Lecture 15: (Chap. 5, cont.)

With the help of the stopping theorem for stochastic integrals, cf. (5.60), we will now extend the definition of stochastic integrals.

For M a continuous local martingale with value 0 at time 0, we define

(5.66) $\Lambda_3(M) = \{H : \mathcal{P}\text{-measurable on } \Omega \times \mathbb{R}_+ \text{ such that}$
$\qquad P\text{-a.s.}, \forall t \geq 0, \int_0^t H_s^2(\omega)\, d\langle M\rangle_s(\omega) < \infty\}.$

One then considers a non-decreasing sequence of stopping time T_n, $n \geq 0$, P-a.s. tending to ∞, such that

(5.67) M^{T_n} is a continuous square integrable martingale for each $n \geq 0$,

(5.68) $1_{[0,T_n]} H \in \Lambda_2(M^{T_n})$.

One such sequence is for instance obtained by setting:

(5.69) $T_n(\omega) = \inf\{s \geq 0, |M_s(\omega)| \geq n \text{ or } \int_0^s H_s^2(\omega)\, d\langle M\rangle_s(\omega) \geq n\}$.

Definition and Theorem:

If $T_n \uparrow \infty$, P-a.s., is a sequence of stopping times satisfying (5.67), (5.68), then the event

(5.70) $N = \bigcup_{n \geq 0} \{\omega \in \Omega;\ \exists t \in \mathbb{Q}_+,\ ((H1_{[0,T_{n+1}]}).M^{T_{n+1}})_{t\wedge T_n} \neq ((H1_{[0,T_n]}).M^{T_n})_{t\wedge T_n}\}$
$\qquad \cup \{\omega \in \Omega;\ \lim_n T_n(\omega) < \infty\}$ is P-negligible and

(5.71) $(H.M)_t(\omega) \overset{\text{def}}{=} ((H1_{[0,T_n]}).M^{T_n})_t(\omega)$, for $0 \leq t \leq T_n(\omega)$, if $\omega \notin N$,
$\qquad = 0$, if $\omega \in N$,

is a continuous local martingale. It is defined in an essentially unique way if one uses different choices of T_n, $n \geq 0$, and of $((H1_{[0,T_n]}).M^{T_n})_t$.

Proof:

By (5.60), letting $M^{T_{n+1}}$ play the role of M, and $H1_{[0,T_{n+1}]}$ of H, we find that P-a.s. for $0 \leq t \leq \infty$,

(5.72) $((H1_{[0,T_{n+1}]}).M^{T_{n+1}})_{t\wedge T_n} = ((H1_{[0,T_n]}).M^{T_n})_t = ((H1_{[0,T_n]}).M^{T_n})_{t\wedge T_n}$;

where the last equality follows from (5.60), with M replaced by M^{T_n} and H by $H1_{[0,T_n]}$. We thus find that $P(N)=0$, and it is immediate from (5.71) that $(H.M)_t$ defines a continuous local martingale.

Now when T_n, T_n', $n \geq 0$, are two sequences of stopping times satisfying the assumptions of the theorem, setting $S_n = T_n \wedge T_n' \uparrow \infty$, P-a.s., we find that P-a.s., for $0 \leq t \leq S_n(\omega)$:

(5.73) $$((H1_{[0,T_n]}).M^{T_n})_t(\omega) \overset{(5.60)}{=} ((H1_{[0,S_n]}).M^{S_n})_t(\omega) \overset{(5.60)}{=} ((H1_{[0,T_n']}).M^{T_n'})_t(\omega),$$

and the claim about the essential uniqueness follows. □

Remark:

When M is a continuous square integrable martingale, with $M_0=0$, and $H \in \Lambda_2(M)$, we can take $T_n \equiv \infty$ in the previous definition, so that (5.71) agrees with our previous definition of the stochastic integral. □

We will now give an alternative characterization of $(H.M)_.$ by means of its bracket $\langle (H.M), N\rangle$ with other continuous local martingales. The following will be helpful.

