Lecture 18: (chap. 6, cont.)

We continue our discussion of some first applications of Ito's formula.

Harmonic functions and Brownian motion

When $U \subseteq \mathbb{R}^d$ is a non-empty open set, a C^2-function on U is said harmonic (in U) when $\Delta f(x) = (\sum_1^d \partial_i^2 f(x)) = 0$, $x \in U$. In fact no regularity requirement on f is necessary in the sense that the equation $\Delta f = 0$ on U in the distribution sense implies that f is C^∞ on U and satisfies $\Delta f(x) = 0$, $x \in U$, in the classical sense, cf. Donoghue "Distributions and Fourier transforms", Academic Press, p. 127. Harmonic functions play a very important role in the study of Brownian motion. Here is an example.

Proposition:

When $d \geq 2$ and $x \neq 0$ is a point of $\mathbb{R}^d$, then

(6.50) W_x-a.s., $X_t \neq 0$, for all $t \geq 0$.

(in other words, "Brownian motion does not hit points when $d \geq 2$").

Proof:

When g is a C^2-function on $(0,\infty)$, we can define the radial function $f(x) = g(|x|) = g(\sqrt{x_1^2 + \dots + x_d^2})$, $x \in \mathbb{R}^d \setminus \{0\}$, and we has the identity (exercise!)

(6.51) $\Delta f(x) = g''(r) + \frac{d-1}{r} g'(r)$, with $r = |x|$, for $x \in \mathbb{R}^d \setminus \{0\}$.

When $d \geq 3$, we choose

(6.52) $g(r) = r^{2-d}$,

so that

$$g''(r) + \frac{d-1}{r} g'(r) = (2-d)(1-d)\, r^{-d} + (d-1)(2-d)\, r^{-d} = 0,$$

and therefore

(6.53) $f(x) = \frac{1}{|x|^{d-2}}$, $x \neq 0$, is harmonic in $\mathbb{R}^d \setminus \{0\}$.

If $x \neq 0$, and $a < |x| < b$, we choose f_a, a smooth radial function, equal to $f(z)$ on $\{y \in \mathbb{R}^d; |y| \geq a\}$, so that applying Itô's formula, cf. (6.26), one finds that: W_x-a.s., for $t \geq 0$:

(6.54) $f_a(X_t) = f_a(x) + \int_0^t \nabla f_a(X_s).dX_s + \frac{1}{2}\int_0^t \Delta f_a(X_s)ds$.

We then introduce the stopping time

(6.55) $\tau = \inf\{u \geq 0;\ |X_u| \leq a \text{ or } |X_u| \geq b\}$,

and see that W_x-a.s.,

$$|X_{t\wedge\tau}|^{-(d-2)} = f_a(X_{t\wedge\tau}) \overset{(6.54)}{=} |x|^{-(d-2)} + \int_0^{t\wedge\tau} \nabla f_a(X_s).dX_s + \frac{1}{2}\int_0^{t\wedge\tau} \Delta f_a(X_s)ds,$$

and the last term vanishes since $\Delta f_a(X_s) = 0$, for $0 < s < \tau$.

As a result $|X_{t\wedge\tau}|^{-(d-2)}$, $t \geq 0$, is a local martingale, which is bounded, and hence:

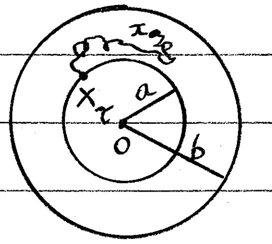
(6.56) $|X_{t\wedge\tau}|^{2-d}$, $t \geq 0$, is a martingale.

As a result we find

(6.57) $E_x[|X_{t\wedge\tau}|^{2-d}] = |x|^{2-d}$, for $t \geq 0$.

Letting $t \to \infty$, and using dominated convergence we find that:

$$|x|^{2-d} = E[|X_\tau|^{2-d}] = a^{2-d}\, W_x(|X_\tau| = a) + b^{2-d}\, W_x(|X_\tau| = b).$$



An illustration of X_s, $0 \leq s \leq \tau$, under W_x.

Since $W_x(|X_\tau| = a) + W_x(|X_\tau| = b) = 1$, we obtain

(6.58) $W_x(|X_\tau| = a) = \dfrac{|x|^{2-d} - b^{2-d}}{a^{2-d} - b^{2-d}}$, $W_x(|X_\tau| = b) = \dfrac{a^{2-d} - |x|^{2-d}}{a^{2-d} - b^{2-d}}$, $a < |x| < b$.

Letting $a \to 0$, with b fixed, we see that

(6.59) $W_x(H_{\{0\}} < T_{B(0,b)}) = 0$, for $0 < b$, with the notation

$T_U = \inf\{s \geq 0;\ X_s \notin U\}$, the "exit time from U".
It now follows that

(6.60) $W_x(H_{\{0\}} < \infty) = W_x(X_t = 0, \text{ for some } t \geq 0)$
$\quad \text{"}\lim_{b\to\infty}\text{"}\ W_x(H_{\{0\}} < T_{B(0,b)}) = 0,$
and this proves (6.50) when $d \geq 3$.

When $d=2$, we choose instead

(6.61) $g(r) = \log \frac{1}{r}$,
so that
$g''(r) + \frac{d-1}{r} g'(r) = \frac{1}{r^2} - \frac{1}{r^2} = 0,$
and

(6.62) $f(x) = \log \frac{1}{|x|}$, $x \neq 0$, is harmonic in $\mathbb{R}^2 \setminus \{0\}$.
The repetition of the above proof now yields that

(6.63) $W_x(|X_\tau| = a) = \dfrac{\log \frac{b}{|x|}}{\log \frac{b}{a}}$, $W_x(|X_\tau| = b) = \dfrac{\log \frac{|x|}{a}}{\log \frac{b}{a}}$,

and one concludes as above by letting $a \to 0$, with b fixed, and then $b \to \infty$.

