Lecture 19: (chap 6, cont.)

Complement

We will now present Novikov's criterion, which refines the condition we gave in (6.29) to ensure that Z_t is a martingale and not merely a continuous local martingale.

Theorem: (Novikov's criterion)

Let $M_t, t \geq 0$, be a continuous local martingale with $M_0 = 0$, such that

(6.72) $E[\exp\{\frac{1}{2} \langle M \rangle_\infty\}] < \infty$,

then

(6.73) $E[\exp\{\frac{1}{2} \sup_{t \geq 0} |M_t|\}] < \infty$,

and

(6.74) $Z_t = \exp\{M_t - \frac{1}{2}\langle M \rangle_t\}, t \geq 0$, is a uniformly integrable continuous martingale,

(and of course $M_t, t \geq 0$, is a continuous martingale as well).

Remark:

If instead of (6.72) we assume that for some $T > 0$,

(6.75) $E[\exp\{\frac{1}{2}\langle M \rangle_T\}] < \infty$,

the above theorem can be applied to $M_{t \wedge T}, t \geq 0$, and we find that

(6.76) $Z_t = \exp\{\frac{1}{2} M_t - \frac{1}{2}\langle M \rangle_t\}, t \leq T$, is a continuous martingale. □

Proof:

We first observe that $E[\langle M \rangle_\infty] < \infty$ implies that

(6.77) $M_t, t \geq 0$, is a continuous martingale bounded in L^2.

Indeed (this is just as in the exercise below (6.30)'), one chooses a sequence $T_n \uparrow \infty$ of stopping times so that $M_{t \wedge T_n}, t \geq 0$, are bounded martingales. Then one has
$E[M^2_{t \wedge T_n}] \overset{(5.36)}{=} E[\langle M \rangle_{t \wedge T_n}] \leq E[\langle M \rangle_\infty]$, and by Fatou's lemma
$E[M_t^2] \leq \liminf_n E[M^2_{t \wedge T_n}] \leq E[\langle M \rangle_\infty]$.

It now also follows with similar considerations as in (4.104) and Doob's inequality that $M_t, t \geq 0$, is a continuous martingale, with $E[\sup_{t\geq 0} |M_t|^2] < \infty$, and that $M_\infty \overset{\text{P-a.s.}}{=} \lim_{t\to\infty} M_t$ is well-defined, by the martingale convergence theorem (6.73). Note that for $0 \leq t \leq \infty$,

(6.78) $$E[\exp\{\tfrac{1}{2} M_t\}] = E[\exp\{\tfrac{1}{2} M_t - \tfrac{1}{4}\langle M\rangle_t\} \exp\{\tfrac{1}{4}\langle M\rangle_t\}]$$
$$\overset{\text{Cauchy-Schwarz}}{\leq} E[\exp\{M_t - \tfrac{1}{2}\langle M\rangle_t\}]^{1/2}\, E[\exp\{\tfrac{1}{2}\langle M\rangle_\infty\}]^{1/2}.$$

Since $Z_t, t \geq 0$, is a non-negative local martingale, it is also a supermartingale, (see exercise below (6.67): for $0 \leq s \leq t$
$$E[Z_t | \mathcal{G}_s] = E[\lim_n Z_{t\wedge T_n} | \mathcal{G}_s] \overset{\text{Fatou}}{\leq} \liminf_n E[Z_{t\wedge T_n} | \mathcal{G}_s]$$
$$= \liminf_n Z_{s\wedge T_n} = Z_s, \text{ thus proving the claim}).$$

As a result $E[Z_t] \leq E[Z_0] = 1$, and coming back to (6.78)

(6.79) $$E[\exp\{\tfrac{1}{2} M_t\}] \leq E[\exp\{\tfrac{1}{2}\langle M\rangle_\infty\}]^{1/2}, \quad 0 \leq t \leq \infty.$$

The same argument applied to $-M$ yields that
$$\sup_{t\geq 0} E[\cosh(\tfrac{1}{2} M_t)] \leq E[\exp\{\tfrac{1}{2}\langle M\rangle_\infty\}]^{1/2}, \text{ so that}$$

