Lecture 17: (Chap. 6, cont.)

We will now describe some first applications of Itô's formula. We recall that $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ is a filtered probability space satisfying the "usual conditions", cf. (4.5), (4.6).

Exponential Martingales:

Theorem:

Let M_t, $t \geq 0$, be a continuous $(\mathcal{G}_t)$-local martingale, with $M_0 = 0$.

(6.27) $Z_t = \exp\{M_t - \frac{1}{2}\langle M\rangle_t\}$, $t \geq 0$, is a continuous $(\mathcal{G}_t)$-local martingale, which satisfies the "stochastic different equation":

(6.28) P-a.s., for all $t \geq 0$, $\quad Z_t = 1 + \int_0^t Z_s \, dM_s$.

Moreover if for some $\varepsilon > 0$ and $0 < T \leq \infty$,

(6.29) $E[\exp\{\frac{(1+\varepsilon)}{2}\langle M\rangle_T\}] < \infty$, then

(6.30) Z_t, $t \leq T$, is a continuous $(\mathcal{G}_t)$-martingale.

Remarks:

1) We will later see that (6.30) is still valid when (6.29) holds with $\varepsilon = 0$. This is the so-called Novikov condition, cf. Karatzas-Shreve, Revuz-Yor. For the time being we discuss this simpler result, which has an elementary proof and can be helpful in a number of situations.

2) If (6.29) holds with $T = \infty$, then the proof below will show that for small $\eta > 0$,

(6.30)' Z_t, $0 \leq t \leq T$, is a continuous $(\mathcal{G}_t)$ martingale bounded in $L^{1+\eta}$. $\square$

Exercise:

Show that when $E[\langle M\rangle_T] < \infty$, ($M_t$, $t \geq 0$, as above), then M_t, $t \leq T$, is a continuous square integrable martingale. $\square$

Proof:

– (6.28): We introduce the function

(6.31) $f(x,t) = \exp\{x - \frac{1}{2}t\}$, $\quad x, t \in \mathbb{R}$.

This function satisfies the equation:

(6.32) $\frac{\partial f}{\partial t}(x,t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,x) = 0.$

We will now apply Ito's formula (6.22) to $Y = (Y^1, Y^2) = (M, \langle M \rangle)$. We first note that
$\langle Y^1, Y^2 \rangle = 0 = \langle Y^2 \rangle$, and $\langle Y^1 \rangle = \langle M \rangle$.
We thus find that P-a.s., for $t \geq 0$:

(6.33) $$f(M_t, \langle M \rangle_t) = f(0,0) + \int_0^t \partial_x f(M_s, \langle M \rangle_s)\, dM_s + \int_0^t \partial_t f(M_s, \langle M \rangle_s)\, d\langle M \rangle_s + \frac{1}{2}\int_0^t \partial_x^2 f(M_s, \langle M \rangle_s)\, d\langle M \rangle_s$$

$$\overset{(6.32)}{=} 1 + \int_0^t f(M_s, \langle M \rangle_s)\, dM_s, \quad (\partial_x f = f).$$

Since $Z_t = f(M_t, \langle M \rangle_t)$, this proves (6.28), as well as the fact that Z_t, $t \geq 0$, is a continuous local martingale.

– (6.30):

We consider $T_n \uparrow \infty$, a sequence of finite stopping times such that

(6.34) $M_{t \wedge T_n}$, $0 \leq t \leq T$, is a bounded martingale for each n.

Observe that $Z_{t \wedge T_n}$ is a bounded continuous local-martingale and hence, cf. (4.104),

(6.35) $Z_{t \wedge T_n}$, $0 \leq t \leq T$, is a bounded martingale for each n.

We will now see that

(6.36) for some $q > 1$, $\sup_{n \geq 0} E[Z^q_{T \wedge T_n}] < \infty$.

From Doob's inequality (4.76), it will follow that:

(6.37) $$E[(\sup_{t \leq T} Z_t)^q] = \lim_{n \to \infty} E[\sup_{t \leq T \wedge T_n} Z_t^q] \leq \varliminf_n \left(\frac{q}{q-1}\right)^q E[Z^q_{T \wedge T_n}] < \infty.$$

Together with (6.35), this will imply by dominated convergence that:

(6.38) Z_t, $0 \leq t \leq T$, is a continuous martingale bounded in L^q.