□

<u>As an application of the same circle of ideas we will discuss recurrence and transience properties of Brownian motion</u> in $\mathbb{R}^d$, when $d \geq 2$.

<u>Theorem</u>: (transience of Brownian motion in $\mathbb{R}^d$, $d \geq 3$)
When $d \geq 3$, then for $x \in \mathbb{R}^d$,

(6.64) W_x-a.s., $\lim_{t\to\infty} |X_t| = \infty$.

<u>Proof:</u>
Since under W_x, $(X_t + z)_{t\geq 0}$ is a Brownian motion starting from $x+z$, it suffices to prove (6.64) for some $x \neq 0$. From (6.54) we know that if $H_{\overline{B}(0,a)}$ stands for the entrance time of x in $\overline{B}(0,a)$:

(6.65) $|X_{t\wedge H_{\overline{B}(0,a)}}|^{2-d}$ is a continuous bounded martingale under W_x.

Using Fatou's lemma for conditional expectations, we find that for $s \le t$, W_x-a.s.

$$E_x[|X_t|^{2-d} \mid F_s] \overset{(6.50)}{=} E_x[\liminf_n |X_{t\wedge H_{\overline{B}(0,\frac{1}{n})}}|^{2-d} \mid F_s] \overset{\text{Fatou}}{\le} \liminf_n E_x[|X_{t\wedge H_{\overline{B}(0,\frac{1}{n})}}|^{2-d} \mid F_s] \overset{(6.65)}{=} \liminf_n |X_{s\wedge H_{\overline{B}(0,\frac{1}{n})}}|^{2-d} \overset{(6.50)}{=} |X_s|^{2-d}$$

In other words we have proved that

(6.66) $|X_t|^{2-d}$, $t \ge 0$, is a continuous supermartingale under W_x.

Since this supermartingale is non-negative, it follows from the convergence theorem that

(6.67) W_x-a.s., $|X_t|^{2-d}$ has a finite limit as $t \to \infty$.

On the other hand, looking at one of the components of X_t, we already know that

W_x-a.s., $\limsup_t |X_t| = \infty$.

This observation combined with (6.67) implies that the finite limit in (6.67) is 0. □

Exercise:

Show that a non-negative continuous local martingale is a supermartingale. □

We now turn to the two-dimensional situation.

Theorem: (recurrence of Brownian motion in $\mathbb{R}^2$)

When $d = 2$, for any $x \in \mathbb{R}^2$,

(6.68) W_x-a.s., for any non-empty open set $O \subseteq \mathbb{R}^2$, $\{t \ge 0;\ X_t \in O\}$ is unbounded.

Proof:

From (6.63), we see by letting $b \to \infty$, that when $a < |x|$, W_x-a.s., $H_{\overline{B}(0,a)} < \infty$. Of course this remains true when $|x| \le a$, so that

(6.69) for any $x \in \mathbb{R}^2$, $a > 0$, W_x-a.s., $H_{\overline{B}(0,a)} < \infty$.

One can then define the sequence of $(\mathcal{F}_t^+)$-stopping times, cf. (2.33), $S_1 = H_{\overline{B}(0,a)}$, $S_2 = S_1 \circ \theta_{S_1+1} + S_1 + 1$, and by induction for $i \ge 1$: $S_{i+1} = S_1 \circ \theta_{S_i+1} + S_i + 1$, so that $S_i \uparrow \infty$.

Using the strong Markov property, cf. (2.46), we see that for any $y \in \mathbb{R}^2$, for $i \ge 1$,

$$
\begin{aligned}
(6.70)\quad W_y[S_{i+1} < \infty] &= W_y[S_i < \infty \text{ and } \theta^{-1}_{S_i+1}(S_1 < \infty)] \\
&\overset{(2.46)}{=} E_y[S_i < \infty, \underbrace{P_{X_{S_i+1}}[S_1 < \infty]}_{=1 \text{ by } (6.69)}] = W_y[S_i < \infty]] \\
&\overset{\text{induction}}{=} W_y[S_1 < \infty] = 1.
\end{aligned}
$$

Note also that by construction for $i \ge 1$,

W_y-a.s., on $\{S_i < \infty\}$, $X_{S_i} \in \overline{B}(0,a)$.

It thus follows from (6.70) and $S_i \uparrow \infty$, that for any $y \in \mathbb{R}^2$, W_y-a.s., for any $a = \frac{1}{n}$, $\{t \ge 0; X_t \in \overline{B}(0, \frac{1}{n})\}$ is unbounded.

Since W_y is the law of $(X_t + y)_{t \ge 0}$, under W_0, the above property implies that (setting $z = -y$):

(6.71) W_0-a.s., for all $z \in \mathbb{Q}^2$, $n \ge 1$, $\{t \ge 0; X_t \in \overline{B}(z, \frac{1}{n})\}$ is unbounded.

This proves (6.68) when $x = 0$. The case of a general x follows since $(X_t)_{t \ge 0}$ under W_x has the law of $(X_t + x)_{t \ge 0}$ under W_0, as was already used in the proof. □

Exercise:

Give a proof of (6.68) using (6.69) and (6.50), (without the introduction of the stopping times S_i, $i \ge 1$). □