\geq 0$, with $T_n(\omega)\geq t$, if $\omega\notin N$,
$= 0$, if $\omega\in N$.

Note that $\langle M\rangle_t$, $t\geq 0$, is continuous, non-decreasing, $(\mathcal{G}_t)$-adapted, and (5.36) holds. Moreover $\langle M\rangle^{T_n}_{\cdot}$ and $\langle M^{T_n}\rangle_{\cdot}$ are indistinguishable, so that $M^2_{t\wedge T_n} - M_0^2 - \langle M\rangle_{t\wedge T_n}$ is a continuous martingale with value 0 at time 0. The argument below (4.104) shows that (5.37) holds. As for (5.38) it directly follows from the previous existence and uniqueness result, and the fact that $M^2_{t\wedge T} - M^2_0 - \langle M\rangle_{t\wedge T}$ is a continuous local martingale. $\square$

Notation:

When M, N are continuous local martingales we write:

(5.42) $\langle M,N\rangle_t = \frac{1}{4}(\langle M+N\rangle_t - \langle M-N\rangle_t)$, $t\geq 0$.

Corollary:

When M,N are continuous local martingales, $\langle M,N\rangle_t$, $t\geq 0$, is a continuous adapted process with bounded variation on finite intervals, essentially unique such that

(5.43) $\langle M,N\rangle_0 = 0$,

(5.44) $M_tN_t - M_0N_0 - \langle M,N\rangle_t$, $t\geq 0$, is a continuous local martingale.

Proof:

We only have to prove the uniqueness, the other properties being immediate. To this end observe that when C_t, $t\geq 0$, is a continuous adapted process with finite variation on finite intervals, then

(5.45) $V_t = \lim_{n\to\infty} \sum_{\frac{k+1}{2^n}\le t} \left|C_{\frac{k+1}{2^n}} - C_{\frac{k}{2^n}}\right|$, $t\geq 0$,

is a continuous, non-decreasing, adapted process, and

(5.46) $V_t - C_t$, $t\geq 0$, is non-decreasing as well.

We apply this observation to the difference of $\langle M,N\rangle_t$ with D_t, some other continuous adapted process with finite variation on finite intervals, satisfying similar conditions as in (5.43), (5.44). By (5.2) we conclude that

(5.47) P-a.s., for all $t\geq 0$, $\langle M,N\rangle_t = D_t$. □

We now turn to the *construction of the stochastic integrals with respect to continuous local martingales*. This construction involves several steps, which often are very similar to what has been done in the previous chapter, (such steps will be merely briefly discussed below).

For $H_s(\omega)$ a *basic process*, (i.e. $H_s(\omega) = C(\omega)\,1\{a<s\leq b\}$, with $C\in b\mathcal{G}_a$, cf. (4.26)), and M_s, $s\geq 0$, a *continuous square integrable martingale* one defines in the spirit of (4.27):

(5.48) $\int_0^\infty H_s\,dM_s \overset{\text{def}}{=} C(\omega)\,(M_b(\omega) - M_a(\omega))$,

and for $0\leq t\leq\infty$:

(5.49) $\int_0^t H_s\,dM_s \overset{\text{def}}{=} \int_0^\infty (H1_{[0,t]})_s\,dM_s \overset{(5.48)}{=} C(\omega)\,(M_{b\wedge t}(\omega) - M_{a\wedge t}(\omega))$.

One immediately extends the definition to $H\in\Lambda_1$, i.e. $H = H^1+\dots+H^n$, with H^i basic processes, for $1\leq i\leq n$, cf. (4.36), by the formula

(5.50) $\int_0^\infty H_s\,dM_s \overset{\text{def}}{=} \sum_{i=1}^n \int_0^\infty H^i_s\,dM_s$,

and one checks that this is well-defined and that as in (4.39), one has

(5.51) $E\left[\int_0^\infty H_s\,dM_s \int_0^\infty K_s\,dM_s\right] = E\left[\int_0^\infty H_s(\omega)K_s(\omega)\,d\langle M\rangle_s(\omega)\right]$, for $H,K\in\Lambda_1$.

