Lecture 20: (Chap. 7, cont.)

We now begin with the discussion of stochastic differential equations. The next theorem provides a basic result.

Theorem: (Picard's iteration method)

Assume that $b(\cdot): \mathbb{R}^d \to \mathbb{R}^d$, and $\sigma(\cdot): \mathbb{R}^d \to M_{d\times n}$ satisfy the Lipschitz condition

(7.3) $|b(y) - b(z)| + |\sigma(y) - \sigma(z)| \le K|y-z|$, for $y, z \in \mathbb{R}^d$.

Then for any $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\ge 0}, P)$ and $B_t, t\ge 0$, as on p. 122, and any x in $\mathbb{R}^d$, there exists an essentially unique continuous $(\mathcal{G}_t)$-adapted $(X_t)_{t\ge 0}$ with values in $\mathbb{R}^d$, such that P-a.s., for $t \ge 0$,

(7.4) $X_t = x + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s).dB_s$.

Proof:

- uniqueness:

Consider $X_., Y_.$ two solutions. For $M > |x|$, we define

(7.5) $T = \inf\{u \ge 0;\ |X_u| \text{ or } |Y_u| \ge M\}$,

so that P-a.s., for all $t \ge 0$,

$$X_{t\wedge T} - Y_{t\wedge T} = \int_0^{t\wedge T} (b(X_u) - b(Y_u))\,du + \int_0^{t\wedge T} (\sigma(X_u) - \sigma(Y_u)).dB_u.$$

As a result we see that for $t_0 > 0$:

$$E\Big[\sup_{t\le t_0} |X_{t\wedge T} - Y_{t\wedge T}|^2\Big] \le 2\,E\Big[\sup_{t\le t_0} \Big|\int_0^{t\wedge T} (\sigma(X_u) - \sigma(Y_u)).dB_u\Big|^2\Big]$$

$$+ 2t_0\,E\Big[\int_0^{t_0\wedge T} |b(X_u) - b(Y_u)|^2\,du\Big].$$

Using Doob's inequality (4.76), with $p=2$, for each component of the $\mathbb{R}^d$-valued stochastic integral we find that

(7.6) $$E\Big[\sup_{t\le t_0} |X_{t\wedge T} - Y_{t\wedge T}|^2\Big] \le 8\,E\Big[\Big|\int_0^{t_0\wedge T} (\sigma(X_u) - \sigma(Y_u)).dB_u\Big|^2\Big]$$

$$+ 2t_0\,E\Big[\int_0^{t_0\wedge T} |b(X_u) - b(Y_u)|^2\,du\Big].$$

On the other hand one has:

(7.7) $E\left[\left|\int_0^{t_0\wedge T}(\sigma(X_u)-\sigma(Y_u)).dB_u\right|^2\right]=\sum_{i=1}^{d} E\left[\left(\sum_{j=1}^{n}\int_0^{t_0\wedge T}(\sigma_{i,j}(X_u)-\sigma_{i,j}(Y_u))dB_u^j\right)^2\right]$

$$=\sum_{i=1}^{d} E\left[\sum_{1\le j,k\le n}\int_0^{t_0\wedge T}(\sigma_{i,j}(X_u)-\sigma_{i,j}(Y_u))dB_u^j\int_0^{t_0\wedge T}(\sigma_{i,k}(X_u)-\sigma_{i,k}(Y_u))dB_u^k\right]$$

$$\overset{(5.90)}{=}\sum_{i=1}^{d}\sum_{1\le j,k\le n} E\left[\int_0^{t_0\wedge T}(\sigma_{i,j}(X_u)-\sigma_{i,j}(Y_u))(\sigma_{i,k}(X_u)-\sigma_{i,k}(Y_u))\, \underbrace{d\langle B^j,B^k\rangle_u}_{=\delta_{j,k}du}\right]$$

$$=\sum_{i=1}^{d}\sum_{j=1}^{n} E\left[\int_0^{t_0\wedge T}(\sigma_{i,j}(X_u)-\sigma_{i,j}(Y_u))^2du\right]$$

$$=E\left[\int_0^{t_0\wedge T}|\sigma(X_u)-\sigma(Y_u)|^2du\right].$$

Inserting (7.7) in the right-hand side of (7.6), and taking (7.3) into account we find that for any $t_0\ge 0$:

(7.8) $E\left[\sup_{s\le t_0}|X_{s\wedge T}-Y_{s\wedge T}|^2\right]\le (8K^2+2t_0K^2)\int_0^{t_0}E\left[|X_{u\wedge T}-Y_{u\wedge T}|^2\right]du$

The next result will be helpful.

Lemma: (Gronwall's lemma)

Let f be a non-negative integrable function on $[0,t]$ such that for some $a,b\ge 0$, and all $0\le u\le t$:

$f(u)\le a+b\int_0^u f(s)ds$,

then

(7.9) $f(u)\le a\,e^{bu}$, for all $0\le u\le t$.

Proof:

Iterating the inequality satisfied by f, we see that for $0\le u\le t$,

$f(u)\le a+b\int_0^u f(s)ds\le a+bau+b^2\int_0^u ds_1\int_0^{s_1}f(s_2)ds_2\le\dots$

$$\le a+bau+\frac{1}{2}b^2au^2+\dots+\frac{1}{n!}b^nau^n+b^{n+1}\int_0^u ds_1\int_0^{s_1}ds_2\dots\int_0^{s_n}f(s_{n+1})ds_{n+1}$$

$$\le a e^{bu}+b^{n+1}\int_0^u\frac{(u-s)^n}{n!}f(s)ds\le a e^{bu}+b^{n+1}\frac{u^n}{n!}\int_0^u f(s)ds.$$

Letting $n\to\infty$, we find (7.9). □

We will now apply the above lemma with the choice

$f(u) = E[\sup_{s \le u} |X_{s\wedge T} - Y_{s\wedge T}|^2]$ $(\ge E[|X_{u\wedge T} - Y_{u\wedge T}|^2])$, $0 \le u \le t$, $t>0$, $a = 0$, $b = K^2(8+2t)$, and find that (with t some positive number)

(7.10) $f(u) = 0$, for $0 \le u \le t$.

Letting M in (7.5) tend to infinity, and then $t \to \infty$,

(7.11) P-a.s., $X_u = Y_u$, for all $u \ge 0$,

(such a statement is called a <u>strong uniqueness</u> result).

— <u>Existence:</u>

We iteratively define for $m \ge 0$, $t \ge 0$,

(7.12)
$$\begin{cases} X^0_t \equiv x, \\ X^1_t = x + \int_0^t b(X^0_s)ds + \int_0^t \sigma(X^0_s).dB_s \\ \quad \vdots \\ X^{m+1}_t = x + \int_0^t b(X^m_s)ds + \int_0^t \sigma(X^m_s).dB_s \end{cases}$$

Then for $m \ge 1$:

(7.13) $X^{m+1}_t - X^m_t = \int_0^t b(X^m_s) - b(X^{m+1}_s)\, ds + \int_0^t (\sigma(X^m_s) - \sigma(X^{m-1}_s)).dB_s$, for $t \ge 0$,

If we now pick $M > |x|$, and define

$T_M = \inf\{u \ge 0;\ |X^m_u| \text{ or } |X^{m+1}_u| \ge M\}$,

the same calculation as for (7.8) yields that for $0 \le t_0 \le t$

(7.14) $E[\sup_{s \le t_0 \wedge T_M} |X^{m+1}_s - X^m_s|^2] \le (8+2t)K^2 \int_0^{t_0} E[|X^m_{u\wedge T_M} - X^{m-1}_{u\wedge T_M}|^2]du$.