This will prove (6.30) (and also (6.30)' in the case $T = \infty$; in this case Z_∞ exists due to the martingale convergence theorem).

There remains to prove (6.36). We pick $q, \alpha > 1$, and write:

(6.39) $E[Z^q_{T\wedge T_n}] = E[\exp\{q M_{T\wedge T_n} - \frac{q}{2} \langle M\rangle_{T\wedge T_n}] =$

$E[\exp\{q M_{T\wedge T_n} - \frac{1}{2}\alpha q^2 \langle M\rangle_{T\wedge T_n} + \frac{1}{2} q(\alpha q - 1)\langle M\rangle_{T\wedge T_n}] \overset{\text{Hölder}}{\leq}$

$E[\exp\{\alpha q M_{T\wedge T_n} - \frac{1}{2}\alpha^2 q^2 \langle M\rangle_{T\wedge T_n}]^{1/\alpha}\, E[\exp\{\frac{1}{2}\frac{\alpha}{\alpha-1} q(\alpha q-1)\langle M\rangle_{T\wedge T_n}\}]^{\frac{\alpha-1}{\alpha}}$

Note that $\exp\{\alpha q M_{t\wedge T_n} - \frac{1}{2}\alpha^2 q^2 \langle M\rangle_{t\wedge T_n}\}$ is a bounded martingale just as in (6.35), and the first term in the last line of (6.39) equals 1.
So we see that for any $n \geq 0$,

$E[Z^q_{T\wedge T_n}] \leq E[\exp\{\frac{1}{2}\frac{\alpha}{\alpha-1} q(\alpha q-1)\langle M\rangle_T\}]^{\frac{\alpha-1}{\alpha}}$.

Note that

$\lim_{\alpha\downarrow 1} \lim_{q\downarrow 1} \frac{\alpha}{\alpha-1} q(\alpha q-1) = 1$,

and hence we can choose $q, \alpha > 1$, so that

$\frac{\alpha}{\alpha-1} q(\alpha q - 1) < 1+\varepsilon$.

Together with (6.29) and the above inequality yields (6.36). □

Example:

X_t, $t \geq 0$, Brownian motion on $\mathbb{R}$. Then for $\lambda \in \mathbb{R}$

(6.40) $\exp\{\lambda X_t - \frac{\lambda^2}{2} t\}$ is a martingale.

Consider $a > 0$ and the entrance time of X in a:

$H_a = \inf\{s \geq 0;\, X_s = a\}$, (the distribution of H_a under W_0 appears in (2.54)).

Then using the stopping theorem, we see that when $\lambda > 0$,

(6.41) $\exp\{\lambda X_{t\wedge H_a} - \frac{\lambda^2}{2}(t\wedge H_a)\}$ is a martingale, which is bounded above by $e^{\lambda a}$.

As a result we obtain that

$1 = E_0[\exp\{\lambda X_{t\wedge H_a} - \frac{\lambda^2}{2}(t\wedge H_a)\}]$, and since $H_a < \infty$, W_0-a.s.,

dominated convergence $\downarrow t \to \infty$

$E_0[\exp\{\lambda H_a - \frac{\lambda^2}{2} H_a\}] = e^{\lambda a} E_0[e^{-\frac{\lambda^2}{2} H_a}]$.

Setting $u = \lambda^2/2$, we obtain by symmetry:

(6.42) $E_0[\exp\{-u H_a\}] = \exp\{-|a|\sqrt{2u}\}$, for $a \in \mathbb{R}$, $u \geq 0$. □

As application of exponential martingales, we will prove Paul Lévy's characterization of Brownian motion. We introduce the following

Definition:

A continuous adapted $\mathbb{R}^d$-valued process $X_t, t \geq 0$, with $X_0 = 0$, is called $(\mathcal{G}_t)$-Brownian motion if for $0 \leq s < t$,

(6.43) $X_t - X_s$ is independent of $\mathcal{G}_s$ and $N(0,(t-s)I)$-distributed.

Remark:

A $(\mathcal{G}_t)$-Brownian motion is then of course in particular a d-dimensional Brownian motion in the sense of the definition (1.1). However the independence assumption (6.43) is a priori a (possibly) more stringent requirement (since $(\mathcal{G}_t)_{t\geq 0}$ contains all the information in $\sigma(X_u, u \leq t), t \geq 0$, and maybe more).