With the help of (4.47) one extends the definition of $\int_0^\infty H_s dM_s$ to H in

(5.52) $\Lambda_2(M) = L^2(\Omega\times\mathbb{R}_+, \mathcal{P}, dP\,d\langle M\rangle_s)$,

so that

(5.53) $H\in\Lambda_2(M) \to \int_0^\infty H_s dM_s \in L^2(P)$ is an isometry.

One chooses a "good version" of $\int_0^t H_s dM_s$, $t\geq 0$, with similar arguments as in the proof of (4.69), (4.70), denoted by $(H.M)_t$, $t\geq 0$, such that

(5.54) $(H.M)_t$, $t\geq 0$, is a continuous square integrable $(\mathcal{G}_t)$-martingale with value 0 at time 0,

(5.55) for each $t\geq 0$, P-a.s., $(H.M)_t = \int_0^t H_s dM_s$,

(5.56) $N_t = (H.M)_t^2 - \int_0^t H_s^2\, d\langle M\rangle_s$, $t\geq 0$, is a continuous martingale, with $\sup_{t\geq 0}|N_t| \in L^1(P)$, (and value 0 at time 0).

In particular, cf. (5.9), (5.2), (5.56),

(5.57) $\langle H.M\rangle_t = \int_0^t H_s^2\, d\langle M\rangle_s$, $t\geq 0$.

The above defined stochastic integral has the following property, (with a similar argument as for the proof of (4.431)): when $H, K\in\Lambda_2(M)$, $G\in\mathcal{G}$ are such that

(5.58) $H_s(\omega) = K_s(\omega)$, for all $s\geq 0$, when $\omega\in G$,

then

(5.59) P-a.s., for $0\leq t\leq\infty$, $(H.M)_t(\omega) = (K.M)_t(\omega)$, for $\omega\in G$.

Then one has (in the notation of (5.35)):

<u>Theorem</u>: (stopping theorem for stochastic integrals)

Let T be a $(\mathcal{G}_t)$-stopping time and $H\in\Lambda_2(M)$, then

(5.60) P-a.s., for $0\leq t\leq\infty$, $((1_{[0,T]}H).M)_t = (H.M)_{t\wedge T} = (H.M^T)_t = ((1_{[0,T]}H).M^T)_t$

<u>Proof</u>:

Note that $M_t^T = M_{T\wedge t}$, $t\geq 0$, is also a continuous square integrable

martingale and, cf. (5.38), $\langle M^T \rangle_\cdot = \langle M \rangle_\cdot^T$, so that $H \in \Lambda_2(M^T)$ as well.
Then we find just as in (4.91) that

(5.61) P-a.s., for $0 \le t \le \infty$, $((1_{[0,T]} H).M)_t = (H.M)_{T \wedge t}$.

Moreover since $\langle M^T \rangle_\cdot = \langle M \rangle_\cdot^T$, we see that

(5.62) $H = 1_{[0,T]} H$ in $L^2(\Omega \times \mathbb{R}_+, \mathcal{P}, dP d\langle M^T \rangle)$,

so that

(5.63) P-a.s., for $0 \le t \le \infty$, $(H.M^T)_t = ((1_{[0,T]} H).M^T)_t$.

Then observe by coming back to (5.49) that for K basic process and then K in Λ_1, one has

(5.64) for $0 \le t \le \infty$, $(K.M)_{t \wedge T} = (K.M^T)_t$.

Then by approximation of $H \in \Lambda_2(M)$ (and hence in $\Lambda_2(M^T)$) one finds that

(5.65) P-a.s., for $0 \le t \le \infty$, $(H.M)_{t \wedge T} = (H.M^T)_t$.

Combining (5.61), (5.63), (5.65), we obtain (5.60). $\square$