Proposition: (Kunita-Watanabe's inequality, 1967)

If H, K are progressively measurable, M, N are continuous local martingales, and

(5.74) P-a.s., $\int_0^\infty H_s^2(\omega)\, d\langle M\rangle_s(\omega) < \infty$, $\int_0^\infty K_s^2(\omega)\, d\langle N\rangle_s(\omega) < \infty$,

then

(5.75) P-a.s., $\int_0^\infty H_s(\omega) K_s(\omega)\, d|\langle M,N\rangle|_s(\omega) \leq \left(\int_0^\infty H_s^2(\omega) d\langle M\rangle_s(\omega)\right)^{1/2} \left(\int_0^\infty K_s^2(\omega) d\langle N\rangle_s(\omega)\right)^{1/2}$,

(with $|\langle M,N\rangle|_s$ denoting the total variation process of $\langle M,N\rangle_s$, cf. (5.45)).

Proof:

From (5.43), (5.44), we see that

(5.76) P-a.s., for all $\lambda \in \mathbb{Q}$, all $t \geq 0$, $\langle M+\lambda N\rangle_t = \langle M\rangle_t + 2\lambda \langle M,N\rangle_t + \lambda^2 \langle N\rangle_t$.

Hence P-a.s., for $s \le t$, $\lambda \in \mathbb{Q}$, (with $\langle\ \rangle_s^t \overset{def}{=} \langle\ \rangle_t - \langle\ \rangle_s$)

(5.77) $\langle M\rangle_s^t + 2\lambda \langle M,N\rangle_s^t + \lambda^2 \langle N\rangle_s^t \geq 0$,

and thus looking at the discriminant (in λ), we find that

(5.78) P-a.s., for $0 \le s \le t$, $|\langle M,N\rangle_s^t(\omega)| \le (\langle M\rangle_s^t(\omega))^{1/2} (\langle N\rangle_s^t(\omega))^{1/2} \le \frac{1}{2}\langle M\rangle_s^t(\omega) + \frac{1}{2}\langle N\rangle_s^t(\omega)$.

This shows that on the above set of ω of full measure

(5.79) $d|\langle M,N\rangle_s| \le \frac{1}{2} d\langle M\rangle + \frac{1}{2} d\langle N\rangle \overset{def}{=} d\nu_\omega(s)$

and we can then introduce $f_s^M(\omega)$, $f_s^N(\omega)$, $f_s^{M,N}(\omega)$ the respective densities of $d\langle M\rangle_s(\omega)$, $d\langle N\rangle_s(\omega)$ and $d\langle M,N\rangle_s(\omega)$ with $d\nu_\omega(s)$.

Coming back to (5.76), we see that for ν_ω-a.e. s,

$f_s^M(\omega) + 2\lambda f_s^{M,N}(\omega) + \lambda^2 f_s^N(\omega) \geq 0$, for $\lambda \in \mathbb{Q}$, and hence $\lambda \in \mathbb{R}$,

We first assume, $H, K \geq 0$, bounded and compactly supported in s.

Setting $sign(x) = 1\{x \ge 0\} - 1\{x < 0\}$, $\tilde{H}_s = H_s\, sign(f_s^{M,N})$, and

$\lambda = \gamma K_s \tilde{H}_s^{-1} 1\{H_s \neq 0\}$, we see multiplying (5.80) by H_s^2 and considering the set of s where $H_s = 0$ separately that,

(5.80) P-a.s., for ν_ω-a.e. s, for all $\gamma \in \mathbb{R}$,

$$H_s^2(\omega)\, f_s^M(\omega) + 2\gamma \tilde{H}_s(\omega) K_s(\omega)\, f_s^{M,N}(\omega) + \gamma^2 K_s^2(\omega)\, f_s^N(\omega) \geq 0$$