(6.80) $$E[\cosh(c M_t)] \leq E[\exp\{\tfrac{1}{2}\langle M\rangle_\infty\}]^{1/2}, \text{ for } 0 \leq c \leq \tfrac{1}{2}, 0 \leq t \leq \infty,$$

and in particular since $\cosh x \geq \frac{1}{2} e^x$,

(6.81) $$\sup_{0\leq t\leq\infty} E[e^{cM_t}] < \infty, \text{ for } 0 \leq c \leq \tfrac{1}{2}.$$

Jensen's inequality implies that $e^{cM_t}, t \geq 0$, is a non-negative submartingale. It then follows from Doob's inequality (4.76) with $p = \frac{1}{2c}$, for $0 < c < \frac{1}{2}$, that

(6.82) $$E[\sup_{t\geq 0} \exp\{\tfrac{1}{2} M_t\}] = E[(\sup_{t\geq 0} \exp\{c M_t\})^p]$$
$$\overset{(4.76)}{\leq} \left(\frac{p}{p-1}\right)^p \sup_{t\geq 0} E[\exp\{pcM_t\}]$$
$$= \left(\frac{p}{p-1}\right)^p \sup_{t\geq 0} E[\exp\{\tfrac{1}{2} M_t\}] \overset{(6.81)}{<} \infty.$$

Of course a similar bound holds for $-M$ in place of M. Note also that $\sup_{t\geq 0}(\cosh(\frac{1}{2}M_t)) \leq \frac{1}{2}\sup_{t\geq 0} e^{\frac{1}{2}M_t} + \frac{1}{2}\sup_{t\geq 0} e^{-\frac{1}{2}M_t}$, and hence

(6.83) $$E[\sup_{t\geq 0} \cosh(\tfrac{1}{2} M_t)] < \infty,$$

from which follows that $E[\exp\{\frac{1}{2}\sup_{t\geq 0}|M_t|\}] < \infty$, and (6.73) holds

– (6.74):

We will use the next

Lemma:

(6.84) If $E[Z_\infty] = 1$, then Z_t, $0 \le t \le \infty$, is a uniformly integrable martingale.

Proof:

From the supermartingale property of $Z_t, t \ge 0$, follows that $1 = E[Z_\infty] \le E[Z_t] \le E[Z_0]$, for $0 \le t \le \infty$, so that $E[Z_t] = 1$, for $0 \le t \le \infty$.

Note that P-a.s., $Z_{t \wedge T_n} \xrightarrow{n \to \infty} Z_t$, and $E[Z_{t \wedge T_n}] = E[Z_t] = 1$, and these variables are non negative.

It now follows that they are uniformly integrable and

(6.85) $Z_{t \wedge T_n} \underset{n \to \infty}{\xrightarrow{L^1}} Z_t$, for $0 \le t \le \infty$,

see for instance Durrett "Probability: Theory and Examples", p. 224.

This now implies that $Z_t, t \ge 0$, is a martingale. Moreover since P-a.s., $Z_t \to Z_\infty$, as $t \to \infty$, and $E[Z_t] = E[Z_\infty] = 1$, the above argument shows that

(6.86) $Z_t \xrightarrow{L^1} Z_\infty$, as $t \to \infty$,

so that

(6.87) $Z_t = E[Z_\infty \mid \mathcal{G}_t]$, for $t \ge 0$,

and the conclusion of the lemma follows, cf. Durrett, p. 223. $\square$

We will now show that

(6.88) $E[Z_\infty] \ge 1$,

since we already know that $E[Z_\infty] \le 1$, the claim (6.74) will now follow from the lemma.

By (6.29), (6.30), with $T = \infty$, we know from (6.72) that $E\left[\exp\{a M_\infty - \frac{a^2}{2} \langle M \rangle_\infty\}\right] = 1$, for $a \in [0,1)$.

Note that the following equality holds:

$$\exp\{aM_\infty - \frac{a^2}{2}\langle M\rangle_\infty\} = \exp\{M_\infty - \frac{1}{2}\langle M\rangle_\infty\}^{a^2} \exp\{\frac{a}{1+a} M_\infty\}^{1-a^2}.$$

Using Hölder's inequality with $p = a^{-2}$ and $q = (1-a^2)^{-1}$, we find:

(6.89) $1 \leq E[Z_\infty]^{a^2} E[\exp\{\frac{a}{1+a} M_\infty\}]^{1-a^2}.$

Using (6.73), we can use dominated convergence to argue that $\lim_{a\to 1} E[\exp\{\frac{a}{1+a} M_\infty\}] = E[\exp\{\frac{1}{2} M_\infty\}] \in (0,\infty)$.