Theorem: (P. Lévy's characterization of Brownian motion)

If $X_t, t \geq 0$, is a d-dimensional continuous $(\mathcal{G}_t)$-local martingale, such that $X_0 = 0$, and

(6.44) $\langle X^i, X^j \rangle_t = \delta_{ij}\, t$, for $1 \leq i,j \leq d$,

then

(6.45) $X_t, t \geq 0$, is a d-dimensional $(\mathcal{G}_t)$-Brownian motion.

Proof:

The same calculation as in (6.33) shows that for $\xi \in \mathbb{R}^d$,

(6.46) $Z_t = \exp\{i\, \xi \cdot X_t + \frac{1}{2}|\xi|^2 t\}, \; t \geq 0,$

is a complex-valued continuous $(\mathcal{G}_t)$ local martingale, which is bounded when t remains bounded. Hence it is a continuous martingale and for $0 \leq s \leq t$:

$E[Z_t \mid \mathcal{G}_s] \overset{P\text{-a.s.}}{=} Z_s$ so that (since $|Z_s| > 0$)

$1 \overset{P\text{-a.s.}}{=} E[Z_t Z_s^{-1} \mid \mathcal{G}_s] = E[\exp\{i\,\xi\cdot(X_t - X_s) + \frac{|\xi|^2}{2}(t-s)\} \mid \mathcal{G}_s].$

As a result for $0 \le s < t$, $\xi \in \mathbb{R}^d$,

(6.47) $E[\exp\{i\xi\cdot(X_t - X_s)\} \mid \mathcal{G}_s] \overset{P\text{-a.s.}}{=} \exp\{-\frac{1}{2}|\xi|^2(t-s)\}$.

This implies that for $0 \le s < t$

$X_t - X_s$ is independent of $\mathcal{G}_s$ and $N(0,(t-s)I)$-distributed.

The claim (6.45) now readily follows.

$\square$

Remark:

If we now look again at the assumptions (4.20), (4.21), when we began the discussion of stochastic integrals, we see that they are equivalent to the fact that $X_t - X_0$ is a $(\mathcal{G}_t)$-Brownian motion and $X_0 \in L^2(\mathcal{G}_0, P)$. This link with Brownian motion was not clear at the time we introduced (4.20), (4.21).

$\square$

We will now give a further application of exponential martingales.

Proposition: (Bernstein inequality)

If $M_t, t \ge 0$, is a continuous local martingale with $M_0 = 0$, and $\langle M\rangle_t \le ct$, for $t \ge 0$, then $M_t, t \ge 0$, is a martingale and

(6.48) $P[\sup_{t \le T} M_t \ge a] \le \exp\{-\frac{a^2}{2cT}\}$, for $a, T > 0$,

(and hence $P[\sup_{t\le T} |M_t| \ge a] \le 2\exp\{-\frac{a^2}{2cT}\}$, for $a, T > 0$).

Proof:

For $\lambda \in \mathbb{R}$, thanks to (6.27), (6.30)

$Z_t = \exp\{\lambda M_t - \frac{\lambda^2}{2}\langle M\rangle_t\}$, $t \ge 0$,

is a continuous martingale. We can pick $\lambda > 0$, and we have thanks to Doob's inequality, cf. (4.67):

$$(6.49)\quad P\Big[\sup_{t\le T} M_t \ge a\Big] \le P\Big[\sup_{t\le T} Z_t \ge \exp\{\lambda a - \tfrac{\lambda^2}{2} c T\}\Big]$$

$$\overset{(4.67)}{\le} \exp\{-\lambda a + \tfrac{\lambda^2}{2} cT\} \underbrace{E[Z_T]}_{=E[Z_0]=1}.$$

We can optimize over $\lambda > 0$, and choose $\lambda = \frac{a}{cT}$, so that $-\lambda a + \frac{\lambda^2}{2} cT = -\frac{a^2}{2cT}$. As a result we find that
$P\big[\sup_{t\le T} M_t \ge a\big] \le \exp\{-\frac{a^2}{2cT}\}$,
that is (6.48) holds. Applying this inequality to $-M$, we thus see that M_t, $t \ge 0$, is a continuous local martingale, which is square integrable. It is therefore a martingale, cf. (4.104), (alternatively use the exercise below (6.30)'). □