Integrating over s, with respect to $d\nu_\omega(s)$, we find that

P-a.s. for all $\gamma \in \mathbb{R}$,

(5.81) $\int_0^\infty H_s^2(\omega)\, d\langle M\rangle_s + 2\gamma \int_0^\infty H_s(\omega) K_s(\omega)\, d|\langle M,N\rangle|_s + \gamma^2 \int_0^\infty K_s^2(\omega)\, d\langle N\rangle_s \geq 0$,

and looking at the discriminant in γ we find that P-a.s.,

$$\int_0^\infty H_s K_s\, d|\langle M,N\rangle|_s \le \Big(\int_0^\infty H_s^2\, d\langle M\rangle_s\Big)^{1/2} \Big(\int_0^\infty K_s^2\, d\langle N\rangle_s\Big)^{1/2}.$$

The case of non-negative H, K satisfying (5.74) follows by monotone convergence, and then the general case is immediate. $\square$

We now have the following characterization of $(H.M)$ for $H \in \Lambda_3(M)$, Theorem:

Let M be a continuous local martingale with $M_0=0$, $H\in\Lambda_3(M)$, then $(H.M)$ is the unique continuous local martingale vanishing in 0, such that for all continuous local martingales N, P-a.s.

(5.82) $\langle (H.M),N\rangle_t = \int_0^t H_s(\omega)\,d\langle M,N\rangle_s$, for all $t\geq 0$,

(note that (5.75) implies that P-a.s. the right-hand side is well-defined)

Proof:

— Uniqueness:

If $I,\tilde{I}$ are continuous local martingales vanishing at time 0 such that $\langle I,N\rangle=\langle \tilde{I},N\rangle$ for all continuous local martingales N, we have $\langle I-\tilde{I}\rangle=0$. Hence we can find $T_n\uparrow\infty$ stopping times such that $(I-\tilde{I})^2_{t\wedge T_n}$, $t\geq 0$, are bounded continuous martingales, cf. remark p. 79-80. Hence we see that

(5.83) $E[(I-\tilde{I})^2_{t\wedge T_n}]=0,\quad t\geq 0,\ n\geq 0,$

so that P-a.s., for $t\in\mathbb{Q}_+$, $n\geq 0$, $I_{t\wedge T_n}=\tilde{I}_{t\wedge T_n}$. Since $T_n\uparrow\infty$, using continuity as well we find that

(5.84) P-a.s., for all $t\geq 0$, $I_t=\tilde{I}_t$.

— (5.82):

When $H=C\,1_{(a,b]}$, with $0\leq a<b$, $C\in b\mathcal{G}_a$, is a basic process and M,N are continuous square integrable martingales

(5.85) $J_t=(H.M)_t\,N_t-\int_0^t H\,d\langle M,N\rangle_s = C\left[(M_{b\wedge t}-M_{a\wedge t})N_t-\langle M,N\rangle_{a\wedge t}^{b\wedge t}\right]$

is a continuous martingale.

For instance when $a\leq s\leq b$, $s<t$:

$$E[J_t\,|\,\mathcal{G}_s]=C\,E\left[E\left[(M_{b\wedge t}-M_a)N_t-\langle M,N\rangle_a^{b\wedge t}\,|\,\mathcal{G}_{b\wedge t}\right]|\,\mathcal{G}_s\right]$$
$$=C\,E\left[(M_{b\wedge t}-M_a)N_{b\wedge t}-\langle M,N\rangle_a^{b\wedge t}\,|\,\mathcal{G}_s\right]$$

and since $N_{b\wedge t}$, $t\geq 0$, and $M_{b\wedge t}N_{b\wedge t}-\langle M,N\rangle_{b\wedge t}$ are martingales

$$=C\left[(M_{b\wedge s}-M_a)N_{b\wedge s}-\langle M,N\rangle_a^{s\wedge t}\right]=J_s,$$

and the other cases are easier to check. We then find that (5.82) holds for $H\in\Lambda_1$, M,N continuous square integrable martingales.