Now $\sup_{s\le t} |X^0_s| = |x|$, $\sup_{s \le t} |X^1_s| \in L^2(P)$, by (7.12).
We now see from (7.14), with $m=1$, letting $M \to \infty$, that $\sup_{s \le t} |X^2_s| \in L^2(P)$, and repeating the argument that

(7.15) $\sup_{s \le t} |X^m_s| \in L^2(P)$, for any $m \ge 0$, and $t \ge 0$.

Coming back to (7.14) we can thus let $M \to \infty$ and find that for $0 \le t_0 \le t$:

$$(7.16)\quad E\Big[\sup_{s\le t_0}|X_s^{m+1}-X_s^m|^2\Big]\le K^2(8+2t)\int_0^{t_0}E[|X_u^m-X_u^{m-1}|^2]\,du$$

and iterating

$$\le \{K^2(8+2t)\}^m\int_0^{t_0}dt_1\int_0^{t_1}dt_2\dots\int_0^{t_{m-1}}dt_m\,E[|X^1_{t_m}-X^0_{t_m}|^2].$$

With (7.12) we also have:

$$(7.17)\quad E[|X^1_{t_m}-X^0_{t_m}|^2]\le 2\,|b(x)|^2t_m^2+2E[|\sigma(x).B_{t_m}|^2]$$
$$\le K'(x,t)\,t_m,\ \text{for } 0\le t_m\le t.$$

Hence we (7.16) we obtain that

$$(7.18)\quad E\Big[\sup_{s\le t}|X_s^{m+1}-X_s^m|^2\Big]\le K'(x,t)\,\frac{(8K^2+2tK^2)^m}{(m+1)!}\,t^{m+1},\ \text{for } t>0,\ m\ge 0.$$

We have thus proved that for $t>0$,

$$(7.19)\quad \sum_{m\ge 0}E\Big[\sup_{s\le t}|X_s^{m+1}-X_s^m|^2\Big]^{1/2}<\infty.$$

As a consequence P-a.s., $X^m_.$ converges uniformly on bounded time intervals to $X^\infty_.$, which can be chosen $(\mathcal{G}_t)$-adapted continuous, (see for instance (4.75)), and

$$(7.20)\quad E\Big[\sup_{s\le t}|X_s^\infty-X_s^m|^2\Big]^{1/2}\overset{\text{Fatou}}{\le}\liminf_{p\to\infty}E\Big[\sup_{s\le t}|X_s^p-X_s^m|^2\Big]^{1/2}\le$$

$$\sum_{k=m}^\infty E\Big[\sup_{s\le t}|X_s^{k+1}-X_s^k|^2\Big]^{1/2}\xrightarrow[m\to\infty]{}0,\ \text{for } t>0.$$

Hence we see that for $t>0$, P-a.s.

$$\begin{array}{ccccc}
X_t^{m+1} & = x+ & \int_0^t b(X_u^m)\,du & + & \int_0^t\sigma(X_u^m).dB_u\\
L^2\ \big\downarrow m\to\infty & & m\to\infty\big\downarrow L^2 & & m\to\infty\big\downarrow L^2\\
X_t^\infty & = x+ & \int_0^t b(X_u^\infty)\,du & + & \int_0^t\sigma(X_u^\infty).dB_u,
\end{array}$$

and in view of the continuity of $X^\infty_.$, we see that P-a.s.

$$(7.21)\quad X_t^\infty=x+\int_0^t b(X_u^\infty)\,du+\int_0^t\sigma(X_u^\infty).dB_u,\ \text{for } u\ge 0.$$

Hence $X^\infty_.$ is a solution of (7.4). □

<u>Remark:</u>

From the definition of the $X^m_.$ in (7.12), and the fact that

P-a.s., $X^m_\cdot$ converges uniformly to $X^\infty_\cdot$ on compact time intervals, we see that for each $t \geq 0$:

(7.22) $F_t^{X^\infty_\cdot} \overset{\text{def}}{=}$ the smallest σ-algebra containing all negligible sets of $\mathcal{G}$ and making X^∞_s measurable for $s \leq t$
$\subseteq F_t^{B_\cdot}$, (defined analogously, with $B_\cdot$ in place of $X^\infty_\cdot$).