As a result we obtain that

(6.90) $\lim_{a\to 1} E[\exp\{\frac{a}{1+a} M_\infty\}]^{1-a^2} = 1$,

and the claim (6.88) now follows from (6.89) and (6.90).

This concludes the proof of (6.74). □

CHAPTER 7: STOCHASTIC DIFFERENTIAL EQUATIONS AND MARTINGALE PROBLEMS

We begin with some heuristic considerations.
In this chapter we want to construct processes, which "locally near a point x in $\mathbb{R}^d$", move like the process

$x + t\, b(x) + \sigma(x) B_t$,

where $b(x) \in \mathbb{R}^d$, $\sigma(x)$ is a $(d \times n)$-matrix and B_t is an n-dimensional Brownian motion. We want that "infinitesimally" the increment of the process we construct behaves as a Gaussian variable with

mean: $b(x)dt$, and

covariance matrix: $a(x)dt$,

where for $\xi \in \mathbb{R}^d$

$$
\begin{aligned}
{}^t\xi\, a(x)\, \xi\, dt \text{ " = " } & E\left[\left({}^t\xi . \sigma(x) . (B_{t+dt} - B_t)\right)^2\right] \\
= \; & E\left[\left({}^t(\underbrace{{}^t\sigma(x)\xi}_{n\text{-vector}}) . (B_{t+dt} - B_t)\right)^2\right] = |{}^t\sigma(x)\xi|^2 dt \\
= \; & {}^t\xi\, \sigma(x)\, {}^t\sigma(x)\, \xi\, dt,
\end{aligned}
$$

i.e. in other words $a(x) = \sigma(x)\, {}^t\sigma(x)$ ($d \times d$-matrix).

We will consider two approaches to build such processes. The first approach will rely on solving stochastic differential equations (SDE):

$$X_t^i = x_0^i + \int_0^t b_i(X_s)\,ds + \sum_{j=1}^n \int_0^t \sigma_{ij}(X_s)\,dB_s^j, \quad i = 1, \dots, d,$$

or in vector notation:

$$(7.1) \quad X_t = x + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s).dB_s, \quad x \in \mathbb{R}^d.$$

The second approach will be based on a martingale problem, i.e. finding on $(C(\mathbb{R}_+, \mathbb{R}^d), \mathcal{F})$ a probability P_x, such that

$$
(7.2)\begin{cases}
M_t^f \overset{\text{def}}{=} f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds,\ t \geq 0, \text{ is an } (\mathcal{F}_t)\text{-martingale under } P_x, \\
\text{for all } f \in C_c^2(\mathbb{R}^d), \text{ with } Lf(y) \overset{\text{def}}{=} \frac{1}{2}\sum_{i,j=1}^d a_{ij}(y)\,\partial^2_{ij} f(y) + \sum_{i=1}^d b_i(y)\,\partial_i f(y),\ y \in \mathbb{R}^d, \\
P_x[X_0 = x] = 1.
\end{cases}
$$

This latter approach shares the same spirit as Lévy's characterization of Brownian motion, cf (6.44), (6.45).

<u>Notation:</u>

- For $b \in \mathbb{R}^d$, $|b| = \left(\sum_{i=1}^{d} b_i^2\right)^{1/2}$,
- For $\sigma \in M_{d\times n}$, $|\sigma| = \left(\sum_{\substack{1\le i\le d\\ 1\le j\le n}} \sigma_{ij}^2\right)^{1/2} = \{\mathrm{Trace}(\underbrace{\sigma\,{}^t\sigma}_{\in M_{d\times d}})\}^{1/2} = \{\mathrm{Trace}(\underbrace{{}^t\sigma\,\sigma}_{\in M_{n\times n}})\}^{1/2}$
- $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, P)$ a probability space satisfying the usual conditions, cf. (4.5), (4.6).
- $B_t, t\ge 0$, an n-dimensional $(\mathcal{G}_t)$- Brownian motion, or equivalently in view of (6.44), (6.45), for all $1\le i,j\le n$, $B_t^i, t\ge 0$, are continuous local martingales with $B_0 = 0$, and $B_t^i B_t^j - \delta_{ij} t$, $t\ge 0$, are continuous local martingales.