Then keeping M,N as above, for $H \in \Lambda_2(M)$ we can choose H^n in Λ_1 approximating H in $\Lambda_2(M)$, so that $(H^n.M)_t \xrightarrow{L^2(P)} (H.M)_t$, for $t \geq 0$.
Then as a result of (5.75), for $t \geq 0$,

(5.87) $E\left[\int_0^t |H_s(\omega) - H^n_s(\omega)|\, d|\langle M,N\rangle|_s\right] \leq E\left[\left(\int_0^\infty (H-H^n)^2_s(\omega)\, d\langle M\rangle_s(\omega)\right)^{1/2} \langle N\rangle_t^{1/2}\right] \underset{\text{Schwarz}}{\overset{\text{Cauchy}}{\leq}}$

$$\|H-H^n\|_{L^2(dP\, d\langle M\rangle)} E[\langle N\rangle_t]^{1/2} \underset{n\to\infty}{\longrightarrow} 0.$$

As a result we see that for $t \geq 0$,

(5.88) $(H^n.M)_t N_t - \int_0^t H^n_s\, d\langle M,N\rangle_s \underset{n\to\infty}{\overset{L^1(P)}{\longrightarrow}} (H.M)_t N_t - \int_0^t H_s\, d\langle M,N\rangle_s$,

and the limit is a martingale as well.
Thus we have proved that (5.82) holds when M,N are continuous square integrable martingales and $H \in \Lambda_2(M)$.

Now in the general case of the theorem, when $H \in \Lambda_3(M)$, we choose stopping times $T_n \uparrow \infty$, so that M^{T_n}, N^{T_n} are continuous square integrable martingales, and $H 1_{[0,T_n]} \in \Lambda_2(M^{T_n})$, for each $n \geq 0$. Then we find from (5.71) that P-a.s., for all $t \geq 0$

(5.89) $(H.M)_{t\wedge T_n} N_{t\wedge T_n} - \int_0^{t\wedge T_n} H_s(\omega)\, d\langle M,N\rangle_s =$

$((H 1_{[0,T_n]}).M^{T_n})_t\, N^{T_n}_t - \int_0^t (H 1_{[0,T_n]})_s\, d\langle M,N\rangle_{s\wedge T_n} \underset{(5.42)}{\overset{(5.38)}{=}}$

$((H 1_{[0,T_n]}).M^{T_n})_t\, N^{T_n}_t - \int_0^t (H 1_{[0,T_n]})_s\, d\langle M^{T_n}, N^{T_n}\rangle_s$,

which is a continuous martingale.
This proves that
$(H.M)_t N_t - \int_0^t H_s\, d\langle M,N\rangle_s$, $t \geq 0$, is a continuous local martingale and by (5.43), (5.44), (recall that $(H.M)_0 = 0$), the claim (5.82) follows in the general case. □

<u>Corollary:</u>
For M,N continuous local martingale, vanishing at time 0, $H \in \Lambda_3(M)$, $K \in \Lambda_3(N)$,
(5.90) P-a.s., $\langle (H.M),(K.N)\rangle_t = \int_0^t H_s(\omega) K_s(\omega)\, d\langle M,N\rangle_s$, for $t \geq 0$.

<u>Proof:</u> From (5.82) we find that P-a.s.,
$d\langle (H.M),(K.N)\rangle = H\, d\langle M,(K.N)\rangle = HK\, d\langle M,N\rangle$. □

We have the following very useful consequence of this result:

Corollary:

For M continuous local martingale with $M_0=0$, $H\in\Lambda_3(M)$, $K\in\Lambda_3((H.M))$, one has

(5.91) $HK\in\Lambda_3(M)$, and

(5.92) P-a.s., for $t\geq 0$, $(K.(H.M))_t = ((KH).M)_t$.

Proof:

- (5.91):

From (5.90), we have P-a.s., $d\langle(H.M)\rangle = H^2 d\langle M\rangle$, so that $K\in\Lambda_3((H.M))$ means that K is $\mathcal{P}$-measurable and P-a.s., $\int_0^t K_s^2 H_s^2\, d\langle M\rangle_s < \infty$, and therefore $HK\in\Lambda_3(M)$.

- (5.92):

By (5.82), for N continuous local martingale, P-a.s.

(5.93) $d\langle(K.(H.M)),N\rangle = K\, d\langle(H.M),N\rangle = KH\, d\langle M,N\rangle$,

and (5.92) now follows from the uniqueness part of (5.82). $\square$