Due to (7.22), $X^\infty_\cdot$ is called a <u>strong solution</u> of (7.4), (intuitively $X^\infty_\cdot$ is a function of the noise $B_\cdot$).
The above theorem shows that for any $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$, $(B_t)_{t\geq 0}$, we have a strong solution of (7.4), which is strongly unique, cf. (7.11).

□

We will now see that <u>solutions of stochastic differential equations (SDE's)</u> can be used to <u>represent solutions of</u> certain partial differential equations (<u>PDE's</u>).

We begin with a result that will also be helpful in the subsequent discussion of martingale problems.

<u>Proposition:</u>

Assume that $b(\cdot): \mathbb{R}^d \to \mathbb{R}^d$, $\sigma(\cdot): \mathbb{R}^d \to M_{d\times n}$ are measurable locally bounded functions, $x \in \mathbb{R}^d$, and on some $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{t\geq 0}, P)$ endowed with an n-dimensional $(\mathcal{G}_t)$-Brownian motion $(B_t)_{t\geq 0}$, a continuous adapted $\mathbb{R}^d$-valued, $(X_t)_{t\geq 0}$, satisfies P-a.s., for $t \geq 0$:

(7.23) $X_t = x + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\cdot dB_s$,

then for any $f \in C^2(\mathbb{R}^d, \mathbb{R})$,

(7.24) $M_t^f \overset{\text{def}}{=} f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds$, $t \geq 0$, is a continuous local martingale

where we used the notation

(7.25) $Lf(y) = \frac{1}{2}\sum_{1\leq i,j\leq d} a_{ij}(y)\,\partial^2_{ij} f(y) + \sum_{1\leq i\leq d} b_i(y)\,\partial_i f(y)$, $a(y) = \underbrace{\sigma(y)\,{}^t\sigma(y)}_{\in M_{d\times d}}$, for $y \in \mathbb{R}^d$.

Proof:

We apply Ito's formula and find that P-a.s., for $t \geq 0$:

$$(7.26)\quad f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t \partial_i f(X_s)\, dX_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \partial^2_{i,j}(X_s)\, d\langle X^i, X^j\rangle_s.$$

Note that from (5.90) and (7.23) follows

$$(7.27)\quad \langle X^i, X^j\rangle_t = \Big\langle \sum_{k=1}^{n} \int_0^t \sigma_{i,k}(X_u)\, dB_u^k, \sum_{\ell=1}^{n} \int_0^t \sigma_{j,\ell}(X_u).\, dB_u^\ell \Big\rangle$$

$$= \sum_{k=1}^{n} \int_0^t \sigma_{i,k}(X_u)\, \sigma_{j,k}(X_u)\, du = \int_0^t a_{i,j}(X_u)\, du.$$

Hence coming back to (7.26), we find that P-a.s., for $t \geq 0$,

$$(7.28)\quad f(X_t) = f(X_0) + \sum_{i=1}^{d} \int_0^t \partial_i f(X_s)\, b_i(X_s)\, ds + \sum_{i=1}^{d} \sum_{k=1}^{n} \int_0^t \partial_i f(X_s)\, \sigma_{i,k}(X_s)\, dB_s^k$$

$$+ \frac{1}{2} \sum_{1 \leq i,j \leq d} \int_0^t a_{ij}(X_s)\, \partial^2_{ij} f(X_s)\, ds$$

$$= f(X_0) + \int_0^t Lf(X_s)\, ds + \int_0^t \nabla f(X_s) \underset{\substack{\uparrow \\ \text{scalar product}}}{.} \underbrace{\sigma(X_s).\, dB_s}_{\text{d-vector}}.$$

The claim (7.24) now follows. $\square$